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Research article

On the time decay for a thermoelastic laminated beam with microtemperature effects, nonlinear weight, and nonlinear time-varying delay

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Abstract: This article examines the joint impacts of microtemperature, nonlinear structural damping, along with nonlinear time-varying delay term, and time-varying coefficient on a thermoelastic laminated beam, where, the equation representing the dynamics of slip is affected by the last three mentioned terms. A general decay result was established regarding the system concerned given equal wave speeds and particular assumptions related to nonlinear terms.

Keywords: laminated beam; nonlinear damping; microtemperature effects; general decay; nonlinear weight; time-varying delay
Mathematics Subject Classification: 35B40, 35L56, 74F05, 93D15, 93D20

1. Introduction

This study focuses on examining the following thermoelastic laminated beam along with microtemperature effects, nonlinear structural damping, nonlinear time-varying delay, and time-varying coefficients:

$$\begin{cases} \varrho\psi_{tt} + G(u - \psi_x)_x = 0, \\ I_\varrho(3\phi - u)_{tt} - D(3\phi - u)_{xx} - G(u - \psi_x) = 0, \\ 3I_\varrho\phi_{tt} - 3D\phi_{xx} + 3G(u - \psi_x) + \gamma\theta_x + dr_x + 4\delta\phi \\ \quad + \beta\mathfrak{b}(t)\mathfrak{h}_1(\phi_t(x,t)) + \mu\mathfrak{b}(t)\mathfrak{h}_2(\phi_t(x, t - \varsigma(t))) = 0, \\ c\theta_t - k_0\theta_{xx} + \gamma\phi_{tx} + k_1 r_x = 0, \\ \alpha r_t - k_2 r_{xx} + k_3 r + d\phi_{tx} + k_1\theta_x = 0, \end{cases} \tag{1.1}$$

where

$$x \in (0, 1), t \in (0, \infty).$$

System Eq (1.1) rests on the below listed initial and boundary conditions:

$$\begin{cases} \psi(x,0) = \psi_0,\ \phi(x,0) = \phi_0,\ u(x,0) = u_0,\ \theta(x,0) = \theta_0,\ r(x,0) = r_0,\ x \in (0,1), \\ \psi_t(x,0) = \psi_1,\ \phi_t(x,0) = \phi_1,\ u_t(x,0) = u_1, \quad x \in (0,1), \\ \psi_x(0,t) = \phi(0,t) = u(0,t) = \theta(0,t) = r(0,t) = 0, \quad t > 0, \\ \phi_x(1,t) = u_x(1,t) = \psi(1,t) = \theta(1,t) = r(1,t) = 0, \quad t > 0, \\ \phi_t(x, t - \varsigma(t)) = f_0(x, t - \varsigma(0)), \quad (x,t) \in (0,1) \times (0, \varsigma(0)). \end{cases} \tag{1.2}$$

Here, ψ, u, ϕ, θ, and r stand for the transverse displacement, the rotation angle, the amount of slip along the interface, the difference temperature, and the microtemperature vector, respectively. The coefficients β, δ, ϱ, I_ϱ, G, and D, are positive and represent the adhesive damping weight, the adhesive stiffness, the density, the shear stiffness, the flexural rigidity, and the mass moment of inertia, respectively. We denote by the positive constants c, k_0, k_1, k_2, k_3, d γ and α, the physical parameters describing the coupling between the various constituents of the materials.

Herein $\varsigma(t) > 0$ is the time-varying delay and μ denotes a positive damping constant, while the function $\mathfrak{b}(t)$ stands for the nonlinear weight.

Structural beams play a crucial role in numerous engineering applications, as some machines are relying on a multitude of them, making them indispensable. As they need to withstand diverse challenges, and adapt to various scenarios, these beams have evolved into a sophisticated technology, embodying cutting-edge engineering concepts. Researchers have proposed various theories to explain their behavior, including the popular Euler-Bernoulli beam theory and the Timoshenko beam theory, which excels in dealing with thick beams under the influence of shear forces and rotatory inertia.

Frequently, the unwanted vibrations of these beams are caused by internal or external forces, which compel scientists to find efficient ways to rapidly mitigate these vibrations. To achieve this objective, numerous types of dampers have been developed.

Time delays can result in lags among input and output processing as well as in achieving or restoring the stability of the coveted system, after internal or external perturbations. The presence of these lags is due to the nature of transportation and processing of information of control systems. Delay differential equations are the most efficient method for explicitly analyzing the impact of delays on stability in control systems. Even though including delays may support system control in some cases, as indicated in [1], researches suggest that delays can also cause instability and degrade the system efficiency. Regarding the time-varying delay along with nonlinear weight, we should invoke the research of

Mukiawa et al. [2], in which a thermoelastic Timoshenko beam with suspenders together with time-varying delay and nonlinear weight was considered, and a general stability result was demonstrated, with convenient assumptions regarding incorporated nonlinear terms.

When it comes to boundary stabilization study, Wang et al. were the pioneers in providing results. They demonstrated an exponential decay result for a laminated beams with structural damping, mixed homogeneous boundary conditions, and unequal wave speeds in their study [3]. Later on, Tatar enhanced upon the work of [3] in [4] by also proving a similar exponential decay result, but supposing that $\varrho G < I_\varrho$.

In the matter of microtemperature effects, we bring up the study of Khochemane [5], where he investigated a theromelastic porous problem, together with microtemperature effects. When the thermal conductivity equals zero, he managed to establish that the dissipation due solely to microtemperature is adequate to stabilize the system exponentially, regardless of the system's wave velocities, and any possible assumption concerning the coefficients.

Newly, a thermoelastic laminated beam along with structural damping was examined by Fayssal in [6] and he came to the conclusion that an exponential stability result is achievable if

$$\frac{\varrho}{G} = \frac{I_\varrho}{D}. \tag{1.3}$$

The coupled system we've described involves several complex physical phenomena, including thermoelasticity, laminated beams, microtemperature effects, nonlinear structural damping, and nonlinear time-varying delay. Let's break down each component and its physical background:

Thermoelastic laminated beam: A laminated beam consists of multiple layers of different materials bonded together. Thermoelasticity refers to the combined behavior of thermal and elastic effects in a material. When the beam is subjected to temperature changes or thermal gradients, it experiences thermal expansion/contraction, which induces mechanical stresses and deformations due to the elastic properties of the materials.

Microtemperature effects: This refers to the consideration of temperature variations at a very small scale, such as at the microstructural level of the materials. At this scale, temperature gradients can lead to localized effects, such as material phase changes, microstructural alterations, or thermal stresses, which can influence the overall behavior of the coupled system.

Nonlinear structural damping: Damping is a phenomenon that dissipates energy from a vibrating system. Nonlinear damping implies that the damping force is not linearly proportional to the velocity of the system. This can occur due to various reasons, such as material hysteresis, contact friction, or fluid-structure interactions. Nonlinear damping can significantly affect the dynamic response of the system.

Nonlinear time-varying delay: A time delay occurs when an effect is not instantaneous and takes some time to propagate through a system. Nonlinear and time-varying delays mean that the delay itself changes based on the current state of the system, and this delay may also have nonlinear effects on the overall behavior. Time delays can lead to instability, oscillations, or even chaos in dynamic systems. The physical background of this coupled system involves the intricate interplay of these phenomena. It requires a sophisticated mathematical and computational approach to model and analyze the system's behavior accurately. Researchers and engineers studying such systems aim to understand how these factors interact and influence each other to predict the system's response to different inputs, boundary conditions, and environmental changes. Such analyses are crucial in various fields, including material science, structural engineering, and advanced manufacturing, where a deep understanding of complex coupled systems is essential for designing reliable and efficient systems.

The remnant of the article is arranged in the following manner: In Section 2, we give necessary assumptions and resources for our study, then bring out our major results. In Section 3, we present useful lemmas, which are indispensable later in the proof. In Section 4, we establish, by means of the energy approach our coveted stability results.

2. Preliminaries

This section is devoted to revealing our major results and setting the necessary assumptions supporting the proof later.

Similar to [7–9], we set the ensuing assumptions:

- ($\mathbf{A}_1$) The function $\mathfrak{h}_1 : \mathbb{R} \to \mathbb{R}$ is increasing and of class C^0, moreover, there exist constants $\lambda_1,\ \lambda_2,\ \varepsilon > 0$, and a convex increasing function $T \in C^1([0,+\infty)) \cap C^2((0,+\infty))$, fulfilling $T(0) = 0$, or the latter is nonlinear strictly convex of class C^2 on $(0,\varepsilon]$, and $T'', T'(0) > 0$, in a way that we have
$$\begin{cases} z^2 + \mathfrak{h}_1^2(z) \le T^{-1}(z\mathfrak{h}_1(z)), & |z| \le \varepsilon, \\ \lambda_1 z^2 \le z\mathfrak{h}_1(z) \le \lambda_2 z^2, & |z| \ge \varepsilon. \end{cases} \tag{2.1}$$
The function $\mathfrak{h}_2 : \mathbb{R} \to \mathbb{R}$ is odd and increasing, with $\mathfrak{h}_2 \in C^1(\mathbb{R})$, in addition, there exist $\vartheta_1,\ \vartheta_2,\ \lambda_3 > 0$, such that
$$z\vartheta_1\mathfrak{h}_2(z) \le \xi(z) \le z\vartheta_2\mathfrak{h}_1(z), \tag{2.2}$$
$$|\mathfrak{h}_2'(z)| \le \lambda_3, \tag{2.3}$$
where
$$\xi(z) = \int_0^z \mathfrak{h}_2(y)dy.$$
This function and conditions are found in many related papers, for example see reference [10], page 1521.
- ($\mathbf{A}_2$) The function $\mathfrak{b} : [0,+\infty) \to (0,+\infty)$ is decreasing, and of class C^1, furthermore
$$\begin{cases} |\mathfrak{b}'(t)| \le \Gamma\mathfrak{b}(t), & \Gamma > 0, \\ \int_0^\infty \mathfrak{b}(t)dt = +\infty. \end{cases} \tag{2.4}$$
- ($\mathbf{A}_3$) The time-varying delay fulfils
$$0 < \varsigma_0 \le \varsigma(t) \le \varsigma_1, \quad \varsigma_0,\ \varsigma_1 > 0,\ \forall t > 0, \tag{2.5}$$
$$\varsigma'(t) \le d_0 < 1,\ d_0 > 0,\ \forall t > 0, \tag{2.6}$$
$$\varsigma \in W^{2,\infty}(0,S),\ \forall S > 0. \tag{2.7}$$
- ($\mathbf{A}_4$) Regarding coefficients β, μ, they satisfy
$$\vartheta_2\mu(1 - d_0\vartheta_1) < (1 - d_0)\vartheta_1\beta. \tag{2.8}$$

Remark 2.1. *The mean value theorem for integrals, together with the monotonicity of* $\mathfrak{h}_2$, *provides us with*

$$\xi(z) \leq z\mathfrak{h}_2(z), \tag{2.9}$$

and estimate Eq (2.2) leads us to announce that

$$\vartheta_1 < 1.$$

Following the lead of [11], we shall begin by introducing

$$\mathcal{Y}(x, p, t) = \phi_t(x, t - p\varsigma(t)) \quad \text{in } (0, 1) \times (0, 1) \times (0, \infty). \tag{2.10}$$

Thereby, $\mathcal{Y}$ certainly fulfills

$$\varsigma(t)\mathcal{Y}_t(x, p, t) + (1 - p\varsigma'(t))\mathcal{Y}_p(x, p, t) = 0. \tag{2.11}$$

Then, we are capable of rewriting system Eq (1.1) as

$$\begin{cases} \varrho\psi_{tt} + G(u - \psi_x)_x = 0, \\ I_\varrho(3\phi - u)_{tt} - D(3\phi - u)_{xx} - G(u - \psi_x) = 0, \\ 3I_\varrho\phi_{tt} - 3D\phi_{xx} + 3G(u - \psi_x) + \gamma\theta_x + dr_x + 4\delta\phi \\ \quad + \beta\mathfrak{b}(t)\mathfrak{h}_1(\phi_t(x, t)) + \mu\mathfrak{b}(t)\mathfrak{h}_2(\mathcal{Y}(x, 1, t)) = 0, \\ c\theta_t - k_0\theta_{xx} + \gamma\phi_{tx} + k_1 r_x = 0, \\ \alpha r_t - k_2 r_{xx} + k_3 r + d\phi_{tx} + k_1\theta_x = 0, \\ \varsigma(t)\mathcal{Y}_t(x, p, t) + (1 - p\varsigma'(t))\mathcal{Y}_p(x, p, t) = 0. \end{cases} \tag{2.12}$$

Surely, system Eq (2.12) depends on the below listed initial and boundary conditions:

$$\begin{cases} \psi(x, 0) = \psi_0,\ \phi(x, 0) = \phi_0,\ u(x, 0) = u_0,\ \theta(x, 0) = \theta_0, r(x, 0) = r_0,\ x \in (0, 1), \\ \psi_t(x, 0) = \psi_1,\ \phi_t(x, 0) = \phi_1,\ u_t(x, 0) = u_1, \quad x \in (0, 1), \\ \psi_x(0, t) = \phi(0, t) = u(0, t) = \theta(0, t) = r(0, t) = 0, \quad t > 0, \\ \phi_x(1, t) = u_x(1, t) = \psi(1, t) = \theta(1, t) = r(1, t) = 0, \quad t > 0, \\ \mathcal{Y}(x, 0, t) = \phi_t(x, t), \quad \mathcal{Y}(x, p, 0) = f_0(x, -\varsigma(0)p), \quad (x, p) \in ((0, 1))^2,\ t > 0. \end{cases} \tag{2.13}$$

Demonstrating the existence and uniqueness result is attainable, if we pursue the Faedo-Galerkin approach, as elucidated in [12].

To address problem Eq (2.12) properly, we shall consider the following positive constant:

$$\frac{\mu(1 - \vartheta_1)}{(1 - d_0)\vartheta_1} < \tilde{\nu} < \frac{\beta - \vartheta_2\mu}{\vartheta_2}, \tag{2.14}$$

along with

$$\nu(t) = \tilde{\nu}\mathfrak{b}(t),$$

furthermore, $\mathcal{Y}(p)$ will serve to denote $\mathcal{Y}(x, p, t)$.

We present the energy of the concerned system Eqs (2.12) and (2.13) by

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\int_0^1 \left\{\varrho\psi_t^2 + I_\varrho(3\phi_t - u_t)^2 + D(3\phi_x - u_x)^2 + 3I_\varrho\phi_t^2 + 3D\phi_x^2\right\}\, dx \\ &+ \frac{1}{2}\int_0^1 \left\{G(u-\psi_x)^2 + 4\delta\phi^2 + c\theta^2 + \alpha r^2\right\}\, dx \\ &+ \varsigma(t)\nu(t)\int_0^1\int_0^1 \xi(\mathcal{Y}(x,p,t))\, dpdx.\end{aligned} \tag{2.15}$$

We then give the ensuing stability result.

Theorem 2.1. *Let $(\psi, u, \phi, \theta, r, \mathcal{Y})$ be the solution of Eqs (2.12) and (2.13), and let ($\boldsymbol{A}_1$)-($\boldsymbol{A}_4$), and Eq (1.3) hold. Then, there exist positive constants $\varkappa_0,\ \varkappa_1,\ \varkappa_2$, and ε_0 such that*

$$\mathcal{E}(t) \le \varkappa_0 T_2^{-1}\left(\varkappa_1\int_0^t \mathfrak{b}(z)dz + \varkappa_2\right), \quad t \ge 0, \tag{2.16}$$

where

$$\begin{cases} T_2(t) = \int_t^1 \frac{1}{T_1(z)}dz, \\ T_1(t) = tT'(\varepsilon_0 t). \end{cases}$$

As numerous examples related to the already defined assumptions and our stability result were explored in earlier works, the reader may reference [2] for more information.

For more details, the existence and uniqueness of the solution of our problem can be established by continuing the arguments of the Faedo-Galerkin method as in reference [12].

3. Technical lemmas

Establishing the practical lemmas necessary to support our stability results proof is the primary goal of this section. We use a particular method known as the multiplier technique, the latter allows us to demonstrate the stability result of problem Eq (2.12). To make matters simpler, we will utilize $\chi,\ \Upsilon_* > 0$ to symbolize a generic constants that may vary from one line to another (including within the same line).

Lemma 3.1. *Consider $(\psi, u, \phi, \theta, r, \mathcal{Y})$ the solution of Eqs (2.12) and (2.13), then, the energy functional satisfies*

$$\begin{aligned}\mathcal{E}'(t) \le& -\mathfrak{b}(t)(\tilde{\nu}(1-\varsigma'(t))\vartheta_1 - \mu(1-\vartheta_1))\int_0^1 \mathcal{Y}(1)\mathfrak{h}_2(\mathcal{Y}(1))\, dx \\ &- k_0\int_0^1 \theta_x^2\, dx - k_2\int_0^1 r_x^2 dx - k_3\int_0^1 r^2 dx \\ &- \mathfrak{b}(t)(\beta - \tilde{\nu}\vartheta_2 - \mu\vartheta_2)\int_0^1 \phi_t\mathfrak{h}_1(\phi_t)\, dx \le 0, \quad \forall t \ge 0.\end{aligned} \tag{3.1}$$

Proof. To start with, we multiply the first five equations of system Eq (2.12) by ψ_t, $(3\phi_t - u_t)$, ϕ_t, θ, and r, respectively. After that, we integrate over $(0,1)$ and employ integration by parts along with boundary conditions Eq (2.13), to establish

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{\varrho\psi_t^2 + I_\varrho(3\phi_t - u_t)^2 + D(3\phi_x - u_x)^2 + 3I_\varrho\phi_t^2 + 3D\phi_x^2 + 4\delta\phi^2\right\}\, dx \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 \left\{G(u-\psi_x)^2 + c\theta^2 + \alpha r^2\right\}\, dx \\
&= -k_0\int_0^1 \theta_x^2\, dx - k_2\int_0^1 r_x^2 dx - k_3\int_0^1 r^2 dx - \beta\mathfrak{b}(t)\int_0^1 \phi_t\mathfrak{h}_1(\phi_t)\, dx - \mu\mathfrak{b}(t)\int_0^1 \phi_t\mathfrak{h}_2(\mathcal{Y}(1))\, dx.
\end{aligned} \tag{3.2}$$

Then, we need to multiply Eq $(2.12)_6$ by $\nu(t)\mathfrak{h}_2(\mathcal{Y}(p))$, and integrate over $(0,1)\times(0,1)$, to achieve

$$\nu(t)\varsigma(t)\int_0^1\int_0^1 \mathfrak{h}_2(\mathcal{Y}(p))\mathcal{Y}_t(p)\, dpdx = -\nu(t)\int_0^1\int_0^1 (1-p\varsigma'(t))\partial_p\xi(\mathcal{Y}(p))dpdx. \tag{3.3}$$

Thereby,

$$\begin{aligned}
\frac{d}{dt}\left[\nu(t)\varsigma(t)\int_0^1\int_0^1 \xi\left(\mathcal{Y}(p)\right) dpdx\right] &= -\nu(t)\int_0^1\int_0^1 \partial_p\left((1-p\varsigma'(t))\xi(\mathcal{Y}(p))\right) dpdx \\
&\quad + \nu'(t)\varsigma(t)\int_0^1\int_0^1 \xi(\mathcal{Y}(p))dpdx \\
&= \nu(t)\int_0^1 [\xi(\mathcal{Y}(0)) - \xi(\mathcal{Y}(1))]dx \\
&\quad + \varsigma'(t)\nu(t)\int_0^1 \xi(\mathcal{Y}(1))dx \\
&\quad + \varsigma(t)\nu'(t)\int_0^1\int_0^1 \xi(\mathcal{Y}(p))dpdx \\
&= -\nu(t)(1-\varsigma'(t))\int_0^1 \xi(\mathcal{Y}(1))dx \\
&\quad + \varsigma(t)\nu'(t)\int_0^1\int_0^1 \xi(\mathcal{Y}(p))dpdx \\
&\quad + \nu(t)\int_0^1 \xi(\phi_t)dx,
\end{aligned}$$

which accompanied with Eq (3.2), $(\mathbf{A}_2)$ and Eq (2.2), leads to

$$\begin{aligned}
\mathcal{E}'(t) \le &-k_0\int_0^1 \theta_x^2\, dx - k_2\int_0^1 r_x^2 dx - k_3\int_0^1 r^2 dx - \mu\mathfrak{b}(t)\int_0^1 \phi_t\mathfrak{h}_2(\mathcal{Y}(1))dx \\
&- (\beta\mathfrak{b}(t) - \vartheta_2\nu(t))\int_0^1 \phi_t\mathfrak{h}_1(\phi_t)dx - \nu(t)(1-\varsigma'(t))\int_0^1 \xi(\mathcal{Y}(1))dx.
\end{aligned} \tag{3.4}$$

We shall now define the convex conjugate function of ξ,

$$\xi^*(z) = z(\xi')^{-1}(z) - \xi\left[(\xi')^{-1}(z)\right], \quad \forall z \ge 0. \tag{3.5}$$

This makes, the relation listed below valid by means of the general Young's inequality (see [13, 14]):

$$zv \leq \xi^*(z) + \xi(v), \quad \forall z, v \geq 0. \tag{3.6}$$

We employ the definition of ξ as well as Eq (2.3), to obtain

$$\xi^*(z) = z\mathfrak{h}_2^{-1}(z) - \xi\left(\mathfrak{h}_2^{-1}(z)\right), \quad \forall z \geq 0, \tag{3.7}$$

and the simple combination of Eqs (3.7) and (2.2) results in

$$\begin{aligned}\xi^*(\mathfrak{h}_2(\mathcal{Y}(1))) =& \mathcal{Y}(1)\mathfrak{h}_2(\mathcal{Y}(1)) - \xi(\mathcal{Y}(1)) \\ \leq& (1-\vartheta_1)\mathcal{Y}(1)\mathfrak{h}_2(\mathcal{Y}(1)).\end{aligned} \tag{3.8}$$

Then, we benefit of Eqs (3.4), (3.6) and (3.8), to be in position to write

$$\begin{aligned}\mathcal{E}'(t) \leq& -k_0\int_0^1 \theta_x^2\, dx - k_2\int_0^1 r_x^2 dx - k_3\int_0^1 r^2 dx \\ &- (\beta\mathfrak{b}(t) - \vartheta_2\nu(t) - \vartheta_2\mu\mathfrak{b}(t))\int_0^1 \phi_t\mathfrak{h}_1(\phi_t)dx \\ &- (\nu(t)(1-\varsigma'(t))\vartheta_1 - \mu\mathfrak{b}(t)(1-\vartheta_1))\int_0^1 \mathcal{Y}(1)\mathfrak{h}_2(\mathcal{Y}(1))dx \\ \leq& -k_0\int_0^1 \theta_x^2\, dx - k_2\int_0^1 r_x^2 dx - k_3\int_0^1 r^2 dx \\ &- \mathfrak{b}(t)(\beta - \vartheta_2(\tilde{\nu}+\mu))\int_0^1 \phi_t\mathfrak{h}_1(\phi_t)dx \\ &- \mathfrak{b}(t)(\tilde{\nu}\vartheta_1(1-\varsigma'(t)) - \mu(1-\vartheta_1))\int_0^1 \mathcal{Y}(1)\mathfrak{h}_2(\mathcal{Y}(1))dx.\end{aligned} \tag{3.9}$$

We finally prove estimate Eq (3.1), with the aid of Eqs (2.14) and (2.6). □

Lemma 3.2. *Consider the functional*

$$\mathcal{I}_1(t) := -\varrho D\int_0^1 \psi_t(3\phi_x - u_x)dx + 3I_\varrho G\int_0^1 \phi_t(3\phi - u)dx - I_\varrho G\int_0^1 \psi_x(3\phi_t - u_t)dx. \tag{3.10}$$

Then, it satisfies

$$\begin{aligned}\mathcal{I}_1'(t) \leq& -\frac{GD}{2}\int_0^1 (3\phi_x - u_x)^2 dx + \epsilon_1\int_0^1 (3\phi_t - u_t)^2 dx + \Upsilon_*\int_0^1 \phi_x^2 dx \\ &+ \frac{\Upsilon_*}{\epsilon_1}\int_0^1 \phi_t^2 dx + \Upsilon_*\int_0^1 (u-\psi_x)^2 dx + \Upsilon_*\int_0^1 \theta_x^2 dx + \Upsilon_*\int_0^1 r_x^2 dx \\ &+ \Upsilon_*\int_0^1 |\mathfrak{h}_1(\phi_t)|^2 dx + \Upsilon_*\int_0^1 |\mathfrak{h}_2(\mathcal{Y}(1))|^2 dx, \quad \textit{for any } \epsilon_1 > 0.\end{aligned} \tag{3.11}$$

Proof. Here, we directly differentiate $\mathcal{I}_1$. After that we take advantage of Eqs $(2.12)_{1,2,3}$, integrate by parts, and $\psi_x = -(u-\psi_x)+u$, to find

$$\begin{aligned}\mathcal{I}_1'(t) =& -GD\int_0^1 (3\phi_x-u_x)^2dx + 3I_\varrho G\int_0^1 \phi_t(3\phi_t-u_t)dx - 3G^2\int_0^1 (u-\psi_x)(3\phi-u)dx\\ &- 4\delta G\int_0^1 \phi(3\phi-u)dx - \gamma G\int_0^1 \theta_x(3\phi-u)dx - dG\int_0^1 r_x(3\phi-u)dx\\ &-\beta G\mathfrak{b}(t)\int_0^1 (3\phi-u)\mathfrak{h}_1(\phi_t)dx - \mu G\mathfrak{b}(t)\int_0^1 (3\phi-u)\mathfrak{h}_2\left(\mathcal{Y}(1)\right)dx\\ &- G^2\int_0^1 (u-\psi_x)\psi_x dx - (\varrho D - I_\varrho G)\int_0^1 (3\phi-u)_{xt}\psi_t dx.\end{aligned}$$

Once again, observing that $\psi_x = -(u-\psi_x)-(3\phi-u)+3\phi$ and maintaining Eq (1.3), gives

$$\begin{aligned}\mathcal{I}_1'(t) =& -GD\int_0^1 (3\phi_x-u_x)^2dx + 3I_\varrho G\int_0^1 \phi_t(3\phi_t-u_t)dx - 2G^2\int_0^1 (u-\psi_x)(3\phi-u)dx\\ &- 4\delta G\int_0^1 \phi(3\phi-u)dx - \gamma G\int_0^1 \theta_x(3\phi-u)dx - dG\int_0^1 r_x(3\phi-u)dx\\ &-\beta G\mathfrak{b}(t)\int_0^1 (3\phi-u)\mathfrak{h}_1(\phi_t)dx - \mu G\mathfrak{b}(t)\int_0^1 (3\phi-u)\mathfrak{h}_2\left(\mathcal{Y}(1)\right)dx\\ &+ G^2\int_0^1 (u-\psi_x)^2dx - 3G^2\int_0^1 \phi(u-\psi_x)dx.\end{aligned}\tag{3.12}$$

Since $(\mathbf{A}_2)$ implies that $\mathfrak{b}(t) \le \mathfrak{b}(0)$, estimate Eq (3.11) is established, if we consider Young and Poincaré's inequalities. □

Lemma 3.3. *Consider the functional*

$$\mathcal{I}_2(t) := -3\varrho D\int_0^1 \psi_t\phi_x dx + 3I_\varrho G\int_0^1 (u-\psi_x)\phi_t dx.$$

Then, it satisfies

$$\begin{aligned}\mathcal{I}_2'(t) \le& - G^2\int_0^1 (u-\psi_x)^2dx + \Upsilon_*\int_0^1 \phi_x^2dx + \Upsilon_*\int_0^1 \theta_x^2dx\\ &+ \epsilon_2\int_0^1 (3\phi_t-u_t)^2dx + \Upsilon_*\left(1+\frac{1}{\epsilon_2}\right)\int_0^1 \phi_t^2dx + \Upsilon_*\int_0^1 r_x^2dx\\ &+ \Upsilon_*\int_0^1 |\mathfrak{h}_1(\phi_t)|^2dx + +\Upsilon_*\int_0^1 \left|\mathfrak{h}_2\left(\mathcal{Y}(1)\right)\right|^2, \quad \textit{for any } \epsilon_2 > 0.\end{aligned}\tag{3.13}$$

Proof. Employing Eqs $(2.12)_1$ and $(2.12)_3$, together with integration by parts, shows that

$$\begin{aligned}\mathcal{I}_2'(t) =& -3G^2\int_0^1 (u-\psi_x)^2dx - 3(I_\varrho G-\varrho D)\int_0^1 \psi_{xt}\phi_t dx + 3I_\varrho G\int_0^1 u_t\phi_t dx\\ &- 4\delta G\int_0^1 \phi(u-\psi_x)dx - \gamma G\int_0^1 \theta_x(u-\psi_x)dx - dG\int_0^1 r_x(u-\psi_x)dx\\ &-\beta G\mathfrak{b}(t)\int_0^1 (u-\psi_x)\mathfrak{h}_1(\phi_t)dx - \mu G\mathfrak{b}(t)\int_0^1 (u-\psi_x)\mathfrak{h}_2\left(\mathcal{Y}(1)\right)dx.\end{aligned}$$

Then, by Eq (1.3) and $u_t = -(3\phi_t - u_t) + 3\phi_t$, we achieve

$$\begin{aligned}\mathcal{I}_2'(t) &= -3G^2\int_0^1 (u-\psi_x)^2dx + 9I_\varrho G\int_0^1 \phi_t^2dx - 3I_\varrho G\int_0^1 (3\phi_t - u_t)\phi_t dx\\ &\quad - 4\delta G\int_0^1 \phi(u-\psi_x)dx - \gamma G\int_0^1 \theta_x(u-\psi_x)dx - dG\int_0^1 r_x(u-\psi_x)dx\\ &\quad - \beta G\mathfrak{b}(t)\int_0^1 (u-\psi_x)\mathfrak{h}_1(\phi_t)dx - \mu G\mathfrak{b}(t)\int_0^1 (u-\psi_x)\mathfrak{h}_2\left(\mathcal{Y}(1)\right)dx,\end{aligned}$$

and by means of $(\mathbf{A}_2)$ and Young and Poincaré's inequalities, we terminate the proof of Eq (3.13). □

Lemma 3.4. *Consider the functional*

$$\mathcal{I}_3(t) := 3I_\varrho\int_0^1 \phi\phi_t\, dx - 3\varrho\int_0^1 \phi\int_0^x \psi_t(y)dydx. \tag{3.14}$$

Then, it satisfies

$$\begin{aligned}\mathcal{I}_3'(t) &\le -3D\int_0^1 \phi_x^2dx - \delta\int_0^1 \phi^2dx + \Upsilon_*\int_0^1 \theta_x^2dx + \epsilon_3\int_0^1 \psi_t^2dx\\ &\quad + \Upsilon_*\left(1+\frac{1}{\epsilon_3}\right)\int_0^1 \phi_t^2dx + \Upsilon_*\int_0^1 r_x^2dx\\ &\quad + \Upsilon_*\int_0^1 |\mathfrak{h}_1(\phi_t)|^2\,dx + \Upsilon_*\int_0^1 |\mathfrak{h}_2\left(\mathcal{Y}(1)\right)|^2\,dx, \quad \textit{for any } \epsilon_3 > 0.\end{aligned} \tag{3.15}$$

Proof. Employing Eqs $(2.12)_1$ and $(2.12)_3$, together with integration by parts, shows that

$$\begin{aligned}\mathcal{I}_3'(t) &= 3I_\varrho\int_0^1 \phi_t^2dx - 3D\int_0^1 \phi_x^2dx - 4\delta\int_0^1 \phi^2dx - \gamma\int_0^1 \theta_x\phi dx - d\int_0^1 r_x\phi dx\\ &\quad - \beta\mathfrak{b}(t)\int_0^1 \phi\mathfrak{h}_1(\phi_t)dx - \mu\mathfrak{b}(t)\int_0^1 \phi\mathfrak{h}_2\left(\mathcal{Y}(1)\right)dx - 3\varrho\int_0^1 \phi_t\int_0^x \psi_t(y)dydx.\end{aligned}$$

Establishing Eq (3.15) is achievable once considering $(\mathbf{A}_2)$ along with Young and Poincaré's inequalities. □

Lemma 3.5. *Consider functional*

$$\mathcal{I}_4(t) := -\varrho\int_0^1 \psi_t\psi dx. \tag{3.16}$$

Then, it satisfies

$$\mathcal{I}_4'(t) \le -\varrho\int_0^1 \psi_t^2dx + D\int_0^1 (3\phi_x - u_x)^2dx + \Upsilon_*\int_0^1 (u-\psi_x)^2dx + \Upsilon_*\int_0^1 \phi_x^2dx. \tag{3.17}$$

Proof. To start with, we differentiate $\mathcal{I}_4$, and consider Eq $(2.12)_1$ with integration by parts, to achieve

$$\mathcal{I}_4'(t) = -\varrho\int_0^1 \psi_t^2dx - G\int_0^1 \psi_x(u-\psi_x)dx, \tag{3.18}$$

and observing that $\psi_x = -(u-\psi_x) - (3\phi - u) + 3\phi$, we obtain

$$\begin{aligned}\mathcal{I}_4'(t) = &-\varrho\int_0^1 \psi_t^2 dx + G^2\int_0^1 (u-\psi_x)^2 dx + G\int_0^1 (u-\psi_x)(3\phi-u)dx\\ &-3G\int_0^1 \phi(u-\psi_x)dx.\end{aligned} \tag{3.19}$$

By means of Young and Poincaré's inequalities, we have

$$G\int_0^1 (u-\psi_x)(3\phi-u)dx \le \frac{G^2}{4D}\int_0^1 (u-\psi_x)^2 dx + D\int_0^1 (3\phi_x - u_x)^2 dx, \tag{3.20}$$

and

$$-3G\int_0^1 \phi(u-\psi_x)dx \le \frac{3G}{2}\int_0^1 (u-\psi_x)^2 dx + \frac{3G}{2}\int_0^1 \phi_x^2 dx, \tag{3.21}$$

once we replace Eqs (3.20) and (3.21) into Eq (3.19), the estimate (3.17) is easily proved. □

Lemma 3.6. *Consider the functional*

$$\mathcal{I}_5(t) := -I_\varrho\int_0^1 (3\phi-u)_t(3\phi-u)\, dx. \tag{3.22}$$

Then, it satisfies

$$\mathcal{I}_5'(t) \le -I_\varrho\int_0^1 (3\phi_t - u_t)^2 dx + 2D\int_0^1 (3\phi_x - u_x)^2\, dx + \Upsilon_*\int_0^1 (u-\psi_x)^2\, dx. \tag{3.23}$$

Proof. We advance by differentiating $\mathcal{I}_5$, employing Eq $(2.12)_2$ accompanied with integration by parts, which results in

$$\begin{aligned}\mathcal{I}_5'(t) &= -I_\varrho\int_0^1 (3\phi-u)_{tt}(3\phi-u)\, dx - I_\varrho\int_0^1 (3\phi_t-u_t)^2\, dx\\ &= -I_\varrho\int_0^1 (3\phi_t-u_t)^2\, dx + D\int_0^1 (3\phi_x-u_x)^2\, dx - G\int_0^1 (3\phi-u)(u-\psi_x)\, dx.\end{aligned} \tag{3.24}$$

We terminate our proof, once Young and Poincaré's inequalities are used. □

Lemma 3.7. *Consider the functional*

$$\mathcal{I}_6(t) := \tilde{\nu}\varsigma(t)\int_0^1\int_0^1 e^{-2p\varsigma(t)}\xi(\mathcal{Y}(p))dpdx. \tag{3.25}$$

Then, it satisfies

$$\mathcal{I}_6'(t) \le \frac{\vartheta_2\tilde{\nu}}{2}\int_0^1 \left(|\mathfrak{h}_1(\phi_t)|^2 + \phi_t^2\right)dx - 2\mathcal{I}_6(t), \quad \forall t \ge 0. \tag{3.26}$$

Proof. We take here the derivative of $\mathcal{I}_6$, to find

$$\begin{aligned}\mathcal{I}_6'(t) &= \tilde{\nu}\varsigma'(t)\int_0^1\int_0^1 e^{-2\varsigma(t)p}\ \xi(\mathcal{Y}(p))\ dpdx\\ &+\tilde{\nu}\varsigma(t)\int_0^1\int_0^1 e^{-2\varsigma(t)p}\mathcal{Y}_t(p)\mathfrak{h}_2\left(\mathcal{Y}(p)\right)dpdx\\ &-2\tilde{\nu}\varsigma'(t)\varsigma(t)\int_0^1\int_0^1 pe^{-2\varsigma(t)p}\ \xi(\mathcal{Y}(p))\ dpdx.\end{aligned} \tag{3.27}$$

Equation $(2.12)_6$ enables us to write

$$\begin{aligned}&\varsigma(t)\int_0^1\int_0^1 e^{-2\varsigma(t)p}\mathcal{Y}_t(p)\mathfrak{h}_2\left(\mathcal{Y}(p)\right)dpdx\\ &=\int_0^1\int_0^1 e^{-2\varsigma(t)p}(p\varsigma'(t)-1)\mathcal{Y}_p(p)\mathfrak{h}_2\left(\mathcal{Y}(p)\right)dpdx\\ &=\int_0^1\int_0^1 \partial_p\left(e^{-2\varsigma(t)p}(p\varsigma'(t)-1)\xi(\mathcal{Y}(p))\right)dpdx\\ &-\varsigma'(t)\int_0^1\int_0^1 e^{-2\varsigma(t)p}\xi(\mathcal{Y}(p))dpdx\\ &+2\varsigma(t)\int_0^1\int_0^1 e^{-2\varsigma(t)p}(p\varsigma'(t)-1)\xi(\mathcal{Y}(p))dpdx\\ &=\int_0^1 \xi\left(\phi_t\right)dx-(1-\varsigma'(t))e^{-2\varsigma(t)}\int_0^1 \xi(\mathcal{Y}(1))dx\\ &-\varsigma'(t)\int_0^1\int_0^1 e^{-2\varsigma(t)p}\xi(\mathcal{Y}(p))dpdx\\ &+2\varsigma(t)\int_0^1\int_0^1 e^{-2\varsigma(t)p}(p\varsigma'(t)-1)\xi(\mathcal{Y}(p))dpdx,\end{aligned} \tag{3.28}$$

which together with Eq (3.27) leads to

$$\begin{aligned}\mathcal{I}_6'(t) = \tilde{\nu}\int_0^1 \xi\left(\phi_t\right)dx &- 2\tilde{\nu}\varsigma(t)\int_0^1\int_0^1 e^{-2\varsigma(t)p}\xi(\mathcal{Y}(p))dpdx\\ &-(1-\varsigma'(t))e^{-2\varsigma(t)}\int_0^1 \xi(\mathcal{Y}(1))dx.\end{aligned} \tag{3.29}$$

To prove Eq (3.26), it is convenient to consider Young's inequality accompanied by Eq (2.2). □

4. Stability result

Here, we exploit lemmas from Section 3 to demonstrate our stability results.

Proof of Theorem 2.1. We advance by introducing a Lyapunov functional

$$\mathcal{R}(t) = N\mathcal{E}(t) + \sum_{i=1}^{6} N_i\mathcal{I}_i(t), \quad \forall t \geq 0, \tag{4.1}$$

where constants $N,\ N_i > 0,\ i = 1 \cdots 6$, will be fixed later.

By Eq (4.1), we are in position to write

$$
\begin{aligned}
|\mathcal{R}(t) - N\mathcal{E}(t)| \leq{}& \varrho D N_1 \int_0^1 |\psi_t(3\phi_x - u_x)|\, dx + 3I_\varrho G N_1 \int_0^1 |\phi_t(3\phi - u)|\, dx \\
&+ I_\varrho G N_1 \int_0^1 |\psi_x(3\phi_t - u_t)|\, dx + 3\varrho D N_2 \int_0^1 |\psi_t \phi_x|\, dx \\
&+ 3I_\varrho G N_2 \int_0^1 |(u - \psi_x)\phi_t|\, dx + 3I_\varrho N_3 \int_0^1 |\phi\phi_t|\, dx \\
&+ 3\varrho N_3 \int_0^1 \left|\phi \int_0^x \psi_t(y)dy\right| dx + \varrho N_4 \int_0^1 |\psi_t \psi|\, dx \\
&+ I_\varrho N_5 \int_0^1 |(3\phi - u)_t(3\phi - u)|\, dx + \tilde{\nu}\varsigma(t) N_6 \int_0^1 \int_0^1 e^{-2p\varsigma(t)} \xi(\mathcal{Y}(p)) dp dx.
\end{aligned}
$$

By means of the energy definition accompanied with Young, Cauchy-Schwarz, and Poincaré's, we achieve

$$
|\mathcal{R}(t) - \mathcal{E}(t)| \leq \mathfrak{d}\mathcal{E}(t), \qquad \text{where } \mathfrak{d} > 0,
$$

i.e.,

$$
(N - \mathfrak{d})\mathcal{E}(t) \leq \mathcal{R}(t) \leq (N + \mathfrak{d})\mathcal{E}(t). \tag{4.2}
$$

We now differentiate the Lyapunov functional $\mathcal{R}$, consider Eqs (3.1), (3.11), (3.13), (3.15), (3.17), (3.23), (3.26) and let

$$
N_1 = \frac{8}{G},\ N_4 = N_5 = N_6 = 1, \quad \epsilon_1 = \frac{I_\varrho}{4N_1},\ \epsilon_2 = \frac{I_\varrho}{4N_2},\ \epsilon_3 = \frac{\varrho}{2N_3},
$$

to get

$$
\begin{aligned}
\mathcal{R}'(t) \leq{}& -\frac{\varrho}{2} \int_0^1 \psi_t^2 dx - [3DN_3 - \Upsilon_* N_2 - \Upsilon_*] \int_0^1 \phi_x^2 dx - \frac{I_\varrho}{2} \int_0^1 (3\phi_t - u_t)^2 dx \\
&- \delta N_3 \int_0^1 \phi^2 dx - \left[G^2 N_2 - \Upsilon_*\right] \int_0^1 (u - \psi_x)^2 dx - D \int_0^1 (3\phi_x - u_x)^2 dx \\
&- [k_0 N - \Upsilon_* N_2 - \Upsilon_* N_3 - \Upsilon_*] \int_0^1 \theta_x^2 dx - [k_2 N - \Upsilon_* N_2 - \Upsilon_* N_3 - \Upsilon_*] \int_0^1 r_x^2 dx \\
&- k_3 N \int_0^1 r^2 dx - \frac{2e^{-2\varsigma_1}}{\mathfrak{b}(0)} \nu(t)\varsigma(t) \int_0^1 \int_0^1 \xi\left(\mathcal{Y}(p)\right) dp dx \\
&+ \left[\Upsilon_* N_2 + \Upsilon_* N_3 + \Upsilon_* + \frac{\tilde{\nu}\vartheta_2}{2}\right] \int_0^1 |\mathfrak{h}_1(\phi_t)|^2 dx \\
&+ [\Upsilon_* N_2 + \Upsilon_* N_3 + \Upsilon_*] \int_0^1 |\mathfrak{h}_2\left(\mathcal{Y}(1)\right)|^2 dx \\
&+ \left[\Upsilon_* N_2(1 + N_2) + \Upsilon_* N_3(1 + N_3) + \Upsilon_* + \frac{\tilde{\nu}\vartheta_2}{2}\right] \int_0^1 \phi_t^2 dx.
\end{aligned} \tag{4.3}
$$

Subsequently, we select coefficients in Eq (4.3) such that all of them (excluding the final three) turn negative. To this end, we opt to take N_2 to be sufficiently large so that

$$G^2 N_2 - \Upsilon_* > 0,$$

which makes us opt to take N_3 enough large to have

$$3DN_3 - \Upsilon_* N_2 - \Upsilon_* > 0,$$

and we finish by taking N to be fairly huge to obtain both Eq (4.2) and

$$\begin{cases} k_0 N - \Upsilon_* N_2 - \Upsilon_* N_3 - \Upsilon_* > 0, \\ k_2 N - \Upsilon_* N_2 - \Upsilon_* N_3 - \Upsilon_* > 0. \end{cases}$$

Now, it is convenient to consider definition Eq (2.15) along with the above selection of constants, and Poincaré's inequality, to find

$$\mathcal{R}'(t) \leq -\Lambda \mathcal{E}(t) + \chi \int_0^1 \left(\phi_t^2 + |\mathfrak{h}_1(\phi_t)|^2\right) dx + \chi \int_0^1 |\mathfrak{h}_2 \left(\mathcal{Y}(1)\right)|^2 dx, \quad \Lambda, \chi > 0. \tag{4.4}$$

As a part of this proof, we shall distinguish two cases:

Case 1: Suppose that T is linear. Hypothesis ($\mathbf{A}_1$) enables us to write

$$\lambda_1 |z| \leq |\mathfrak{h}_1(z)| \leq \lambda_2 |z|, \ \forall z \in \mathbb{R},$$

hence,

$$\mathfrak{h}_1^2(z) \leq \lambda_2 u \mathfrak{h}_1(z), \ \forall z \in \mathbb{R}. \tag{4.5}$$

If we multiply Eq (4.4) by $\mathfrak{b}(t)$ and take advantage of both Eqs (3.1) and (4.5), we easily come to

$$\begin{aligned} \mathfrak{b}(t)\mathcal{R}'(t) &\leq -\Lambda \mathfrak{b}(t)\mathcal{E}(t) + \chi \mathfrak{b}(t) \int_0^1 \phi_t \mathfrak{h}_1(\phi_t) dx + \chi \mathfrak{b}(t) \int_0^1 \mathcal{Y}(1) \mathfrak{h}_2 \left(\mathcal{Y}(1)\right) dx \\ &\leq -\Lambda \mathfrak{b}(t)\mathcal{E}(t) - \chi \mathcal{E}'(t), \ t \in \mathbb{R}_+. \end{aligned}$$

Now, we continue by introducing

$$R(t) := \mathfrak{b}(t)\mathcal{R}(t) + \chi \mathcal{E}(t). \tag{4.6}$$

If we consider Eq (4.2) together with ($\mathbf{A}_1$), it is obvious to observe that

$$R(t) \sim \mathcal{E}(t), \tag{4.7}$$

and

$$R'(t) \leq -\Lambda_1 \mathfrak{b}(t) R(t), \quad \Lambda_1 > 0, \ \forall t \geq 0. \tag{4.8}$$

Finally, we simply integrate Eq (4.8) and exploit Eq (4.7), to be able to get

$$\mathcal{E}(t) \leq \varkappa_0 \exp\left(-\varkappa_1 \int_0^t \mathfrak{b}(z) dz\right) = \varkappa_0 T_2^{-1} \left[\varkappa_1 \int_0^t \mathfrak{b}(z) dz\right], \ \forall t \geq 0. \tag{4.9}$$

Case 2: Suppose that T is nonlinear on $(0,\varepsilon]$. Following the lead of [10], we take $0<\varepsilon_1\leq\varepsilon$, in a way that

$$z\mathfrak{h}_1(z)\leq\min\{\varepsilon,T(\varepsilon)\},\quad \forall|z|\leq\varepsilon_1. \tag{4.10}$$

With the help of $(\mathbf{A}_1)$ accompanied with $\mathfrak{h}_1$ being continuous, and observing that $|\mathfrak{h}_1(z)|>0,\ z\neq0$, we establish

$$\begin{cases} z^2+\mathfrak{h}_1^2(z)\leq T^{-1}(z\mathfrak{h}_1(z)), & |z|\leq\varepsilon_1,\\ \lambda_1'|z|\leq|\mathfrak{h}_1(z)|\leq\lambda_2'|z|, & |z|\geq\varepsilon_1. \end{cases} \tag{4.11}$$

If we take the below partitions

$$\mathcal{J}_1=\{x\in(0,1)\ :\ |\phi_t|\leq\varepsilon_1\},\quad \mathcal{J}_2=\{x\in(0,1)\ :\ |\phi_t|>\varepsilon_1\},$$

$$\mathcal{J}_3=\{x\in(0,1)\ :\ |\mathcal{Y}(1)|\leq\varepsilon_1\},\quad \mathcal{J}_3=\{x\in(0,1)\ :\ |\mathcal{Y}(1)|>\varepsilon_1\},$$

then, the Jensen inequality along with the concavity of T^{-1}, gives us

$$T^{-1}(\mathcal{J}(t))\geq\chi\int_{\mathcal{J}_1}T^{-1}(\phi_t\mathfrak{h}_1(\phi_t))\,dx, \tag{4.12}$$

where

$$\mathcal{J}(t)=\int_{\mathcal{J}_1}\phi_t\mathfrak{h}_1(\phi_t)dx.$$

Employing the above estimates, we are in position to write

$$\begin{aligned} \mathfrak{b}(t)\int_0^1\left(\phi_t^2+\mathfrak{h}_1^2(\phi_t)\right)\,dx&=\mathfrak{b}(t)\int_{\mathcal{J}_1}\left(\phi_t^2+\mathfrak{h}_1^2(\phi_t)\right)\,dx+\mathfrak{b}(t)\int_{\mathcal{J}_2}\left(\phi_t^2+\mathfrak{h}_1^2(\phi_t)\right)\,dx\\ &\leq\mathfrak{b}(t)\int_{\mathcal{J}_1}T^{-1}(\phi_t\mathfrak{h}_1(\phi_t))\,dx+\chi\mathfrak{b}(t)\int_{\mathcal{J}_2}(\phi_t\mathfrak{h}_1(\phi_t))\,dx\\ &\leq\chi\mathfrak{b}(t)T^{-1}(\mathcal{J}(t))-\chi\mathcal{E}'(t), \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \mathfrak{b}(t)\int_0^1\mathfrak{h}_2^2(\mathcal{Y}(1))\,dx&=\mathfrak{b}(t)\int_{\mathcal{J}_3}\mathfrak{h}_2^2(\mathcal{Y}(1))\,dx+\mathfrak{b}(t)\int_{\mathcal{J}_4}\mathfrak{h}_2^2(\mathcal{Y}(1))\,dx\\ &\leq\chi\mathfrak{b}(t)\int_{\mathcal{J}_3}\mathcal{Y}(1)\mathfrak{h}_2(\mathcal{Y}(1))\,dx+\mathfrak{b}(t)\int_{\mathcal{J}_4}\mathcal{Y}(1)\mathfrak{h}_2(\mathcal{Y}(1))\,dx\\ &\leq-\chi\mathcal{E}'(t). \end{aligned} \tag{4.14}$$

Let us now multiply Eq (4.4) by $\mathfrak{b}(t)$ and then apply both estimates Eqs (4.13) and (4.14) to achieve

$$\mathfrak{b}(t)\mathcal{R}'(t)+\chi\mathcal{E}'(t)\leq-\Lambda\mathfrak{b}(t)\mathcal{E}(t)+\chi\mathfrak{b}(t)T^{-1}(\mathcal{J}(t)).$$

We shall next introduce

$$\mathcal{R}_0(t):=\mathfrak{b}(t)\mathcal{R}(t)+\chi\mathcal{E}(t). \tag{4.15}$$

Taking relation Eq (4.2) into account, we readily obtain

$$\mathcal{R}_0(t)\sim\mathcal{E}(t), \tag{4.16}$$

but then according to $(\mathbf{A}_2)$,

$$\mathcal{R}_0'(t) \le -\Lambda \mathfrak{b}(t)\mathcal{E}(t) + \chi \mathfrak{b}(t) T^{-1}(\mathcal{J}(t)). \tag{4.17}$$

We then take the functional

$$\mathcal{R}_1(t) := T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{R}_0(t) + \Lambda_0 \mathcal{E}(t), \quad \varepsilon_0 < \varepsilon, \ \ \Lambda_0 > 0, \tag{4.18}$$

along with the fact that $\mathcal{E}' \le 0, \quad T' > 0, \ T'' > 0, \ \text{on} \ (0, \varepsilon]$, to reach

$$\bar{\ }_1 \mathcal{R}_1(t) \le \mathcal{E}(t) \le \bar{\ }_2 \mathcal{R}_1(t), \quad \bar{\ }_1, \bar{\ }_2 > 0. \tag{4.19}$$

Moreover, once we utilize Eq (4.17), we see that

$$\begin{aligned}\mathcal{R}_1'(t) &= \varepsilon_0 \frac{\mathcal{E}'(t)}{\mathcal{E}(0)} T''\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{R}_0(t) + T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{R}_0'(t) + \Lambda_0\mathcal{E}'(t)\\ &\le -\Lambda \mathfrak{b}(t) T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{E}(t) + \chi \mathfrak{b}(t) T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) T^{-1}(\mathcal{J}(t)) + \Lambda_0 \mathcal{E}'(t).\end{aligned} \tag{4.20}$$

Set

$$\mathcal{Z} = \chi \mathfrak{b}(t) T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) T^{-1}(\mathcal{J}(t)).$$

Similar to what we did earlier, we shall now estimate $\mathcal{Z}$ by letting T^* be the convex conjugate of T given by

$$T^*(z) = z(T')^{-1}(z) - T\left[(T')^{-1}(z)\right] \le z(T')^{-1}(z), \quad \text{where} \ \ z \in (0, T'(\varepsilon)). \tag{4.21}$$

Moreover, applying the general Young's inequality, we notice that

$$zv \le T^*(z) + T(v), \quad \text{where} \ \ z \in (0, T'(\varepsilon)), \ v \in (0, \varepsilon]. \tag{4.22}$$

Let us also set

$$z = T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right), \quad \text{and} \ \ v = T^{-1}(\mathcal{J}(t)),$$

exploiting Eqs (4.20)–(4.22), (4.10), along with Lemma 3.1, yields

$$\begin{aligned}\mathcal{R}_1'(t) &\le -\Lambda \mathfrak{b}(t) T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{E}(t) + \chi \mathfrak{b}(t)\left(T^*\left[T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\right] + T\left[T^{-1}(\mathcal{J}(t))\right]\right) + \Lambda_0\mathcal{E}'(t)\\ &= -\Lambda \mathfrak{b}(t) T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{E}(t) + \chi \mathfrak{b}(t) T^*\left[T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\right] + \chi \mathfrak{b}(t)\mathcal{J}(t) + \Lambda_0\mathcal{E}'(t)\\ &\le -\Lambda \mathfrak{b}(t) T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{E}(t) + \varepsilon_0 \chi \mathfrak{b}(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) - \chi\mathcal{E}'(t) + \Lambda_0\mathcal{E}'(t)\\ &\le -(\Lambda\mathcal{E}(0) - \varepsilon_0\chi)\mathfrak{b}(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)} T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) + (\Lambda_0 - \chi)\mathcal{E}'(t).\end{aligned} \tag{4.23}$$

Next, we shall pick $\varepsilon_0 = \frac{\Lambda\mathcal{E}(0)}{2\chi}$, $\Lambda_0 = 2\chi$, and notice that $\mathcal{E}'(t) \le 0$ to achieve estimate

$$\mathcal{R}_1'(t) \le -\Lambda_1 \mathfrak{b}(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} T'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) = -\Lambda_1 \mathfrak{b}(t) T_1\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right), \tag{4.24}$$

where

$$\Lambda_1 > 0, \quad \text{and} \quad T_1(z) = zT'(\varepsilon_0 z).$$

Because T is strictly convex on $(0, \varepsilon]$, one can notice that $T_1(z),\ T_1'(z) > 0$ on $(0, 1]$. Therefore, letting

$$\mathcal{R}_{1*}(t) := \frac{\bar{\ }_1\mathcal{R}_1(t)}{\mathcal{E}(0)}, \tag{4.25}$$

and employing both (4.19) and (4.24), we obviously have

$$\mathcal{R}_{1*}(t) \sim \mathcal{E}(t), \quad \text{and} \quad \mathcal{R}_{1*}'(t) \leq -\varkappa_1 \mathrm{b}(t) T_1\left(\mathcal{R}_{1*}(t)\right),\ \varkappa_1 > 0. \tag{4.26}$$

Thereby, if we let

$$T_2(t) = \int_t^1 \frac{1}{T_1(z)} dz, \quad t \in (0, 1],$$

we decisively reach

$$\left[T_2\left(\mathcal{R}_{1*}(t)\right)\right]' \geq \varkappa_1 \mathrm{b}(t), \tag{4.27}$$

we then integrate Eq (4.27) over $[0, t]$, and make sure that $T_2'(z) < 0,\ z \in (0, 1]$, along with T_1 and its properties, we accomplish what follows:

$$\mathcal{R}_{1*}(t) \leq T_2^{-1}\left(\varkappa_1 \int_0^t \mathrm{b}(z) dz + \varkappa_2\right), \quad \varkappa_2 > 0,\ \forall t \in \mathbb{R}_+. \tag{4.28}$$

The existence of functions $\mathcal{R}_{1*}(t) \in (0, 1]$ and $T_2(\mathcal{R}_{1*}(t))$ are assured. See the reference [9] in Sections 4 and 5.

The use of relation Eq (4.26) eventually concludes our proof. □

5. Conclusions

This paper investigates the energy decay of the solutions for the coupled system of a thermoelastic laminated Timoshenko beam with nonlinear damping, microtemperature effects, nonlinear weight, and nonlinear time-varying delay, together with the Dirichlet boundary condition for θ, r and mixed boundary condition for u, ψ, ϕ.

We examined the joint impacts of microtemperature, nonlinear structural damping, along with nonlinear time-varying delay term, and time-varying coefficient on a thermoelastic laminated beam, where, the equation representing the dynamics of slip is affected by the last three mentioned terms. The impact of different terms is outlined and their impact on the stability of the solution is shown. Our results extend the recent related results [15–18].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

References

1. H. Suh, Z. Bien, Use of time-delay actions in the controller design, *IEEE Trans. Automat. Contr.*, **25** (1980), 600–603. https://doi.org/10.1109/TAC.1980.1102347
2. S. E. Mukiawa, C. D. Enyi, S. A. Messaoudi, Stability of thermoelastic Timoshenko beam with suspenders and time-varying feedback, *Adv. Contin. Discret. Models*, **2023** (2023), 7. https://doi.org/10.1186/s13662-023-03752-w
3. J. M. Wang, G. Q. Xu, S. P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, *SIAM J. Control Optim.*, **44** (2005), 1575–1597. https://doi.org/10.1137/040610003
4. N. E. Tatar, Stabilization of a laminated beam with interfacial slip by boundary controls, *Bound. Value Probl.*, **2015** (2015), 169. https://doi.org/10.1186/s13661-015-0432-3
5. H. E. Khochemane, Exponential stability for a thermoelastic porous system with microtemperature effects, *Acta Appl. Math.*, **173** (2021), 8. https://doi.org/10.1007/s10440-021-00418-1
6. D. Fayssal, Well posedness and stability result for a thermoelastic laminated beam with structural damping, *Ric. Mat.*, 2022.
7. A. Benaissa, A. Benaissa, S. A. Messaoudi, Global existence and energy decay of solutions for the wave equation with a time-varying delay term in the weakly nonlinear internal feedbacks, *J. Math. Phys.*, **53** (2012), 123514. https://doi.org/10.1063/1.4765046
8. L. Djilali, A. Benaissa, A. Benaissa, Global existence and energy decay of solutions to a viscoelastic Timoshenko beam system with a nonlinear delay term, *Appl. Anal.*, **95** (2016), 2637–2660. https://doi.org/10.1080/00036811.2015.1105961
9. K. Mpungu, T. A. Apalara, Asymptotic behavior of a laminated beam with nonlinear delay and nonlinear structural damping, *Hacet. J. Math. Stat.*, **51** (2022), 1517–1534. https://doi.org/10.15672/hujms.947131
10. I. Lasiecka, D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, *Diff. Integ. Equ.*, **6** (1993), 507–533.
11. S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, **45** (2006), 1561–1585. https://doi.org/10.1137/060648891
12. T. A. Apalara, A. Soufyane, Energy decay for a weakly nonlinear damped porous system with a nonlinear delay, *Appl. Anal.*, **101** (2022), 6113–6135. https://doi.org/10.1080/00036811.2021.1919642
13. V. I. Arnol'd, *Mathematical methods of classical mechanics*, New York: Springer, 1989.
14. A. Benaissa, M. Bahlil, Global existence and energy decay of solutions to a nonlinear Timoshenko beam system with a delay term, *Taiwanese J. Math.*, **18** (2014), 1411–1437. https://doi.org/10.11650/tjm.18.2014.3586

15. C. S. Zhu, X. Q. Fang, J. X. Liu, H. Y. Li, Surface energy effect on nonlinear free vibration behavior of orthotropic piezoelectric cylindrical nano-shells, *Eur. J. Mech. A-Solid.*, **66** (2017), 423–432. https://doi.org/10.1016/j.euromechsol.2017.08.001

16. J. Xie, Z. Zhang, The high-order multistep ADI solver for two-dimensional nonlinear delayed reaction-diffusion equations with variable coefficients, *Comput. Math. Appl.*, **75** (2018), 3558–3570. https://doi.org/10.1016/j.camwa.2018.02.017

17. J. Xie, X. Yan, M. Aamir Ali, Z. Hammouch, A linear decoupled physical-property-preserving difference method for fractional-order generalized Zakharov system, *J. Comput. Appl. Math.*, **426** (2023), 115044. https://doi.org/10.1016/j.cam.2022.115044

18. X. Q. Fang, H. W. Ma, C. S. Zhu, Non-local multi-fields coupling response of a piezoelectric semiconductor nanofiber under shear force, *Mech. Adv. Mater. Struc.*, 2023. https://doi.org/10.1080/15376494.2022.2158503