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Research article

Incomplete exponential type of *R*-matrix functions and their properties

Ahmed Bakhet[1,*] **and Mohra Zayed**[2,*]

[1] Department of Mathematics, Faculty of Science Al-Azhar University, Assiut 71524, Egypt
[2] Mathematics Department, College of Science, King Khalid University, Abha 61413, Saudi Arabia

* **Correspondence:** Email: kauad_2006@azhar.edu.eg, mzayed@kku.edu.sa.

Abstract: In the present paper, we establish the incomplete exponential type (IEF) of *R*-matrix functions and identify some properties of the incomplete exponential matrix functions including integral representation, some derivative formula and generating functions of the incomplete exponential of *R*-matrix functions. Finally, special cases of the presented results are pointed out.

Keywords: incomplete exponential functions; special functions; special matrix functions; R-matrix function; integral representation
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1. Introduction

The incomplete exponential functions (IEF) introduced by Chaudhry and Qadir [1] considered two classes of functions:

$$e\Big[(\theta;(x,z))\Big] := \sum_{n=0}^{\infty} \frac{\gamma(\theta+n;x)}{\Gamma(\theta+n)} \frac{z^n}{n!} \tag{1.1}$$

and

$$E\Big[[\theta;(x,z)]\Big] = \sum_{n=0}^{\infty} \frac{\Gamma(\theta+n;x)}{\Gamma(\theta+n)} \frac{z^n}{n!}, \tag{1.2}$$

such that the incomplete gamma functions $\gamma(\theta;x)$ and $\Gamma(\theta;x)$ defined by Srivastava, Chaudhry and Agarwal [2] as

$$\gamma(\theta;x) = \int_0^x t^{\theta-1}e^{-t}dt \tag{1.3}$$

and

$$\Gamma(\theta;x) = \int_x^{\infty} t^{\theta-1}e^{-t} \quad dt, \tag{1.4}$$

respectively, and they achieved the following decomposition:

$$\gamma(\theta; x) + \Gamma(\theta; x) = \Gamma(\theta). \tag{1.5}$$

The ${}_pR_q(\phi, \varphi; z)$ function [3] is defined as:

$$\begin{aligned} {}_pR_q(\phi, \varphi; z) &= {}_pR_q\left(\begin{matrix} \theta_1, \ldots, \theta_p \\ \eta_1, \ldots, \eta_q \end{matrix} \middle| \phi, \varphi; z \right) \\ &= \sum_{n\geq 0} \frac{1}{\Gamma(\phi n + \varphi)} \frac{(\theta_1)_n \ldots (\theta_p)_n}{(\eta_1)_n \ldots (\eta_q)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.6}$$

where $p, q \in \mathbf{Z}^+$, $\phi, \varphi \in \mathbf{C}$ and $\mathbf{Re}(\phi), \mathbf{Re}(\varphi), \mathbf{Re}(\theta_i), \mathbf{Re}(\eta_j) > 0$, for $\forall i = 1, 2, \ldots, p, \forall j = 1, 2, \ldots, q$, and $(\theta)_n$ denotes the Pochhammer symbol which defined by:

$$(\theta)_n = \begin{cases} \theta(\theta + 1) \ldots (\theta + n - 1) = \frac{\Gamma(\theta+n)}{\Gamma(\theta)}, & n \geq 1, \\ 1, & n = 0. \end{cases} \tag{1.7}$$

Exploring the extension of classical matrix functions and matrix polynomials has recently become a prominent topic. Special matrix functions such as Gamma, Beta were studied by Jódar and Cortès who studied matrix analogues of gamma, beta and Gauss hypergeometric functions [4–6] and other contributions have been directed to discuss the polynomials in two variables such as the 2-variables Shivley's matrix polynomials [7], the 2-variables Laguerre matrix polynomials [8], the 2-variables Hermite generalized matrix polynomials [9–12], the 2-variables Gegenbauer matrix polynomials [13] and the second kind of Chebyshev matrix polynomials with two variables [14].

In the current study, we intend to establish incomplete exponential matrix functions. Involving the ${}_pR_q(P, Q; z)$ functions of matrix parameters, we investigate some properties of an incomplete exponential of type of R -matrix functions . Furthermore, we provide generating formulas for the incomplete exponential type of R -matrix functions.

The paper is organized as follows. In Section 2, we review basic definitions and previous results which will be mandatory through the following sections. Section 3 introduces the definition of the incomplete exponential of type of R -matrix functions and states some theorems about integral and derivative formula of the incomplete exponential of type of R -matrix functions. Some generating matrix relations incomplete exponential of type of R -matrix functions are provided in Section 4. In Section 5, we discuss some special cases of the incomplete exponential of type of R-matrix functions. The paper is appended with conclusions in Section 6.

2. Preliminaries

Throughout this paper, we consider a matrix $L \in \mathbb{C}^{h\times h}$ and its spectrum $\sigma(L)$ represents the collection of all eigenvalues L . Let $\mathbb{C}^h$ denote the h-dimensional complex vector space and $\mathbb{C}^{h\times h}$ denote all square matrices with h rows and h columns with complex entries. As usual, let $\mathbf{Re}(z)$ and $\mathbf{Im}(z)$ be referring to the real and imaginary parts of a complex number z, respectively. The two-norm of L is defined on $\mathbb{C}^{h\times h}$ as follows

$$||L||_2 = \sup_{x\neq 0} \frac{||Lx||_2}{||x||_2} = \max\{ \sqrt{\lambda} : \lambda \in \sigma(L^*L)\}, \quad \forall x \in \mathbb{C}^h, \tag{2.1}$$

where for a vector $x \in \mathbb{C}^h$, $\|x\|_2 = (x^* x)^{\frac{1}{2}}$ is the Euclidean norm of x such that L^* denotes the transposed conjugate of L. Let us denote the real numbers $\alpha(L)$ and $\beta(L)$ as in the following

$$\alpha(L) = \max\{\mathbf{Re}(z) : z \in \sigma(L)\}, \quad \beta(L) = \min\{\mathbf{Re}(z) : z \in \sigma(L)\}. \tag{2.2}$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable established in an open set Ω of the complex plane and L, M are matrices in $\mathbb{C}^{h\times h}$ with $\sigma(L) \subset \Omega$ and $\sigma(M) \subset \Omega$, such that $LM = ML$, then it follows from the matrix functional calculus properties in [7]), that $f(L)g(M) = g(M)f(L)$.

We recall that the reciprocal Gamma function, given by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$, is an entire function of the complex variable, and thus $\Gamma^{-1}(L)$ is a well defined matrix for any matrix L in $\mathbb{C}^{h\times h}$. In addition, if L is a matrix, then

$$L + nI \quad \text{is invertible for all integers } n \geq 0, \tag{2.3}$$

where I is the identity matrix in $\mathbb{C}^{h\times h}$. Then $\Gamma(L)$ is invertible and its inverse coincides with $\Gamma^{-1}(L)$. The Pochhammer symbol of a matrix argument is given by (see [5]):

$$(L)_n = \begin{cases} L(L+I)\ldots(L+(n-1)I) = \Gamma^{-1}(L)\Gamma(L+nI), & n \geq 1, \\ I, & n = 0. \end{cases} \tag{2.4}$$

Assume that L and M are positive stable matrices in $\mathbb{C}^{h\times h}$. The Gamma matrix function $\Gamma(L)$ and Beta matrix function $\mathfrak{B}(L, M)$ have been defined in [5, 15], as follows:

$$\Gamma(L) = \int_0^\infty e^{-t} t^{L-I} dt, \quad \mathfrak{B}(L, M) = \int_0^1 t^{L-I}(1-t)^{M-I} dt, \tag{2.5}$$

where $t^{L-I} = \exp((L - I)\ln t)$. Jódar and Cortés showed in [5] that

$$\Gamma(L) = \lim_{n\longrightarrow\infty} (n-1)![(L)_n]^{-1} n^L, \tag{2.6}$$

where $n \geq 1$ is an integer.

Now, the incomplete matrix gamma is defined as follows: [15, 16]. Assume that L is a positive stable matrix in $\mathbb{C}^{h\times h}$ and y be a positive real number. Then, the incomplete matrix gamma function $\gamma(L, y)$ and its complement $\Gamma(L, y)$ are defined by

$$\gamma(L, y) = \int_0^y e^{-t} t^{L-I} dt, \tag{2.7}$$

$$\Gamma(L, y) = \int_y^\infty e^{-t} t^{L-I} dt, \tag{2.8}$$

and we have the following decomposition formula (see [15]):

$$\gamma(L, y) + \Gamma(L, y) = \Gamma(L). \tag{2.9}$$

The following provides the hypergeometric matrix function ${}_2F_1(L, M; N; z)$ as, assume that L, M and N are matrices in $\mathbb{C}^{h\times h}$ and N satisfy condition (2.3), then the hypergeometric matrix function of 2-numerator and 1-denominator for $|z| < 1$ is defined by the matrix power series (see [5, 6])

$$ {}_2F_1(L, M; N; z) = \sum_{n\geq 0} \frac{(L)_n(M)_n[(N)_n]^{-1}}{n!} z^n. \tag{2.10}$$

The Bessel matrix function $J_L(z)$ of the first kind associated to L is given in the following form: (see [15, 16])

$$J_L(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k)!}\, \Gamma^{-1}\left(L + (k + 1)I\right)\, (\frac{z}{2})^{L+2kI} \tag{2.11}$$

and the modified Bessel matrix function $I_L(z)$ has been defined in the form:

$$I_L(z) = \sum_{k=0}^{\infty} \frac{1}{(k)!}\, \Gamma^{-1}(L + (k + 1)I)\, (\frac{z}{2})^{L+2kI}, \tag{2.12}$$

where L is a matrix in $\mathbb{C}^{h\times h}$ satisfying the condition (2.3). We may rewrite the Bessel and modified Bessel matrix functions as

$$J_L(z) = (\frac{z}{2})^A\, \Gamma^{-1}(L + I)\, {}_0F_1(-; L + I, \frac{-z^2}{4}), \tag{2.13}$$

and

$$I_L(z) = (\frac{z}{2})^A\, \Gamma^{-1}(L + I)\, {}_0F_1(-; L + I, \frac{z^2}{4}), \tag{2.14}$$

where ${}_0F_1(-; L + I, \frac{-z^2}{4})$ is a hypergeometric matrix function of 1-denominator

$${}_0F_1(-; L + I; \frac{-z^2}{4}) = \sum_{n\geq 0} \frac{[(L + I)_n]^{-1}}{n!} \Big(\frac{-z^2}{4}\Big)^n.$$

Recently, the authors of [17] introduced an extension of the generalized hypergeometric matrix function ${}_pR_q(P, Q; z)$ with regard to the matrices occurring in its series representation. Furthermore, they provided integral representations, contiguous matrix function relations, and differential formulas satisfied by the matrix function ${}_pR_q(P, Q; z)$ and they used the notation (P) to denote the array of $p \times p$ matrices $P_1, P_2, \ldots, P_k$ for some $k \in \mathbb{N}$.

For $1 \leq i \leq p, 1 \leq j \leq q$, suppose that P, Q, S_i and D_j are positive stable matrices in $\mathbb{C}^{h\times h}$ such that $D_j + kI$ are invertible for all integers $k \geq 0$, then the matrix function denoted by ${}_pR_q(P, Q : (S), (D); z)$ is defined as

$$\begin{aligned} {}_pR_q(P, Q : (S), (D); z) &= {}_pR_q\left(\begin{matrix} S_1, \ldots, S_p \\ D_1, \ldots, D_q \end{matrix} \middle| P, Q; z\right) \\ &= \sum_{n\geq 0} \Gamma^{-1}(nP + Q)\, (S_1)_n \ldots \big(S_p\big)_n \\ &\quad \times (D_1)_n^{-1} \ldots \big(D_q\big)_n^{-1} \frac{z^n}{n!} = {}_pR_q\left[\begin{matrix}\mathbf{S_p}\\ \mathbf{D_q}\end{matrix}\middle| P; Q; z\right], \end{aligned} \tag{2.15}$$

whenever the series converges absolutely and $\mathbf{S_p} = S_1, \ldots, S_p$, $\mathbf{D_q} = D_1, \ldots, D_q$.

3. The incomplete exponential matrix functions

Let L be a matrix in $\mathbb{C}^{h\times h}$. We define the incomplete exponential matrix functions (IEMFs) as follows:

$$e\left[(L;(x,z))\right] = \sum_{n=0}^{\infty} \Gamma^{-1}(L+nI)\gamma(L+nI;x)\frac{z^n}{n!} \tag{3.1}$$

and

$$E\Big[[L;(x,z)]\Big] = \sum_{n=0}^{\infty} \Gamma^{-1}(L+nI)\,\Gamma(L+nI;x)\frac{z^n}{n!}, \tag{3.2}$$

so that

$$e\Big[(L;(x,z))\Big] + E\Big[[L;(x,z)]\Big] = e^{zI}. \tag{3.3}$$

Next, some important properties of the IEMF are listed below.

Lemma 3.1. *Let L be a matrix in $\mathbb{C}^{h\times h}$. For the two IEMFs; $e\left[(L;(x,z))\right]$ and $E\Big[[L;(x,z)]\Big]$ defined in* (3.1) *and* (3.2)*, respectively, then the following integral representations hold:*

(i)

$$\begin{aligned} e\Big[(L;(x,z))\Big] &= \Gamma^{-1}(L)\int_0^x u^{L-I}e^{-u}\Big(\sum_{n=0}^{\infty}\Big[(L)_n\Big]^{-1}\frac{(uz)^n}{n!}\Big)du \\ &= \Gamma^{-1}(L)\int_0^x u^{L-I}e^{-u}{}_0F_1(-,L;zu)du. \end{aligned}$$

(ii)

$$\begin{aligned} E\Big[(L;(x,z))\Big] &= \Gamma^{-1}(L)\int_x^{\infty} u^{L-I}e^{-u}\Big(\sum_{n=0}^{\infty}\Big[(L)_n\Big]^{-1}\frac{(tz)^n}{n!}\Big)dt \\ &= \Gamma^{-1}(L)\int_x^{\infty} u^{L-I}e^{-u}\ {}_0F_1(-,L;zt)dt. \end{aligned}$$

Proof. By replacing the incomplete gamma matrix functions in (3.1) and (3.2), respectively, by their integral representations, we obtain the integral representations of (i) and (ii). □

Lemma 3.2. *Let L be a matrix in $\mathbb{C}^{h\times h}$, then the two IEMFs; $e\left[(L;(x,z))\right]$ and $E\Big[[L;(x,z)]\Big]$ defined in* (3.1) *and* (3.2)*, respectively, satisfy the differential properties:*

(i) $\frac{\partial}{\partial z}e\Big[(L;(x,z))\Big] = \sum_{n=0}^{\infty}\Gamma^{-1}(L+nI)\gamma(L+nI;x)\frac{z^{n-1}}{(n-1)!}$,

(ii) $\frac{\partial}{\partial z}E\Big[(L;(x,z))\Big] = \sum_{n=0}^{\infty}\Gamma^{-1}(L+nI)\,\Gamma(L+nI;x)\frac{z^{n-1}}{(n-1)!}$,

(iii) $\frac{\partial}{\partial z}e\Big[(L;(x,z))\Big] = e\Big[(L+I;(x,z))\Big]$,

(iv) $\frac{\partial}{\partial z}E\Big[(L;(x,z))\Big] = E\Big[(L+I;(x,z))\Big]$.

Proof. We differentiate (3.1) and (3.2) with respect to z, to conclude (i) and (ii) respectively. To verify (iii), we replace n by $n-1$ in (i) and L by $L+I$, Changing n to $n-1$ in (ii) and L to $L+I$, immediately imply (iv). □

Remark 3.3. *By the integral representation in Lemma 3.2, we define the incomplete matrix exponential by using Bessel matrix function in the form:*

$$\begin{aligned} e\Big[(L;(x,z))\Big] &= z^{-\frac{L}{2}} \int_0^x t^{\frac{L}{2}} e^{-t} I_{(L)}\left(2\sqrt{2t}\right) dt, \\ E\Big[(L;(x,z))\Big] &= z^{-\frac{L}{2}} \int_x^\infty t^{\frac{L}{2}} e^{-t} I_{(L)}\left(2\sqrt{2t}\right) dt, \\ e\Big[(L+I;(x,-z))\Big] &= z^{-\frac{L}{2}} \int_0^x t^{\frac{L}{2}} e^{-t} J_L\left(2\sqrt{2t}\right) dt, \end{aligned}$$

and

$$E\Big[(L+I;(x,-z))\Big] = z^{-\frac{L}{2}} \int_x^\infty t^{\frac{L}{2}} e^{-t} J_L\left(2\sqrt{2t}\right) dt. \tag{3.4}$$

Now, we provide the definition of the incomplete exponential of R-matrix function as

Definition 3.4. *Let $L,\ M,\ \mathbf{C_p},\ \mathbf{D_q}$ in $\mathbb{C}^{h\times h}$ such that $\mathbf{D_q}+\mathbf{I}$ satisfying the condition (2.3), then we define the incomplete exponential of R-matrix functions as*

$$\begin{aligned} {}_pe_q\Big[((L,M;x);z)\Big] &= {}_pe_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle| L,M;x;z\right] \\ &= \sum_{n=0}^{\infty} \Gamma^{-1}(nL+M)\gamma(nL+M;x)(\mathbf{C_p})_n\Big[(\mathbf{D_q})_n\Big]^{-1}\frac{z^n}{n!} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} {}_pE_q\Big[(L,M;x;z)\Big] &= {}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle| L,M;x;z\right] \\ &= \sum_{n=0}^{\infty} \Gamma^{-1}(nL+M)\Gamma(nL+M;x)(\mathbf{C_p})_n\Big[(\mathbf{D_q})_n\Big]^{-1}\frac{z^n}{n!}. \end{aligned} \tag{3.6}$$

Using (3.5) and (3.6), we obtain the following decomposition formula:

$${}_pe_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle| L,M;x;z\right] + {}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle| L,M;x;z\right] = {}_pF_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle| z\right], \tag{3.7}$$

where ${}_pF_q(.)$ is the generalized hypergeometric matrix function defined in [18].

Remark 3.5. *For $p=0,\ q=0,\ L=I$, the expressions* (3.5) *and* (3.6) *reduce to the incomplete exponential matrix functions in* (3.1) *and* (3.2) *as*

$$\begin{aligned} {}_pe_q\Big[(I,M;x;z)\Big] &= {}_0e_0\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle| I,M;x;z\right] \\ &= \sum_{n=0}^{\infty} \Gamma^{-1}(nI+M)\gamma(nI+M;x)\frac{z^n}{n!} \\ &= e\Big[((M,x);z)\Big] \end{aligned} \tag{3.8}$$

and

$$
\begin{aligned}
{}_0E_0\Big[(I,M;x;z)\Big] =&{}_0E_0\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| I,M;x;z\right]\\
=&\sum_{n=0}^{\infty}\Gamma^{-1}(nI+B)\Gamma(nI+M;x)\frac{z^n}{n!}\\
=&E\Big[(M,x;z)\Big].
\end{aligned}
\tag{3.9}
$$

3.1. Integral representations of the IEMFs

In the current section, we deduce several integral representations of the incomplete exponential of R-matrix functions.

Theorem 3.6. *The incomplete exponential of R-matrix function* ${}_pE_q\Big[(L,M;x;z)\Big]$ *matrix function satisfies the following integral representations:*

$$
\begin{aligned}
{}_pE_q\Big[(L,M;x;z)\Big] =&{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| L,M;x;z\right]\\
=&\int_x^{\infty} t^{M-I}e^{-t}{}_pR_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| L,M;zt^L\right]dt,
\end{aligned}
\tag{3.10}
$$

where $L, M, \mathbf{C_p}$ *and* $\mathbf{D_q}$ *are commuting matrices in* $\mathbb{C}^{h\times h}$*,* $\beta(M) > 0, \beta(nL+M) > 0, \beta(\mathbf{C_p}) > 0$ *and* $\mathbf{D_p}+\mathbf{I}$ *satisfies the condition (2.3).*

Proof. By using the definition of complement of gamma matrix function defined by (2.8), we obtain

$$
\begin{aligned}
{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| L,M;x;z\right] =&\int_x^{\infty} t^{nL+M-I}e^{-t}\\
&\left[\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{z^n}{n!}\right]dt.
\end{aligned}
\tag{3.11}
$$

By reversing the order of summation and integration and using Lemma 6 in [19], we find

$$
\begin{aligned}
{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| L,M;x;z\right] =&\int_x^{\infty} t^{M-I}e^{-t}\\
&\left[\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{(zt^L)^n}{n!}\right]dt,
\end{aligned}
$$

and this completes the proof of Theorem 3.6. □

From the proof of the previous Theorem 3.6, we conclude the following result.

Corollary 3.7.

(i) *By setting* $L = I$*,* $M = C$ *and* $p = 1$*,* $q = 0$*, then,* $C_1 = \mathbf{A}$ *in* (3.11) *and noting that all matrices are commutative, we get*

$$
{}_1E_0\left[\begin{array}{c}\mathbf{A}\\ -\end{array}\Big| I,C;x;z\right] = \Gamma^{-1}(C)\int_x^{\infty}\left[u^{C-I}e^{-u}{}_1F_1\left[\begin{array}{c}\mathbf{A}\\ \mathbf{C}\end{array}\Big| zu\right]\right]du,
\tag{3.12}
$$

where ${}_1F_1$ *is hypergeometric matrix function in* (2.10)

(ii) *From R-matrix function, we have the integral matrix representation as*

$$ {}_pR_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| L,M;z\right]=\Gamma^{-1}(C_1)\int_0^\infty\left[u^{C_1-I}e^{-u}\,{}_{p-1}R_q\left[\begin{array}{c}\mathbf{C_{p-1}}\\ \mathbf{C_q}\end{array}\Big| L,M;u^z\right]\right]du, \tag{3.13}$$

where $p\le q+1$.

Theorem 3.8. *Let $L, M, \mathbf{C_p}$ and $\mathbf{D_q}$ be commuting matrices in $\mathbb{C}^{h\times h}$, $\beta(M)>0, \beta(nL{+}M)>0, \beta(\mathbf{C_p})>0$ and $\mathbf{D_q}+\mathbf{I}$ satisfying the condition (2.3), then the incomplete exponential of R-matrix function ${}_pE_q\Big[(L,M;x;z)\Big]$ matrix function have the following integral representation:*

$$\begin{aligned}{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| L,M;x,z\right]=&\Gamma^{-1}(C_1)\Gamma^{-1}(D_1-C_1)\Gamma(D_1)\int_0^1\Bigg[t^{C_1-I}(1-t)^{D_1-C_1-I}\\ &{}_{p-1}E_{q-1}\left[\begin{array}{c}\mathbf{C_{p-1}}\\ \mathbf{C_{q-1}}\end{array}\Big| L,M;x,zt\right]\Bigg]dt.\end{aligned}\tag{3.14}$$

Proof. By using integral definition of Beta matrix function

$$(\mathbf{M})_n\,[(\mathbf{C})_n]^{-1}=\Gamma^{-1}(M)\Gamma^{-1}(C-M)\Gamma(C)\int_0^1 t^{M+(n-1)I}(1-u)^{C-BM-I}dt. \tag{3.15}$$

By substituting (3.15) in (3.14), it follows that

$$\begin{aligned}{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| L,M;x,z\right]=&\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\Gamma(nL+M;x)\\ &(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{z^n}{n!}\\ =&\Gamma^{-1}(C_1)\Gamma^{-1}(D_1-C_1)\Gamma(D_1)\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\\ &\Gamma(nL+M;x)(\mathbf{C_{p-1}})_n\left[(\mathbf{D_{q-1}})_n\right]^{-1}\frac{z^n}{n!}\\ &\int_0^1 t^{C_1+(n-1)I}(1-u)^{D_1-C_1-I}dt.\end{aligned}\tag{3.16}$$

Further simplification and reversing the order and integration leads to the required result. □

3.2. Differential formulas of the IEMFs

Theorem 3.9. *For R-matrix function the incomplete exponential have the following derivative formula.*

$$\begin{aligned}\frac{d^n}{dz^n}\left\{{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\Big| L,M;x,z\right]\right\}=&(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\\ &{}_pE_q\left[\begin{array}{c}\mathbf{C_p}+nI\\ \mathbf{D_q}+nI\end{array}\Big| L,L+M;x,z\right],\end{aligned}\tag{3.17}$$

where $L,\ M, \mathbf{C_p}, \mathbf{D_q}$ are commuting matrices $\in\mathbb{C}^{h\times h}$.

Proof. From Eq (3.6) by differentiating with respect to z and replacing n by $n+1$, we get

$$\begin{aligned}\frac{d}{dz}\left\{{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle|; L, M; x, z\right]\right\} &= \sum_{n=0}^{\infty}\Gamma^{-1}\Big(L(n+1)+M\Big)\\ &\Gamma\Big(L(n+1)+M; x\Big)(\mathbf{C_p})_{n+1}\left[(\mathbf{D_q})_{n+1}\right]^{-1}\frac{z^n}{n!},\end{aligned}\tag{3.18}$$

using the relation $(L)_{n+1} = L(L+I)_n$, we find that

$$\begin{aligned}\frac{d}{dz}\left\{{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle|; L, BM; x, z\right]\right\} &= (\mathbf{C_p})\left[(\mathbf{D_q})\right]^{-1}\\ &{}_pE_q\left[\begin{array}{c}\mathbf{C_p}+I\\ \mathbf{D_q}+I\end{array}\middle| L, L+M; x, z\right].\end{aligned}\tag{3.19}$$

By repeating above procedure n-times yields the R.H.S. of assertion (3.17). □

Theorem 3.10. *Suppose that $L, M, \mathbf{C_p}$ and $\mathbf{D_q}$ are commuting matrices in $\mathbb{C}^{h\times h}$ and $\mathbf{D_q}+\mathbf{I}$ satisfying the condition (2.3), then, the incomplete exponential of R-matrix function ${}_pE_q\Big[(L, M; x; z)\Big]$ matrix function have the following partial derivatives holds true:*

$$\frac{\partial}{\partial z}\left\{{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle|; L, M; x, z\right]\right\} = (\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\ {}_pE_q\left[\begin{array}{c}\mathbf{C_p}+I\\ \mathbf{D_q}+I\end{array}\middle| L, L+M; x, z\right]\tag{3.20}$$

and

$$\frac{\partial}{\partial z}\left\{{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle|; L, M; x, z\right]\right\} = e^{-x}x^{B-I}{}_pE_q\left[\begin{array}{c}\mathbf{C_p}+I\\ \mathbf{D_q}+I\end{array}\middle| L, L+M; x, zx^A\right].\tag{3.21}$$

Proof. Differentiating partially (3.6) with respect to z, it follows that:

$$\begin{aligned}&\frac{\partial}{\partial z}\left\{{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle|; L, M; x, z\right]\right\}\\ &= \frac{\partial}{\partial z}\left\{\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\ \ \Gamma(nL+M; x)\ (\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{z^n}{n!}\right\}\\ &= \sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\ \Gamma(nL+M; x)(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{z^{n-1}}{(n-1)!}.\end{aligned}\tag{3.22}$$

By replacing n by $n+1$ in the Eq (3.21). For the proof (3.22), By using differentiate partially first integral representation (3.11) with respect to x. □

4. Generating functions of the incomplete exponential of R-matrix functions

This section is devoted to exploring some generating functions of the incomplete exponential of R-matrix functions. Furthermore, several linear generating relations of the R-matrix function of the incomplete exponential function is deduced.

Theorem 4.1. *Let $L, M, \mathbf{C_p}$, and $\mathbf{D_q}$ are commuting matrices in $\mathbb{C}^{h\times h}$ such that $D_j + kI, \quad 1 < j < q$ are inevitable for all integers $k \geq 0$. Then, the generating function of the incomplete exponential of R-matrix functions is given as:*

$$\begin{aligned}\int_0^u t^{L-I}(u-t)^{M-I}{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle|; L, M; x, \lambda t^k\right]dt = \mathfrak{B}(L, M)\ \ u^{L+M-I}\\ {}_{p+k}E_{q+k}\left[\begin{array}{c}\mathbf{C_p}, \Delta(k, L)\\ \mathbf{D_q}, \Delta(k, L+M)\end{array}\middle|; L, M; x, \lambda u^k\right],\end{aligned} \tag{4.1}$$

where k is a positive integer and $\Delta(k, L)$ represents the sequence matrix of k parameters as

$$\frac{L}{k},\ \ \frac{L+I}{k},\ \ \frac{L+2I}{k}, \ldots,\ \ \frac{L+(k-1)I}{k}.$$

Proof. Let w_1 be the left hand side of Eq (4.1). Then, by using (3.6), this gives

$$\begin{aligned}w_1 = \int_0^u t^{L-I}(u-t)^{M-I}\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\ \ \Gamma(nL+M;x)\\ \times(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{(\lambda t^k)^n}{n!}dt.\end{aligned} \tag{4.2}$$

By substituting $t = ux$, we have

$$\begin{aligned}w_1 =& u^{L+M-I}\int_0^1 x^{L+(kn-1)I}(1-t)^{M-I}\Gamma^{-1}(nL+M)\\ &\Gamma(nL+M;x)(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{(\lambda t^k)^n}{n!}dx\\ =& u^{L+M-I}\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\ \ \Gamma(nL+M;x)\\ &\times(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\ \ \Gamma^{-1}(L+M+knI)\ \ \Gamma(L+knI)\ \ \Gamma(M)\ \ \frac{(\lambda t^k)^n}{n!}.\end{aligned} \tag{4.3}$$

Therefore, using the property Pochhammer matrix symbol leads directly to the right hand side of Eq (4.1).

□

Theorem 4.2. *Let $L, M, \mathbf{C_p}$, and $\mathbf{D_q}$ are commuting matrices in $\mathbb{C}^{h\times h}$ such that $D_j + kI, \quad 1 < j < q$ are inevitable for all integers $k \geq 0$. A linear generating relation for the R-matrix function of incomplete exponential can be given as:*

$$\begin{aligned}&\int_t^x (x-u)^{E-I}(u-t)^{M-I}{}_pE_q\left[\begin{array}{c}\mathbf{C_p}\\ \mathbf{D_q}\end{array}\middle|; L, M; x, \lambda(u-t)^k\right]du\\ &= \Gamma(E)\ \Gamma(M)\ \Gamma^{-1}(M+E)(x-t)^{E+M-I}\\ &{}_{p+k}E_{q+k}\left[\begin{array}{c}\mathbf{C_p}, \Delta(k, L)\\ \mathbf{D_q}, \Delta(k, E+M)\end{array}\middle|; L, M; x, \lambda(x-t)^k\right].\end{aligned} \tag{4.4}$$

Proof. Let w_2 be the left hand side of Eq (4.4) and by using (3.6), we obtain

$$\begin{aligned} w_2 =&\Gamma^{-1}(M)\,\Gamma^{-1}(E)\,\Gamma(M+E)\int_t^x (x-u)^{E-I}(u-t)^{M-I} \\ &\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\,\Gamma(nL+M;x)(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{(\lambda(u-t)^k)^n}{n!}dt. \end{aligned} \tag{4.5}$$

Now, by substituting $m=\frac{u-t}{x-t}$, it follows that

$$\begin{aligned} w_2 =&\Gamma^{-1}(M)\,\Gamma^{-1}(E)\,\Gamma(M+E)(x-t)^{E+M-I}\int_0^1 m^{M+(kn-1)I}(1-m)^{E-I} \\ &\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\,\Gamma(nL+M;x)(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{(\lambda(x-t)^k)^n}{n!}dm \\ =&\Gamma^{-1}(M)\,\Gamma^{-1}(E)\,\Gamma(M+E)(x-t)^{E+M-I}\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\,\Gamma(nL+M;x) \\ &\Gamma^{-1}(E+M+knI)\,\Gamma(M+knI)\,\Gamma(E)(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{(\lambda(x-t)^k)^n}{n!}. \end{aligned} \tag{4.6}$$

Thus, using the properties of Pochhammer matrix implies the right-hand side of (4.4). □

Now, we provide some linear generating relations for the R-matrix function of generalized incomplete exponential as follows.

Theorem 4.3. *Let $L, M, \mathbf{C_p}$, and $\mathbf{D_q}$ are commuting matrices in $\mathbb{C}^{h\times h}$ such that $D_j+kI,\quad 1<j<q$ are inevitable for all integers $k\geq 0$. Then, the following linear generating relation for the R-matrix function of incomplete exponential hold:*

$$\begin{aligned} &\sum_{n=0}^{\infty}\begin{bmatrix}(k-1)I-E\\ k\end{bmatrix}{}_pe_{q+1}\left[\begin{matrix}\mathbf{C_p}\\ \mathbf{D_q}, E-(k-1)I\end{matrix}\middle|; L, M; x, z\right]t^k \\ &= (1-t)^E\;{}_pe_{q+1}\left[\begin{matrix}\mathbf{C_p}\\ \mathbf{D_q}, I-E\end{matrix}\middle|; L, M; x, z(1-t)\right], \end{aligned} \tag{4.7}$$

where $|z|<1$.

Proof. Let w_3 be the left-hand side of (4.7) and by applying (3.5), we obtain that

$$\begin{aligned} w_3 =& \sum_{k=0}^{\infty}\begin{bmatrix}(k-1)I-E\\ k\end{bmatrix}\Bigg(\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\,\Gamma(nL+M;x) \\ &\left[\Big(E-(k-1)I\Big)_n\right]^{-1}(\mathbf{C_p})_n\left[(\mathbf{D_q})_n\right]^{-1}\frac{z^n}{n!}\Bigg)t^k. \end{aligned} \tag{4.8}$$

By reversing the order of summation and using the relation

$$\begin{aligned} \Big(E-(k-1)I\Big)_n =&(E+I)_n\begin{bmatrix}(k-1)I-E\\ k\end{bmatrix}.\begin{bmatrix}(k-n-1)I-E\\ k\end{bmatrix}^{-1} \\ &k,n\in N, \end{aligned} \tag{4.9}$$

where

$$\left(\begin{array}{c} E \\ k \end{array} \right) = \Gamma^{-1}(k+1)\, \Gamma^{-1}(E-(k-1)I))\, \Gamma(E+I), \tag{4.10}$$

we obtain

$$\begin{aligned} w_3 = \sum_{n=0}^{\infty} &\Gamma^{-1}(nL+M)\, \Gamma(nL+M;x)\, [(E-k)_n]^{-1}\, (\mathbf{C_p})_n \left[(\mathbf{D_q})_n\right]^{-1} \\ &\frac{z^n}{n!} \sum_{k=0}^{\infty} \left[\begin{array}{c} (k-n-1)I - E \\ k \end{array} \right] t^k. \end{aligned} \tag{4.11}$$

Moreover, we find the inner sum in (4.11), by using the relation

$$\sum_{k=0}^{\infty} \left[\begin{array}{c} (k-1)I - E \\ k \end{array} \right] t^k = (1-t)^E. \tag{4.12}$$

From (4.11) and (4.12), the right hand side of (4.7) yields. □

Theorem 4.4. *Let $L, M, \mathbf{C_p}$, and $\mathbf{D_q}$ are commuting matrices in $\mathbb{C}^{h\times h}$ such that $D_j + kI, \quad 1 < j < q$ are inevitable for all integers $k \geq 0$. Then*

$$\begin{aligned} &\sum_{k=0}^{\infty} \left[\begin{array}{c} (k-1)I - E \\ k \end{array} \right] {}_pE_{q+1} \left[\begin{array}{c} \mathbf{C_p} \\ \mathbf{D_q}, E-(k-1)I \end{array} \middle| ; L, M; x, z \right] t^k \\ &= (1-t)^E\ {}_pE_{q+1} \left[\begin{array}{c} \mathbf{C_p} \\ \mathbf{D_q}, I-E \end{array} \middle| ; L, M; x, z(1-t) \right], \end{aligned} \tag{4.13}$$

where $|z| < 1$.

Proof. The proof here runs similarly to the the proof of Theorem 4.3. The details are omitted. □

Remark 4.5. *If we add the generating Eqs* (4.7)*,* (4.13) *and using* (3.7)*, then we have the following generating form as*

$$\begin{aligned} &\sum_{k=0}^{\infty} \left[\begin{array}{c} (k-1)I - E \\ k \end{array} \right] {}_pF_{q+1} \left[\begin{array}{c} \mathbf{C_p} \\ \mathbf{D_q}, E-(k-1)I \end{array} \middle| z \right] t^k \\ &= (1-t)^E\ {}_pF_{q+1} \left[\begin{array}{c} \mathbf{C_p} \\ \mathbf{D_q}, E-(k-1)I \end{array} \middle| z(1-t) \right]. \end{aligned} \tag{4.14}$$

5. Special cases

In this section, we discuss some special cases of generalized incomplete exponential matrix functions as stated in the following theorems:

Theorem 5.1. *Let $L, M, \mathbf{C_p}$, and $\mathbf{D_q}$ are commuting matrices in $\mathbb{C}^{h\times h}$ such that $D_j + kI, \quad 1 < j < q$ are inevitable for all integers $k \geq 0$. Then*

$$\begin{aligned} {}_2E_1 \left[\begin{array}{c} \mathbf{C_1}, \mathbf{C_2} \\ \mathbf{D_1} \end{array} \middle| L, M; x; 1 \right] &= \Gamma^{-1}(D_1 - C_1)\, \Gamma^{-1}(D_1 - C_2) \Gamma(D_1) \\ &\Gamma(D_1 - C_1 - C_2)\ - \gamma(nL+M;x)\ {}_2R_1 \left[\begin{array}{c} - \\ - \end{array} \middle| L, M; z \right]. \end{aligned} \tag{5.1}$$

Proof. Putting $z = 1,\ p = 2,\ q = 1$ in the decomposition formula (3.7) implies that

$$
\begin{aligned}
&{}_2E_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|L,M;x;1\right]\\
&= {}_2F_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|1\right] - {}_2e_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|L,M;x,z\right]\\
&= {}_2F_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|1\right] - \int_0^x t^{M-I}e^{-t}\ {}_2R_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|L,M;zt^L\right]dt \qquad (5.2)\\
&= {}_2F_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|1\right] - \int_0^x t^{M-I}e^{-t}\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\,[(\mathbf{D_1})_n]^{-1}\\
&\quad x(\mathbf{C_1})_n(\mathbf{C_2})_n\frac{(zt^L)^n}{n!}dt.
\end{aligned}
$$

□

By using the relation of Gauss matrix summation [18] and reversing the order of summation and integration, we find that

$$
\begin{aligned}
&{}_2E_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|L,M;x;1\right] = \Gamma^{-1}(D_1-C_1)\,\Gamma^{-1}(D_1-C_2)\,\Gamma(D_1)\\
&\Gamma(D_1-C_1-C_2)\ \sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\,[(\mathbf{D_1})_n]^{-1} \qquad (5.3)\\
&(\mathbf{C_1})_n(\mathbf{C_2})_n\frac{z^n}{n!}\int_0^x t^{nL+M-1}e^{-t}dt.
\end{aligned}
$$

Further simplification leads to the right-hand side of (5.1).

Theorem 5.2. *Let $L, M, \mathbf{C_p}$, and $\mathbf{D_q}$ be commuting matrices in $\mathbb{C}^{h\times h}$ such that $D_j + kI$, $\quad 1 < j < q$ are inevitable for all integers $k \geq 0$. Then*

$$
\begin{aligned}
&(C_1 - D_1 + I)\ {}_2E_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|L,M;x;z\right]\\
&= {}_2E_1\left[\begin{matrix}\mathbf{C_1}+\mathbf{I},\mathbf{C_2}\\ \mathbf{D_1}\end{matrix}\ \middle|L,M;x;z\right] - (D_1 - I)\ {}_2E_1\left[\begin{matrix}\mathbf{C_1},\mathbf{C_2}\\ \mathbf{D_1}-\mathbf{I}\end{matrix}\ \middle|L,M;x;z\right].
\end{aligned} \qquad (5.4)
$$

Proof. Let w_4 be the L.H.S of (5.4), then by using (3.6), we find that

$$
\begin{aligned}
w_4 = &\sum_{n=0}^{\infty}\Gamma^{-1}(nL+M)\ \ \Gamma[nL+M;x]\,[(D_1)_n]^{-1}\,C_1\,(C_1+I)_n\\
&(C_2)_n\,(D_1-I)\,[(D_1-I)_n]^{-1}\,\frac{z^n}{n!}.
\end{aligned} \qquad (5.5)
$$

Employing the relation of Pochhammer matrix implies that

$$C_1\,(C_1+I)_n = (C_1+nI)\,(C_1)_n$$

and

$$(D_1-I)\,(D_1)_n = (D_1-I)_n\,(D_1+(n-1)I)\,.$$

This yields the left side of (5.4). □

6. Conclusions

The incomplete exponential type of R-matrix function is exhibited in the current study. Several characterizations of the proposed incomplete exponential ${}_pR_q(P, Q; z)$ matrix functions such that the integral representation, the derivative formulas and generating functions of the incomplete exponential of R-matrix functions. We conclude our study by presenting special cases of the obtained results. The findings of the present paper can be extended to obtain some interesting new results by fitting some suitable parameters.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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