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Research article

Homotopic morphisms between weighted digraphs

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Abstract: In this paper, we consider weighted digraphs and prove that homotopic morphisms between weighted digraphs induce the identical homomorphisms on weighted path homology groups with field coefficients.

Keywords: weighted digraph; weighted path homology; cartesian product; one-step homotopic
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1. Introduction

Digraph is an important topology model of complex networks. For example, taking Alipay users as the vertices, we determine the directed edges by the relationship of paying, so the complex network composed by Alipay users can be regarded as a digraph.

In 2012, A. Grigor'yan, Y. Lin, Y. Muranov and S.T. Yau first defined and studied the path homology of digraphs [1]. Subsequently, there are many references about path (co)homology of digraphs [2–6] and meanwhile homotopy of digraphs is developed [7, 8].

Weighted structure can provide more information about mathematical objects. We can take the degree of each vertex in the digraph as the vertex weight. In the actual complex network, the weight of nodes shows the importance of nodes in the entire digraph. For example, if we regard the urban traffic network of a country as a digraph, the communication between each city and other cities reflects the importance of its traffic hub to some extent.

In this paper, we mainly consider homotopy invariance of weighted digraphs in Theorem 3.1 based on the definitions in [9].

2. Preliminaries

In this section, we review some relative definitions of path homology of vertex-weighted digraphs in [9], and similar to [7], we give the notion of homotopy of morphisms between vertex-weighted digraphs. Let R be an integral domain with unit 1.

2.1. Weighted path homology and morphisms between weighted digraphs

A digraph G is a pair determined by a finite set V and an ordered binary subset E of $V \times V$ where V is called the vertex set of G and E is called the directed edge set of G. For each $n \geq 0$, an *elementary n-path* (abbreviated as n-path) on V is a sequence $v_0 v_1 \cdots v_n$ of vertices in V. If all pairs $(v_i, v_{i+1}) \in E$ where v_i and v_{i+1} are assumed to be distinct for each $0 \leq i \leq n-1$, then the n-path is called *allowed*. Let $\Lambda_n(V)$ be the free R-module consisting of all the formal linear combinations of the n-paths on V. Let $\mathcal{A}_n(G)$ be the free R-module consisting of all the formal linear combinations of allowed elementary n-paths on G. Then $\mathcal{A}_n(G)$ is a sub-R-module of $\Lambda_n(V)$.

A *weighted digraph* is a digraph G with an R-valued weight function $w : V \longrightarrow R$, which is simply denoted as G if there is no danger of confusion. The weighted boundary map $\partial_n^w : \Lambda_n(V) \longrightarrow \Lambda_{n-1}(V)$ is defined as

$$\partial_n^w(v_0 v_1 \ldots v_n) = \sum_{i=0}^{n} w(v_i)(-1)^i d_i(v_0 v_1 \ldots v_n)$$

where d_i is the face map given by

$$d_i(v_0 v_1 \ldots v_n) = v_0 v_1 \ldots \hat{v}_i \ldots v_n.$$

Note that ∂_n^w is an R-linear map from $\Lambda_n(V)$ to $\Lambda_{n-1}(V)$ satisfying $\partial_n^w \partial_{n+1}^w = 0$ for each $n \geq 0$ (cf. [1–6]). Hence $\{\Lambda_n(V), \partial_n^w\}_{n\geq 0}$ is a chain complex. We define

$$\begin{aligned} \Omega_n^w(G) &= \mathcal{A}_n(G) \cap (\partial_n^w)^{-1} \mathcal{A}_{n-1}(G), \\ \Gamma_n^w(G) &= \mathcal{A}_n(G) + \partial_{n+1}^w \mathcal{A}_{n+1}(G). \end{aligned}$$

Then as graded R-modules,

$$\Omega_*^w(G) \subseteq \mathcal{A}_*(G) \subseteq \Gamma_*^w(G) \subseteq \Lambda_*(V).$$

And as chain complexes,

$$\{\Omega_n^w(G), \partial_n^w \mid_{\Omega_n^w(G)}\}_{n\geq 0} \subseteq \{\Gamma_n^w(G), \partial_n^w \mid_{\Gamma_n^w(G)}\}_{n\geq 0} \subseteq \{\Lambda_n(V), \partial_n^w\}_{n\geq 0}.$$

By [10], the canonical inclusion

$$\iota : \Omega_n^w(G) \longrightarrow \Gamma_n^w(G), \quad n \geq 0$$

of chain complexes induces an isomorphism between the homology groups

$$\iota_* : H_m(\{\Omega_n^w(G), \partial_n^w \mid_{\Omega_n^w(G)}\}_{n\geq 0}) \xrightarrow{\cong} H_m(\{\Gamma_n^w(G), \partial_n^w \mid_{\Gamma_n^w(G)}\}_{n\geq 0}), \quad m \geq 0.$$

This isomorphism gives the *weighted path homology* of G, denoted as $H_m(G, w; R)$, $m \geq 0$.

Remark 2.1. *When all weights are* 1*, the weighted path homology is the usual path homology.*

Definition 2.1. (cf. [7, Definition 2.2]) A *morphism* from a digraph G to a digraph G' is a map $f : V(G) \longrightarrow V(G')$ such that for any directed edge $u \to v$ on G we have $f(u) \overrightarrow{=} f(v)$ on G' (that is, either $f(u) \to f(v)$ or $f(u) = f(v)$). We will refer to such morphisms also as digraphs maps (sometimes simply maps) and denote them shortly by $f : G \longrightarrow G'$.

Let $G = (V, E, w)$ and $G' = (V', E', w')$ be two weighted digraphs. A *weighted morphism* of weighted digraphs from (G, w) to (G', w') is a morphism of digraphs $f : G \to G'$ such that for any $v_i \in V$, $w(v_i) = w'(f(v_i))$.

2.2. Weighted homotopy

A *line digraph* I_n is a digraph with the vertex set $\{v_0, v_1, \dots, v_n\}$ and the directed edge set $\{v_i \to v_{i+1}$ or $v_{i+1} \to v_i, i = 0, 1, \dots, n-1\}$(cf. [7, p. 632]). Note that a path is a special line digraph with all the directed edges $v_i \to v_{i+1}$. For $0 \le i \le n$, we sometimes write v_i as i for short.

The *Cartesian product* of vertex-weighted digraph G and line digraph I_1 is a vertex-weighted digraph $G \times I_1$ such that

$$\begin{aligned} V_{G\times I_1} &= \{(v,0)\,\text{and}\,(v,1), v \in V(G)\}\text{and} \\ E_{G\times I_1} &= \{(v,0) \to (v,1)\,\text{or}\,(v,0) \to (u,0)\,\text{or}\,(v,1) \to (u,1) \mid (v,u) \in E(G)\}, \text{with} \\ w_{G\times I_1} &: V_{G\times I_1} \to R \text{ such that } w_{G\times I_1}(v,0) = w_{G\times I_1}(v,1) = w(v). \end{aligned}$$

Here I_1 is the digraph $I_1 = (0 \to 1)$ (the case $I_1^- = (1 \to 0)$ is similar).

Let $f, g : (G, w) \to (G', w')$ be two morphisms of weighted digraphs (G, w) and (G', w'). Then f, g are called *one-step homotopic* if there exists a digraph map

$$F : G \times I_1 \to G'$$

such that

$$\begin{aligned} &F\,|_{G\times\{0\}} = f \text{ and } F|_{G\times\{1\}} = g \\ &w_{G\times I_1}(v,0) = w'(F(v,0)) = w'(f(v)) \\ &w_{G\times I_1}(v,1) = w'(F(v,1)) = w'(g(v)). \end{aligned}$$

For simplicity, we denote $(v, 0)$ as v and $(v, 1)$ as v' respectively in this paper. Let $R = \mathbb{F}$ be a field. Consider the case in which the weight is a non-vanishing function on the vertex set of G. For any element $x = \sum_{i=1}^{n} a^i\sigma^i \in \Omega_p(G, w)$ where $\sigma^i = v_0^i \cdots v_j^i \cdots v_p^i \in \mathcal{A}_p$. Define

$$\begin{aligned} x \times I_1 &= \sum_{i=1}^{n} a^i \sum_{m=0}^{p} (-1)^{p-m} (w(v_m^i))^{-1} v_0^i \cdots v_m^i (v_m^i)'(v_{m+1}^i)' \cdots (v_p^i)' \\ x \times \{1\} &= \sum_{i=1}^{n} a^i (v_0^i)' ... (v_j^i)' ... (v_p^i)', \quad x \times \{0\} = \sum_{i=1}^{n} a^i v_0^i ... v_j^i ... v_p^i \end{aligned} \tag{2.1}$$

where $w(v_m^i)$ is the weight of vertex v_m^ior $(v_m^i)'$, $a^i \in \mathbb{F}$.

3. Main theorem

In this section, we will give some auxiliary results for main theorem in this paper and prove the main theorem.

Let $\mathbb{F}$ be a field. Firstly, by the definition (2.1), we can prove the following proposition.

Proposition 3.1. *Let (G, w) be a vertex-weighted digraph where $w : V(G) \longrightarrow \mathbb{F} \setminus \{0\}$ is a weight function on G. Then for any $x \in \Omega_p^w(G)$, $x \times I_1 \in \Omega_{p+1}^w(G \times I_1)$.*

Proof. Suppose $x = \sum_{i=1}^{n} a^i \sigma^i \in \Omega_p^w(G)$ where $\sigma^i = v_0^i \cdots v_p^i$ are allowed elementary p-paths on G. Denote

$$\sigma_m^i = v_0^i \cdots v_m^i (v_m^i)'(v_{m+1}^i)' \cdots (v_p^i)'.$$

According to the action of weighted boundary operator ∂^w, it is sufficient to consider the following cases.

Case 1. $j = m$. If $m = 0$, then

$$d_j(\sigma_m^i) = d_0(\sigma_0^i) = (v_0^i)'(v_1^i)' \cdots (v_p^i)' \in \mathcal{A}_p(G \times I_1);$$

If $m \geq 1$, then for each

$$d_j(\sigma_m^i) \;=\; v_0^i \cdots v_{m-1}^i (v_m^i)' \cdots (v_p^i)' \notin \mathcal{A}_p(G \times I_1),$$

we have that

$$d_m(\sigma_{m-1}^i) \;=\; d_m(\sigma_m^i)$$

and the coefficient of $d_j(\sigma_m^i)$ in $\partial^w(\sigma^i \times I_1)$ is

$$(-1)^{p-(m-1)}(w(v_{m-1}^i))^{-1}(-1)^m w(v_{m-1}^i) + (-1)^{p-m}(w(v_m^i))^{-1}(-1)^m w(v_m^i) = 0.$$

Case 2. $j = m + 1$. If $m = p$, then

$$\begin{aligned} \sigma_p^i &= v_0^i \cdots v_p^i (v_p^i)', \\ d_j(\sigma_p^i) &= v_0^i \cdots v_p^i \in \mathcal{A}_p(G \times I_1); \end{aligned}$$

If $m < p$, then for each

$$d_j(\sigma_m^i) \;=\; v_0^i \cdots v_m^i (v_{m+1}^i)' \cdots (v_p^i)' \notin \mathcal{A}_p(G \times I_1),$$

we have that

$$d_{m+1}(\sigma_{m+1}^i) \;=\; d_{m+1}(\sigma_m^i)$$

and the coefficient of $d_j(\sigma_m^i)$ in $\partial^w(\sigma^i \times I_1)$ is

$$(-1)^{p-(m+1)}(w(v_{m+1}^i))^{-1}(-1)^{m+1} w(v_{m+1}^i) + (-1)^{p-m}(w(v_m^i))^{-1}(-1)^{m+1} w(v_m^i) = 0.$$

CASE 3. $j < m$. The coefficient of $d_j(\sigma^i)$ in $\partial^w u$ is

$$\sum_{\{(k,l)|d_l(\sigma^k)=d_j(\sigma^i)\}} a^k(-1)^l w(v_l^k) = 0.$$

On the other hand, the coefficient of $d_j(\sigma_m^i)$ in $\partial^w(x \times I_1)$ is

$$\begin{aligned}
&\sum_{\{(k,l,r)|d_l(\sigma_r^k)=d_j(\sigma_m^i)\}} a^k(-1)^{p-r}(-1)^l(w(v_r^k))^{-1}w(v_l^k) \\
&= \sum_{\{(k,l)|d_l(\sigma_m^k)=d_j(\sigma_m^i)\}} a^k(-1)^{p-m}(-1)^l(w(v_m^k))^{-1}w(v_l^k) \\
&= (w(v_m^i))^{-1} \sum_{\{(k,l)|d_l(\sigma_m^k)=d_j(\sigma_m^i)\}} a^k(-1)^{p-m}(-1)^l w(v_l^k) \\
&= (-1)^{p-m}(w(v_m^i))^{-1} \sum_{\{(k,l)|d_l(\sigma_k)=d_j(\sigma_i)\}} a^k(-1)^l w(v_l^k).
\end{aligned}$$

Hence,

$$d_j(\sigma_m^i) \notin \mathcal{A}_p(G \times I_1) \Leftrightarrow d_j(\sigma^i) \notin \mathcal{A}_p(G) \text{ for } j < m.$$

Since $x \in \Omega_p^w(G)$ and $\partial^w x \in \mathcal{A}_{p-1}^w(G)$, it follows that if $d_j(\sigma_m^i) \notin \mathcal{A}_p(G \times I_1)$, its coefficient in $\partial^w(x \times I_1)$ is zero.

CASE 4. $j > m + 1$. Similar to the analysis in Case 3, we have that

$$d_j(\sigma_m^i) \notin \mathcal{A}_p(G \times I_1) \Leftrightarrow d_{j-1}(\sigma^i) \notin \mathcal{A}_p(G),$$

Then for $d_j(\sigma_m^i) \notin \mathcal{A}_p(G \times I_1)$, $d_{j-1}(\sigma^i) \notin \mathcal{A}_p(G)$. Since $x \in \Omega_p^w(G)$ and $\partial^w x \in \mathcal{A}_{p-1}^w(G)$, the coefficient of $d_{j-1}(\sigma^i)$ in $\partial^w x$ is

$$\sum_{\{(k,l)|d_{l-1}(\sigma^k)=d_{j-1}(\sigma^i)\}} a^k(-1)^{l-1}w(v_{l-1}^k) = 0.$$

Hence, the coefficient of $d_j(\sigma_m^i)$ in $\partial^w(x \times I_1)$ is

$$\begin{aligned}
&\sum_{\{(k,l,r)|d_l(\sigma_r^k)=d_j(\sigma_m^i)\}} a^k(-1)^{p-r}(-1)^l(w(v_r^k))^{-1}w(v_{l-1}^k) \\
&= (w(v_m^i))^{-1} \sum_{\{(k,l)|d_l(\sigma_m^k)=d_j(\sigma_m^i)\}} a^k(-1)^{p-m}(-1)^l w(v_{l-1}^k) \\
&= (-1)^{p-m}(w(v_m^{(i)}))^{-1} \sum_{\{(k,l)|d_{l-1}(\sigma^k)=d_{j-1}(\sigma^i)\}} a^k(-1)^l w(v_{l-1}^k) \\
&= 0.
\end{aligned}$$

Combing Case 1-Case 4, we have that for any $x \in \Omega_p^w(G)$, $\partial^w(x \times I_1) \in \mathcal{A}_p(G \times I_1)$. Therefore, $x \times I_1 \in \Omega_{p+1}^w(G \times I_1)$. □

Remark 3.1. *In Case 3 of the proof of Proposition 3.1, since* $d_l(\sigma_r^k) = d_j(\sigma_m^i)$*, it follows that* $r = m$ *and* $(w(v_m^i))^{-1} = (w(v_m^k))^{-1}$.

Secondly, by Proposition 3.1, we have that

Proposition 3.2. *(Weighted Product Rule)*

$$\partial^w(x \times I_1) = (\partial^w x) \times I_1 + (-1)^p (x \times \partial I_1) \tag{3.1}$$

where $x \in \Omega_p^w(G)$.

Proof. By Proposition 3.1, we have that $x \times I_1 \in \Omega_{p+1}^w(G \times I_1)$. By (2.1), it is sufficient to consider the following cases.

Case 1. $0 < m < p$.

Subcase 1.1. $m \geqslant 1$ and $j < m$. Then for each

$$d_j(\sigma_m^i) = v_0^i \cdots \widehat{v_j^i} \cdots v_m^i (v_m^i)' \cdots (v_p^i)' \in \mathcal{A}_p(G \times I_1),$$

the coefficient of $d_j(\sigma_m^i)$ in $\partial^w(\sigma^i \times I_1)$ is

$$(-1)^{p-m}(w(v_m^i))^{-1}(-1)^j w(v_j^i) a^i.$$

On the other hand, since $d_j(\sigma^i) = v_0^i \cdots v_{j-1}^i \widehat{v_j^i} v_{j+1}^i \cdots v_p^i$, the coefficient of $d_j(\sigma_m^i)$ in $\partial^w \sigma^i \times I_1$ is

$$(-1)^j w(v_j^i)(-1)^{(p-1)-(m-1)}(w(v_m^i))^{-1} a^i = (-1)^{p-m}(w(v_m^i))^{-1}(-1)^j w(v_j^i) a^i.$$

Subcase 1.2. $j > m+1$ and $m < p$. Then for any

$$d_j(\sigma_m^i) = v_0^i \cdots v_m^i (v_m^i)' \cdots \widehat{(v_{j-1}^i)'} \cdots (v_p^i)' \in \mathcal{A}_p(G \times I_1),$$

the coefficient of $d_j(\sigma_m^i)$ on the left side of (3.1) is

$$(-1)^{p-m}(w(v_m^i))^{-1}(-1)^j w(v_{j-1}^i)$$

while its coefficient on the right side of (3.1) is

$$(-1)^{j-1} w(v_{j-1}^i)(-1)^{p-1-m}(w(v_m^i))^{-1} = (-1)^{p-m}(w(v_m^i))^{-1}(-1)^j w(v_{j-1}^i).$$

Subcase 1.3. $j = m$ and $0 < m < p$. Then

$$\begin{aligned} d_j(\sigma_m^i) &= (-1)^j w(v_m^i)(-1)^{p-m}(w(v_m^i))^{-1} v_0^i \cdots v_{m-1}^i \widehat{v_m^i} (v_m^i)' \cdots (v_p^i)', \\ d_j(\sigma_{m-1}^i) &= (-1)^j w(v_{m-1}^i)(-1)^{p-(m-1)}(w(v_{m-1}^i))^{-1} v_0^i \cdots v_{m-1}^i \widehat{(v_{m-1}^i)'} (v_m^i)' \cdots (v_p^i)'. \end{aligned}$$

Hence, $d_j(\sigma_m^i)$ can be cancelled out in $\partial^w(x \times I_1)$.

Subcase 1.4. $j = m+1$ and $0 < m < p$. Then

$$\begin{aligned} d_j(\sigma_m^i) &= (-1)^j w(v_m^i)(-1)^{p-m}(w(v_m^i))^{-1} v_0^i \cdots v_m^i \widehat{(v_m^i)'} (v_{m+1}^i)' \cdots (v_p^i)', \\ d_j(\sigma_{m+1}^i) &= (-1)^j w(v_{m+1}^i)(-1)^{p-(m+1)}(w_{m+1}^i)^{-1} v_0^i \cdots v_m^i \widehat{(v_{m+1}^i)} (v_{m+1}^i)' \cdots (v_p^i)'. \end{aligned}$$

Hence, $d_j(\sigma^i_m)$ can be cancelled out in $\partial^w(x\times I_1)$.

CASE 2. $m=0$.

SUBCASE 2.1. $j=0$. Then the coefficient of $d_0(\sigma^i_0)=(\sigma^i)'=(v^i_0)'\cdots(v^i_p)'$ in $\partial^w(\sigma^i\times I_1)$ is

$$(-1)^{p-0}(w(v^i_0))^{-1}w(v^i_0)(-1)^0=(-1)^p$$

which consistents with the coefficient of $(\sigma^i)'$ in $(-1)^p(\sigma_i\times\partial I_1)$.

SUBCASE 2.2. $j=1$. Then

$$\begin{aligned} d_j(\sigma^i_0) &= (-1)^1 w(v^i_0)(-1)^{p-0}(w(v^i_0))^{-1}v^i_0\widehat{(v^i_0)'}(v^i_1)'\cdots(v^i_p)' \\ d_j(\sigma^i_1) &= (-1)^1 w(v^i_1)(-1)^{p-1}(w(v^i_1))^{-1}v^i_0\widehat{v^i_1}(v^i_1)'\cdots(v^i_p)' \end{aligned}$$

which implies that $d_1(\sigma^i_0)$ is cancalled out in $\partial^w(x\times I_1)$.

SUBCASE 2.3. $j>1$. Then for any

$$d_j(\sigma^i_0)=v^i_0(v^i_0)'\cdots\widehat{(v^i_{j-1})'}\cdots(v^i_p)'\in\mathcal{A}_p(G\times I_1),$$

its coefficients in $\partial^w(\sigma^i\times I_1)$ and $\partial^w\sigma^i\times I_1$ are

$$(-1)^{p-0}(w(v^i_0))^{-1}(-1)^j w(v^i_{j-1})$$

and

$$(-1)^{p-1-0}(w(v^i_0))^{-1}(-1)^{j-1}w(v^i_{j-1})$$

respectively. Hence, they are the same.

For any

$$d_j(\sigma^i_0)=v^i_0(v^i_0)'\cdots\widehat{(v^i_{j-1})'}\cdots(v^i_p)'\notin\mathcal{A}_p(G\times I_1),$$

by the proof of Proposition 3.1, it must be cancelled out in $\partial^w(x\times I_1)$.

CASE 3. $m=p$.

SUBCASE 3.1. $j=p+1$. Then the coefficient of $d_{p+1}(\sigma^i_p)=\sigma^i=v^i_0\cdots v^i_p$ in $\partial^w(\sigma^i\times I_1)$ is

$$(-1)^{p-p}(w(v^i_p))^{-1}w(v^i_p)(-1)^{p+1}\quad=(-1)^{p+1}$$

which consistents with the coefficient of σ^i in $(-1)^p(\sigma^i\times\partial I_1)$.

SUBCASE 3.2. $j=p$. Then

$$\begin{aligned} d_j(\sigma^i_p) &= (-1)^p w(v^i_p)(-1)^{p-p}(w(v^i_p))^{-1}v^i_0\cdots v^i_{p-1}\widehat{(v^i_p)}(v^i_p)' \\ d_j(\sigma^i_{p-1}) &= (-1)^p w(v^i_{p-1})(-1)^{p-(p-1)}(w(v^i_{p-1}))^{-1}v^i_0\cdots v^i_{p-1}\widehat{(v^i_{p-1})'}(v^i_p)' \end{aligned}$$

which implies that $d_p(\sigma^i_p)$ is cancalled out in $\partial^w(x\times I_1)$.

SUBCASE 3.3. $j < p$. Then for any

$$d_j(\sigma^i_p) = v^i_0 \cdots \widehat{v^i_j} \cdots v^i_p (v^i_p)' \in \mathcal{A}_p(G \times I_1),$$

its coefficients in $\partial^w(\sigma^i \times I_1)$ and $\partial^w \sigma^i \times I_1$ are

$$(-1)^{p-p}(w(v^i_p))^{-1}(-1)^j w(v^i_j)$$

and

$$(-1)^j w(v^i_j)(-1)^{(p-1)-(p-1)}(w(v^i_p))^{-1}$$

respectively. Obviously, they are the same.

For any

$$d_j(\sigma^i_p) = v^i_0 \cdots \widehat{v^i_j} \cdots v^i_p (v^i_p)' \notin \mathcal{A}_p(G \times I_1),$$

by the proof of Proposition 3.1, it must be cancelled out in $\partial^w(x \times I_1)$.

Summarizing Case 1 to Case 3, the proposition follows. □

Finally, by Proposition 3.2, we can prove the main theorem of this paper.

Theorem 3.1. *Let $(G, w)(G', w')$ be two vertex weighted digraphs. Let $f, g : G \to G'$ be two weighted homotopic morphisms between G and G'. Then they can induce identical homomorphisms of weighted homology groups of G and G'. More precisely, the following maps*

$$(f_*)_p : H_p(G, w) \to H_p(G', w') \qquad (g_*)_p : H_p(G, w) \to H_p(G', w').$$

are identical for each $p \geq 0$.

Proof. Suppose f and g are one-step weighted homotopic. Let F be a homotopy between f and g. Denote the morphisms of chain complexes induced by f and g as $f_\sharp$ and $g_\sharp$, respectively. Let $F_\sharp : \Omega_*(G \times I_1) \to \Omega_*(G')$ be the morphism induced by F (cf. [7, Theorem 3.3]). By [11], it is sufficient to construct a chain homotopy between the chain complexes $\Omega^w_*(G)$ and $\Omega^{w'}_*(G')$, that is, an $\mathbb{F}-$linear mapping

$$L_p : \Omega^w_p(G) \to \Omega^{w'}_{p+1}(G')$$

such that

$$\partial^w L_p + L_{p-1}\partial^w = g_\sharp - f_\sharp.$$

Define L_p as follows

$$L_p(x) = F_\sharp((-1)^p x \times I_1),$$

where $x = \sum_{i=1}^{n} a^i\sigma^i \in \Omega_p^w(G)$ and $x \times I_1 \in \Omega_{p+1}^w(G \times I_1)$. By the product rule in Proposition 3.2 and $\partial^w F_\sharp = F_\sharp \partial^w$ in [7, Theorem 2.10], we obtain that

$$\begin{aligned}
(\partial^w L_p + L_{p-1}\partial^w)(x) &= \partial^w F_\sharp((-1)^p x \times I_1) + F_\sharp((-1)^{p-1}\partial^w x \times I_1) \\
&= F_\sharp((-1)^p \partial^w(x \times I_1) + (-1)^{p-1}\partial^w x \times I_1) \\
&= F_\sharp\Big((-1)^p(\partial^w x \times I_1 + (-1)^p x \times \partial I_1) + (-1)^{p-1}\partial^w x \times I_1\Big) \\
&= F_\sharp(x \times \{1\} - x \times \{0\}) \\
&= F_\sharp(\sum_{i=1}^{n} a^i(\sigma^i)' - \sum_{i=1}^{n} a^i\sigma^i) \\
&= \sum_{i=1}^{n} a^i g(\sigma^i) - \sum_{i=1}^{n} a^i f(\sigma^i) \\
&= g_\sharp(x) - f_\sharp(x).
\end{aligned} \tag{3.2}$$

Moreover, by the induction on the homotopic step of f and g, the theorem is proved. □

At last, we give an example to illustrate the main theorem above.

Example 3.1. *Let G be a digraph with the vertex set $V(G) = \{v_0, v_1, v_2\}$ and the directed edge set $E(G) = \{v_0v_1, v_0v_2, v_1v_2\}$. Let G' be a digraph with the vertex set $V(G') = \{w_0, w_2\}$ and the directed edge set $E(G') = \{w_0w_2\}$. Let w, w' be the non-vanished weighted functions on G and G', respectively. Suppose f, g_1 are two weighted digraph maps from G to G' such that $f(v_0) = f(v_1) = w_0$, $f(v_2) = w_2$ and $g_1(v_0) = w_0$, $g_1(v_1) = g_1(v_2) = w_2$. Then w, w' can induce a weighted function $w_{G\times I_1}$ on $G \times I_1$ and there exists a weighted digraph map $F : G \times I_1 \longrightarrow G'$ such that*

$$\begin{aligned}
F\,|_{G\times 0} &= f \\
F\,|_{G\times 1} &= g_1 \\
F(w(v_i, 0)) &= w'(f(v_i)) \\
F(w(v_i, 1)) &= w'(g_1(v_i)).
\end{aligned}$$

Specifically, for $\sigma = v_0v_1v_2 \in \Omega(G)$, we have that

$$\begin{aligned}
\sigma \times I_1 &= (w(v_0))^{-1} v_0v_0'v_1'v_2' - (w(v_1))^{-1} v_0v_1v_1'v_2' + (w(v_2))^{-1} v_0v_1v_2v_2' \\
F(\sigma \times I_1) &= \Big((w(v_0))^{-1} - (w(v_1))^{-1} + (w(v_2))^{-1}\Big) w_0w_2.
\end{aligned}$$

Moreover, let $g_2 : G \longrightarrow G'$ such that $g_2(v_0) = g_2(v_1) = g_2(v_2) = w_2$. Then f and g_2 are two-step weighted homotopic.

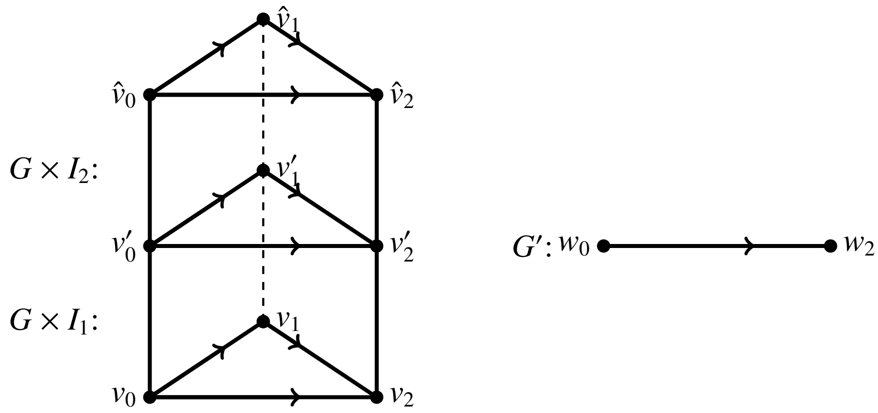

Figure 1. Example 3.1.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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