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Research article

On a common fixed point theorem in vector-valued b-metric spaces: Its consequences and application

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Abstract: We introduce a Ćirić type contraction principle in a vector-valued b-metric space that generalizes Perov's contraction principle. We investigate the possible conditions on the mappings $W, E : G \to G$ (G is a non-empty set), for which these mappings admit a unique common fixed point in G subject to a nonlinear operator $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$. We illustrate the hypothesis of our findings with examples. We consider an infectious disease model represented by the system of delay integro-differential equations and apply the obtained fixed point theorem to show the existence of a solution to this model.

Keywords: common fixed point; $(\zeta, \mathbf{F})$-Ćirić contraction; ζ-complete vector-valued b-metric space
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1. Introduction

In the field of metric fixed point theory, the Banach contraction principle [3] is regarded as a key fixed point theorem. The introduction of a new contraction principle is the core part of the metric fixed-point theory. Such a new contraction principle not only contributes in theory but also ensures the existence of solutions to mathematical models. In an effort to introduce a new contraction principle, Rakotch [27] introduced a new contraction principle that involved a function instead of a Lipschitz constant in the Banach contraction principle. Then, Boyd and Wong [5] generalized the Rakotch contraction principle. In a similar effort, Kannan [16] introduced a contraction principle that characterizes the metric completeness and gave a new direction in metric fixed point theory that led many mathematicians to introduce various contraction principles. Among the classical contraction principles, most famous are: the Meir and Keeler contraction principle [19], Chatterjea contraction principle, Reich contraction principle [28], Hardy and Rogers contraction principle [14], Ćirić contraction principle [9] and Caristi contraction principle [7].

Wardowski [32] (2012) generalized the Banach contraction principle by using an auxiliary nonlinear function $F : (0, \infty) \to (-\infty, \infty)$ that satisfied three conditions. In the literature, this new contraction principle is known as the F-contraction principle. This idea proved another milestone in metric fixed point theory. The F-contraction principle has been revisited and generalized in many abstract spaces (see [8, 10, 13, 21, 23, 24, 29] and references therein).

On the other hand, metric generalization also has a significant impact on metric fixed point theory. Many mathematicians have contributed in this direction, producing many generalizations of a metric space (see [18]). Perov [26], by extending the co-domain of the metric function from $\mathbb{R}$ to $\mathbb{R}^n$, gave a vector version of the metric and hence produced another generalization of the Banach contraction principle. Altun, et al. (2020) [2] obtained an extension of Wardowski [32] fixed point theorem in the vector-valued metric spaces as follows:

Theorem 1.1. *[2] Every self-mapping T on a complete vector-valued metric space (X, d) that satisfies the inequality*

$$\mathbf{d}(T(q), T(h)) > \mathbf{0} \Rightarrow \mathbf{I} \oplus \mathbf{F}(\mathbf{d}(T(q), T(h))) \leq \mathbf{F}(\mathbf{d}(q, h)) \, \forall \, q, h \in X,$$

admits a unique fixed point, provided $\mathbf{F}$ *satisfies* $(AF_1) - (AF_3)$ *and* $\mathbf{I} = (\tau_i)_{i=1}^m \geq \mathbf{0}$*;*

the operator $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ satisfies the following conditions:

(AF_1) $\forall \, Q, W \in \mathbb{P}^m$ with $Q \leq W$, and we have $\mathbf{F}(Q) \leq \mathbf{F}(W)$;
(AF_2) $\forall \, \{\mathbf{v}_n : n \in \mathbb{N}\} \subset \mathbb{P}^m$, and we have

$$\lim_{n\to\infty} v_n^{(i)} = 0 \text{ if and only if } \lim_{n\to\infty} u_n^{(i)} = -\infty, \text{ for each } i;$$

(AF_3) $\exists \, \kappa \in (0, 1)$ satisfying $\lim_{v_i \to 0^+} (v_i)^\kappa u_i = 0$.

Recently, Mínak, et al. [20] and Cosentino, et al. [12] have established some fixed point theorems for the existence of fixed points of Ćirić type and Hardy-Rogers type F-contractions, respectively. The significance of the fixed point results on Ćirić type and Hardy-Rogers type F-contractions [12, 20] requires more research work in generalized metric spaces. The vector valued distance function (being a column matrix) has many applications in Matrix Analysis and hence in Engineering, so, to broaden the scope of the fixed point results on Ćirić type and Hardy-Rogers type F-contractions, in this paper, we decided to revisit these notions in a vector-valued b-metric space $(G, \mathbf{A}, s)$ (defined in next section) and to investigate the possible conditions on the mappings W, and E for which these mappings admit a unique common fixed point. Note that for $s = 1$, we have a vector valued metric space. We will see that the Theorem 1.1 and some results in [22] are special cases of results presented in this paper.

Integro-differential equations have found applications in epidemiology, the mathematical modeling of epidemics, particularly when the models contain age-structure [6]. The Kermack-McKendrick theory of infectious disease transmission is one particular example where age-structure in the population is incorporated into the modeling framework. Following the work in [25], we will show the existence of a solution to the system of delay integro-differential equations that represent an Infectious Disease Model:

$$l(t) = \int_{t-L}^{t} W(h, l(h), l'(h)) \, dh,$$

$$q(t) = \int_{t-L}^{t} E(h, q(h), q^{'}(h))\, dh,$$

where

(a) $l(t), q(t)$: the prevalence of infection at time t in the population.
(b) $0 < L$: the amount of time a person can still spread disease.
(c) $l^{'}(h), q^{'}(h)$: the current rate of infectivity.
(d) $W(h, l(h), l^{'}(h)), E(h, q(h), q^{'}(h))$: the rate of newly acquired infections per unit of time.

We are aimed at applying Theorem 5.1 to show the existence of a solution to a system of delay integro-differential equations.

2. Preliminaries and related results

In this section, we present a summary of prerequisites and notations to be considered in the sequel. Let $\mathbb{R}^m = \{\mathbf{v} = (x_i)_{i=1}^m = (x_1, x_2, \cdots, x_m) \mid \forall i \;\; x_i \in \mathbb{R}\}$ represents all matrices of order $m \times 1$ (which will be called vectors), and then $(\mathbb{R}^m, \oplus, \odot)$ is a linear space with $\oplus$ and $\odot$ defined by

$$\mathbf{v} \oplus \mathbf{w} = (x_i + y_i)_{i=1}^m \text{ for all } \mathbf{v} = (x_i)_{i=1}^m \text{ and } \mathbf{w} = (y_i)_{i=1}^m \in \mathbb{R}^m,$$

$$k \odot \mathbf{v} = (k \cdot x_i)_{i=1}^m \text{ for all } \mathbf{v} = (x_i)_{i=1}^m \in \mathbb{R}^m \text{ and } k \in \mathbb{R}.$$

Note that $+$ and $\cdot$ represent the usual addition and multiplication of scalars. By using the above operations, we can define the difference of vectors as $\mathbf{v} \ominus \mathbf{w} = \mathbf{v} \oplus (-1) \odot \mathbf{w}$. Define the relations $\preceq$ and $\prec$ on $\mathbb{R}^m$ by

$$\mathbf{v} \preceq \mathbf{w} \Leftrightarrow x_i \leq y_i \text{ and } \mathbf{v} \prec \mathbf{w} \Leftrightarrow x_i < y_i, \;\; \forall i. \tag{2.1}$$

The relation $\preceq$ defines a partial-order on $\mathbb{R}^m$. Let $\mathbb{P}^m$ denote the set of positive definite vectors, that is, if $\mathbf{v} = (x_i)_{i=1}^m \succ \mathbf{0}$ (zero vector of order $m \times 1$), and $\mathbf{v} = (x_i)_{i=1}^m \in \mathbb{R}^m$, then $\mathbf{v} = (x_i)_{i=1}^m \in \mathbb{P}^m$. Also, let $\mathbb{R}_0^m = \{\mathbf{v} = (x_i)_{i=1}^m = (x_1, x_2, \cdots, x_m) \mid \forall i \;\; x_i \in [0, \infty)\}$. The two vectors are considered equal if their corresponding coordinates are equal.

Definition 2.1. *[31] (1) Let $V = [v_{ij}]$ be an $m \times m$ complex matrix having eigenvalues λ_i, $1 \leq i \leq n$. Then, the spectral radius $\rho(V)$ of matrix V is defined by $\rho(V) = \max_{1 \leq i \leq m} |\lambda_i|$.*

(2) The matrix V converges to zero, if the sequence $\{V^n; n \in \mathbb{N}\}$ converges to zero matrix O.

Theorem 2.1. *[31] Let $\mathbf{V}$ be any complex matrix of order $m \times m$, then $\mathbf{V}$ is convergent if and only if $\rho(V) < 1$.*

Perov [26] applied Theorem 2.1 to obtain the following result in vector-valued metric spaces.

Theorem 2.2. *[26] Every self-mapping J defined on a complete vector-valued metric space $(X, \mathbf{d})$ satisfying the inequality*

$$\mathbf{d}(J(g), J(h)) \preceq A\mathbf{d}(g, h) \, \forall \, (g, h) \in X \times X,$$

admits a unique fixed point provided $\rho(A) < 1$ and that A is a positive square matrix of order m.

By a vector-valued metric, we mean a mapping $d : X \times X \to \mathbb{R}^m$ obeying all the axioms of the metric. The object $d(x, y)$ is an m-tuple or a column matrix. Let

$$\mathbf{v} = (v_i)_{i=1}^m = (v_1, v_2, v_3, \cdots, v_m) \in \mathbb{P}^m,$$
$$\mathbf{v}_n = \left(v_n^{(i)}\right)_{i=1}^m = \left(v_n^{(1)}, v_n^{(2)}, v_n^{(3)}, \cdots, v_n^{(m)}\right) \in \mathbb{P}^m,$$
$$\mathbf{F}(\mathbf{v}) = (u_i)_{i=1}^m = (u_1, u_2, u_3, \cdots, u_m) \in \mathbb{R}^m \text{ and}$$
$$\mathbf{F}(\mathbf{v_n}) = \left(u_n^{(i)}\right)_{i=1}^m = \left(u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, \cdots, u_n^{(m)}\right) \in \mathbb{R}^m,$$
$$(v)_1^m = (v, v, v, \cdots, v) \in \mathbb{R}^m.$$

The following concepts and results will be required in the sequel.

3. The vector-valued b-metric space

In light of the definitions of b-metric and vector-valued metric given by Czerwik [11] and Perov [2], respectively, Boriceanu [4] introduced a vector-valued b-metric as follows:

Definition 3.1. *(vector-valued b-metric) [4] Let G be a non-empty set. The operator* $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ *satisfying the axioms* $(\mathbf{A}_1) - (\mathbf{A}_3)$ *given below is known as a vector-valued b-metric.*

$(\mathbf{A}_1)$ $q = t$ *if and only if* $\mathbf{A}(q, t) = \mathbf{0}$, *for all* $q, t \in G$.
$(\mathbf{A}_2)$ $\mathbf{A}(q, t) = \mathbf{A}(t, q)$, *for all* $q, t \in G$.
$(\mathbf{A}_3)$ $\mathbf{A}(q, g) \preceq s \odot \left[\mathbf{A}(q, t) \oplus \mathbf{A}(t, g)\right]$; $s \geq 1$, *for all* $q, t, g \in G$.

The triplet $(G, \mathbf{A}, s)$ *represents a vector-valued b-metric-space.*

For $s = 1$, $(G, \mathbf{A}, s)$ is a vector-valued metric-space, but this is not true when $s > 1$. Thus, it can be remarked that every vector-valued metric-space is a vector-valued b-metric-space but not conversely.

Example 3.1. *Let* $G = \mathbb{R}$ *and the operator* $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ *be defined by*

$$\mathbf{A}(l, q) = \left(|H|^2, |H|^3, \cdots, |H|^{m+1}\right) \forall l, q \in G,$$

where $H = |l - q|$. *Then,* $(G, \mathbf{A}, 2^m)$ *is a vector-valued b-metric space. Note that it is not a vector-valued-metric space.*

Example 3.2. *Let* $G \neq \emptyset$ *and* $d_i : G \times G \to [0, \infty)$ *be a b-metric for each i with respective constant* $s_i \geq 1$ $(1 \leq i \leq m)$ *for each positive integer i. The mapping* $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ *defined by*

$$\mathbf{A}(l_1, l_2) = (d_1(l_1, l_2), d_2(l_1, l_2), \cdots, d_m(l_1, l_2)) \text{ for all } l_1, l_2 \in G$$

defines a vector-valued b-metric on G.

The axioms $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold trivially. To prove* $(\mathbf{A}_3)$, *consider*

$$\begin{aligned}
\mathbf{A}(l, l_2) &= (d_1(l, l_2), d_2(l, l_2), \cdots, d_m(l, l_2)) \\
&\preceq (s_1(d_1(l, l_1) + d_1(l_1, l_2)), s_2(d_2(l, l_1) + d_2(l_1, l_2)), \cdots, s_m(d_m(l, l_1) + d_m(l_1, l_2))) \\
&\preceq s \odot (d_1(l, l_1) + d_1(l_1, l_2), d_2(l, l_1) + d_2(l_1, l_2), \cdots, d_m(l, l_1) + d_m(l_1, l_2)) \\
&= s \odot ((d_1(l, l_1), d_2(l, l_1), \cdots, d_m(l, l_1)) \oplus (d_1(l_1, l_2), d_2(l_1, l_2), \cdots, d_m(l_1, l_2))) \\
&= s \odot (\mathbf{A}(l, l_1) \oplus \mathbf{A}(l_1, l_2)) \,;\, s = \max\{s_1, s_2, \cdots, s_m\}.
\end{aligned}$$

Since the vector-valued b-metric is a discontinuous operator, fixed point theorems in the vector-valued b-metric spaces need to be supported by an auxiliary convergence result. We provide the following lemma (Lemma 3.1) for this purpose.

Lemma 3.1. *Let $(G, \mathbf{A}, s)$ be a vector-valued b-metric space. If $l^*, g^* \in G$, and $\{l_n\}_{n\in\mathbb{N}}$ is such that $\lim_{n\to\infty} l_n = l^*$, then*

$$\frac{1}{s} \odot \mathbf{A}(l^*, g^*) \preceq \liminf_{n\to\infty} \mathbf{A}(l_n, g^*) \preceq \limsup_{n\to\infty} \mathbf{A}(l_n, g^*) \preceq s \odot \mathbf{A}(l^*, g^*).$$

Proof. By $(\mathbf{A}_3)$, we have

$$\frac{1}{s} \odot \mathbf{A}(l^*, g^*) \preceq \mathbf{A}(l^*, l_n) \oplus \mathbf{A}(l_n, g^*) \frac{1}{s} \odot \mathbf{A}(l^*, g^*) \ominus \mathbf{A}(l_n, l^*) \preceq \mathbf{A}(l_n, g^*).$$

By $\liminf$, we have

$$\frac{1}{s} \odot \mathbf{A}(l^*, g^*) \preceq \liminf_{n\to\infty} \mathbf{A}(l_n, g^*). \tag{3.1}$$

By $(\mathbf{A}_3)$, we get

$$\mathbf{A}(l_n, g^*) \preceq s \odot (\mathbf{A}(l_n, l^*) \oplus \mathbf{A}(l^*, g^*)).$$

This implies

$$\limsup_{n\to\infty} \mathbf{A}(l_n, g^*) \preceq s \odot \mathbf{A}(l^*, g^*). \tag{3.2}$$

We know that

$$\liminf_{n\to\infty} \mathbf{A}(l_n, g^*) \preceq \limsup_{n\to\infty} \mathbf{A}(l_n, g^*). \tag{3.3}$$

Combining (3.1)–(3.3), we get the required result.

Apart from Lemma 3.1, to fulfill the objective of this paper, the following compatibility condition is required:

(AF_4): for every positive term sequence $\mathbf{v}_n = \left(x_n^{(i)}\right)_{i=1}^m$, $\exists\, \mathbf{I} = (\tau_i)_{i=1}^m \succeq \mathbf{0}$ satisfying

$$\mathbf{I} \oplus \mathbf{F}(s\mathbf{v}_n) \preceq \mathbf{F}(\mathbf{v}_{n-1}) \text{ implies } \mathbf{I} \oplus \mathbf{F}(s^n \mathbf{v}_n) \preceq \mathbf{F}(s^{n-1}\mathbf{v}_{n-1}).$$

Our findings rely mostly on the class of vector-valued nonlinear operators satisfying (AF_1), (AF_3) and (AF_4) denoted by Π_s^b.

Remark 3.1. The collection of vector-valued nonlinear operators Π_s^b is non-empty.

Let $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ be defined by $\mathbf{F}((x_i)_{i=1}^m) = \left(\log_e(x_i + 1)\right)_{i=1}^m$ for all $\mathbf{v} \in \mathbb{P}^m$, and then (AF_1) and (AF_3) are obvious.

We establish (AF_4): Let $\mathbf{I} \oplus \mathbf{F}(s\mathbf{v}_n) \preceq \mathbf{F}(\mathbf{v}_{n-1})$, and then for m-tuple $\mathbf{I} = \left(\log_e(s^{n-1}), \log_e(s^{n-1}), \cdots, \log_e(s^{n-1})\right) = \left(\log_e(s^{n-1})\right)_1^m$, we have

$$\begin{aligned}
\left(\log_e(s^{n-1})\right)_1^m &\oplus \mathbf{F}\left(s\left(x_n^{(i)}\right)_{i=1}^m\right) \preceq \mathbf{F}\left(\left(x_{n-1}^{(i)}\right)\right)_{i=1}^m \\
\left(\log_e(s^{n-1})\right)_1^m &\oplus \left(\log_e\left(sx_n^{(i)} + 1\right)\right)_{i=1}^m \preceq \left(\log_e\left(x_{n-1}^{(i)} + 1\right)\right)_{i=1}^m \\
&\Rightarrow \left(\log_e\left(s^n x_n^{(i)} + s^{n-1}\right)\right)_{i=1}^m \preceq \left(\log_e\left(x_{n-1}^{(i)} + 1\right)\right)_{i=1}^m
\end{aligned}$$

$$\Rightarrow \quad \log_e\left(s^n x_n^{(i)} + s^{n-1}\right) \leq \log_e\left(x_{n-1}^{(i)} + 1\right) \text{ for each } i$$
$$\Rightarrow \quad s^n x_n^{(i)} \leq x_{n-1}^{(i)} + 1 - s^{n-1} \text{ for each } i.$$

Now, consider

$$\begin{aligned}
\mathbf{I} \oplus \mathbf{F}(s^n \mathbf{v}_n) &= \left(\log_e(s^{n-1})\right)_1^m \oplus \left(\log_e(s^n x_n^{(i)} + 1)\right)_{i=1}^m \\
&\leq \left(\log_e(s^{n-1})\right)_1^m \oplus \left(\log_e(x_{n-1}^{(i)} + 1 - s^{n-1} + 1)\right)_{i=1}^m \\
&= \left(\log_e(s^{n-1} x_{n-1}^{(i)} - s^{2n-2} + 2s^{n-1})\right)_{i=1}^m = \left(\log_e(s^{n-1} x_{n-1}^{(i)} + s^{n-1}(2 - s^{n-1}))\right)_{i=1}^m \\
&\leq \left(\log_e(s^{n-1} x_{n-1}^{(i)} + 1)\right)_{i=1}^m = \mathbf{F}\left(\left(s^{n-1} x_{n-1}^{(i)}\right)_{i=1}^m\right) = \mathbf{F}\left(s^{n-1}\mathbf{v}_{n-1}\right).
\end{aligned}$$

Hence, $\mathbf{F} \in \Pi_s^b$.

Example 3.3. Let $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ be defined by

(a) $\mathbf{F}(\mathbf{v}) = (\log_e(x_i))_{i=1}^m$;
(b) $\mathbf{F}(\mathbf{v}) = (x_i + \log_e(x_i))_{i=1}^m$;
(c) $\mathbf{F}(\mathbf{v}) = \log_e(x_i^2 + x_i)_{i=1}^m$;
(d) $\mathbf{F}(\mathbf{v}) = \left(-\frac{1}{\sqrt{x_i}}\right)_{i=1}^m$;
(e) $\mathbf{F}(\mathbf{v}) = \left(x_i^a\right)_{i=1}^m$; $a > 0$;
(f) $\mathbf{F}(\mathbf{v}) = (\log_e(x_i + 1))_{i=1}^m$.

Note that the Definitions (a)–(d) satisfy (AF_1)–(AF_3), while (e), (f) belong to the family Π_s^b.
Define $\mathbf{F} : \mathbb{R}_+^2 \to \mathbb{R}^2$ *by* $\mathbf{F}((g_1, g_2)) = \left(g_1^t, \log_e(g_2 + 1)\right)$, $t > 0$, *and then* $\mathbf{F} \in \Pi_s^b$.

The following lemma explains the reasons to omit axiom (AF_2).

Lemma 3.2. *Let* $\mathbf{F}$ *satisfy* (AF_1), *and* $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset \mathbb{P}^m$ *is a decreasing sequence satisfying* $\lim_{n\to\infty} u_n^{(i)} = -\infty$. *Then* $\lim_{n\to\infty} v_n^{(i)} = 0$ *for each* $i \in \{1, 2, \cdots, m\}$.

4. Common fixed points of $(\zeta, \mathbf{F})$-contractions

Recently, Altun, et al. [2] obtained an existence theorem involving a vector-valued nonlinear operator and explained it through nontrivial examples. We will introduce and investigate the notion of $(\zeta, \mathbf{F})$-contractions where the operator $\mathbf{F}$ is taken from Π_s^b and ζ is defined below.

Definition 4.1. *Let* $\mathbf{F} \in \Pi_s^b$ *and* $\mathbf{I} > \mathbf{0}$. *The mapping* $T : (G, \mathbf{A}, s) \to (G, \mathbf{A}, s)$ *is said to be an* $(s, \mathbf{F})$-*contraction, if it satisfies the following inequality:*

$$\mathbf{A}(T(l), T(q)) > \mathbf{0} \Rightarrow \mathbf{I} \oplus \mathbf{F}(s \odot \mathbf{A}(T(l), T(q)) \leq \mathbf{F}(\mathbf{A}(l, q)), \textit{ for all } l, q \in G. \tag{4.1}$$

Remark 4.1. *Note that for* $s = 1$, *Definition 4.1 is identical to Perov's type* F-*contraction introduced by Altun, et al. [2]. Thus, the class of* $(s, \mathbf{F})$-*contractions (defined in Definition 4.1) is wider compared to that of Perov's type* F-*contraction introduced by Altun, et al. [2]. Now, we explain inequality (4.1) with the following example (Example 4.1).*

Example 4.1. *Let $G = \{l_n = 2^{\frac{n}{2}} n | n \in \mathbb{N}\}$. Define $\mathbf{A}\colon G \times G \to \mathbb{P}^m$ by $\mathbf{A}(l, q) = \left(|l-q|^2\right)_1^m$, and then $(G, \mathbf{A}, s = 2)$ is a vector-valued b-metric space. Define the mapping $\phi : G \to G$ by*

$$\phi(l) = \begin{cases} 2^{\frac{n-1}{2}}(n-1) & \text{if } l = l_n; \\ l_0 & \text{if } l = l_0. \end{cases}$$

Take $(1)_1^m = \mathbf{I} > \mathbf{0}$ and define $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ by $\mathbf{F}\left((g_i)_{i=1}^m\right) = (g_i)_{i=1}^m$. Then, for every $l, q \in G$ such that $\phi(l) \neq \phi(q)$, we have

$$\mathbf{F}\left(2 \odot \mathbf{A}(\phi(l), \phi(q))\right) \ominus \mathbf{F}\left(\mathbf{A}(l, q)\right) \leq \ominus \mathbf{I}.$$

Indeed, for $l = l_{n+k}$ and $q = l_n$, consider

$$\begin{aligned} 2 \odot \mathbf{A}(\phi(l_{n+k}), \phi(l_n)) \ominus \mathbf{A}(l_{n+k}, l_n) &= \left(\left(2^{\frac{n+k}{2}}(n+k-1) - 2^{\frac{n}{2}}(n-1)\right)^2 - \left(2^{\frac{n+k}{2}}(n+k) - 2^{\frac{n}{2}}(n)\right)^2\right)_1^m \\ &= \left(2^n\left(1 - 2^{\frac{k}{2}}\right)\left(2^{\frac{k}{2}}(2n+2k-1) - (2n-1)\right)\right)_1^m \\ &\leq (-1)_1^m = \ominus (1)_1^m. \end{aligned}$$

We notice that $\mathbf{F} \in \Pi_s^b$. Indeed, for $\mathbf{F}\left((g_i)_{i=1}^m\right) = (g_i)_{i=1}^m$, axioms (AF_1) and (AF_3) hold. For axiom (AF_4), we proceed as follows: Let $\mathbf{I} \oplus \mathbf{F}\left(s \odot \left(g_n^{(i)}\right)_{i=1}^m\right) \leq \mathbf{F}\left(\left(g_{n-1}^{(i)}\right)\right)_{i=1}^m$, that is, $1 + \ell g_n^{(i)} \leq g_{n-1}^{(i)}$ for each $i \in \{1, 2, \cdots, m\}$. Now, consider

$$\begin{aligned} \mathbf{I} \oplus \mathbf{F}\left(\ell^n \odot \left(g_n^{(i)}\right)_{i=1}^m\right) &= \mathbf{I} \oplus \ell^n \odot \left(g_n^{(i)}\right)_{i=1}^m \\ &= \mathbf{I} \oplus \ell^{n-1} \odot \left(\ell g_n^{(i)}\right)_{i=1}^m \leq \mathbf{I} \oplus \ell^{n-1} \odot \left(g_{n-1}^{(i)} - 1\right)_{i=1}^m \\ &= 1 + \ell^{n-1} g_{n-1}^{(i)} - \ell^{n-1} = 1 - \ell^{n-1} + \ell^{n-1} g_{n-1}^{(i)} \text{ for each } i \\ &\leq \left(\ell^{n-1} g_{n-1}^{(i)}\right)_{i=1}^m = \mathbf{F}\left(\ell^{n-1} \odot \left(g_{n-1}^{(i)}\right)_{i=1}^m\right). \end{aligned}$$

This shows that for $\mathbf{I} = (1)_1^m$, ϕ is an $\mathbf{F}$-contraction.

Remark 4.2. *We observe that the function α_s (defined in [1]) is superficial because we can always have a function $\zeta : G \times G \to [0, \infty)$ defined by $\zeta(l, q) = \dfrac{\alpha_s(l, q)}{s^2}$ with the following properties:*

(1) (ϕ is ζ-admissible)

$$\zeta(l, q) \geq 1 \text{ implies } \zeta(\phi(l), \phi(q)) \geq 1 \text{ for all } l, q \in G,$$

(2) the α_s-completeness implies ζ-completeness and vice versa.

Definition 4.2. *Let G be a non-empty set, and $\zeta : G \times G \to [0, \infty)$. The function $\phi : G \to G$ is said to be ζ-admissible if*

$$\zeta(l, q) \geq 1 \text{ implies } \zeta(\phi(l), \phi(l)) \geq 1 \text{ for all } l, q \in G \text{ and}$$

triangular ζ-admissible if in addition ζ follows

$$\zeta(l, j) \geq 1, \quad \zeta(j, q) \geq 1, \text{ imply } \zeta(l, q) \geq 1.$$

Definition 4.3. *The mappings $W, E : G \to G$ defined on the space $(G, \mathbf{A}, s)$ and satisfying the inequality*

$$\zeta(l,q) \geq 1 \textit{ implies } \zeta(W(l), EW(l)) \geq 1 \textit{ and } \zeta(E(q), WE(q)) \geq 1 \textit{ for all } l, q \in G$$

are called weakly ζ-admissible. Moreover, W, E are called triangular weakly ζ-admissible mappings if

(1) $\zeta(l,q) \geq 1$ implies $\zeta(W(l), EW(l)) \geq 1$ and $\zeta(E(q), WE(q)) \geq 1$ for all $l, q \in G$,

(2) $\zeta(l,u) \geq 1$, $\zeta(u,q) \geq 1$, imply $\zeta(l,q) \geq 1$,

for all $l, q, u \in G$.

Example 4.2. Let $G = [0, \infty)$, and

$$W(g) = \begin{cases} g & \text{if } g \in [0,1); \\ 1 & \text{if } g \in [1,\infty), \end{cases} \quad E(t) = \begin{cases} t^{\frac{1}{3}} & \text{if } t \in [0,1); \\ 1 & \text{if } t \in [1,\infty). \end{cases}$$

Define $\zeta : G \times G \to \mathbb{R}_0^+$ by $\zeta(l,q) = \begin{cases} 1+q-l & \text{if } l,q \in [0,1); \\ 0 & \text{if } l,q \in [1,\infty). \end{cases}$ If $\begin{cases} \zeta(l,q) \geq 1; \\ \beta(q,u) \geq 1, \end{cases}$ then $\begin{cases} l-q \leq 0; \\ q-u \leq 0, \end{cases}$ which implies that $l-u \leq 0$. Hence, $\zeta(l,u) = 1+u-l \geq 1$,

$$\zeta(W(l), EW(l)) = \zeta(l, l^{\frac{1}{3}}) \geq 1, \text{ and } \zeta(E(q), WE(q)) = \zeta(q^{\frac{1}{3}}, q^{\frac{1}{3}}) \geq 1,$$

for all $l, q \in [0,1)$.

Definition 4.4. *Let $(G, \mathbf{A}, s)$ be a vector-valued b-metric space, $\zeta : G \times G \to [0,\infty)$, $l \in G$, and sequence $\{l_n\} \subseteq G$. A mapping $q : G \to G$ is ζ-continuous at $l = l_0$ if whenever*

$$\lim_{n\to\infty} \mathbf{A}(l_n, l) = \mathbf{0} \textit{ and } \zeta(l_n, l_{n+1}) \geq 1, \textit{ we have } \lim_{n\to\infty} \mathbf{A}(q(l_n), q(l)) = \mathbf{0}.$$

Example 4.3. *Let $G = [0,\infty)$ and define $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ by*

$$\mathbf{A}(l,q) = \left(|H|^2, |H|^3, \cdots, |H|^{m+1}\right) \forall\, l, q \in G,$$

where $H = |l-q|$, and let $q : G \to G$ be defined by

$$q(l) = \begin{cases} \sin(\pi l) & \textit{if } l \in [0,1]; \\ \cos(\pi l) + 2 & \textit{if } l \in (1,\infty), \end{cases} \quad \zeta(l,q) = \begin{cases} l+q+1 & \textit{if } l,q \in [0,1]; \\ 0 & \textit{otherwise}. \end{cases}$$

Obviously, q is not continuous at $l_0 = 1$; however, q is a ζ-continuous mapping at this point. Indeed, the assumption $\lim_{n\to\infty} \mathbf{A}(l_n, l_0) = \mathbf{0}$ leads us to choose, $l_n = 1 - \frac{1}{n} \subseteq [0,1]$ and $\zeta(l_n, l_{n+1}) \geq 1$ directs us to choose $[0,1]$ as the domain of mapping q. Thus,

$$\lim_{n\to\infty} |q(l_n) - q(l)|^i = \lim_{n\to\infty} \left(\sin\left(\pi\left(1 - \frac{1}{n}\right)\right)\right)^i = 0 \textit{ for each } i; \quad 2 \leq i \leq m+1.$$

Hence, $\lim_{n\to\infty} \mathbf{A}(q(l_n), q(l)) = \mathbf{0}$.

Definition 4.5. *If an arbitrary Cauchy sequence $\{l_n\} \subseteq G$ satisfying $\zeta(l_n, l_{n+1}) \geq 1$ converges in G, the space $(G, \mathbf{A}, s)$ is called ζ-complete.*

Remark 4.3. *Every complete vector-valued b-metric space is a ζ-complete vector-valued b-metric space but not conversely.*

Look at the following example.

Example 4.4. *Let $G = (0, \infty)$ and define the vector-valued b-metric $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ by*

$$\mathbf{A}(l, q) = \left(|H|^2, |H|^3, \cdots, |H|^{m+1}\right) \text{ for all } l, q \in G,$$

where $H = |l - q|$. Define $\zeta \colon G \times G \to [0, \infty)$ by

$$\zeta(l, q) = \begin{cases} l^2 + q^2 & \text{if } l, q \in [2, 5]; \\ 0 & \text{if not in } [2, 5]. \end{cases}$$

We observe that the space $(G, \mathbf{A}, s)$ is not complete, but it satisfies ζ-completeness criteria.

Definition 4.6. *If an arbitrary sequence $\{l_n\} \subset G$ satisfies the condition*

$$\zeta(l_n, l_{n+1}) \geq 1 \text{ and } \mathbf{A}(l_n, l) \to \mathbf{0} \Rightarrow \zeta(l_n, l) \geq 1,$$

$\forall\, n \in \mathbb{N}$, *then the space $(G, \mathbf{A}, s)$ is known as a ζ -regular space.*

Let $G = [2, 5]$ and define vector-valued b-metric as in Example 3.1 and ζ as in Example 4.4. Let $l_n = 2 + \frac{3}{n}$ be n^{th} term of a sequence in G. Then, $(G, \mathbf{A}, s)$ is a ζ-regular space.

Suzuki [30] established the following lemma.

Lemma 4.1. [30] If there is a number $C > 0$ such that the sequence $\{x_n\} \subset (G, d)$ satisfies the inequality

$$d(x_n, x_{n+1}) \leq Cn^{-\nu} \text{ for every } \nu > 1 + \log_2 s,$$

then $\{x_n\}$ is a Cauchy sequence.

The following Lemma extends Lemma 4.1.

Lemma 4.2. *If there is a number $C > 0$ such that the sequence $\{x_n\} \subset (G, \mathbf{A}, s)$ satisfies the inequality*

$$\mathbf{A}(x_n, x_{n+1}) \preceq (Cn^{-\nu})_1^m \text{ for every } \nu > 1 + \log_2 s \text{ and for every positive integer } n,$$

then $\{x_n\}$ is a Cauchy sequence.

Proof. Let G be any non-empty set and $s = \max\{s_i : \quad 1 \leq i \leq m\}$. Let $d_i : G \times G \to [0, \infty)$ be a b-metric for every $i \in \{1, 2, 3, \cdots, m\}$ and $s_i \geq 1$. Define the vector-valued b-metric $\mathbf{A}$ by

$$\mathbf{A}(q, t) = (d_i(q, t))_{i=1}^m \text{ for all } q, t \in G.$$

Let $\{x_n\}$ be a sequence in G and assume that

$$\mathbf{A}(x_n, x_{n+1}) \preceq (Cn^{-\nu})_1^m \text{ for every } \nu > 1 + \log_2 s \text{ and for every } n \in \mathbb{N}.$$

Then, by definition of partial order $\preceq$ defined by (2.1), we have for each i

$$d_i(x_n, x_{n+1}) \leq Cn^{-\nu} \text{ for every } \nu > 1 + \log_2 s_i \text{ and for every } n \in \mathbb{N}.$$

Lemma 4.1 does not depend on a particular b-metric; therefore, Lemma 4.1 can be applied for each d_i $(1 \leq i \leq m)$. Thus, $\{x_n\}$ is a Cauchy sequence with respect to every d_i $(1 \leq i \leq m)$. Thus,

$$d_i(x_n, x_m) \to 0 \text{ as } n, m \to \infty \text{ for each } i.$$

This leads us to write that

$$\mathbf{A}(x_n, x_m) \to (0, 0, \cdots, 0) = O \text{ as } n, m \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence in $(G, \mathbf{A}, s)$.

Now, we have an analogue of Lemma 4.2 subject to $\mathbf{F}$ contraction.

Lemma 4.3. *Let $\{D_n\}$ be a sequence in $\mathbb{P}^m$ where $D_n := (j_n^{(i)})_{i=1}^m$. Assume that there exist a mapping $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$, $\mathbf{I} = (\tau_i)_{i=1}^m > O$ and $\kappa \in (0, \ell) : \ell = 1/1 + \log_2 s$ satisfying (AF_3) and the following:*

$$n \odot \mathbf{I} \oplus \mathbf{F}(s^n \odot D_n) \leq \mathbf{F}(D_0). \tag{4.2}$$

Then, $D_n \leq \left(Cn^{-\frac{1}{\kappa}}\right)_1^m$.

Proof. The inequality (4.2) implies $\lim_{n\to\infty} \mathbf{F}(s^n \odot D_n) = (-\infty)_1^m$, and by Lemma 3.2, we get $\lim_{n\to\infty} s^n \odot D_n = \mathbf{0}$. By (AF_3),

$$\lim_{n\to\infty} (s^n j_n^{(i)})^\kappa \vartheta_n^{(i)} = 0 \text{ for each } i\,; \quad \mathbf{F}(s^n \odot D_n) := (\vartheta_n^{(i)})_{i=1}^m \in \mathbb{R}^m.$$

By (4.2), we also have the following information for each i.

$$\left(s^n j_n^{(i)}\right)^\kappa \vartheta_n^{(i)} - (s^n j_n^{(i)})^\kappa \vartheta_0^{(i)} \leq -\left(s^n j_n^{(i)}\right)^\kappa n\tau_i \leq 0. \tag{4.3}$$

As $n \to \infty$ in (4.3), we have

$$\lim_{n\to\infty} n\left(s^n j_n^{(i)}\right)^\kappa = 0 \text{ for each } i.$$

Equivalently there exists a positive integer N_1 such that $n\left(s^n j_n^{(i)}\right)^\kappa \leq 1$ for $n \geq N_1$. It then follows for each i that

$$s^n j_n^{(i)} \leq \frac{1}{n^{\frac{1}{\kappa}}} \Rightarrow j_n^{(i)} \leq \frac{1}{s^n} n^{-\frac{1}{\kappa}} \leq \frac{1}{s} n^{-\frac{1}{\kappa}}.$$

This implies $j_n^{(i)} \leq Cn^{-\frac{1}{\kappa}}$ for $n \geq N_1$ and for each i, where $C = s^{-1}$. Hence, $D_n \leq \left(Cn^{-\frac{1}{\kappa}}\right)_1^m$.

5. Fixed point theorems

Recently, Mínak et al. [20] and Cosentino et al. [12] have established some fixed point theorems for the existence of fixed points of Ćirić type and Hardy-Rogers type F-contractions, respectively. In this section, we revisit these notions in a vector-valued b-metric space $(G, \mathbf{A}, s)$. For this purpose, we consider the operator $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ that is an element of the collection Π_s^b. We introduce the notions of $(\zeta, \mathbf{F})$-Ćirić contraction and $(\zeta, \mathbf{F})$-Hardy-Rogers contraction defined as follows:

Definition 5.1. Let $(G, \mathbf{A}, s)$ be a vector-valued b metric space. The mappings $W, E : G \to G$ are said to form $(\zeta, \mathbf{F})$-Ćirić contraction if there exist $\mathbf{F} \in \Pi_s^b$ and $\mathbf{I} \succ \mathbf{0}$ such that

$$\mathbf{A}(W(l), E(q)) \succ \mathbf{0}\, \forall\, l, q \in G,\ \zeta(l, q) \geq 1 \text{ imply}$$

$$\mathbf{I} \oplus \mathbf{F}(s\zeta(l, q) \odot \mathbf{A}(W(l), E(q)) \preceq \mathbf{F}(\mathbf{M}(l, q)), \tag{5.1}$$

where

$$\|\mathbf{M}(l, q)\| = \max \left\{ \begin{array}{l} \|\mathbf{A}(l, q)\|, \|\mathbf{A}(l, W(l))\|, \|\mathbf{A}(q, E(q))\|, \\ \\ \dfrac{\|\mathbf{A}(l, E(q))\| + \|\mathbf{A}(q, W(l))\|}{2s} \end{array} \right\}, \text{ and}$$

$$\forall x, y \in \mathbb{P}^m,\ x \preceq y \Leftrightarrow \|x\| \leq \|y\|.$$

Recall that the norm on $\mathbb{P}^m$ is defined by

$$\|x\| = \sqrt{\sum_{i=1}^{m} |x_i|^2}\ \forall\ x = (x_1, x_2, ..., x_m) \in \mathbb{P}^m.$$

Also, we have assumed that

$$\left\{ \mathbf{A}(l, q), \mathbf{A}(l, W(l)), \mathbf{A}(q, E(q)), \frac{\mathbf{A}(l, E(q)) \oplus \mathbf{A}(q, W(l))}{2s} \right\} \subset \mathbb{P}^m.$$

Definition 5.2. Let $(G, \mathbf{A}, s)$ be a vector-valued b metric space. The mappings $W, E : G \to G$ are said to form $(\zeta, \mathbf{F})$-Hardy-Rogers contraction if there exist $a_i \geq 0$ $(i = 1, 2, 3, 4)$ such that $a_1 + a_2 + a_3 + 2sa_4 = 1$, $\mathbf{F} \in \Pi_s^b$ and $\mathbf{I} \succ \mathbf{0}$ such that

$$\mathbf{A}(W(l), E(q)) \succ \mathbf{0}\, \forall\, l, q \in G,\ \zeta(l, q) \geq 1 \text{ imply}$$

$$\mathbf{I} \oplus \mathbf{F}(s\zeta(l, q) \odot \mathbf{A}(W(l), E(q)) \preceq \mathbf{F}(\mathbf{H}(l, q)), \tag{5.2}$$

where

$$\mathbf{0} \prec \mathbf{H}(l, q) = a_1 \odot \mathbf{A}(l, q) \oplus a_2 \odot \mathbf{A}(l, W(l)) \oplus a_3 \odot \mathbf{A}(q, E(q))) \oplus a_4 \odot [\mathbf{A}(l, E(q)) \oplus \mathbf{A}(W(l), q)].$$

Remark 5.1. *Every $(\zeta, \mathbf{F})$-Hardy-Rogers contraction implies $(\zeta, \mathbf{F})$-Ćirić contraction but not conversely.*

The following main result states the requirements that ensure the existence of common fixed points of $(\zeta, \mathbf{F})$-Ćirić contraction.

Theorem 5.1. *Let $W, E : G \to G$ be a pair of ζ-continuous and weakly ζ-admissible mappings forming $(\zeta, \mathbf{F})$-Ćirić contraction defined on ζ-complete space $(G, \mathbf{A}, s)$. If $\exists\, \kappa \in \left(0, \dfrac{1}{1 + \log_2 s}\right)$ and g_0 in G such that $\zeta(g_0, W(g_0)) \geq 1$, then h is a common fixed point of W, E provided $\zeta(h, h) \geq 1$. Moreover, if W, E are not ζ-continuous, then assuming that G is ζ-regular space and the operator $\mathbf{F}$ is continuous guarantees the existence of a fixed point.*

Proof. Uniqueness of the common fixed point: Suppose that h and v are two different common fixed points of W and E. Then, $W(h) = h \neq v = E(v)$. It follows that $\mathbf{A}(W(h), E(v)) = \mathbf{A}(h, v) \succ 0$. Since $\zeta(h, v) \geq 1$, the contractive condition (5.1) implies

$$\begin{aligned}
\mathbf{I} \;\oplus\; & \mathbf{F}(s\zeta(h,v)\mathbf{A}(W(h),E(v))) \\
\preceq\; & \mathbf{F}\left(\max\left\{\mathbf{A}(h,v),\mathbf{A}(h,W(v)),\mathbf{A}(v,E(v)),\frac{\mathbf{A}(h,E(v))\oplus\mathbf{A}(v,W(h))}{2s}\right\}\right) \\
\preceq\; & \mathbf{F}\left(\max\left\{\mathbf{A}(h,v),\mathbf{A}(h,h),\mathbf{A}(v,v),\frac{\mathbf{A}(h,v)\oplus\mathbf{A}(v,h)}{2s}\right\}\right) \\
=\; & \mathbf{F}\left(\mathbf{A}(h,v)\right) \preceq W(s\zeta(h,v)\mathbf{A}(h,v)).
\end{aligned}$$

This shows that $\mathbf{I} \preceq 0$, a contradiction. Hence, the pair (W, E) has at most one common fixed point.

(a) We note that for all $l \neq q$, $\mathbf{M}(l, q) \succ 0$. Let $g_0 \in G$ be as in (2). We construct an iterative sequence $\{g_n\}$ of points in G such that $g_1 = W(g_0)$, $g_2 = E(g_1)$, and generally $g_{2n+1} = W(g_{2n})$, $g_{2n} = E(g_{2n-1})$ for all $n \in \{0, 1, 2, 3, ...\}$. By assumption (1), we have

$$\zeta(W(g_0), EW(g_0)) = \zeta(g_1, g_2) \geq 1 \text{ and } \zeta(E(g_1), WE(g_1)) = \zeta(g_2, g_3) \geq 1,$$

$$\zeta(W(g_2), EW(g_2)) = \zeta(g_3, g_4) \geq 1 \text{ and } \zeta(E(g_3), WE(g_3)) = \zeta(g_4, g_5) \geq 1,$$

and continuing on the same pattern, we have

$$\zeta(W(g_{2n}), EW(g_{2n})) = \zeta(g_{2n+1}, g_{2n+2}) \geq 1 \text{ and } \zeta(E(g_{2n-1}), WE(g_{2n-1})) = \zeta(g_{2n}, g_{2n+1}) \geq 1.$$

Hence, $\zeta(g_n, g_{n+1}) \geq 1$ for all n. If $\mathbf{A}(W(g_{2n}), E(g_{2n+1})) = \mathbf{0}$, then g_{2n} is a common fixed point of mappings W, E. Let $\mathbf{A}(W(g_{2n}), E(g_{2n+1})) \succ \mathbf{0}$, and then by contractive condition (5.1), we get

$$\mathbf{F}\left(s \odot \mathbf{A}(g_{2n+1}, g_{2n+2})\right) \preceq \mathbf{F}\left(s\zeta(g_{2n}, g_{2n+1}) \odot \mathbf{A}(W(g_{2n}), E(g_{2n+1}))\right) \prec \mathbf{F}\left(\mathbf{M}(g_{2n}, g_{2n+1})\right) \ominus \mathbf{I},$$

for all $n = 0, 1, 2, ...$ where

$$\begin{aligned}
\mathbf{M}(g_{2n}, g_{2n+1}) &= \max\left\{\begin{array}{l} \mathbf{A}(g_{2n}, g_{2n+1}), \mathbf{A}(g_{2n}, W(g_{2n})), \mathbf{A}(g_{2n+1}, E(g_{2n+1})), \\ \dfrac{\mathbf{A}(g_{2n}, E(g_{2n+1})) + \mathbf{A}(g_{2n+1}, W(g_{2n}))}{2s} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} \mathbf{A}(g_{2n}, g_{2n+1}), \mathbf{A}(g_{2n}, g_{2n+1}), \mathbf{A}(g_{2n+1}, g_{2n+2}), \\ \dfrac{\mathbf{A}(g_{2n}, g_{2n+2}) + \mathbf{A}(g_{2n+1}, g_{2n+1})}{2s} \end{array}\right\} \\
&\preceq \max\left\{\mathbf{A}(g_{2n}, g_{2n+1}), \mathbf{A}(g_{2n+1}, g_{2n+2})\right\}.
\end{aligned}$$

If $\|\mathbf{M}(g_{2n}, g_{2n+1})\| = \|\mathbf{A}(g_{2n+1}, g_{2n+2})\|$, then

$$\mathbf{F}\left(s \odot \mathbf{A}(g_{2n+1}, g_{2n+2})\right) \prec \mathbf{F}\left(\mathbf{A}(g_{2n+1}, g_{2n+2})\right) \ominus \mathbf{I},$$

which is a contradiction to (AF_1). Therefore,

$$\mathbf{F}\left(s \odot \mathbf{A}(g_{2n+1}, g_{2n+2})\right) \prec \mathbf{F}\left(\mathbf{A}(g_{2n}, g_{2n+1})\right) \ominus \mathbf{I}, \tag{5.3}$$

for all $n \in \{0, 1, 2, 3, ...\}$. Similarly, we can have

$$\mathbf{F}\left(s \odot \mathbf{A}(g_{2n+2}, g_{2n+3})\right) \prec \mathbf{F}\left(\mathbf{A}(g_{2n+1}, g_{2n+2})\right) \ominus \mathbf{I}, \tag{5.4}$$

for all $n \in \{0, 1, 2, 3, ...\}$. Hence, from (5.3) and (5.4), we have

$$\mathbf{F}\left(s \odot \mathbf{A}(g_n, g_{n+1})\right) \prec \mathbf{F}\left(\mathbf{A}(g_{n-1}, g_n)\right) \ominus \mathbf{I}, \tag{5.5}$$

for all $n \in \{0, 1, 2, 3, ...\}$. Let $\mathbf{b}_n = \mathbf{A}(g_n, g_{n+1})$ for each $n \in \{0, 1, 2, 3, ...\}$, and by (5.5) and (AF_4), we have

$$\mathbf{I} \oplus \mathbf{F}\left(s^n \odot \mathbf{b}_n\right) \preceq \mathbf{F}\left(s^{n-1} \odot \mathbf{b}_{n-1}\right), \qquad n \in \mathbb{N}.$$

Repeating the above process, we obtain

$$\mathbf{F}\left(s^n \odot \mathbf{b}_n\right) \prec \mathbf{F}\left(\mathbf{b}_0\right) \ominus n \odot \mathbf{I}, \qquad n \in \mathbb{N}. \tag{5.6}$$

By Lemma 4.3, $\{\mathbf{b}_n\} \in O(n^{-\frac{1}{\kappa}})$. Since $\frac{1}{\kappa} \in (1 + \log_2 s, \infty)$, by Lemma 4.1, we infer that $\{g_n\}$ is a Cauchy sequence. Since G is a ζ-complete vector-valued b-metric space, there exists (say) $h \in G$ such that $g_{2n+1} \to h$ and $g_{2n+2} \to h$ as $n \to \infty$. The ζ-continuity of E implies

$$h = \lim_{n\to\infty} g_n = \lim_{n\to\infty} g_{2n+1} = \lim_{n\to\infty} g_{2n+2} = \lim_{n\to\infty} E(g_{2n+1}) = E(\lim_{n\to\infty} g_{2n+1}) = E(h).$$

If $\mathbf{A}(h, W(h)) \succ \mathbf{0}$, and $\zeta(h, h) \geq 1$, then by contractive condition (5.1), we have

$$\mathbf{I} \oplus \mathbf{F}(s \odot \mathbf{A}(W(h), h)) \preceq \mathbf{I} \oplus \mathbf{F}(s\zeta(h, h) \odot \mathbf{A}(W(h), E(h))) \preceq \mathbf{F}(\mathbf{M}(h, h)) = \mathbf{F}(\mathbf{A}(W(h), h)),$$

a contradiction. Thus, $\mathbf{A}(W(h), h) = \mathbf{0}$ and $(\mathbf{A}_1)$ imply $h = W(h)$. Thus, we have $W(h) = E(h) = h$. Hence, (W, E) has a common fixed point h.

(b) We have two different cases.

Case 1. if there exists a subsequence $\{g_{n_j}\}_{j\in\mathbb{N}} \subset \{g_n\}_{n\in\mathbb{N}}$ such that

$$g_{n_j} = W(h) \text{ for all even } j \text{ and } g_{n_j} = E(h) \text{ for all odd } j,$$

then

$$h = \lim_{j\to\infty} g_{n_j} = \lim_{j\to\infty} W(h) = W(h) \text{ and } h = \lim_{j\to\infty} g_{n_j} = \lim_{j\to\infty} E(h) = E(h).$$

So, we are done.

Case 2. if there is no such subsequence of $\{g_{n_j}\}_{n\in\mathbb{N}}$ then there exists a natural number L_0 such that for every $n \geq L_0$, we have $\mathbf{A}(W(g_{2n}), E(h)) \succ \mathbf{0}$ and $\mathbf{A}(E(g_{2n+1}), W(h)) \succ \mathbf{0}$. It is given that the space G is ζ-regular, and thus $\zeta(g_{2n+1}, h) \geq 1$, $\zeta(g_{2n}, h) \geq 1$. By contractive condition (5.1), we have

$$\begin{aligned} &\mathbf{I} \oplus \mathbf{F}(s\zeta(g_{2n}, \upsilon) \odot \mathbf{A}(W(g_{2n}), E(h))) \\ &\quad \preceq \mathbf{F}\left(\max\left\{\begin{array}{c} \mathbf{A}(g_{2n}, h), \mathbf{A}(g_{2n}, W(g_{2n})), \mathbf{A}(h, E(h)), \\ \dfrac{\mathbf{A}(g_{2n}, E(h)) \oplus \mathbf{A}(h, W(g_{2n}))}{2s} \end{array}\right\}\right). \end{aligned} \tag{5.7}$$

We show that $\mathbf{A}(h, E(h)) = 0$. Suppose on the contrary that $\mathbf{A}(h, E(h)) = \mathbf{u} \succ \mathbf{0}$. Put $\mathbf{y}_n = \mathbf{A}(g_n, h)$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} g_n = h$, there exists $L_1 \in \mathbb{N}$ such that for every $n \geq L_1$ both $\mathbf{y}_n \prec \frac{\mathbf{u}}{2}$ and $\mathbf{b}_n \prec \frac{\mathbf{u}}{2}$ hold. Consequently, by (5.7), we have

$$
\begin{aligned}
&\mathbf{I} \oplus \mathbf{F}(s\zeta(g_{2n}, h)\mathbf{A}(W(g_{2n}), E(h))) \\
&\leq \mathbf{F}\left(\max\left\{\mathbf{y}_{2n}, \mathbf{b}_{2n}, \mathbf{u}, \frac{\mathbf{A}(g_{2n}, E(h)) \oplus \mathbf{y}_{2n+1}}{2s}\right\}\right) \\
&\leq \mathbf{F}\left(\max\left\{\mathbf{y}_{2n}, \mathbf{b}_{2n}, \mathbf{u}, \frac{s \odot \mathbf{y}_{2n} \oplus s \odot \mathbf{u} \oplus \mathbf{y}_{2n+1}}{2s}\right\}\right) \\
&\leq \mathbf{F}\left(\max\left\{\frac{\mathbf{u}}{2}, \frac{\mathbf{u}}{2}, \mathbf{u}, \frac{\frac{s}{2} \odot \mathbf{u} \oplus s \odot \mathbf{u} \oplus \frac{\mathbf{u}}{2}}{2s}\right\}\right) \\
&= \mathbf{F}(\mathbf{u}).
\end{aligned}
$$

Thus, for every $n \geq \max\{L_0, L_1\}$, we obtain

$$
\mathbf{I} \oplus \mathbf{F}(s\zeta(g_{2n}, h)\mathbf{A}(W(g_{2n}), E(h))) \leq W(\mathbf{A}(h, E(h))). \tag{5.8}
$$

Since $\mathbf{F}$ is continuous and increasing, by Lemma 3.1 and inequality (5.8), we have

$$
\begin{aligned}
\mathbf{I} \oplus \mathbf{F}(\mathbf{A}(h, E(h))) &\leq \mathbf{I} \oplus \mathbf{F}(s\zeta(g_{2n}, h) \odot \liminf_{n\to\infty} \mathbf{A}(W(g_{2n}), E(h))) \\
&\leq \mathbf{I} \oplus \liminf_{n\to\infty} \mathbf{F}(s\zeta(g_{2n}, h) \odot \mathbf{A}(W(g_{2n}), E(h))) \\
&\leq \mathbf{F}(\mathbf{A}(h, E(h))).
\end{aligned}
$$

The above inequality shows that $\mathbf{I} \leq 0$, which is a contradiction. Thus, $\mathbf{A}(E(h), h) = \mathbf{0}$, and hence $h = E(h)$. Similarly, we can prove that $h = W(h)$, and consequently, h is a common fixed point of W and E.

Note that for $W \equiv E$, $s = 1$ and $\zeta(l, q) = 1$, $\forall\, l, q \in G$, Theorem 5.1 reduces to Altun's fixed point theorem [2].
The following example illustrates Theorem 5.1.

Example 5.1. Let $G = [0, \infty)$ and define $\mathbf{A} \colon G \times G \to \mathbb{R}^m$ by $\mathbf{A}(l, q) = \left(|l - q|^j\right)_{j=2}^{m+1}$.

$$
\text{Define } \zeta \colon G \times G \to [0, \infty) \text{ by } \zeta(l, q) = \begin{cases} e^{\|\mathbf{A}(l,q)\|} & \text{for all } l, q \in G \text{ with } l \geq q; \\ 0 & \text{for all } l, q \in G \text{ with } l < q, \end{cases}
$$

so, $(G, \mathbf{A}, s)$ is a ζ-complete vector-valued b-metric space with $s = 2^m$. Define the mappings $W, E \colon G \to G$, for all $g \in G$, by

$$
W(g) = \log_e(1 + \frac{g}{6}), \quad E(g) = \log_e(1 + \frac{g}{7}).
$$

W is ζ-continuous self-mapping: Indeed, consider the sequence $l_n = \frac{K}{n^2}$ for all positive integers n, $\frac{K}{n^2} \geq \frac{K}{(n+1)^2}$, so $\zeta(l_n, l_{n+1}) \geq 1$, and $\lim_{n\to\infty} \mathbf{A}(l_n, l) = \mathbf{0}$ implies $(l^2, l^3, \cdots, l^{m+1}) = \mathbf{0}$. This is true for $l = 0$.

$$
\text{Now, } \lim_{n\to\infty} \mathbf{A}(W(l_n), W(l)) = \lim_{n\to\infty} \left(\left(\log_e\left(1 + \frac{K}{6n^2}\right)\right)^j\right)_{j=2}^{m+1} = \mathbf{0}_W.
$$

Thus, whenever $\zeta(l_n, l_{n+1}) \geq 1$, and $\lim_{n\to\infty} \mathbf{A}(l_n, l) = \mathbf{0}$, we have $\lim_{n\to\infty} \mathbf{A}(W(l_n), W(l)) = \mathbf{0}_W$. Similarly, E is a ζ-continuous self-mapping. To prove that (W, E) is a weakly ζ-admissible pair of mappings, let $l, q \in G$ be such that $q = W(l)$, and thus, we have $q = \log_e\left(1 + \frac{l}{6}\right)$.

$$W(l) = \log_e\left(1 + \frac{l}{6}\right) \geq \log_e\left(1 + \frac{\log_e(1 + \frac{l}{6})}{7}\right) = \log_e\left(1 + \frac{q}{7}\right) = E(q) = EW(l).$$

Thus, $\zeta(Wl, EW(l)) \geq 1$. Again, let $q, t \in G$ be such that $t = E(q)$, and thus, we have $t = \log_e\left(1 + \frac{q}{7}\right)$.

$$E(q) = \log_e\left(1 + \frac{q}{7}\right) \geq \log_e\left(1 + \frac{\log_e\left(1 + \frac{q}{7}\right)}{6}\right) = \log_e\left(1 + \frac{t}{6}\right) = W(t) = WE(q).$$

Thus, $\zeta(Eq, WE(q)) \geq 1$. Hence, (W, E) is a weakly ζ-admissible pair of mappings. Now, for each $l, q \in G$ with $l \geq q$ and choosing z such that $\frac{z}{2e^{\|\mathbf{A}(l,q)\|}} > 1 + \log_2 s$, we have

$$\begin{aligned} s\zeta(l, q) \odot \left(|W(l) - E(q)|^j + 1\right)_{j=2}^{m+1} &= \left(se^{\|\mathbf{A}(l,q)\|}|W(l) - E(q)|^j + 1\right)_{j=2}^{m+1} \\ &= \left(se^{\|\mathbf{A}(l,q)\|}\left|\log_e\left(1 + \frac{l}{6}\right) - \log_e\left(1 + \frac{q}{7}\right)\right|^j + 1\right)_{j=2}^{m+1} \\ &\leq \left(se^{\|\mathbf{A}(l,q)\|}\left|\frac{l}{6} - \frac{q}{7}\right|^j + 1\right)_{j=2}^{m+1}. \end{aligned}$$

Define the function $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ by $\mathbf{F}(\mathbf{v}) = (\log_e(v_i + 1))_{i=2}^{m+1}$, for all $\mathbf{v} = (v_i)_{i=2}^{m+1} \in \mathbb{P}^m$, and then $\mathbf{F} \in \Pi_s^b$ (as shown above). Hence, for all $l, q \in G$ such that $\mathbf{A}(W(l), E(q)) \succ \mathbf{0}$, $\mathbf{I} = \left(\log_e\left(\frac{z}{se^{\|\mathbf{A}(l,q)\|}}\right)\right)_1^m$, and we obtain

$$\mathbf{I} \oplus \mathbf{F}\left(s\zeta(l, q) \odot \mathbf{A}(W(l), E(q))\right) \preceq \mathbf{F}\left(\mathbf{M}(l, q)\right).$$

This shows that the inequality (5.1) holds true for all $l, q \in G$. Thus, the mappings W, E fulfill all the requirements of Theorem 5.1; moreover, W, E have a unique common fixed point $h = 0$.

Corollary 5.1. *Let $(G, \mathbf{A}, s)$ be a ζ-complete vector-valued b-metric space. Suppose that $W, E : G \to G$ are self-mappings such that*

$$s^3 \odot \mathbf{A}(W(l), E(q)) \preceq Q\left(\max\left\{\begin{array}{c} \mathbf{A}(l, q), \mathbf{A}(l, W(l)), \mathbf{A}(q, E(q)), \\ \dfrac{\mathbf{A}(l, E(q)) \oplus \mathbf{A}(q, W(l))}{2s} \end{array}\right\}\right),$$

for all $l, q \in G$, $\rho(Q) \in \left(0, \dfrac{1}{1 + \log_2 s}\right)$. If W or E is continuous, then W, E have a unique common fixed point in G.

Proof. Define $\zeta(l, q) = s^2$ for all $l, q \in G$ and let $\mathbf{I} \succ \mathbf{0}$ be such that Q is a scalar matrix of order m having every non-zero entry equal to e^{-t}, $t > 0$. Then, for $\mathbf{F}(\mathbf{v}) = (\log_e(v_i))_{i=2}^{m+1}$ and applying Theorem 5.1, we have the required result. Note that for $s = 1$ and $W \equiv E$, Corollary 5.1 reduces to Perov's fixed point theorem [26].

The weak ζ-admissibility.

Definition 5.3. *The mapping $W : G \to G$ defined on a space $(G, \mathbf{A}, s)$ and satisfying the inequality*

$$\zeta(l, q) \geq 1 \textit{ implies } \zeta(W(l), W^2(l)) \geq 1 \textit{ and } \zeta(W(q), W^2(q)) \geq 1 \textit{ for all } l, q \in G$$

is called weakly ζ-admissible.

Definition 5.4. *The weakly ζ-admissible mapping $W : G \to G$ defined on a space $(G, \mathbf{A}, s)$ is called a $(\zeta, \mathbf{F})$-weak contraction, if there exist $\mathbf{F} \in \Pi_s^b$, $\mathbf{I} > \mathbf{0}$ such that*

$$\mathbf{I} \oplus \mathbf{F}(s\zeta(l, q) \odot \mathbf{A}(W(l), W(q))) \leq \mathbf{F}(\mathbf{M}(l, q)),$$

for all $l, q \in G$, whenever $\min\{\mathbf{A}(W(l), W(q)), \mathbf{M}(l, q)\} > \mathbf{0}$.

The following corollary extends the results in Mínak et al. [20].

Corollary 5.2. *Let $W : G \to G$ be a ζ-continuous and $(\zeta, \mathbf{F})$-weak contraction defined on ζ-complete space $(G, \mathbf{A}, s)$. If $\exists\, \kappa \in \left(0, \dfrac{1}{1 + \log_2 s}\right)$ and g_0 in G such that $\zeta(g_0, W(g_0)) \geq 1$, then h is a fixed point of W provided $\zeta(h, h) \geq 1$. Moreover, if W is not ζ-continuous, then assuming that G is ζ-regular space and the operator $\mathbf{F}$ is continuous guarantees the existence of a fixed point.*

Proof. Set $E \equiv W$ in Theorem 5.1.

The following theorem is on the $(\zeta, \mathbf{F})$-Hardy-Rogers contraction.

Theorem 5.2. *Let $W, E : G \to G$ be a pair of ζ-continuous and weakly ζ-admissible mappings forming a $(\zeta, \mathbf{F})$-Hardy-Rogers contraction defined on ζ-complete space $(G, \mathbf{A}, s)$. If $\exists\, \kappa \in \left(0, \dfrac{1}{1 + \log_2 s}\right)$ and g_0 in G such that $\zeta(g_0, W(g_0)) \geq 1$, then h is a common fixed point of W, E provided $\zeta(h, h) \geq 1$. Moreover, if W, E are not ζ-continuous, then assuming that G is ζ-regular space and the operator $\mathbf{F}$ is continuous, guarantees the existence of a fixed point.*

Proof. Since

$$\begin{aligned}
\mathbf{H}(l, q) &= a_1 \odot \mathbf{A}(l, q) \oplus a_2 \odot \mathbf{A}(l, W(l)) \oplus a_3 \odot \mathbf{A}(q, E(q)) \\
&\oplus a_4 \odot [\mathbf{A}(l, E(q)) \oplus \mathbf{A}(q, W(l))] \\
&= a_1 \odot \mathbf{A}(l, q) \oplus a_2 \odot \mathbf{A}(l, W(l)) \oplus a_3 \odot \mathbf{A}(q, E(q)) \\
&\oplus 2sa_4 \odot \left[\frac{\mathbf{A}(l, E(q)) \oplus \mathbf{A}(q, W(l))}{2s}\right] \\
&\leq a_1 \odot \mathbf{M}(l, q) \oplus a_2 \odot \mathbf{M}(l, q) \oplus a_3 \odot \mathbf{M}(l, q) \\
&\oplus 2sa_4 \odot \mathbf{M}(l, q) \\
&= (a_1 + a_2 + a_3 + 2sa_4) \odot \mathbf{M}(l, q) = \mathbf{M}(l, q),
\end{aligned}$$

the inequality (5.2) implies the inequality (5.1), so the proof of Theorem 5.2 follows from Theorem 5.1.

6. Results subject to binary relation

Let $\mathbf{M}(l,q)$ and $\mathbf{H}(l,q)$ represent the same vector as in the above section. Let R represent a binary relation over G. We need the following definitions.

Definition 6.1. *We say the mappings W, E weakly increasing subject to R if for all $l, q \in G$, we have*

$$lRq \text{ implies } W(l)REW(l) \text{ and } E(q)RWE(q).$$

Let $\zeta : G \times G \to \mathbb{R}_0^+$ be given by

$$\zeta(l,q) = \begin{cases} 1 & \text{if } lRq; \\ 0 & \text{otherwise.} \end{cases}$$

Note that Definition 6.1 seems to be a particular case of Definition 4.2.

Definition 6.2. *The vector-valued b-metric space $(G, \mathbf{A}, s)$ is said to be R-regular if for any sequence, $\{g_n\} \subset G$ such that $g_n R g_{n+1}$ and $g_n \to h$ as $n \to \infty$, we have $g_n R h$ for all $n \in \mathbb{N}$.*

Now, we are able to revisit Theorems 5.1 and 5.2.

Theorem 6.1. *Let $W, E : G \to G$ be R-continuous and R-weakly increasing mappings defined on R-complete space $(G, \mathbf{A}, s, \preceq)$. If there exist $g_0 \in G$ such that $g_0 R W(g_0)$, $\mathbf{F} \in \pi_s^b$, $\mathbf{I} \succ \mathbf{0}$ and $\kappa \in \left(0, \dfrac{1}{1+\log_2 s}\right)$ such that*

$$\mathbf{I} \oplus \mathbf{F}(s^3 \mathbf{A}(W(l), E(q))) \preceq \mathbf{F}\left(\mathbf{M}(l,q)\right),$$

for all $l, q \in G$ with lRq, whenever $\min\{\mathbf{A}(W(l), E(q)), \mathbf{M}(l,q)\} \succ \mathbf{0}$, then W and E admit a common fixed point. Moreover, if W, E are not R-continuous, then assuming that G is R-regular space and the operator $\mathbf{F}$ is continuous guarantees the existence of a common fixed point.

Proof. Define

$$\zeta(l,q) = \begin{cases} s^2 & \text{if } lRq; \\ 0 & \text{otherwise,} \end{cases}$$

and follow the proof of Theorem 5.1.

Theorem 6.2. *Let $W, E : G \to G$ be R-continuous and R-weakly increasing mappings defined on R-complete space $(G, \mathbf{A}, s, \preceq)$. If there exist $g_0 \in G$ such that $g_0 R W(g_0)$, $\mathbf{F} \in \pi_s^b$, $\mathbf{I} \succ \mathbf{0}$ and $\kappa \in \left(0, \dfrac{1}{1+\log_2 s}\right)$ such that*

$$\mathbf{I} \oplus \mathbf{F}(s^3 \mathbf{A}(W(l), E(q))) \preceq \mathbf{F}\left(\mathbf{H}(l,q)\right),$$

for all $l, q \in G$ with lRq, whenever $\min\{\mathbf{A}(W(l), E(q)), \mathbf{H}(l,q)\} \succ \mathbf{0}$, then W and E admit a common fixed point. Moreover, if W, E are not R-continuous, then assuming that G is R-regular space and the operator $\mathbf{F}$ is continuous guarantees the existence of a common fixed point.

Proof. Define

$$\zeta(l,q) = \begin{cases} s^2 & \text{if } lRq; \\ 0 & \text{otherwise,} \end{cases}$$

and follow the proof of Theorem 5.2.

7. Results in graph theory

Jachymski [15] recently presented a very intriguing approach to the theory of fixed points in some generic structures utilizing the context of metric spaces furnished with a graph. Let $(G, \mathbf{A}, s)$ be a vector-valued b-metric space, and δ denotes the diagonal of the Cartesian product $G \times G$. Assume that the set $V(B)$ of the vertices in a directed graph B coincides with G, where $e(B)$ is the set of edges in the graph. Assume that B does not have any parallel edges. Then, B can be associated with the pair $(V(B), e(B))$. Now we give some results that generalize the results in Jachymski [15].

Definition 7.1. Let W and E be two self-mappings on a vector-valued b-metric space $(V(B), \mathbf{A}, s)$ endowed with graph B. A pair $[W, E]$ is said to be,

(i) weakly B-connected if $(W(g), EW(g)) \in e(B)$ and $(E(g), WE(g)) \in e(B)$ for all $g \in G$,
(ii) partially weakly B-connected if $(W(g), EW(g)) \in e(B)$ for all $g \in G$.

Let $(V(B), \mathbf{A}, s)$ be a vector-valued b-metric space associated with graph B and let

$$\zeta(g_1, g_2) = \begin{cases} 1 & \text{if } (g_1, g_2) \in e(B); \\ 0 & \text{otherwise.} \end{cases}$$

Then, the above definitions are special cases of the definition of weak ζ-admissibility and partially weak ζ-admissibility.

Definition 7.2. Let $(V(B), \mathbf{A}, s)$ be a vector-valued b-metric space associated with graph B. We say it is B-complete if and only if every Cauchy sequence $\{g_n\}$ in G satisfying $(g_n, g_{n+1}) \in e(B)\ \forall\ n \in \mathbb{N}$ converges in G.

Definition 7.3. Let $(V(B), \mathbf{A}, s)$ be a vector-valued b-metric space associated with graph B and $W : G \to G$ be a mapping. We say that W is a B-continuous mapping on $(G, \mathbf{A}, s)$ if for given $g \in G$ and sequence $\{g_n\}$, $\lim_{n\to\infty} \mathbf{A}(g_n, g) = \mathbf{0}$ and $(g_n, g_{n+1}) \in e(B)\ \forall\ n \in \mathbb{N}$ imply $\lim_{n\to\infty} \mathbf{A}(W(g_n), W(g)) = \mathbf{0}$.

Definition 7.4. Let $(V(B), \mathbf{A}, s)$ be a vector-valued b-metric space associated with graph B. The pair $[W, E]$ is said to be an B-compatible if and only if $\lim_{n\to\infty} \mathbf{A}(WE(g_n), EW(g_n)) = \mathbf{0}$, whenever $\{g_n\}$ is a sequence in G satisfying $(g_n, g_{n+1}) \in e(B)$ and

$$\lim_{n\to\infty} W(g_n) = \lim_{n\to\infty} E(g_n) = g \text{ for some } g \in G.$$

Now we are able to revisit Theorems 5.1 and 5.2 in the framework of vector-valued b-metric space associated with graph B

Theorem 7.1. *Let $W, E : V(B) \to V(B)$ be B-continuous and weakly B-increasing self-mappings defined on B-complete vector-valued b-metric space $(V(B), \mathbf{A}, s)$ endowed with graph. If there exist $g_0 \in V(B)$ such that $(g_0, W(g_0)) \in e(B)$, $\mathbf{F} \in \Pi_s^b$, $\mathbf{I} > \mathbf{0}$ and $\kappa \in \left(0, \dfrac{1}{1 + \log_2 s}\right)$ such that*

$$\mathbf{I} \oplus \mathbf{F}(s^3 \odot \mathbf{A}(W(l), E(q))) \leq \mathbf{F}(\mathbf{M}(l, q)),$$

for all $l, q \in V(B)$ with $(l, q) \in e(B)$, whenever $\min\{\mathbf{A}(W(l), E(q)), \mathbf{M}(l, q)\} > \mathbf{0}$, then W and E admit a common fixed point.

Proof. Define

$$\zeta(l,q) = \begin{cases} s^2 & \text{if } (l,q) \in e(B), \\ 0 & \text{otherwise,} \end{cases}$$

and follow the proof of Theorem 5.1.

Theorem 7.2. *Let $W, E : V(B) \to V(B)$ be B-continuous and weakly B-increasing self-mappings defined on B-complete vector-valued b-metric space $(V(B), \mathbf{A}, s)$ endowed with graph. If there exist $g_0 \in V(B)$ such that $(g_0, W(g_0)) \in e(B)$, $\mathbf{F} \in \Pi_s^b$, $\mathbf{I} \succ \mathbf{0}$ and $\kappa \in \left(0, \dfrac{1}{1+\log_2 s}\right)$ such that*

$$\mathbf{I} \oplus \mathbf{F}(s^3 \odot \mathbf{A}(W(l), E(q))) \preceq \mathbf{F}\left(\mathbf{H}(l,q)\right),$$

for all $l, q \in V(B)$ with $(l,q) \in e(B)$, whenever $\min\{\mathbf{A}(W(l), E(q)), \mathbf{H}(l,q)\} \succ \mathbf{0}$, then, W and E admit a common fixed point.

Proof. Define

$$\zeta(l,q) = \begin{cases} s^2 & \text{if } (l,q) \in e(B); \\ 0 & \text{otherwise,} \end{cases}$$

and follow the proof of Theorem 5.2.

8. Application of Theorem 5.1

A tool used to explore the processes of disease transmission, forecast the trajectory of an outbreak, and assess epidemic control measures is infectious disease modeling. The progression of infectious diseases can be predicted mathematically to demonstrate the likely course of an epidemic (even in plants) and to guide public health and plant health actions. In order to determine parameters for different infectious diseases and utilize those parameters to calculate the impact of various treatments, such mass vaccination programs, models use basic assumptions or collected statistics together with mathematics. The models can assist in selecting the interventions to try and those to avoid, or it can forecast future growth trends. In this section, we apply Theorem 5.1 to the following system of delay integro-differential equations that represents an infectious model:

$$l(t) = \int_{t-L}^{t} W(h, l(h), l'(h))\, dh, \tag{8.1}$$

$$q(t) = \int_{t-L}^{t} E(h, q(h), q'(h))\, dh, \tag{8.2}$$

where

(a) $l(t), q(t)$: the prevalence of infection at time t in the population.
(b) $0 < L$: the amount of time a person can still spread disease.
(c) $l'(h), q'(h)$: the current rate of infectivity.
(d) $W(h, l(h), l'(h)), E(h, q(h), q'(h))$: the rate of newly acquired infections per unit of time.

Now, we look for existence and uniqueness of the positive, periodic solution to (8.1) and (8.2) by the application of Theorem 5.1.

Let $\exists\ p > 0$ and $W, E \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ satisfying

$$W(t + p, l, q) = W(t, l, q)\,\forall\,(t, l, q) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

$$E(t + p, l, q) = E(t, l, q)\,\forall\,(t, l, q) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

Let us define the functional spaces by

$$\begin{aligned}\mathcal{F}(p) &= \{J \in C^1(\mathbb{R}) : \quad J(t + p) = J(t), \quad t \in \mathbb{R}\}.\\ \mathcal{F}_+(p) &= \{J \in \mathcal{F}(p) : \quad J(t) \geq 0 \quad t \in \mathbb{R}\}.\end{aligned}$$

Let $V = \mathcal{F}_+(p) \times \mathcal{F}(p)$, and define a metric $\mathbf{A} : V \times V \to \mathbb{R}^2$ by

$$\mathbf{A}((l_1, q_1), (l_2, q_2)) = (\|l_1 - l_2\|^2\,, \|q_1 - q_2\|^2),$$

where $\|\mathbf{l}\| = \max\{|l(t)| : \quad t \in [0, p], \quad \mathbf{l} \in \mathcal{F}(p)\}$. The function $\zeta : V \times V \to [1, \infty)$ is defined by $\zeta(l, q) = K^2$ for all $l, q \in V$. Then, $(V, \mathbf{A}, s)$ is a ζ-complete vector-valued b-metric space. Now, we develop the structure to apply Theorem 5.1. Let $g(t) = l^{'}(t)$, and $h(t) = q^{'}(t)$. Then, we have

$$g(t) = W(t, l(t), g(t)) - W(t - L, l(t - L), g(t - L)).$$

$$q(t) = E(t, q(t), h(t)) - E(t - L, q(t - L), h(t - L)).$$

Thus, Eqs (8.1) and (8.2) can be written as

$$\begin{cases} l(t) = \int_{t-L}^{t} W(h, l(h), g(h))\, dh \\ \\ g(t) = W(t, l(t), g(t)) - W(t - L, l(t - L), g(t - L)). \end{cases}$$

$$\begin{cases} q(t) = \int_{t-L}^{t} E(h, q(h), h(h))\, dh \\ \\ h(t) = E(t, q(t), h(t)) - E(t - L, q(t - L), h(t - L)). \end{cases}$$

Let $\mathbb{Y} : V \to C(\mathbb{R}) \times C(\mathbb{R})$ be a mapping defined by

$$\mathbb{Y}(l, \sigma) = (\mathbb{Y}_1(l, \sigma), \mathbb{Y}_2(l, \sigma) \text{ for all } (l, \sigma) \in V,$$

where $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are defined by the following matrix equation:

$$\begin{pmatrix} \mathbb{Y}_1(l, \sigma)(t) \\ \\ \mathbb{Y}_2(l, \sigma)(t) \end{pmatrix} = \begin{pmatrix} \int_{t-L}^{t} W(h, l(h), \sigma(h))\, dh \\ \\ W(t, l(t), \sigma(t)) - W(t - L, l(t - L), \sigma(t - L)) \end{pmatrix}.$$

Similarly, define the mapping $\mathbb{X} : V \to C(\mathbb{R}) \times C(\mathbb{R})$ by

$$\mathbb{X}(q, r) = (\mathbb{X}_1(q, r), \mathbb{X}_2(q, r) \text{ for all } (q, r) \in V,$$

where $\mathbb{X}_1$ and $\mathbb{X}_2$ are defined by the following matrix equation:

$$\begin{pmatrix} \mathbb{X}_1(q, r)(t) \\ \\ \mathbb{X}_2(q, r)(t) \end{pmatrix} = \begin{pmatrix} \int_{t-L}^{t} E(h, q(h), r(h))\, dh \\ \\ E(t, q(t), r(t)) - E(t - L, q(t - L), r(t - L)) \end{pmatrix}.$$

(I_1) Suppose that the mappings W, E are bounded and periodic having period p and there exist $\omega, K > 0$ so that $\forall t \in \mathbb{R}, l(t), u(t) \in \mathbb{R}_+$ and $q(t), \epsilon(t) \in \mathbb{R}$

$$|W(t, l(t), q(t)) - E(t, u(t), \epsilon(t))| \leq \frac{e^{-\omega}}{sK} |l(t) - u(t)|,$$

(I_2) there exists $l_0 \in V$ such that $\zeta(l_0, \mathbb{Y}(l_0)) = K^2$.

Theorem 8.1. *Let the mappings $W, E \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ satisfy conditions $(I_1) - (I_2)$ and $\frac{e^{-2\omega}}{s^2K^2} < \frac{1}{4+L^2}$. Then, Eqs (8.1) and (8.2) admit a common solution in $\mathcal{F}_+(p)$.*

Proof. We note that the conditions (1) and (2) of Theorem 5.1 can be verified by using (I_2) and continuity of mappings W, E respectively. In the following, we prove the contractive condition (5.1). By definition,

$$\begin{aligned}
\mathbb{Y}_1(l, g)(t + p) &= \int_{t+p-L}^{t+p} W(h, l(h), g(h))\, dh \\
&= \int_{t-L}^{t} W(u - p, l(u - p), g(u - p))\, du \\
&= \int_{t-L}^{t} W(u - p + p, l(u - p + p), g(u - p + p))\, du \\
&= \int_{t-L}^{t} W(u, l(u), g(u))\, du \\
&= \mathbb{Y}_1(l, q)(t) \text{ for all } t \in \mathbb{R}, (l, g) \in V.
\end{aligned}$$

This shows that $\mathbb{Y}_1(V) \subseteq \mathcal{F}_+(p)$. Similarly, we have that $\mathbb{Y}_2(V), \mathbb{X}_1(V)$ and $\mathbb{X}_2(V)$ are subsets of $\mathcal{F}_+(p)$. Let $(l_1, q_1), (l_2, q_2) \in V$ and consider

$$\begin{aligned}
&|\mathbb{Y}_1(l_1, q_1)(t) - \mathbb{X}_1(l_2, q_2)(t)|^2 \\
= &\left| \int_{t-L}^{t} W(h, l_1(h), q_1(h))\, dh - \int_{t-L}^{t} E(h, l_2(h), q_2(h))\, dh \right|^2 \\
\leq &\left(\int_{t-L}^{t} |W(h, l_1(h), q_1(h)) - E(h, l_2(h), q_2(h))|\, dh \right)^2 \\
\leq &\left(\int_{t-L}^{t} \left(\frac{e^{-\omega}}{sK} (|l_1(h) - l_2(h)|) \right) dh \right)^2 \\
\leq &\frac{e^{-2\omega} L^2}{s^2 K^2} \|l_1 - l_2\|^2
\end{aligned}$$

and

$$\begin{aligned}
&|\mathbb{Y}_2(l_1, q_1)(t) - \mathbb{X}_2(l_2, q_2)(t)|^2 \\
= &\left| \begin{array}{l} W(t, l_1(t), q_1(t)) - W(t - L, l_1(t - L), q_1(t - L)) - \\ E(t, l_2(t), q_2(t)) + E(t - L, l_2(t - L), q_2(t - L)) \end{array} \right|^2 \\
\leq &\left(\begin{array}{l} |W(t, l_1(t), q_1(t)) - E(t, l_2(t), q_2(t))| + \\ |W(t - L, l_1(t - L), q_1(t - L)) - E(t - L, l_2(t - L), q_2(t - L))| \end{array} \right)^2
\end{aligned}$$

$$\leq \left(\frac{e^{-\omega}}{sK}\left(|l_1(t)-l_2(t)|\right)+\frac{e^{-\omega}}{sK}\left(|l_1(t-L)-l_2(t-L)|\right)\right)^2$$
$$\leq \frac{4e^{-2\omega}}{s^2K^2}\left\|l_1-l_2\right\|^2.$$

Consequently, we obtain the following matrix inequality:

$$\begin{pmatrix}\|\mathbb{Y}_1(l_1,q_1)-\mathbb{X}_1(l_2,q_2)\|^2\\ \|\mathbb{Y}_2(l_1,q_1)-\mathbb{X}_2(l_2,q_2)\|^2\end{pmatrix} \leq \begin{pmatrix}\frac{e^{-2\omega}L^2}{s^2K^2}\|l_1-l_2\|^2\\ \frac{4e^{-2\omega}}{s^2K^2}\|l_1-l_2\|^2\end{pmatrix} = \begin{pmatrix}\frac{e^{-2\omega}L^2}{s^2K^2} & 0\\ 0 & \frac{4e^{-2\omega}}{s^2K^2}\end{pmatrix}\begin{pmatrix}\|l_1-l_2\|^2\\ \|l_1-l_2\|^2\end{pmatrix}.$$

Now, define the mappings $\mathbf{F}:\mathbb{R}^2_+\to\mathbb{R}^2$ by $\mathbf{F}(l,q)=(\log_e(l),\log_e(q))$ and $\mathbf{I}=\left(\frac{s}{e^{-2\omega}L^2},\frac{s}{4e^{-2\omega}}\right)$, and we obtain

$$\mathbf{I}\oplus\mathbf{F}\left(s\zeta((l_1,q_1),(l_2,q_2))\odot\mathbf{A}(\mathbb{Y}(l_1,q_1),\mathbb{X}(l_2,q_2))\right)\leq\mathbf{F}\left(\mathbf{A}((l_1,q_1),(l_2,q_2))\right).$$

Finally, keeping in mind the definition of mapping ζ and the above inequality, we say that the mappings $\mathbb{Y},\mathbb{X}$ satisfy all the requirements of Theorem 5.1 and hence admit a common fixed point in their domain. Consequently, the Eqs (8.1) and (8.2) have a positive, periodic solution.

9. Conclusions

The findings and analysis discussed here could inspire further investigation into this topic by interested academics. A fundamental finding in vector-valued b-metric space, the main result (Theorem 5.1) concerns an F-contraction in vector-valued b-metric spaces. In order to demonstrate the presence of solutions to various linear and nonlinear equations reflecting models of the associated real-world issues, the application approach is also discussed. We conclude that: (1) the classical approach only addresses the functions while the new approach involves operators (linear and nonlinear). (2) The classical approach only involves real numbers on both sides of inequalities (called F-contractions). However, this approach also allows matrices and vectors on both sides of inequalities and hence makes it applicable in engineering and optimization problems.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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