



Research article

An application of generalized Morrey spaces to unique continuation property of the quasilinear elliptic equations

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Abstract: In this paper, we study nonnegative weak solutions of the quasilinear elliptic equation $\operatorname{div}(A(x, u, \nabla u)) = B(x, u, \nabla u)$, in a bounded open set Ω , whose coefficients belong to a generalized Morrey space. We show that $\log(u + \delta)$, for u a nonnegative solution and δ an arbitrary positive real number, belongs to $\operatorname{BMO}(B)$, where B is an open ball contained in Ω . As a consequence, this equation has the strong unique continuation property. For the main proof, we use approximation by smooth functions to the weak solutions to handle the weak gradient of the composite function which involves the weak solutions and then apply Fefferman’s inequality in generalized Morrey spaces, recently proved by Tumulun et al. [1].

Keywords: quasilinear elliptic equations; generalized Morrey spaces; Fefferman’s inequality; bounded mean oscillation; strong unique continuation property

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1. Introduction

Zamboni, in his paper [2], proved that the equation $Lu = -\operatorname{div}(M\nabla u) + G \cdot \nabla u + Vu = 0$, in an open bounded set Ω , has the strong unique continuation property, where M is $n \times n$ bounded elliptic matrix, G^2 and V belong to the Morrey space $L^{q, n-2q}(\mathbb{R}^n)$. By this it is meant that every nonnegative weak solution u of $Lu = 0$ which vanishes with infinite order at a point in Ω satisfies $u = 0$ in a ball contained in Ω . Independently, Chanillo and Sawyer [3] proved the strong unique continuation property holds for the inequality $|\Delta u| \leq |V||u|$, assuming V belongs to the Morrey space $L^{q, n-2q}(\mathbb{R}^n)$. Recently, Tumulun et al. [1] generalized these results by proving the equation $Lu = 0$ has the strong unique continuation property, where G^2 and V belong to the generalized Morrey space $L^{q, \Phi}(\mathbb{R}^n)$, where Φ satisfies some certain conditions.

In 2001, Zamboni [4] obtained the strong unique continuation property for nonnegative solutions

of the quasilinear elliptic equation of the form $\operatorname{div}(A(x, u, \nabla u)) = B(x, u, \nabla u)$, assuming that suitable powers of the coefficients belong to the Morrey space $L^{q, n-2q}(\mathbb{R}^n)$. The special case of this Zamboni's result can be seen in [5], where they assume that the suitable powers of the coefficients belong to the Lebesgue Spaces $L^{\frac{n}{p}}(\mathbb{R}^n)$. There are also some results regarding to the strong unique continuation property with different setting elliptic equations or the function spaces contain coefficients of the equations (see [6, 7] for example).

One of the important tools used by the above authors is Fefferman's inequality (see Theorem 2.1). In his paper [8], Fefferman proved a weighted embedding, that is now known as Fefferman's inequality, where the potential belongs to the $L^{q, n-2q}(\mathbb{R}^n)$. Chiarenza and Frasca [9] then proved the inequality assuming the potential in $L^{q, n-pq}(\mathbb{R}^n)$ for $1 < p < n$. For the general case, Tumalun et al. [1] recently proved the inequality assuming the potential belongs to the generalized Morrey space $L^{q, \Phi}(\mathbb{R}^n)$. Fefferman's inequality also holds for potential belongs to some function spaces called Stummel-Kato classes [1, 2, 4]. However, the Morrey spaces are generally independent to the Stummel-Kato classes and contain the Stummel-Kato classes in certain cases [1, 10, 11].

In this paper, we will prove the strong unique continuation property for nonnegative solutions of the quasilinear elliptic equation of the form $\operatorname{div}(A(x, u, \nabla u)) = B(x, u, \nabla u)$, assuming that suitable powers of the coefficients belong to the generalized Morrey space $L^{q, \Phi}(\mathbb{R}^n)$. It is important to point out that in [4–7] they started their main proof, regarding to the strong unique continuation property for nonnegative solutions of the (degenerate) quasilinear elliptic equation, by using the test function $\phi^p u^{1-p}$ (for $1 < p < n$) in the weak solution definition, where ϕ is a smooth function and u the nonnegative weak solution belongs to the Sobolev space $W_0^{1,p}(\Omega)$. This arises two problems. The first problem is u^{1-p} may be undefined since u can be equal to zero in a subset of Ω which has non zero Lebesgue measure. Meanwhile, the second problem is that there are no tools to handle the weak derivatives of u^{1-p} . We overcome this difficulties by adding the weak solution with an arbitrary positive real number and approximating the weak solution with a sequence of the smooth functions (see the proof of Theorem 4.4).

2. Morrey spaces and Fefferman's inequality

Let $1 \leq q < \infty$ and $\Phi : (0, \infty) \rightarrow (0, \infty)$. The **generalized Morrey space** $L^{q, \Phi}(\mathbb{R}^n)$ is the collection of all functions $f \in L_{\text{loc}}^q(\mathbb{R}^n)$ satisfying

$$\|f\|_{L^{q, \Phi}} := \sup_{x \in \mathbb{R}^n, r > 0} \left(\frac{1}{\Phi(r)} \int_{|x-y| < r} |f(y)|^q dy \right)^{\frac{1}{q}} < \infty.$$

This spaces were introduced by Nakai [12]. If $\Phi(r) = 1$, then $L^{p, \Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = r^n$, then $L^{p, \Phi}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $\Phi(r) = r^\lambda$, where $0 < \lambda < n$, then $L^{p, \Phi}(\mathbb{R}^n) = L^{p, \lambda}(\mathbb{R}^n)$ is the classical Morrey space introduced in [13]. For the last few years, there are many papers which discuss the inclusion between Morrey spaces and the applications of Morrey spaces in elliptic partial differential equations, that can be seen for example in [14–21].

Let $1 < q < n$ and $1 < p < \frac{n}{q}$. We assume the following conditions for Φ throughout this paper.

There exists a positive $K > 0$ such that:

$$s \leq t \Rightarrow \Phi(s) \leq K\Phi(t) \text{ and } \frac{\Phi(s)}{s^n} \geq K \frac{\Phi(t)}{t^n},$$

and, for every $\delta > 0$,

$$\int_{\delta}^{\infty} \frac{\Phi(t)}{t^{(n+1)-\frac{p}{2}(q+1)}} dt \leq K\delta^{\frac{p}{2}(1-q)}.$$

Using the assumptions on Φ , Tumulun et al. [1] proved the following theorem.

Theorem 2.1. *If $V \in L^{q,\Phi}(\mathbb{R}^n)$, then, there exists a positive $K > 0$ such that for every $\phi \in C_0^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |V(x)||\phi(x)|^p dx \leq K\|V\|_{L^{q,\Phi}} \int_{\mathbb{R}^n} |\nabla\phi(x)|^p dx. \quad (2.1)$$

Theorem 2.1 is called Fefferman's inequality. Setting $\Phi(t) = t^{n-qp}$, $t > 0$ (one can check that Φ satisfies all conditions above), then Theorem 2.1 recovers the results in [8, 9].

3. Bounded mean oscillation space

Let $R > 0$ and $x_0 \in \mathbb{R}^n$. The set $B = B(x_0, R) = \{y \in \mathbb{R}^n : |y - x_0| < R\}$ is called a ball in \mathbb{R}^n . A locally integrable function f on \mathbb{R}^n is said to be of **bounded mean oscillation** on a ball $B \subseteq \mathbb{R}^n$, we write $f \in \text{BMO}(B)$, if there exists a positive constant K such that for every ball $B' \subseteq B$,

$$\frac{1}{|B'|} \int_{B'} |f(y) - f_{B'}| dy \leq K,$$

where $f_{B'} = \frac{1}{|B'|} \int_{B'} f(x) dx$ and $|B'|$ is the Lebesgue measure of the ball B' in \mathbb{R}^n .

The following is known as the John-Nirenberg Theorem. We refer to [22] for its proof.

Theorem 3.1. *Let B be a ball in \mathbb{R}^n . If $f \in \text{BMO}(B)$, then there exist $\beta > 0$ and $K > 0$ such that for every ball $B' \subseteq B$,*

$$\int_{B'} \exp(\beta|f(x) - f_{B'}|) \leq K|B'|.$$

Theorem 3.1 has an application to prove the following property which is stated in [1] without proof. Now, we are going to proof that property for the reader convenience.

Theorem 3.2. *Let $f : \Omega \rightarrow \mathbb{R}$ and $B(x_0, 2R) \subseteq \Omega$. If $\log(f) \in \text{BMO}(B(x_0, R))$, then there exists $M > 0$ such that*

$$\int_{B(x_0, R)} f(y)^\gamma dy \leq M \int_{B(x_0, \frac{R}{2})} f(y)^\gamma dy,$$

for some $0 < \gamma \leq 1$.

Proof. Let $B = B(x_0, R)$. By Theorem 3.1, there exist $\beta > 0$ and $K > 0$ such that

$$\left(\int_B \exp(\beta |\log(f) - \log(f)_B|) dy \right)^2 \leq K^2 |B|^2. \quad (3.1)$$

Assume that $\beta < 1$. Using (3.1), we compute

$$\begin{aligned} & \left(\int_B f(y)^\beta dy \right) \left(\int_B f(y)^{-\beta} dy \right) \\ &= \left(\int_B \exp(\beta \log(f(y))) dy \right) \left(\int_B \exp(-\beta \log(f(y))) dy \right) \\ &= \left(\int_B \exp(\beta(\log(f(y)) - \log(f(y))_B)) dy \right) \left(\int_B \exp(-\beta(\log(f(y)) - \log(f(y))_B)) dy \right) \\ &\leq \left(\int_B \exp(\beta |\log(f(y)) - \log(f(y))_B|) dy \right)^2 \\ &\leq K^2 |B|^2, \end{aligned}$$

which yields

$$\left(\int_B f(y)^{-\beta} dy \right)^{\frac{1}{2}} \leq K |B| \left(\int_B f(y)^\beta dy \right)^{-\frac{1}{2}}. \quad (3.2)$$

Applying Hölder's inequality and (3.2), we obtain

$$\begin{aligned} \left| B \left(x_0, \frac{R}{2} \right) \right| &= \int_{B(x_0, \frac{R}{2})} f(y)^{\frac{\beta}{2}} f(y)^{-\frac{\beta}{2}} dy \\ &\leq \left(\int_{B(x_0, \frac{R}{2})} f(y)^\beta dy \right)^{\frac{1}{2}} \left(\int_{B(x_0, \frac{R}{2})} f(y)^{-\beta} dy \right)^{\frac{1}{2}} \\ &\leq \left(\int_{B(x_0, \frac{R}{2})} f(y)^\beta dy \right)^{\frac{1}{2}} \left(\int_B f(y)^{-\beta} dy \right)^{\frac{1}{2}} \\ &\leq \left(\int_{B(x_0, \frac{R}{2})} f(y)^\beta dy \right)^{\frac{1}{2}} K |B| \left(\int_B f(y)^\beta dy \right)^{-\frac{1}{2}}. \end{aligned} \quad (3.3)$$

From (3.3), we have

$$\int_B f(y)^\beta dy \leq 2^{2n} K^2 \int_{B(x_0, \frac{R}{2})} f(y)^\beta dy. \quad (3.4)$$

By setting $\gamma = \beta$, $M = 2^{2n} K^2$, and observing the inequality (3.4), the theorem has proved. Assume that $\beta \geq 1$. We set $\gamma = 1$ and use (3.1) to get

$$\left(\int_B \exp(\gamma |\log(f) - \log(f)_B|) dy \right)^2 \leq \left(\int_B \exp(\beta |\log(f) - \log(f)_B|) dy \right)^2 \leq K^2 |B|^2. \quad (3.5)$$

Processing the inequality (3.5) as previously method, we have the conclusion of the theorem. \square

4. Quasilinear elliptic equations and strong unique continuation property

Let Ω be an open bounded subset of \mathbb{R}^n and $1 < p < \infty$. Consider the following equation:

$$\begin{cases} \operatorname{div}(A(x, u, \nabla u)) = B(x, u, \nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where $A = A(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B = B(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ are two continuous functions and satisfy:

$$\begin{cases} |A(x, u, \xi)| \leq a|\xi|^{p-1} + b(x)|u|^{p-1} \\ |B(x, u, \xi)| \leq c(x)|\xi|^{p-1} + d(x)|u|^{p-1} \\ \xi A(x, u, \xi) \geq |\xi|^p - d(x)|u|^p, \end{cases} \quad (4.2)$$

for almost all $x \in \Omega$, for all $u \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^n$. In (4.2), we assume $p \in (1, n)$, a is a positive constant and b, c and d are measurable functions defined on Ω whose extensions with zero value outside of Ω are such that

$$b^{p/(p-1)}, c^p, d \in L^{q, \Phi}(\mathbb{R}^n). \quad (4.3)$$

Definition 4.1. A function $u \in W_0^{1,p}(\Omega)$ is a weak solution of (4.1) if

$$\int_{\Omega} (A(x, u(x), \nabla u(x)) \nabla \phi(x) + B(x, u(x), \nabla u(x)) \phi(x)) dx = 0 \quad (4.4)$$

for every $\phi \in C_0^\infty(\Omega)$.

We remark that the integral appearing in Definition 4.1 is finite because of the assumptions (4.2) and (4.3), Theorem 2.1.

Definition 4.2. Let $w \in L^1(\Omega)$ and $w \geq 0$ in Ω . The function w is said to **vanish with infinite order** at $x_0 \in \Omega$ if

$$\lim_{R \rightarrow 0} \frac{1}{|B(x_0, R)|^k} \int_{B(x_0, R)} w(x) dx = 0, \quad \forall k > 0.$$

One interesting example of a strictly positive function that vanishes with infinite order at some point in its domain was given by [1]. More precisely, let $\Omega = B(0, 1) \subseteq \mathbb{R}^n$ and $w : \Omega \rightarrow \mathbb{R}$ defined by

$$w(x) = \begin{cases} \exp(-|x|^{-1})|x|^{-(n+1)}, & x \in \Omega \setminus \{0\} \\ 1, & x = 0. \end{cases}$$

We can show that this function vanishes with infinite order at $0 \in \Omega$.

Definition 4.3. The Eq (4.1) is said to have the **strong unique continuation property** in Ω if for every nonnegative weak solution u which vanishes with infinite order at some $x_0 \in \Omega$ satisfies $u \equiv 0$ in $B(x_0, R) \subseteq \Omega$, for some $R > 0$.

If a function vanishes with infinity order at some $x_0 \in \Omega$ and satisfies the doubling integrability over some neighborhood of x_0 , then the function must be identically to zero in the neighborhood. This property is stated in the following lemma.

Lemma 4.1. Let $w \in L^1(\Omega)$, $w \geq 0$, $B(x_0, R) \subseteq \Omega$, and $0 < \gamma \leq 1$. Assume that there exists a constant $C > 0$ satisfying

$$\int_{B(x_0, R)} w(y)^\gamma dy \leq C \int_{B(x_0, \frac{R}{2})} w(y)^\gamma dy.$$

If w vanishes with infinity order at x_0 , then, $w \equiv 0$ in $B(x_0, R)$.

Proof. Assume that $0 < \gamma < 1$. We note that the proof of $\gamma = 1$ can be done by a similar method. According to the hypothesis, for every $j \in \mathbb{N}$, we have

$$\begin{aligned} \int_{B(x_0, R)} w(y)^\gamma dy &\leq C^1 \int_{B(x_0, 2^{-1}R)} w(y)^\gamma dy \\ &\leq C^2 \int_{B(x_0, 2^{-2}R)} w(y)^\gamma dy \\ &\quad \vdots \\ &\leq C^j \int_{B(x_0, 2^{-j}R)} w(y)^\gamma dy. \end{aligned}$$

Hölder's inequality implies that

$$\left(\int_{B(x_0, R)} w^\gamma(y) dy \right)^{\frac{1}{\gamma}} \leq C^{\frac{j}{\gamma}} |B(x_0, 2^{-j}R)|^{\frac{1}{\gamma}} \frac{|B(x_0, 2^{-j}R)|^k}{|B(x_0, 2^{-j}R)|^{k+1}} \int_{B(x_0, 2^{-j}R)} w(y) dy, \quad (4.5)$$

where we choose $k > 0$ such that $C^{\frac{1}{\gamma}} 2^{-nk} = 1$. Then, (4.5) gives

$$\left(\int_{B(x_0, R)} w^\gamma(y) dy \right)^{\frac{1}{\gamma}} \leq (v_n r^n)^{\frac{1}{\gamma} + k} (2^{-\frac{n}{\gamma}})^j \frac{1}{|B(x_0, 2^{-j}R)|^{k+1}} \int_{B(x_0, 2^{-j}R)} w(y) dy, \quad (4.6)$$

where v_n is the Lebesgue measure of unit ball in \mathbb{R}^n . Letting $j \rightarrow \infty$, we obtain from (4.6) that $w^\gamma \equiv 0$ on $B(x_0, R)$. Therefore, $w \equiv 0$ on $B(x_0, R)$. \square

The following theorem is the main property that will be used to prove the strong unique continuation property of (4.1).

Theorem 4.4. Let $u \geq 0$ be the weak solution of (4.1) and $B(x_0, 2R) \subseteq \Omega$. Then $\log(u + \delta) \in BMO(B(x_0, R))$ for every $\delta > 0$.

Proof. Let u be a non negative weak solution of (4.1) and $\delta > 0$. Since $u \in W_0^{1,p}(\Omega)$, then there exists a sequence $\{u_k\}_{k \in \mathbb{N}}$ in $C_0^\infty(\Omega)$, such that $\lim_{k \rightarrow \infty} \|u_k - u\|_{W^{1,p}(\Omega)} = 0$. Therefore, we may assume that $u_k \rightarrow u$ and $\nabla u_k \rightarrow \nabla u$ a.e. in Ω . Moreover, there exist $g, h \in L^p(\Omega)$ such that $|u_k| \leq g$ and $|\nabla u_k| \leq h$ a.e. in Ω , and $u_k + \delta > u \geq 0$, for every $k \in \mathbb{N}$ (see [23]).

Let $x_0 \in \Omega$, $B(x_0, r) \subseteq B(x_0, R)$, and $p' = p/(p-1)$. Let $\phi \in C_0^\infty(B(x_0, 2r))$. We start to prove the convergent of a sequence whose term is defined by

$$\int_{\Omega} A(x, u(x), \nabla u(x)) \nabla (\phi(x)^p (u_k(x) + \delta)^{1-p}) dx, \quad (4.7)$$

for every $k \in \mathbb{N}$. By expanding the integrand in (4.7), we get

$$\begin{aligned} & A(x, u(x), \nabla u(x)) \nabla \left(\phi(x)^p (u_k(x) + \delta)^{1-p} \right) \\ &= pA(x, u(x), \nabla u(x)) \nabla \phi(x) \phi(x)^{p-1} (u_k(x) + \delta)^{1-p} \\ &\quad - (p-1)A(x, u(x), \nabla u(x)) \nabla (u_k(x) + \delta) (u_k(x) + \delta)^{-p} \phi(x)^p. \end{aligned} \quad (4.8)$$

From (4.8), we have

$$\begin{aligned} & |A(x, u(x), \nabla u(x)) \nabla \left(\phi(x)^p (u_k(x) + \delta)^{1-p} \right)| \\ &\leq p|A(x, u(x), \nabla u(x))| |\nabla \phi(x)| |\phi(x)|^{p-1} |u_k(x) + \delta|^{1-p} \\ &\quad + (p-1)|A(x, u(x), \nabla u(x))| |\nabla (u_k(x) + \delta)| |u_k(x) + \delta|^{-p} |\phi(x)|^p. \end{aligned} \quad (4.9)$$

Now, we will prove that the terms in the right hand side of (4.9) are bounded by an integrable function which is independent from $k \in \mathbb{N}$. Using assumption in (4.2), we have

$$\begin{aligned} & p|A(x, u(x), \nabla u(x))| |\nabla \phi(x)| |\phi(x)|^{p-1} |u_k(x) + \delta|^{1-p} \\ &\leq p\delta^{1-p} (\max |\nabla \phi|) (\max |\phi|)^{p-1} a |\nabla u(x)|^{p-1} \\ &\quad + p(\max |\nabla \phi|) (\max |\phi|)^{p-1} b(x) |u(x)|^{p-1} |u_k(x) + \delta|^{1-p} \\ &\leq p\delta^{1-p} (\max |\nabla \phi|) (\max |\phi|)^{p-1} a |\nabla u(x)|^{p-1} + p(\max |\nabla \phi|) (\max |\phi|)^{p-1} b(x), \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} & (p-1)|A(x, u(x), \nabla u(x))| |\nabla (u_k(x) + \delta)| |u_k(x) + \delta|^{-p} |\phi(x)|^p \\ &\leq (p-1)(\max |\phi|)^p |\nabla (u_k(x) + \delta)| |u_k(x) + \delta|^{-p} a |\nabla u(x)|^{p-1} \\ &\quad + (p-1)(\max |\phi|)^p |\nabla (u_k(x) + \delta)| |u_k(x) + \delta|^{-p} b(x) |u(x)|^{p-1} \\ &\leq (p-1)\delta^{-p} (\max |\phi|)^p a |\nabla u_k(x)| |\nabla u(x)|^{p-1} \\ &\quad + (p-1)\delta^{-1} (\max |\phi|)^p |\nabla u_k(x)| b(x) \\ &\leq (p-1)\delta^{-p} (\max |\phi|)^p a h(x) |\nabla u(x)|^{p-1} + (p-1)\delta^{-1} (\max |\phi|)^p h(x) b(x), \end{aligned} \quad (4.11)$$

a.e. in Ω . Since $b^{p'} \in L^{q,\Phi}$, which means $b \in L^{p'q}(\Omega) \subseteq L^{p'}(\Omega)$, then the right hand side of (4.10) is integrable, that is,

$$\int_{\Omega} \left(K_1 |\nabla u(x)|^{p-1} + K_2 b(x) \right) dx \leq K_1 |\Omega|^{\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}^{p-1} + K_2 |\Omega|^{\frac{1}{p}} \|b\|_{L^{p'}(\Omega)} < \infty,$$

which is obtained by Hölder's inequality, where $K_1 = p\delta^{1-p} (\max |\nabla \phi|) (\max |\phi|)^{p-1} a$ and $K_2 = p(\max |\nabla \phi|) (\max |\phi|)^{p-1}$. Similarly, the right hand side of (4.11) is also integrable, that is,

$$\begin{aligned} & \int_{\Omega} \left(K_3 h(x) |\nabla u(x)|^{p-1} + K_4 h(x) b(x) \right) dx \\ &\leq K_3 \|h\|_{L^p(\Omega)} \|\nabla u\|_{L^p(\Omega)}^{p-1} + K_4 \|h\|_{L^p(\Omega)} \|b\|_{L^{p'}(\Omega)} < \infty, \end{aligned}$$

since $h \in L^p\Omega$, where $K_3 = (p-1)\delta^{-p} (\max |\phi|)^p a$ and $K_4 = (p-1)\delta^{-1} (\max |\phi|)^p$. Therefore, we have proved that the right hand side of (4.9) is bounded by the integrable functions in the right hand side of (4.10) and (4.11). We note that

$$A(x, u(x), \nabla u(x)) \nabla \left(\phi(x)^p (u(x) + \delta)^{1-p} \right)$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} A(x, u(x), \nabla u(x)) \nabla \left(\phi(x)^p (u_k(x) + \delta)^{1-p} \right) \\
&= pA(x, u(x), \nabla u(x)) \nabla \phi(x) \phi(x)^{p-1} (u(x) + \delta)^{1-p} \\
&\quad - (p-1)A(x, u(x), \nabla u(x)) \nabla (u(x) + \delta) (u(x) + \delta)^{-p} \phi(x)^p.
\end{aligned} \tag{4.12}$$

By using (4.8). We can use the Lebesgue Dominated Convergent Theorem (LDCT), by observing (4.12) and using the fact that $|A(x, u(x), \nabla u(x)) \nabla (\phi(x)^p (u_k(x) + \delta)^{1-p})|$ is bounded by the integrable functions, to obtain

$$\begin{aligned}
&\int_{\Omega} A(x, u(x), \nabla u(x)) \nabla \left(\phi(x)^p (u(x) + \delta)^{1-p} \right) dx \\
&= \lim_{k \rightarrow \infty} \int_{\Omega} A(x, u(x), \nabla u(x)) \nabla \left(\phi(x)^p (u_k(x) + \delta)^{1-p} \right) dx \\
&= p \int_{\Omega} A(x, u(x), \nabla u(x)) \nabla \phi(x) \phi(x)^{p-1} (u(x) + \delta)^{1-p} dx \\
&\quad - (p-1) \int_{\Omega} A(x, u(x), \nabla u(x)) \nabla (u(x) + \delta) (u(x) + \delta)^{-p} \phi(x)^p dx.
\end{aligned} \tag{4.13}$$

Now, we will prove the convergent of a sequence whose term is defined by

$$\int_{\Omega} B(x, u(x), \nabla u(x)) \phi(x)^p (u_k(x) + \delta)^{1-p} dx, \tag{4.14}$$

for every $k \in \mathbb{N}$. By the assumption in (4.2), we have

$$\begin{aligned}
&|B(x, u(x), \nabla u(x)) \phi(x)^p (u_k(x) + \delta)^{1-p}| \\
&\leq c(x) |\nabla u(x)|^{p-1} |\phi(x)|^p |u_k(x) + \delta|^{1-p} + d(x) |u(x)|^{p-1} |\phi(x)|^p |u_k(x) + \delta|^{1-p}.
\end{aligned} \tag{4.15}$$

The first and second term in the right hand side of (4.15) are respectively bounded by $K_5 c(x) |\nabla u(x)|^{p-1}$ and $d(x) |\phi(x)|^p$, where $K_5 = \delta^{1-p} (\max |\phi|)^p$. Hölder's inequality implies

$$\int_{\Omega} K_5 c(x) |\nabla u(x)|^{p-1} \leq K_5 \|c\|_{L^p(\Omega)} \|\nabla u\|_{L^p(\Omega)}^{p-1} < \infty,$$

since $c^p \in L^{q,\Phi}$, which means $c \in L^{pq}(\Omega) \subseteq L^p(\Omega)$. Meanwhile, Fefferman's inequality implies

$$\int_{\Omega} d(x) |\phi(x)|^p \leq K_0 \|d\|_{L^{p,\Phi}} \|\nabla \phi\|_{L^p(\Omega)}^p < \infty,$$

since $d \in L^{q,\Phi}$. It is clear that

$$B(x, u(x), \nabla u(x)) \phi(x)^p (u(x) + \delta)^{1-p} = \lim_{k \rightarrow \infty} B(x, u(x), \nabla u(x)) \phi(x)^p (u_k(x) + \delta)^{1-p}.$$

Thus, we can use the LDCT to get

$$\lim_{k \rightarrow \infty} \int_{\Omega} B(x, u(x), \nabla u(x)) \phi(x)^p (u_k(x) + \delta)^{1-p} dx$$

$$= \int_{\Omega} B(x, u(x), \nabla u(x)) \phi(x)^p (u(x) + \delta)^{1-p} dx. \quad (4.16)$$

Using $\phi^p(x)(u_k(x) + \delta)^{1-p}$ as a test function in (4.4), we have

$$\begin{aligned} & - \int_{\Omega} A(x, u(x), \nabla u(x)) \nabla (\phi^p(x)(u_k(x) + \delta)^{1-p}) dx \\ & = \int_{\Omega} B(x, u(x), \nabla u(x)) \phi^p(x)(u_k(x) + \delta)^{1-p} dx. \end{aligned} \quad (4.17)$$

Taking the limit in (4.17), then, using (4.13) and (4.16), we get

$$\begin{aligned} & (p-1) \int_{\Omega} A(x, u(x), \nabla u(x)) \nabla (u(x) + \delta) (u(x) + \delta)^{-p} \phi(x)^p dx \\ & = \int_{\Omega} B(x, u(x), \nabla u(x)) \phi(x)^p (u(x) + \delta)^{1-p} dx \\ & \quad + p \int_{\Omega} A(x, u(x), \nabla u(x)) \nabla \phi(x) \phi(x)^{p-1} (u(x) + \delta)^{1-p} dx. \end{aligned} \quad (4.18)$$

By using (4.2), the left hand side of (4.18) estimates as follows

$$\begin{aligned} & (p-1) \int_{\Omega} A(x, u(x), \nabla u(x)) \nabla (u(x) + \delta) (u(x) + \delta)^{-p} \phi(x)^p dx \\ & \geq (p-1) \int_{\Omega} |\nabla (u(x) + \delta)|^p |u(x) + \delta|^{-p} |\phi(x)|^p dx \\ & \quad - (p-1) \int_{\Omega} d(x) |u(x)|^p |u(x) + \delta|^{-p} |\phi(x)|^p dx \\ & \geq (p-1) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p |\phi(x)|^p dx - (p-1) \int_{\Omega} d(x) |\phi(x)|^p dx. \end{aligned} \quad (4.19)$$

Substituting (4.19) to (4.18) gives us

$$\begin{aligned} & (p-1) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p |\phi(x)|^p dx \\ & \leq (p-1) \int_{\Omega} d(x) |\phi(x)|^p dx + \int_{\Omega} B(x, u(x), \nabla u(x)) \phi(x)^p (u(x) + \delta)^{1-p} dx \\ & \quad + p \int_{\Omega} A(x, u(x), \nabla u(x)) \nabla \phi(x) \phi(x)^{p-1} (u(x) + \delta)^{1-p} dx. \end{aligned} \quad (4.20)$$

Let $\epsilon > 0$ be fixed latter. By using (4.2), the second term in the right hand side of (4.20) is estimated as follows

$$\begin{aligned} & \int_{\Omega} B(x, u(x), \nabla u(x)) \phi(x)^p (u(x) + \delta)^{1-p} dx \\ & \leq \int_{\Omega} c(x) |\nabla u(x)|^{p-1} |\phi(x)|^p |u(x) + \delta|^{1-p} dx + \int_{\Omega} d(x) |u(x)|^{p-1} |\phi(x)|^p |u(x) + \delta|^{1-p} dx \\ & \leq \int_{\Omega} c(x) |\nabla (u(x) + \delta)|^{p-1} |\phi(x)|^p |u(x) + \delta|^{1-p} dx + \int_{\Omega} d(x) |\phi(x)|^p dx. \end{aligned} \quad (4.21)$$

Young's inequality implies

$$\begin{aligned} & \int_{\Omega} c(x)|\nabla(u(x) + \delta)|^{p-1}|\phi(x)|^p|u(x) + \delta|^{1-p}dx \\ & \leq \epsilon \int_{\Omega} c(x)^p|\phi(x)|^pdx + K_6(\epsilon) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p|\phi(x)|^pdx, \end{aligned} \quad (4.22)$$

where $K_6 = K_6(\epsilon) = (\epsilon^{-\frac{1}{p-1}} p^{-\frac{1}{p-1}})(\frac{p-1}{p})$. We infer from (4.21) and (4.22) that

$$\begin{aligned} & \int_{\Omega} B(x, u(x), \nabla u(x))\phi(x)^p(u(x) + \delta)^{1-p}dx \\ & \leq \epsilon \int_{\Omega} c(x)^p|\phi(x)|^pdx + K_6(\epsilon) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p|\phi(x)|^pdx + \int_{\Omega} d(x)|\phi(x)|^pdx. \end{aligned} \quad (4.23)$$

We remain to estimate the last term of (4.20). We have from (4.2) that

$$\begin{aligned} & p \int_{\Omega} A(x, u(x), \nabla u(x))\nabla\phi(x)\phi(x)^{p-1}(u(x) + \delta)^{1-p}dx \\ & \leq pa \int_{\Omega} |\nabla u(x)|^{p-1}|\nabla\phi(x)||\phi(x)|^{p-1}|u(x) + \delta|^{1-p}dx \\ & \quad + p \int_{\Omega} b(x)|u(x)|^{p-1}|\nabla\phi(x)||\phi(x)|^{p-1}|u(x) + \delta|^{1-p}dx \\ & \leq pa \int_{\Omega} |\nabla u(x)|^{p-1}|\nabla\phi(x)||\phi(x)|^{p-1}|u(x) + \delta|^{1-p}dx \\ & \quad + p \int_{\Omega} b(x)|\nabla\phi(x)||\phi(x)|^{p-1}dx. \end{aligned} \quad (4.24)$$

Again, Young's inequality implies

$$\begin{aligned} & pa \int_{\Omega} |\nabla u(x)|^{p-1}|\nabla\phi(x)||\phi(x)|^{p-1}|u(x) + \delta|^{1-p}dx \\ & \leq \epsilon(pa)^p \int_{\Omega} |\nabla\phi(x)|^pdx + K_6(\epsilon) \int_{\Omega} |\nabla(u(x) + \delta)|^p|u(x) + \delta|^{-p}|\phi(x)|^pdx \\ & = \epsilon(pa)^p \int_{\Omega} |\nabla\phi(x)|^pdx + K_6(\epsilon) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p|\phi(x)|^pdx, \end{aligned} \quad (4.25)$$

and

$$p \int_{\Omega} b(x)|\nabla\phi(x)||\phi(x)|^{p-1}dx \leq \epsilon p^p \int_{\Omega} |\nabla\phi(x)|^pdx + K_6(\epsilon) \int_{\Omega} b(x)^{\frac{p}{p-1}}|\phi(x)|^pdx. \quad (4.26)$$

Substituting (4.25) and (4.26) into (4.24) yields

$$\begin{aligned} & p \int_{\Omega} A(x, u(x), \nabla u(x))\nabla\phi(x)\phi(x)^{p-1}(u(x) + \delta)^{1-p}dx \\ & \leq \epsilon(pa)^p \int_{\Omega} |\nabla\phi(x)|^pdx + K_6(\epsilon) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p|\phi(x)|^pdx \end{aligned}$$

$$\begin{aligned}
& + \epsilon p^p \int_{\Omega} |\nabla \phi(x)|^p dx + K_6(\epsilon) \int_{\Omega} b(x)^{\frac{p}{p-1}} |\phi(x)|^p dx \\
& \leq (\epsilon(pa)^p + \epsilon p^p) \int_{\Omega} |\nabla \phi(x)|^p dx + K_6(\epsilon) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p |\phi(x)|^p dx \\
& \quad + K_6(\epsilon) \int_{\Omega} b(x)^{\frac{p}{p-1}} |\phi(x)|^p dx.
\end{aligned} \tag{4.27}$$

Substituting (4.23) and (4.27) into (4.20), we have

$$\begin{aligned}
& (p-1) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p |\phi(x)|^p dx \\
& \leq \epsilon \int_{\Omega} c(x)^p |\phi(x)|^p dx + 2K_6(\epsilon) \int_{\Omega} |\nabla \log(u(x) + \delta)|^p |\phi(x)|^p dx + p \int_{\Omega} d(x) |\phi(x)|^p dx \\
& \quad + (\epsilon(pa)^p + \epsilon p^p) \int_{\Omega} |\nabla \phi(x)|^p dx + K_6(\epsilon) \int_{\Omega} b(x)^{\frac{p}{p-1}} |\phi(x)|^p dx.
\end{aligned} \tag{4.28}$$

Choose $\epsilon = (\frac{2}{p})^{p-1} \frac{2}{p}$, and set $K_7 = p - 1 - 2K_6(\epsilon)$, $K_8 = (\frac{2}{p})^{p-1} \frac{2}{p}$, $K_9 = p$, and $K_{10} = \epsilon(pa)^p + \epsilon p^p$. Note that $K_7 > 0$. The inequality (4.28) reduces to

$$\begin{aligned}
& K_7 \int_{\Omega} |\nabla \log(u(x) + \delta)|^p |\phi(x)|^p dx \\
& \leq K_8 \int_{\Omega} c(x)^p |\phi(x)|^p dx + K_9 \int_{\Omega} d(x) |\phi(x)|^p dx + K_{10} \int_{\Omega} |\nabla \phi(x)|^p dx \\
& \quad + K_6 \int_{\Omega} b(x)^{\frac{p}{p-1}} |\phi(x)|^p dx.
\end{aligned} \tag{4.29}$$

By applying Theorem 2.1 in the right hand side of (4.29), we get

$$K_7 \int_{\Omega} |\nabla \log(u(x) + \delta)|^p |\phi(x)|^p dx \leq K_{11} \int_{\Omega} |\nabla \phi(x)|^p dx, \tag{4.30}$$

where $K_{11} = K_8 K \|c^p\|_{L^q, \phi} + K_9 K \|d\|_{L^q, \phi} + K_{10} + K_6 K \|b^{p'}\|_{L^q, \phi}$. Now, we choose ϕ such that $\phi = 1$ in $B(x_0, r)$, $0 \leq \phi \leq 1$, and $|\nabla \phi| \leq 2/r$. Then, by (4.30), we have

$$\int_{B(x_0, r)} |\nabla \log(u(x) + \delta)|^p dx \leq \frac{2^p K_{11}}{K_7 r^p} \int_{B(x_0, 2r)} 1 dx = K_{12} \frac{|B(x_0, r)|}{r^p}, \tag{4.31}$$

where $K_{12} = \frac{2^p K_{11}}{K_7}$. By Hölder's inequality and (4.31), we obtain

$$\begin{aligned}
\int_{B(x_0, r)} |\nabla \log(u(x) + \delta)| dx & \leq \left(\int_{B(x_0, r)} |\nabla \log(u(x) + \delta)|^p dx \right)^{\frac{1}{p}} \left(\int_{B(x_0, r)} 1 dx \right)^{1-\frac{1}{p}} \\
& \leq \left(K_{12} \frac{|B(x_0, r)|}{r^p} \right)^{\frac{1}{p}} |B(x_0, r)|^{1-\frac{1}{p}} \\
& = K_{13} \frac{|B(x_0, r)|}{r},
\end{aligned} \tag{4.32}$$

where $K_{13} = K_{12}^{\frac{1}{p}}$. We infer from Poincaré's inequality and (4.32) that

$$\begin{aligned} \int_{B(x_0, r)} |\log(u(x) + \delta) - \log(u + \delta)_{B(x_0, r)}| dx &\leq K_{14} r \int_{B(x_0, r)} |\nabla \log(u(x) + \delta)| dx \\ &\leq K_{14} r K_{13} \frac{|B(x_0, r)|}{r} \\ &= K_{15} |B(x_0, r)|, \end{aligned} \quad (4.33)$$

where $K_{14} = K_{14}(n)$ is the positive constant which appears in the Poincaré inequality and $K_{15} = K_{14} K_{13}$. Since (4.33) holds for arbitrary $B(x_0, r) \subseteq B(x_0, R)$, then, $\log(u + \delta) \in \text{BMO}(B(x_0, R))$. \square

Theorem 4.4 combining with Theorem 3.2 and Lemma 4.1 give us the strong unique continuation property of the Eq (4.1). This property is stated and proved in the next theorem.

Theorem 4.5. *The Eq (4.1) has the strong unique continuation property in Ω .*

Proof. Let $x_0 \in \Omega$, $B(x_0, 2R) \subseteq \Omega$, u be a non negative weak solution of (4.1) which vanishes with infinite order at x_0 , and $\{\delta_j\}$ be a sequence of positive real numbers such that $\delta_j \rightarrow 0$ as $j \rightarrow \infty$. According to Theorem 4.4, we have $\log(u + \delta_j) \in \text{BMO}(B(x_0, R))$. Applying Theorem 3.2, there exists $M > 0$ such that

$$\int_{B(x_0, R)} (u(y) + \delta_j)^\gamma dy \leq M \int_{B(x_0, \frac{R}{2})} (u(y) + \delta_j)^\gamma dy,$$

for some $0 < \gamma \leq 1$. Letting $j \rightarrow \infty$ in the last inequality, then

$$\int_{B(x_0, R)} u(y)^\gamma dy \leq M \int_{B(x_0, \frac{R}{2})} u(y)^\gamma dy.$$

From Lemma 4.1 we conclude that $w \equiv 0$ in $B(x_0, R)$. This completes the proof. \square

5. Conclusions

The strong unique continuation property for the nonnegative weak solutions of the quasilinear elliptic Eq (4.1), where the suitable powers of the coefficients belong to some generalized Morrey spaces, is proved in this paper. We provide the rigorous proof that can be used in many similar situations and may be useful to other audience.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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