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Research article

Chaos control and numerical solution of time-varying fractional Newton-Leipnik system using fractional Atangana-Baleanu derivatives

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Abstract: Nonlinear fractional differential equations and chaotic systems can be modeled with variable-order differential operators. We propose a generalized numerical scheme to simulate variable-order fractional differential operators. Fractional calculus' fundamental theorem and Lagrange polynomial interpolation are used. Two methods, Atangana-Baleanu-Caputo and Atangana-Seda derivatives, were used to solve a chaotic Newton-Leipnik system problem with fractional operators. Our scheme examined the existence and uniqueness of the solution. We analyze the model qualitatively using its equivalent integral through an iterative convergence sequence. This novel method is illustrated with numerical examples. Simulated and analytical results agree. We contribute to real-world mathematical applications. Finally, we applied a numerical successive approximation method to solve the fractional model.

Keywords: fractional-order Newton-Leipnik system; mathematical modeling; stability analysis; computational simulation
Mathematics Subject Classification: 92D30, 92D25, 92C42, 34C60

1. Introduction

Chemistry, astronomy, population growth, turbulence, weather, linguistics and stock markets are some of the scientific fields that exhibit chaotic phenomena. Chaos is understood through linguistics. In [1], Eleonora and Pietro investigated a linguistic approach to chaos understanding. We translate the complexity of chaotic systems into sounds and music, as with Chua's attractors. They found that there is an interaction between the different levels of chaotic language. Furthermore, some traits of the dynamics of language evolution in human infants are found in the major routes to chaos.

In 1981, Leipnik and Newton [2] studied their concept of the gyroscope for chaotic motion, discovering two strange attractors for rigid bodies' motion. In [3], Wang and Tian named this system the Newton-Leipnik system and studied a chaotic Newton-Leipnik system with two strange attractors. Since Leipnik and Newton's work, many scientists have intensively examined the chaotic dynamics of rigid body motion [4–9]. Chaos control is performed by a simple linear controller, and a numerical simulation of the control is provided. In addition, Chen and Lee [4] introduced a novel chaotic system capable of generating dual-role chaos attractors when investigating rigid body motion anti-chaos control. Richter [5] studied the stability and chaos control of Newton-Leipnik systems using static nonlinear feedback laws based on Lyapunov functions. In [6], Long-Jye Sheu et al., investigated the dynamics of the Newton-Leipnik system with fractional order and was studied numerically. The Newton-Leipnik system [3] is described as:

$$\begin{aligned} \dot{\mathrm{f}} &= -a\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h}, \\ \dot{\mathrm{g}} &= -\,\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h}, \\ \dot{\mathrm{h}} &= b\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g}\,. \end{aligned} \tag{1.1}$$

Fractional calculus (FC) allows integration and differentiation of operators in fractional order. FC is considered a popular research topic and has been extensively studied in recent decades [10–34]. Samko proposed a fascinating extension of constant-order FC in [35]. Solís-Pérez et al. [36] introduced fractional operators that consider order as a function of time, space or other variables see also [37, 38]. In addition, applications of these fractional variable-order operators can be found in [39–48].

Variable-order fractional differential equations cannot be solved exactly, so developing numerical schemes for solving these equations is crucial. In [36], Solís-Pérez et al. developed a constant-order numerical scheme that combines fractional calculus and Lagrange polynomials. Using this method, they generalized the numerical schemes for simulating variable-order fractional differential operators with power-law, exponential-law and Mittag-Leffler kernels. See also [49–60].

The memory and genetic properties of Eq (1.1) can be improved by extending the integer-order Newton-Leipnik model to a variable-order Newton-Leipnik model. In this paper, we apply the novel generalized numerical scheme of Mittag-Leffler kernels to simulate a variable-order fractional Newton-Leipnik system. Numerical methods developed by Atangana-Baleanu and Atangana-Seda for solving fractional Newton-Leipnik systems are presented. Moreover, we use numerical successive approximation methods to solve the proposed model. In order to stabilize the proposed system, fractional Routh-Hurwitz conditions must be applied. When nonlinear control functions are chosen carefully, chaos synchronization of Newton-Leipnik systems has also been demonstrated when fractional-order systems are considered. Furthermore, some examples are given, and our numerical simulations reveal novel attractors. Through simulations, the method's accuracy and efficiency have been demonstrated.

2. Preliminaries

A very significant role in fractional calculus was introduced by Humbert and Agarwal in 1953. It is a two-parameter Mittag-Leffler function defined as (Gorenflo et al., 1998):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \quad \beta > 0.$$

For $\beta = 1$ we obtain the Mittag-Leffler function in one parameter (Mittag-Leffler, 1903):

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \equiv E_{\alpha}(z).$$

According to [45], ABC (Atangana-Baleanu-Caputo) may be derived in fractional derivatives with variable orders $g \in C^1(0,T), 0 < T$,

$${}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa} g(t) = \frac{\mathrm{ABC}(\kappa)}{1-\kappa} \int_0^t E_{\kappa}\Big[-\frac{\kappa}{1-\kappa}(t-\eta)^{\kappa}\Big] \frac{dg(\eta)}{d\eta}\, d\eta.$$

In this formula, $\mathrm{ABC}(\kappa) = 1 - \kappa + \eta/\Gamma(\kappa)$ is the normalization function, $\Gamma(\cdot)$ is the Euler Gamma function, and $E_{\kappa}(\cdot)$ is the Mittag-Leffler function.

Theorem 1. *[39, 40] The time-fractional ABC differential equation:*

$${}^{\mathrm{ABC}}\mathscr{D}_t^{\kappa} g(t) = \zeta(t),$$

provides a unique solution

$$g(t) = \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\zeta(t) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \int_0^t \zeta(\kappa)(t-\kappa)^{\kappa-1}\, d\kappa.$$

In [39, 40], ABC fractional derivative has the fractional-integral associate:

$${}^{\mathrm{ABC}}I_t^{\kappa}\big\{\chi(t)\big\} = \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\chi(t) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \int_0^t \chi(\kappa)(t-\kappa)^{\kappa-1}\, d\kappa.$$

3. Model properties

3.1. Existence and uniqueness

ABC fractional derivatives of the Newton-Leipnik system are presented here.

$$\begin{aligned} {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{f} &= -a\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h}, \\ {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{g} &= -\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h}, \\ {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{h} &= b\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g}. \end{aligned} \tag{3.1}$$

Taking the fractional integral operator of ABC and applying it to Eq (3.1), we get

$$\begin{aligned} \mathrm{f} - \mathrm{f}(0) &= {}^{\mathrm{ABC}}I_t^{\kappa}[-a\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h}], \\ \mathrm{g} - \mathrm{g}(0) &= {}^{\mathrm{ABC}}I_t^{\kappa}[-\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h}], \\ \mathrm{h} - \mathrm{h}(0) &= {}^{\mathrm{ABC}}I_t^{\kappa}[b\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g}]. \end{aligned} \tag{3.2}$$

We will assume that f, g and h are nonnegative bounded functions, i.e., $\| \mathrm{f} \| \leq \kappa_1$, $\| \mathrm{g} \| \leq \kappa_2$, $\| \mathrm{h} \| \leq \kappa_3$, where κ_1, κ_2 and κ_3 are some positive constants. Denote

$$\ell_1 = a,\ \ell_2 = 0.4,\ \ell_3 = b. \tag{3.3}$$

When we apply the ABC fractional integral to system (3.1), we obtain

$$\begin{aligned}
\mathrm{f} - \mathrm{f}(0) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_1(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_1(z, \mathrm{f}, \mathrm{g}, \mathrm{h})\, dz,\\
\mathrm{g} - \mathrm{g}(0) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_2(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_2(z, \mathrm{f}, \mathrm{g}, \mathrm{h})\, dz,\\
\mathrm{h} - \mathrm{h}(0) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_3(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_3(z, \mathrm{f}, \mathrm{g}, \mathrm{h})\, dz,
\end{aligned} \tag{3.4}$$

where the kernels $\mathscr{L}_1(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$, $\mathscr{L}_2(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$, $\mathscr{L}_3(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$ are defined as:

$$\begin{aligned}
\mathscr{L}_1(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) &= -a\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h},\\
\mathscr{L}_2(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) &= -\,\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h},\\
\mathscr{L}_3(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) &= b\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g}\,.
\end{aligned}$$

Proposition 1. *If*

$$0 \leq M = max\{\ell_1, \ell_2, \ell_3\} < 1.$$

Thus, the kernels $\mathscr{L}_1(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$*,* $\mathscr{L}_2(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$*,* $\mathscr{L}_3(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$ *satisfy Lipschitz conditions.*

Proof. If f_2 and f_1 are any two functions, then

$$\| \mathscr{L}_1(t, \mathrm{f}_2, \mathrm{g}, \mathrm{h}) - \mathscr{L}_1(t, \mathrm{f}_1, \mathrm{g}, \mathrm{h})\| = \Big\| -a\,\mathrm{f}_1 + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h} + a\,\mathrm{f}_2 - \mathrm{g} - 10\,\mathrm{g}\,\mathrm{h} \Big\| \leq \ell_1 \| \mathrm{f}_2 - \mathrm{f}_1 \|,$$

where ℓ_1 is the Lipschitz constant and is defined in Eq (3.3). Similarly, the kernels $\mathscr{L}_2, \mathscr{L}_3$ can be calculated using $\{\mathrm{g}_2, \mathrm{g}_1\}$, $\{\mathrm{h}_2, \mathrm{h}_1\}$, respectively:

$$\begin{aligned}
\|\mathscr{L}_2(t, \mathrm{f}, \mathrm{g}_2, \mathrm{h}) - \mathscr{L}_2(t, \mathrm{f}, \mathrm{g}_1, \mathrm{h})\| &\leq \ell_2 \| \mathrm{g}_2 - \mathrm{g}_1 \|,\\
\| \mathscr{L}_3(t, \mathrm{f}, \mathrm{g}, \mathrm{h}_2) - \mathscr{L}_3(t, \mathrm{f}, \mathrm{g}, \mathrm{h}_1)\| &\leq \ell_3 \| \mathrm{h}_2 - \mathrm{h}_1 \|.
\end{aligned}$$

Thus, the Lipschitz conditions are satisfied for$\mathscr{L}_1(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$, $\mathscr{L}_2(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$, $\mathscr{L}_3(t, \mathrm{f}, \mathrm{g}, \mathrm{h})$. Additionally, since $0 \leq M = \max\{\ell_1, \ell_2, \ell_3\} < 1$, the kernels are contractions. Equation (3.4) can be used to display state variables as kernels:

$$\begin{aligned}
\mathrm{f} &= \mathrm{f}(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_1(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_1(z, \mathrm{f}, \mathrm{g}, \mathrm{h})\, dz,\\
\mathrm{g} &= \mathrm{g}(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_2(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_2(z, \mathrm{f}, \mathrm{g}, \mathrm{h})\, dz,\\
\mathrm{h} &= \mathrm{h}(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_3(t, \mathrm{f}, \mathrm{g}, \mathrm{h}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_3(z, \mathrm{f}, \mathrm{g}, \mathrm{h})\, dz.
\end{aligned} \tag{3.5}$$

We now introduce the following recursive formulas using Eq (3.5):

$$
\begin{aligned}
\mathrm{f}_n(t) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_1(t,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_1(z,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1})\,dz,\\
\mathrm{g}_n(t) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_2(t,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_2(z,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1})\,dz,\\
\mathrm{h}_n(t) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_3(t,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{L}_3(z,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1})\,dz,
\end{aligned}
$$

with

$$
\mathrm{f}_0(t)=\mathrm{f}(0), \qquad \mathrm{g}_0(t)=\mathrm{g}(0), \qquad \mathrm{h}_0(t)=\mathrm{h}(0).
$$

For the recursive formulas, the difference between consecutive terms is

$$
\begin{aligned}
\text{®}_n(t) = \mathrm{f}_n(t)-\mathrm{f}_{n-1}(t) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\Big(\mathscr{L}_1(t,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) - \mathscr{L}_1(t,\mathrm{f}_{n-2},\mathrm{g}_{n-2},\mathrm{h}_{n-2})\Big)\\
&+ \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \Big(\mathscr{L}_1(z,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) - \mathscr{L}_1(z,\mathrm{f}_{n-2},\mathrm{g}_{n-2},\mathrm{h}_{n-2})\Big)\,dz,\\
\mathfrak{I}_n(t) = \mathrm{g}_n(t)-\mathrm{g}_{n-1}(t) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\Big(\mathscr{L}_2(t,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) - \mathscr{L}_2(t,\mathrm{f}_{n-2},\mathrm{g}_{n-2},\mathrm{h}_{n-2})\Big)\\
&+ \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \Big(\mathscr{L}_2(z,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) - \mathscr{L}_2(z,\mathrm{f}_{n-2},\mathrm{g}_{n-2},\mathrm{h}_{n-2})\Big)\,dz,\\
\aleph_n(t) = \mathrm{h}_n(t)-\mathrm{h}_{n-1}(t) &= \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\Big(\mathscr{L}_3(t,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) - \mathscr{L}_3(t,\mathrm{f}_{n-2},\mathrm{g}_{n-2},\mathrm{h}_{n-2})\Big)\\
&+ \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \Big(\mathscr{L}_3(z,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) - \mathscr{L}_3(z,\mathrm{f}_{n-2},\mathrm{g}_{n-2},\mathrm{h}_{n-2})\Big)\,dz.
\end{aligned}
$$

Note that:

$$
\mathrm{f}_n(t)=\sum_{i=1}^{n}\text{®}_i(t), \quad \mathrm{g}_n(t)=\sum_{i=1}^{n}\mathfrak{I}_i(t), \quad \mathrm{h}_n(t)=\sum_{i=1}^{n}\aleph_i(t).
$$

As a result, the recursive inequalities:

$$
\begin{aligned}
\parallel \text{®}_n(t) \parallel &= \parallel \mathrm{f}_n(t)-\mathrm{f}_{n-1}(t)\parallel\\
&\le \frac{1-\kappa}{\mathrm{ABC}(\kappa)} \parallel \mathscr{L}_1(t,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) - \mathscr{L}_1(t,\mathrm{f}_{n-2},\mathrm{g}_{n-2},\mathrm{h}_{n-2})\parallel\\
&+ \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\int_0^t \parallel \mathscr{L}_1(t,\mathrm{f}_{n-1},\mathrm{g}_{n-1},\mathrm{h}_{n-1}) - \mathscr{L}_1(t,\mathrm{f}_{n-2},\mathrm{g}_{n-2},\mathrm{h}_{n-2}) \parallel \; dz\\
&\le \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\ell_1 |\,\mathrm{f}_{n-1}-\mathrm{f}_{n-2}\,| + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\ell_1\int_0^t |\,\mathrm{f}_{n-1}-\mathrm{f}_{n-2}\,|\,dz.
\end{aligned}
$$

Therefore

$$
\parallel \text{®}_n(t)\parallel \le \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\ell_1 \parallel \text{®}_{n-1}(t)\parallel + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\ell_1\int_0^t \parallel \text{®}_{n-1}(z)\parallel dz.
$$

Similarly

$$\| \Im_n(t)\| \leq \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\ell_2 \parallel \psi_{n-1}(t)\| + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\ell_2 \int_0^t \parallel \psi_{n-1}(z)\| \, dz,$$
$$\| \aleph_n(t)\| \leq \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\ell_3 \parallel \xi_{n-1}(t)\| + \mho(\kappa)\ell_3 \int_0^t \parallel \xi_{n-1}(z)\| \, dz.$$

□

As a result, a solution of a system (3.1) exists if the following condition is satisfied, if $t_0 > 0$:

$$\frac{1-\kappa}{\mathrm{ABC}(\kappa)}\ell_i + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\ell_i t_0 < 1, \quad \textit{for } i = 1,2,3.$$

Proposition 2. *System (3.1) provides a unique solution if the following conditions are met:*

$$\Big(1 - \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\ell_i - \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)}\ell_i t\Big) > 0 \quad \textit{for } i = 1,2,3.$$

4. Newton-Leipnik chaos control

Consider the ABC fractional-order Newton-Leipnik system:

$$\begin{aligned} {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{f} &= -a\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h}, \\ {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{g} &= -\,\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h}, \\ {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{h} &= b\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g}\,. \end{aligned} \tag{4.1}$$

Its Jacobian matrix $J(\mathscr{E}^*)$ is

$$J(\mathscr{E}^*) = \begin{pmatrix} -a & 1 + 10\,\mathrm{h}^* & 10\,\mathrm{g}^* \\ -1 + 5\,\mathrm{h}^* & -0.4 & 5\,\mathrm{f}^* \\ -5\,\mathrm{g}^* & -5\,\mathrm{f}^* & b \end{pmatrix},$$

where $\mathscr{E}^* = (\mathrm{f}^*, \mathrm{g}^*, \mathrm{h}^*)$ is the equilibrium point of the system (4.1). The Jacobian eigenvalues at all equilibrium points indicate that they are all saddle points. All of these eigenvalues have a chaos condition.

Newton-Leipnik controlled fractional-order system is given by:

$$\begin{aligned} {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{f} &= -a\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h} - c_1(\mathrm{f} - \mathrm{f}^*), \\ {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{g} &= -\,\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h} - c_2(\mathrm{g} - \mathrm{g}^*), \\ {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa}\,\mathrm{h} &= b\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g} - c_3(\mathrm{h} - \mathrm{h}^*). \end{aligned} \tag{4.2}$$

In addition, $K = \mathrm{diag}\,(c_1, c_2, c_3)\,, c_1, c_2, c_3 \geq 0$. By selecting the appropriate feedback control gains c_1, c_2, c_3, the controlled system (4.2) approaches the unstable equilibrium point asymptotically. Specifically, we are trying to reduce the trajectory of system (4.1) to one of $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3$ and $\mathscr{E}_4$ which defined in Table 1. A simple feedback gain is $K = \mathrm{diag}\,(c_1, 0.0)$. At $E_0 = (0,0,0)$, the characteristic equation of (4.2) is:

$$\lambda^3 + (c_1 + a + 0.4 - b)\,\lambda^2 + ((c_1 + a)(0.4 - b) - 0.4 - b + 1)\,\lambda - 0.4b(c_1 + a) - b = 0.$$

By choosing satisfactory feedback control gains c_1, c_2, c_3

$$c_1 > max\{b - a - 0.4, \frac{0.4 + b - 1}{(0.4 - b)} - a\},$$
$$(c_1 + a + 0.4 - b)\,((c_1 + a)(0.4 - b) - 0.4b + 1) = -0.4b(c_1 + a) - b.$$

It is necessary to apply the fractional Routh-Hurwitz conditions to control chaos in the (4.1) to its equilibrium. The equilibrium point of (4.1) can be achieved by using specific linear controllers, based on Routh-Hurwitz. When the nonlinear control functions are chosen carefully, chaos synchronization of (4.1) has been demonstrated.

Table 1. Eigenvalues and equilibrium points.

Equilibrium points	Eigenvalues
$\mathscr{E}_0 = (0, 0, 0)$	0.175, $-0.4 \pm i$
$\mathscr{E}_1 = (-0.2390, -0.0308, 0.2103)$	- 0.8, $0.0875 \pm 1.2113i$
$\mathscr{E}_2 = (-0.0315, 0.1224, -0.1103)$	- 0.8, $0.0875 \pm 0.8752i$
$\mathscr{E}_3 = (0.0315, -0.1224, -0.1103)$	- 0.8, $0.0875 \pm 0.8752i$
$\mathscr{E}_4 = (0.2390, 0.0308, 0.2103)$	- 0.8, $0.0875 \pm 1.2113i$

4.1. Chaos synchronization

The following example illustrates synchronizing chaos between two Newton-Leipnik systems with unidirectional linear error feedback. This is the drive system:

$$\begin{aligned} {}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{f}_1 &= -a\,\mathrm{f}_1 + \mathrm{g}_1 + 10\,\mathrm{g}_1\,\mathrm{h}_1, \\ {}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{g}_1 &= -\,\mathrm{f}_1 - 0.4\,\mathrm{g}_1 + 5\,\mathrm{f}_1\,\mathrm{h}_1, \\ {}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{h}_1 &= b\,\mathrm{h}_1 - 5\,\mathrm{f}_1\,\mathrm{g}_1\,. \end{aligned} \tag{4.3}$$

The response system is provided by:

$$\begin{aligned} {}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{f}_2 &= -a\,\mathrm{f}_2 + \mathrm{g}_2 + 10\,\mathrm{g}_2\,\mathrm{h}_2 - \omega_1, \\ {}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{g}_2 &= -\,\mathrm{f}_2 - 0.4\,\mathrm{g}_2 + 5\,\mathrm{f}_2\,\mathrm{h}_2 - \omega_2, \\ {}_{0}{}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{h}_2 &= b\,\mathrm{h}_2 - 5\,\mathrm{f}_2\,\mathrm{g}_2 - \omega_3, \end{aligned} \tag{4.4}$$

where the controller $u = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T$, with

$$\omega_1 = c_1\,(\mathrm{f}_2 - \mathrm{f}_1)\,, \quad \omega_2 = c_2\,(\mathrm{g}_2 - \mathrm{g}_1)\,, \quad \omega_3 = c_3\,(\mathrm{h}_2 - \mathrm{h}_1)\,.$$

By subtracting (4.3) from (4.4) in case $\kappa = 1$, we obtain:

$$\begin{aligned} \dot{e}_{\mathrm{f}} &= -ae_{\mathrm{f}} + e_{\mathrm{g}} + 10\,\mathrm{g}_2\,\mathrm{h}_2 - 10\,\mathrm{g}_1\,\mathrm{h}_1 - c_1 e_{\mathrm{f}}, \\ \dot{e}_{\mathrm{g}} &= -e_{\mathrm{f}} - 0.4e_{\mathrm{g}} + 5\,\mathrm{f}_1\,\mathrm{h}_1 - 5\,\mathrm{f}_2\,\mathrm{h}_2 - c_2 e_{\mathrm{g}}, \\ \dot{e}_{\mathrm{h}} &= be_{\mathrm{h}} - 5\,\mathrm{f}_1\,\mathrm{g}_1 - 5\,\mathrm{f}_2\,\mathrm{g}_2 - c_3 e_{\mathrm{h}}, \end{aligned} \tag{4.5}$$

where $e_{\mathrm{f}} = \mathrm{f}_1 - \mathrm{f}_2, e_{\mathrm{g}} = \mathrm{g}_1 - \mathrm{g}_2, e_{\mathrm{h}} = \mathrm{h}_1 - \mathrm{h}_2$. The matrix form of system (3.1) is as follows:

$$\dot{e} = Ae + M_{\mathrm{f}_1,\mathrm{f}_2}e - Ke,$$

where

$$e = \begin{bmatrix} e_{\mathrm{f}} & e_{\mathrm{g}} & e_{\mathrm{h}} \end{bmatrix}^T, \, A = \begin{pmatrix} -a & 1 & 0 \\ -1 & -0.4 & 0 \\ 0 & 0 & b \end{pmatrix},$$

$$M_{\mathrm{f}_1,\mathrm{f}_2} = \begin{pmatrix} 0 & 10\,\mathrm{h}_2 & 10\,\mathrm{g}_1 \\ 5\,\mathrm{h}_1 & 0 & 5\,\mathrm{f}_2 \\ 5\,\mathrm{g}_1 & 5\,\mathrm{f}_2 & 0 \end{pmatrix}, \, K = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}.$$

As a result, if the feedback control c_1, c_2 and c_3 gains satisfy the following inequalities:

$$\begin{aligned} c_1 &> \frac{1}{2}\left(-2a + 5\,\mathrm{h}_1 + 10\,\mathrm{h}_2 + 15\,\mathrm{g}_1 - \kappa\right), \\ c_2 &> \frac{1}{2}(5\,\mathrm{h}_1 + 10\,\mathrm{h}_2 - 0.8 + 10\,\mathrm{f}_2 - \kappa), \\ c_3 &> \frac{1}{2}(15\,\mathrm{g}_1 + 10\,\mathrm{f}_2 + 2b - \kappa), \end{aligned} \tag{4.6}$$

the synchronization of (4.3) is achieved. Combined Newton-Leipnik systems (4.1) and (4.2) are numerically integrated with parameters $c_1 = 280$, $c_2 = 250$, $c_3 = 100$.

4.2. *Nonlinear control of a proposed system*

The controllers for the drive and response systems (4.1) and (4.2) are as follows:

$$\omega_1 = (c_1 + v k_{\mathrm{f}_1,\mathrm{f}_2})\, e_{\mathrm{f}}, \quad \omega_2 = 0, \quad \omega_3 = 0.$$

As a result, fractional-order error dynamics are characterized by

$$\begin{aligned} {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa} e_{\mathrm{f}} &= -a e_{\mathrm{f}} + e_{\mathrm{g}} + 10\,\mathrm{g}_2\,\mathrm{h}_2 - 10\,\mathrm{g}_1\,\mathrm{h}_1 - c_1 e_{\mathrm{f}}, \\ {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa} e_{\mathrm{g}} &= -e_{\mathrm{f}} - 0.4 e_{\mathrm{g}} + 5\,\mathrm{f}_1\,\mathrm{h}_1 - 5\,\mathrm{f}_2\,\mathrm{h}_2 - c_2 e_{\mathrm{g}}, \\ {}_0^{\mathrm{ABC}}\mathscr{D}_t^{\kappa} e_{\mathrm{h}} &= b e_{\mathrm{h}} - 5\,\mathrm{f}_1\,\mathrm{g}_1 - 5\,\mathrm{f}_2\,\mathrm{g}_2 - c_3 e_{\mathrm{h}}. \end{aligned} \tag{4.7}$$

By using the feedback control gain $c_1 = 69.2111$ as well as the parameters $\kappa = 1.0, \kappa = 0.98$, the error system (4.7) is numerically integrated.

5. Numerical results

The following schemes is taken from [50, 51].

5.1. *Atangana-Baleanu-Caputo numerical scheme*

The fundamental theorem of fractional calculus of (2.1), gives

$$\phi(t) - \phi(0) = \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\zeta(t, \phi(t)) + \frac{\kappa}{\Gamma(\kappa)\,\mathrm{ABC}(\kappa)} \int_0^t \zeta(\kappa, \phi(\kappa))(t-\kappa)^{\kappa-1} d\kappa.$$

At t_{n+1}, we have

$$\phi(t_{n+1})-\phi(0)=\frac{\Gamma(\kappa)(1-\kappa)}{\Gamma(\kappa)(1-\kappa)+\kappa}\zeta(t_n,\phi(t_n))+\frac{\kappa}{\Gamma(\kappa)+\kappa(1-\Gamma(\kappa))}\times \sum_{\nu=0}^{n}\int_{t_\nu}^{t_{\nu+1}}\zeta(\kappa,\phi(\kappa))(t_{n+1}-\kappa)^{\kappa-1}d\kappa. \tag{5.1}$$

Two-step Lagrange polynomial interpolation can be used to approximate $\zeta(\eta,\phi(\eta))$:

$$\mathscr{P}_k(\kappa)\simeq\frac{\zeta(t_\nu,\mathrm{g}_\nu)}{h}(\kappa-t_{\nu-1})-\frac{\zeta(t_{\nu-1},\mathrm{g}_{\nu-1})}{h}(\kappa-t_\nu). \tag{5.2}$$

Equation (5.2) is replaced by Eq (5.1) in order to obtain

$$\begin{aligned}\phi_{n+1}(t)=\phi_0&+\frac{\Gamma(\kappa)(1-\kappa)}{\Gamma(\kappa)(1-\kappa)+\kappa}\zeta(t_n,\phi(t_n))+\frac{\kappa}{\Gamma(\kappa)+\kappa(1-\Gamma(\kappa))}\\&\times\sum_{\nu=0}^{n}\Big(\frac{\zeta(t_\nu,\mathrm{g}_\nu)}{h}\int_{t_\nu}^{t_{\nu+1}}(\kappa-t_{\nu-1})(t_{n+1}-\kappa)^{\kappa-1}d\kappa\\&-\frac{\zeta(t_{\nu-1},\mathrm{g}_{\nu-1})}{h}\int_{t_\nu}^{t_{\nu+1}}(\kappa-t_\nu)(t_{n+1}-\kappa)^{\kappa-1}d\kappa\Big),\end{aligned} \tag{5.3}$$

which implies that

$$\begin{aligned}\mathscr{A}_{\kappa,\nu,1}&=h^{\kappa+1}\frac{(n+1-\nu)^\kappa(n-\nu+2+\kappa)-(n-\nu)^\kappa(n-\nu+2+2\kappa)}{\kappa(\kappa+1)},\\\mathscr{A}_{\kappa,\nu,2}&=h^{\kappa+1}\frac{(n+1-\nu)^{\kappa+1}-(n-\nu)^\kappa(n-\nu+1+\kappa)}{\kappa(\kappa+1)}.\end{aligned} \tag{5.4}$$

The following approximation can be obtained by integrating (5.4) and replacing it with (5.3):

$$\begin{aligned}\phi_{n+1}(t)=\phi_0&+\frac{\Gamma(\kappa)(1-\kappa)}{\Gamma(\kappa)(1-\kappa)+\kappa}\zeta(t_n,\phi(t_n))+\frac{1}{(\kappa+1)((1-\kappa)\Gamma(\kappa)+\kappa}\\&\times\sum_{\nu=0}^{n}(h^\kappa\zeta(t_\nu,\mathrm{g}_\nu)((n+1-\nu)^\kappa(n-\nu+2+\kappa)-(n-\nu)^\kappa(n-\nu+2+2\kappa))\\&-h^\kappa\zeta(t_{\nu-1},\mathrm{g}_{\nu-1})((n+1-\nu)^{\kappa+1}-(n-\nu)^\kappa\times(n-\nu+1+\kappa)).\end{aligned}$$

The following is the numerical representation of the system (2.1):

$$\begin{aligned}\mathrm{f}_{n+1}(t)=\mathrm{f}_0&+\frac{\Gamma(\kappa)(1-\kappa)}{\Gamma(\kappa)(1-\kappa)+\kappa}\mathscr{L}_1(t_n,\mathrm{f}_n(t),\mathrm{g}_n(t),\mathrm{h}_n(t))\\&+\frac{1}{(\kappa+1)((1-\kappa)\Gamma(\kappa)+\kappa}\times\sum_{\nu=0}^{n}(h^\kappa\mathscr{L}_1(t_\nu,\mathrm{f}_\nu(t),\mathrm{g}_\nu(t),\mathrm{h}_\nu(t))\\&\times((n+1-\nu)^\kappa(n-\nu+2+\kappa)-(n-\nu)^\kappa)\times(n-\nu+2+2\kappa))\\&-h^\kappa\mathscr{L}_1(t_{\nu-1},\mathrm{f}_{\nu-1}(t),\mathrm{g}_{\nu-1}(t),\mathrm{h}_{\nu-1}(t))\times((n+1-\nu)^{\kappa+1}-(n-\nu)^\kappa(n-\nu+1+\kappa))),\end{aligned}$$

$$
\begin{aligned}
g_{n+1}(t) = g_0 &+ \frac{\Gamma(\kappa)(1-\kappa)}{\Gamma(\kappa)(1-\kappa)+\kappa}\mathscr{L}_2(t_n, f_n(t), g_n(t), h_n(t)) \\
&+ \frac{1}{(\kappa+1)((1-\kappa)\Gamma(\kappa)+\kappa} \times \sum_{\nu=0}^{n} (h^\kappa \mathscr{L}_2(t_\nu, f_\nu(t), g_\nu(t), h_\nu(t)) \\
&\times ((n+1-\nu)^\kappa (n-\nu+2+\kappa) - (n-\nu)^\kappa \times (n-\nu+2+2\kappa)) \\
&- h^\kappa \mathscr{L}_2(t_{\nu-1}, f_{\nu-1}(t), g_{\nu-1}(t), h_{\nu-1}(t)) \times ((n+1-\nu)^{\kappa+1} - (n-\nu)^\kappa (n-\nu+1+\kappa))),
\end{aligned}
$$

$$
\begin{aligned}
h_{n+1}(t) = h_0 &+ \frac{\Gamma(\kappa)(1-\kappa)}{\Gamma(\kappa)(1-\kappa)+\kappa}\mathscr{L}_3(t_n, f_n(t), g_n(t), h_n(t)) \\
&+ \frac{1}{(\kappa+1)((1-\kappa)\Gamma(\kappa)+\kappa} \times \sum_{\nu=0}^{n} (h^\kappa \mathscr{L}_3(t_\nu, f_\nu(t), g_\nu(t), h_\nu(t)). \\
&\times ((n+1-\nu)^\kappa (n-\nu+2+\kappa) - (n-\nu)^\kappa \times (n-\nu+2+2\kappa)).
\end{aligned}
$$

5.2. Mittag-Leffler kernel for the Atangana-Seda method

Based on the integration of the equations above, we are able to arrive at the result:

$$
\begin{aligned}
k(t) &= k(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_1(t, f, g, h, t) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \times \int_0^t \mathscr{L}_1(t, f, g, h, \eta)\,(t_{n+1}-\eta)^{\kappa-1}\,d\eta, \\
g(t) &= g(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_2(t, f, g, h, t) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \times \int_0^t \mathscr{L}_2(t, f, g, h, \eta)\,(t_{n+1}-\eta)^{\kappa-1}\,d\eta, \\
h(t) &= h(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_3(t, f, g, h, t) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \times \int_0^t \mathscr{L}_3(t, f, g, h, \eta)\,(t_{n+1}-\eta)^{\kappa-1}\,d\eta.
\end{aligned}
$$

At $t = t_{n+1}$,

$$
\begin{aligned}
f(t_{n+1}) &= f(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_1\Big(t_n, f^n, g^n, h^n\Big) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \int_0^{t_{n+1}} \mathscr{L}_1(t, f, g, h, \eta) \times (t_{n+1}-\eta)^{\kappa-1}\,d\eta, \\
g(t_{n+1}) &= g(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_2\Big(t_n, f^n, g^n, h^n\Big) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \int_0^{t_{n+1}} \mathscr{L}_2(t, f, g, h, \eta) \times (t_{n+1}-\eta)^{\kappa-1}\,d\eta, \\
h(t_{n+1}) &= h(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_3\Big(t_n, f^n, g^n, h^n\Big) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \int_0^{t_{n+1}} \mathscr{L}_3(t, f, g, h, \eta) \times (t_{n+1}-\eta)^{\kappa-1}\,d\eta.
\end{aligned}
$$

As a result, we have the following:

$$
\begin{aligned}
f(t_{n+1}) &= f(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_1\Big(t_n, f^n, g^n, h^n\Big) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \sum_{\nu=2}^{n} \int_{t_\nu}^{t_{\nu+1}} \mathscr{L}_1(t, f, g, h, \eta) \times (t_{n+1}-\eta)^{\kappa-1}\,d\eta. \\
g(t_{n+1}) &= g(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_2\Big(t_n, f^n, g^n, h^n\Big) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \sum_{\nu=2}^{n} \int_{t_\nu}^{t_{\nu+1}} \mathscr{L}_2(t, f, g, h, \eta) \times (t_{n+1}-\eta)^{\kappa-1}\,d\eta, \\
h(t_{n+1}) &= h(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_3\Big(t_n, f^n, g^n, h^n\Big) + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \sum_{\nu=2}^{n} \int_{t_\nu}^{t_{\nu+1}} \mathscr{L}_3(t, f, g, h, \eta) \times (t_{n+1}-\eta)^{\kappa-1}\,d\eta.
\end{aligned}
$$

A Newton polynomial is added to the integrals above to yield the following result:

$$
\begin{aligned}
\mathrm{f}\left(t_{n+1}\right) = {} & \mathrm{f}(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)} \mathscr{L}_1\Big(t_n, \mathrm{f}^n, \mathrm{g}^n, \mathrm{h}^n\Big) \\
& + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \sum_{\nu=2}^{n} \int_{t_\nu}^{t_{\nu+1}} \left\{ \begin{array}{c}
\mathscr{L}_1\Big(t_{\nu-2}, \mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}\Big) \\
+\frac{\mathscr{L}_1\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) - \mathscr{L}_1\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)}{\Delta t} \\
\times\left(\eta - t_{\nu-2}\right) \\
+\frac{\mathscr{L}_1\left(\mathrm{f}^{\nu}, \mathrm{g}^{\nu}, \mathrm{h}^{\nu}, t_\nu\right) - 2\mathscr{L}_1\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) + \mathscr{L}_1\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)}{2(\Delta t)^2} \\
\times\left(\eta - t_{\nu-2}\right)\left(\eta - t_{\nu-1}\right)
\end{array} \right\} \\
& \times \left(t_{n+1} - \eta\right)^{\kappa-1} d\eta, \\
\mathrm{g}\left(t_{n+1}\right) = {} & \mathrm{g}(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)} \mathscr{L}_2\Big(t_n, \mathrm{f}^n, \mathrm{g}^n, \mathrm{h}^n\Big) \\
& + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \sum_{\nu=2}^{n} \int_{t_\nu}^{t_{\nu+1}} \left\{ \begin{array}{c}
\mathscr{L}_2\Big(t_{\nu-2}, \mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}\Big) \\
+\frac{\mathscr{L}_2\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) - \mathscr{L}_2\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)}{\Delta t} \\
\times\left(\eta - t_{\nu-2}\right) \\
+\frac{\mathscr{L}_2\left(\mathrm{f}^{\nu}, \mathrm{g}^{\nu}, \mathrm{h}^{\nu}, t_\nu\right) - 2\mathscr{L}_2\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) + \mathscr{L}_2\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)}{2(\Delta t)^2} \\
\times\left(\eta - t_{\nu-2}\right)\left(\eta - t_{\nu-1}\right)
\end{array} \right\} \\
& \times \left(t_{n+1} - \eta\right)^{\kappa-1} d\eta, \\
\mathrm{h}\left(t_{n+1}\right) = {} & \mathrm{h}(0) + \frac{1-\kappa}{\mathrm{ABC}(\kappa)} \mathscr{L}_3\Big(t_n, \mathrm{f}^n, \mathrm{g}^n, \mathrm{h}^n\Big) \\
& + \frac{\kappa}{\mathrm{ABC}(\kappa)\Gamma(\kappa)} \sum_{\nu=2}^{n} \int_{t_\nu}^{t_{\nu+1}} \left\{ \begin{array}{c}
\mathscr{L}_3\Big(t_{\nu-2}, \mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}\Big) \\
+\frac{\mathscr{L}_3\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) - \mathscr{L}_3\left(t_{\nu-2}, \mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}\right)}{\Delta t} \\
\times\left(\eta - t_{\nu-2}\right) \\
+\frac{\mathscr{L}_3\left(\mathrm{f}^{\nu}, \mathrm{g}^{\nu}, \mathrm{h}^{\nu}, t_\nu\right) - 2\mathscr{L}_3\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) + \mathscr{L}_3\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)}{2(\Delta t)^2} \\
\times\left(\eta - t_{\nu-2}\right)\left(\eta - t_{\nu-1}\right)
\end{array} \right\} \\
& \times \left(t_{n+1} - \eta\right)^{\kappa-1} d\eta.
\end{aligned}
$$

Therefore, we can express the Newton-Leipnik system numerically as follows:

$$
\begin{aligned}
\mathrm{f}^{n+1} &= k^0 + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_1\Big(t_n, \mathrm{f}^n, \mathrm{g}^n, \mathrm{h}^n\Big) \\
&+ \frac{\kappa(\Delta t)^\kappa}{\mathrm{F}(\kappa)\Gamma(\kappa+1)}\sum_{\nu=2}^{n}\mathscr{L}_1\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)[(n-\nu+1)^\kappa - (n-\nu)^\kappa] \\
&+ \frac{\kappa(\Delta t)^\kappa}{\mathrm{F}(\kappa)\Gamma(\kappa+2)}\sum_{\nu=2}^{n}\left[\mathscr{L}_1\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) - \mathscr{L}_1\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)\right] \\
&\times\Big[\ (n-\nu+1)^\kappa(n-r+3+2\kappa) - (n-\nu)^\kappa(n-r+3+3\kappa)\ \Big] \\
&+ \frac{\kappa(\Delta t)^\kappa}{2\mathrm{F}(\kappa)\Gamma(\kappa+3)}\sum_{\nu=2}^{n}\left[\begin{array}{c}\mathscr{L}_1\left(\mathrm{f}^\nu, \mathrm{g}^\nu, \mathrm{h}^\nu, t_\nu\right) - 2\mathscr{L}_1\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) \\ +\mathscr{L}_1\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)\end{array}\right] \\
&\times\left[\begin{array}{c}(n-\nu+1)^\kappa\Big[\ 2(n-\nu)^2 + (3\kappa+10)(n-\nu) + 2\kappa^2 + 9\kappa + 12\ \Big] \\ -(n-\nu)^\kappa\Big[\ 2(n-\nu)^2 + (5\kappa+10)(n-\nu) + 6\kappa^2 + 18\kappa + 12\ \Big]\end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{g}^{n+1} &= l^0 + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_2\Big(t_n, \mathrm{f}^n, \mathrm{g}^n, \mathrm{h}^n\Big) \\
&+ \frac{\kappa(\Delta t)^\kappa}{\mathrm{F}(\kappa)\Gamma(\kappa+1)}\sum_{\nu=2}^{n}\mathscr{L}_2\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)[(n-\nu+1)^\kappa - (n-\nu)^\kappa] \\
&+ \frac{\kappa(\Delta t)^\kappa}{\mathrm{F}(\kappa)\Gamma(\kappa+2)}\sum_{\nu=2}^{n}\left[\mathscr{L}_2\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) - \mathscr{L}_2\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)\right] \\
&\times\Big[\ (n-\nu+1)^\kappa(n-r+3+2\kappa) - (n-\nu)^\kappa(n-r+3+3\kappa)\ \Big] \\
&+ \frac{\kappa(\Delta t)^\kappa}{2\mathrm{F}(\kappa)\Gamma(\kappa+3)}\sum_{\nu=2}^{n}\left[\begin{array}{c}\mathscr{L}_2\left(\mathrm{f}^\nu, \mathrm{g}^\nu, \mathrm{h}^\nu, t_\nu\right) - 2\mathscr{L}_2\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) \\ +\mathscr{L}_2\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)\end{array}\right] \\
&\times\left[\begin{array}{c}(n-\nu+1)^\kappa\Big[\ 2(n-\nu)^2 + (3\kappa+10)(n-\nu) + 2\kappa^2 + 9\kappa + 12\ \Big] \\ -(n-\nu)^\kappa\Big[\ 2(n-\nu)^2 + (5\kappa+10)(n-\nu) + 6\kappa^2 + 18\kappa + 12\ \Big]\end{array}\right], \\
\mathrm{h}^{n+1} &= \mathrm{h}^0 + \frac{1-\kappa}{\mathrm{ABC}(\kappa)}\mathscr{L}_3\Big(t_n, \mathrm{f}^n, \mathrm{g}^n, \mathrm{h}^n\Big) \\
&+ \frac{\kappa(\Delta t)^\kappa}{\mathrm{F}(\kappa)\Gamma(\kappa+1)}\sum_{\nu=2}^{n}\mathscr{L}_3\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)[(n-\nu+1)^\kappa - (n-\nu)^\kappa] \\
&+ \frac{\kappa(\Delta t)^\kappa}{\mathrm{F}(\kappa)\Gamma(\kappa+2)}\sum_{\nu=2}^{n}\left[\mathscr{L}_3\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) - \mathscr{L}_3\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)\right] \\
&\times\Big[\ (n-\nu+1)^\kappa(n-r+3+2\kappa) - (n-\nu)^\kappa(n-r+3+3\kappa)\ \Big] \\
&+ \frac{\kappa(\Delta t)^\kappa}{2\mathrm{F}(\kappa)\Gamma(\kappa+3)}\sum_{\nu=2}^{n}\left[\begin{array}{c}\mathscr{L}_3\left(\mathrm{f}^\nu, \mathrm{g}^\nu, \mathrm{h}^\nu, t_\nu\right) - 2\mathscr{L}_3\left(k^{\nu-1}, \mathrm{g}^{\nu-1}, \mathrm{h}^{\nu-1}, t_{\nu-1}\right) \\ +\mathscr{L}_3\left(\mathrm{f}^{\nu-2}, \mathrm{g}^{\nu-2}, \mathrm{h}^{\nu-2}, t_{\nu-2}\right)\end{array}\right] \\
&\times\left[\begin{array}{c}(n-\nu+1)^\kappa\Big[\ 2(n-\nu)^2 + (3\kappa+10)(n-\nu) + 2\kappa^2 + 9\kappa + 12\ \Big] \\ -(n-\nu)^\kappa\Big[\ 2(n-\nu)^2 + (5\kappa+10)(n-\nu) + 6\kappa^2 + 18\kappa + 12\ \Big]\end{array}\right].
\end{aligned}
$$

5.3. Successive approximation method

For a system (3.1) with initial conditions, we use the successive approximation method:

$$
\begin{aligned}
{}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{f} &= -a\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h}, \quad \mathrm{f}(0) = 0.19,\\
{}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{g} &= -\,\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h}, \quad \mathrm{g}(0) = 0,\\
{}^{ABC}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{h} &= b\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g}, \quad \mathrm{h}(0) = -0.18.
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathrm{f}_n(t) &= 0.19 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(-a\,\mathrm{f}_{n-1} + \mathrm{g}_{n-1} + 10\,\mathrm{g}_{n-1}\,\mathrm{h}_{n-1} \Big)(t-u)^{\kappa-1}du,\\
\mathrm{g}_n(t) &= 0 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(-\,\mathrm{f}_{n-1} - 0.4\,\mathrm{g}_{n-1} + 5\,\mathrm{f}_{n-1}\,\mathrm{h}_{n-1} \Big)(t-u)^{\kappa-1}du,\\
\mathrm{h}_n(t) &= -0.18 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(b\,\mathrm{h}_{n-1} - 5\,\mathrm{f}_{n-1}\,\mathrm{g}_{n-1} \Big)(t-u)^{\kappa-1}du.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathrm{f}_1 &= 0.19 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(-a\,\mathrm{f}_0 + \mathrm{g}_0 + 10\,\mathrm{g}_0\,\mathrm{h}_0 \Big)(t-u)^{\kappa-1}du,\\
\mathrm{g}_1 &= 0 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(-\,\mathrm{f}_0 - 0.4\,\mathrm{g}_0 + 5\,\mathrm{f}_0\,\mathrm{h}_0 \Big)(t-u)^{\kappa-1}du,\\
\mathrm{h}_1 &= -0.18 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(b\,\mathrm{h}_0 - 5\,\mathrm{f}_0\,\mathrm{g}_0 \Big)(t-u)^{\kappa-1}du.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{f}_1 &= 0.19 + \frac{-0.19a}{\Gamma(\kappa)}\int_0^t (t-u)^{\kappa-1}du,\\
\mathrm{g}_1 &= 0 + \frac{(-0.19 - 17.1)}{\Gamma(\kappa)}\int_0^t (t-u)^{\kappa-1}du,\\
\mathrm{h}_1 &= -0.18 + \frac{-0.0315}{\Gamma(\kappa)}\int_0^t (t-u)^{\kappa-1}du.
\end{aligned}
$$

Based on the theorem regarding the product of convolution, we obtain the following three iterations:

$$
\begin{aligned}
\mathrm{f}_1 &= 0.19 + \frac{-0.19at^{\kappa}}{\Gamma(\kappa+1)},\\
\mathrm{g}_1 &= 0 + \frac{(-0.19 - 17.1)t^{\kappa}}{\Gamma(\kappa+1)},\\
\mathrm{h}_1 &= -0.18 + \frac{-0.0315t^{\kappa}}{\Gamma(\kappa+1)}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{f}_2 &= 0.19 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(-a\,\mathrm{f}_1 + \mathrm{g}_1 + 10\,\mathrm{g}_1\,\mathrm{h}_1 \Big)(t-u)^{\kappa-1}du,\\
\mathrm{g}_2 &= 0 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(-\,\mathrm{f}_1 - 0.4\,\mathrm{g}_1 + 5\,\mathrm{f}_1\,\mathrm{h}_1 \Big)(t-u)^{\kappa-1}du,\\
\mathrm{h}_2 &= -0.18 + \frac{1}{\Gamma(\kappa)}\int_0^t \Big(b\,\mathrm{h}_1 - 5\,\mathrm{f}_1\,\mathrm{g}_1 \Big)(t-u)^{\kappa-1}du.
\end{aligned}
$$

Then

$$\begin{aligned}
\mathrm{f} &= 0.19 - 0.0766354t^{0.98} + 0.165549t^{1.96} - 0.0200889t^{2.94} - 0.16188t^{3.92} \\
&\quad + 0.141366t^{4.9} - 0.104083t^{5.88} + 0.0145646t^{6.86} + 0.00748789t^{7.84} \\
&\quad - 0.00948982t^{8.82} + 0.00375966t^{9.8} + 0.0000968983t^{10.78}, \\
y &= -0.364018t^{0.98} + 0.134262t^{1.96} - 0.0657408t^{2.94} - 0.027892t^{3.92} \\
&\quad + 0.133288t^{4.9} - 0.0482851t^{5.88} - 0.0112423t^{6.86} + 0.0185251t^{7.84} \\
&\quad - 0.0152564t^{8.82} + 0.00529679t^{9.8} - 0.000271352t^{10.78}, \\
z &= -0.18 - 0.0317634t^{0.98} + 0.175007t^{1.96} - 0.0818753t^{2.94} \\
&\quad + 0.104826t^{3.92} - 0.0217401t^{4.9} - 0.0561595t^{5.88} + 0.0502709t^{6.86} \\
&\quad - 0.0223372t^{7.84} - 0.00164042t^{8.82} + 0.0056128t^{9.8} - 0.00176122t^{10.78}.
\end{aligned}$$

5.4. Numerical simulation and discussions

Example 1. *[6] We deal with the following chaotic problem:*

$$\begin{aligned}
{}^{\mathrm{ABC}}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{f} &= -0.4\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h}, \\
{}^{\mathrm{ABC}}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{g} &= -\,\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h}, \\
{}^{\mathrm{ABC}}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{h} &= 0.175\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g},
\end{aligned} \tag{5.5}$$

with the initial conditions.

The numerical simulations of the ABC fractional Newton-Leipnik system (5.5) are presented in Figures 1, 3, 5 and 7 for $\kappa(t) = 1$, $\kappa(t) = 0.98$, $\kappa(t) = 0.97 + 0.03 \times \tanh(t/10)$, $\kappa(t) = 0.97 - 0.03 \times \sin(t/10)$, respectively. The numerical simulations of the AS fractional Newton-Leipnik system (5.5) are presented in Figures 9, 11, 13 and 15 for $\kappa(t) = 1$, $\kappa(t) = 0.98$, $\kappa(t) = 0.97 + 0.03 \times \tanh(t/10)$, $\kappa(t) = 0.97 - 0.03 \times \sin(t/10)$, respectively.

Example 2. *We deal with the following chaotic problem:*

$$\begin{aligned}
{}^{\mathrm{ABC}}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{f} &= -0.4\,\mathrm{f} + \mathrm{g} + 10\,\mathrm{g}\,\mathrm{h} - 7\,\mathrm{f}, \\
{}^{\mathrm{ABC}}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{g} &= -\,\mathrm{f} - 0.4\,\mathrm{g} + 5\,\mathrm{f}\,\mathrm{h}, \\
{}^{\mathrm{ABC}}_{0}\mathscr{D}^{\kappa}_{t}\,\mathrm{h} &= 0.175\,\mathrm{h} - 5\,\mathrm{f}\,\mathrm{g},
\end{aligned} \tag{5.6}$$

with the initial conditions

The numerical simulations of the ABC of the controlled fractional order Newton-Leipnik system (5.6) are presented in Figures 2, 4, 6 and 8 for $\kappa(t) = 1$, $\kappa(t) = 0.98$, $\kappa(t) = 0.97 + 0.03 \times \tanh(t/10)$, $\kappa(t) = 0.97 - 0.03 \times \sin(t/10)$, respectively. The numerical simulations of the AS of the controlled fractional order Newton-Leipnik system (5.6) are presented in Figures 10, 12, 14 and 16 for $\kappa(t) = 1$, $\kappa(t) = 0.98$, $\kappa(t) = 0.97 + 0.03 \times \tanh(t/10)$, $\kappa(t) = 0.97 - 0.03 \times \sin(t/10)$, respectively.

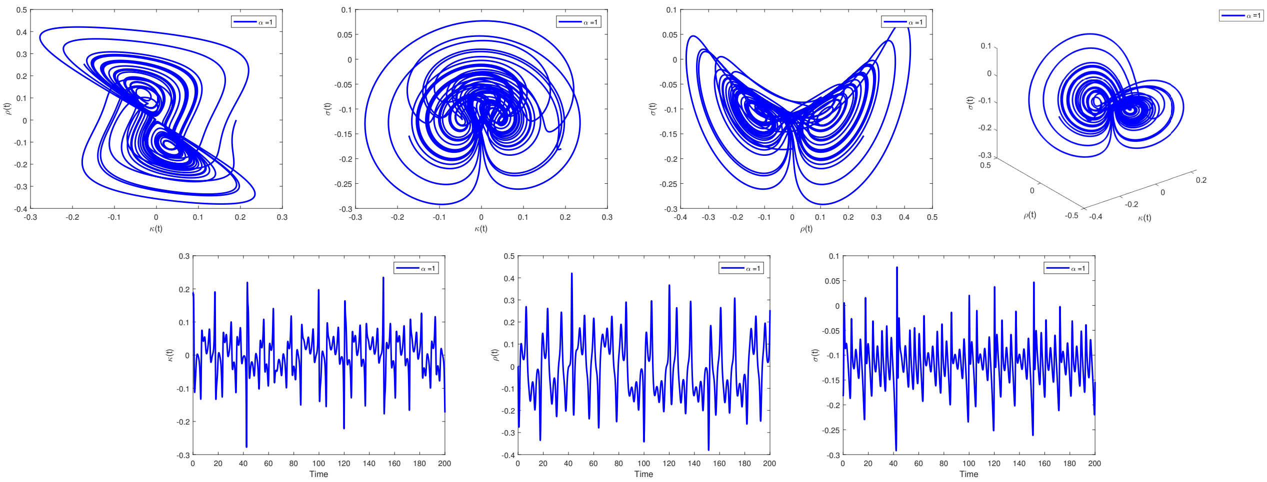

Figure 1. ABC for (5.5) with $\kappa(t) = 1$.

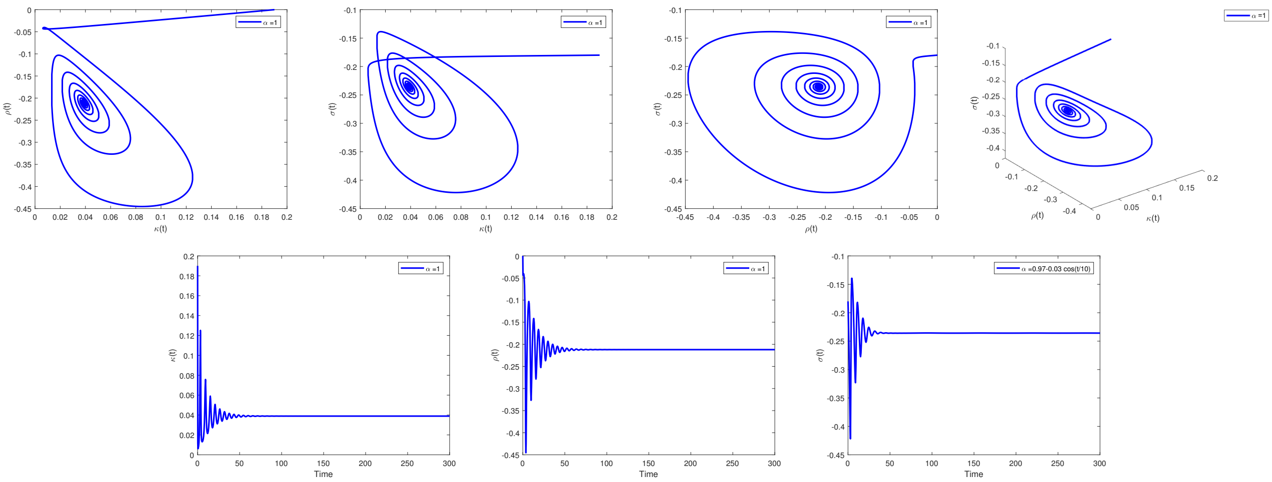

Figure 2. ABC for (5.6) with $\kappa(t) = 1$.

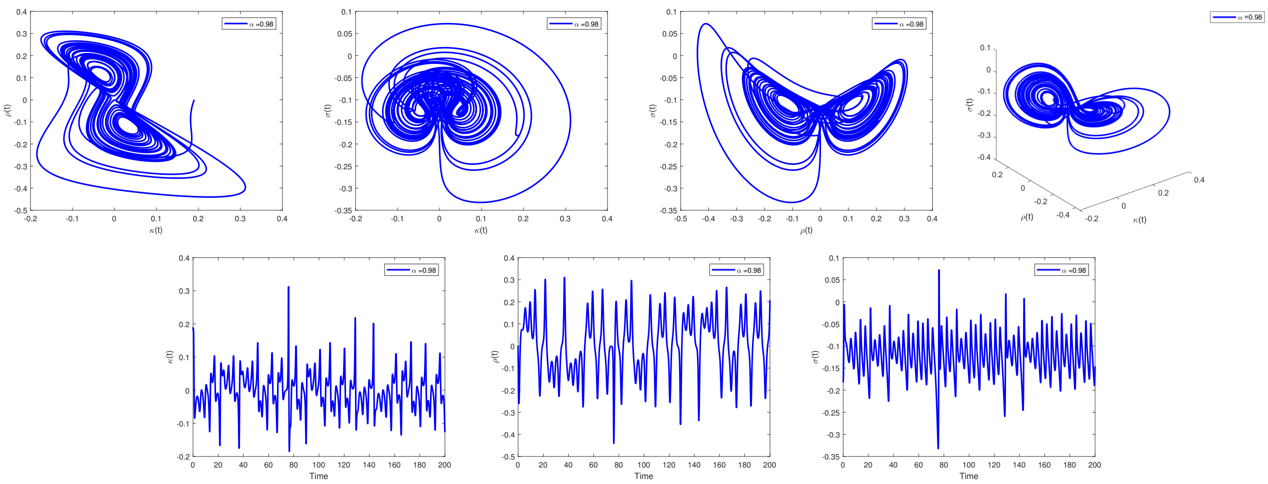

Figure 3. ABC for (5.5) with $\kappa(t) = 0.98$.

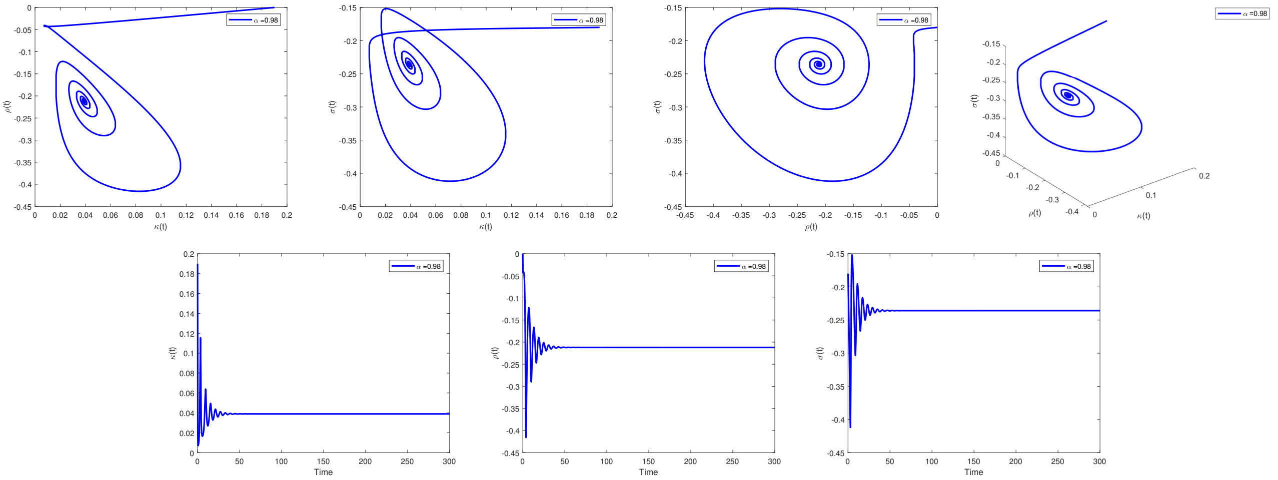

Figure 4. ABC for (5.6) with $\kappa(t) = 0.98$.

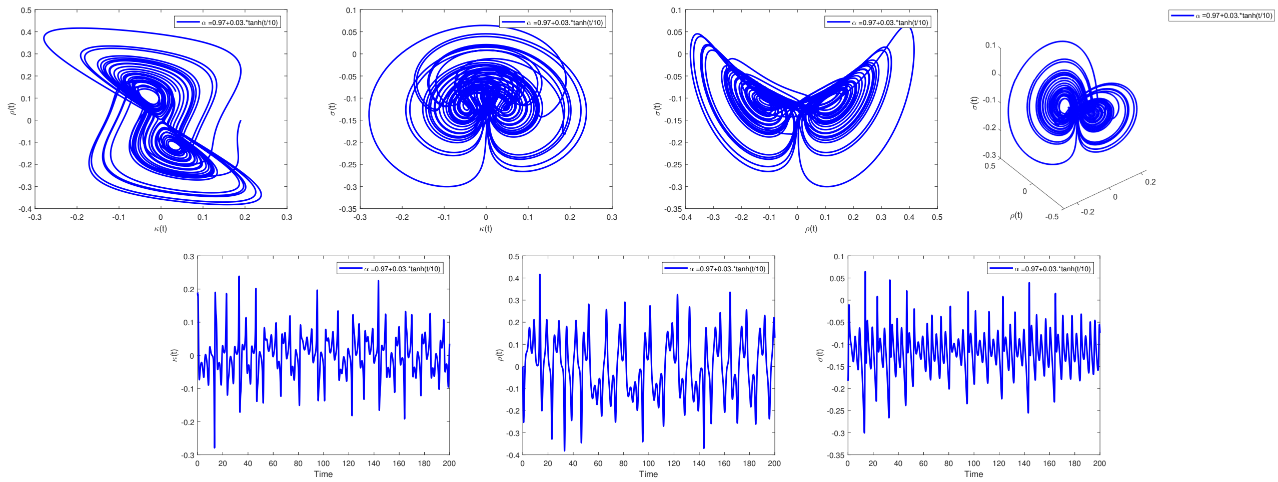

Figure 5. ABC for (5.5) with $\kappa(t) = 0.97 + 0.03 \times tanh(t/10)$.

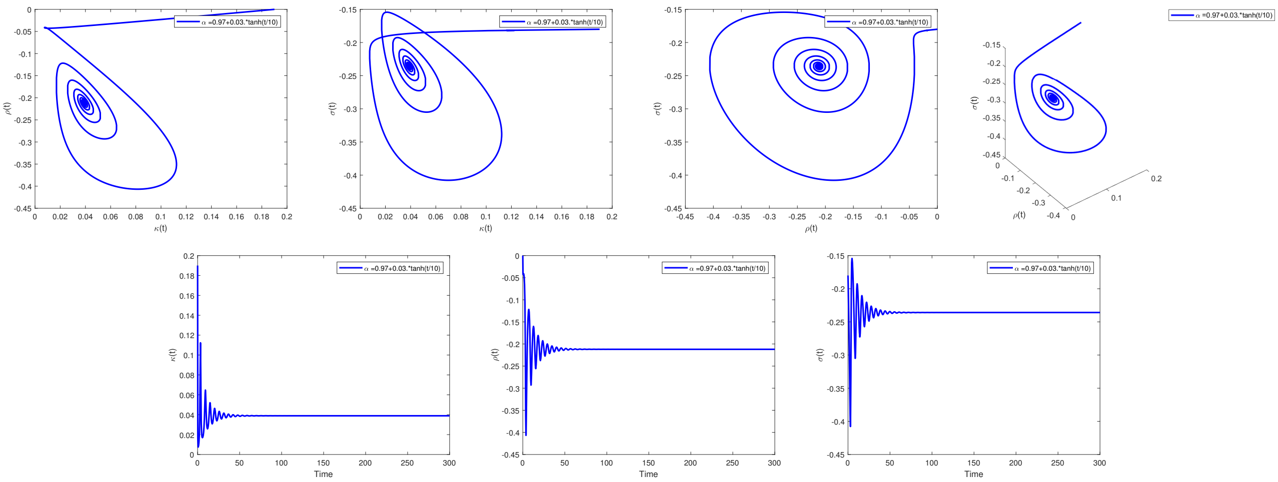

Figure 6. ABC for (5.6) with $\kappa(t) = 0.97 + 0.03 \times tanh(t/10)$.

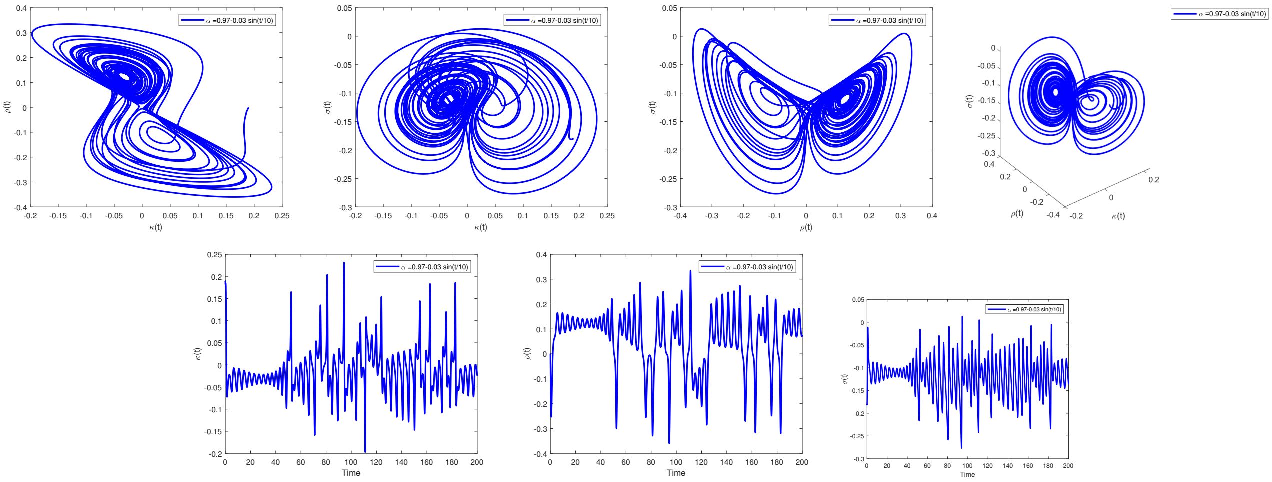

Figure 7. ABC for (5.5) with $\kappa(t) = 0.97 - 0.03 \times sin(t/10)$.

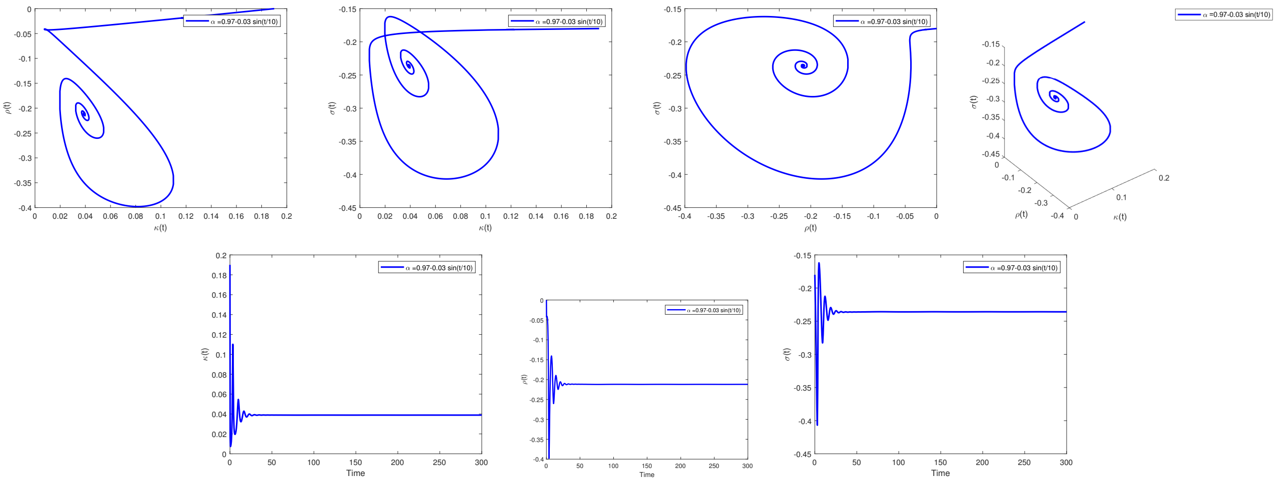

Figure 8. ABC for (5.6) with $\kappa(t) = 0.97 - 0.03 \times sin(t/10)$.

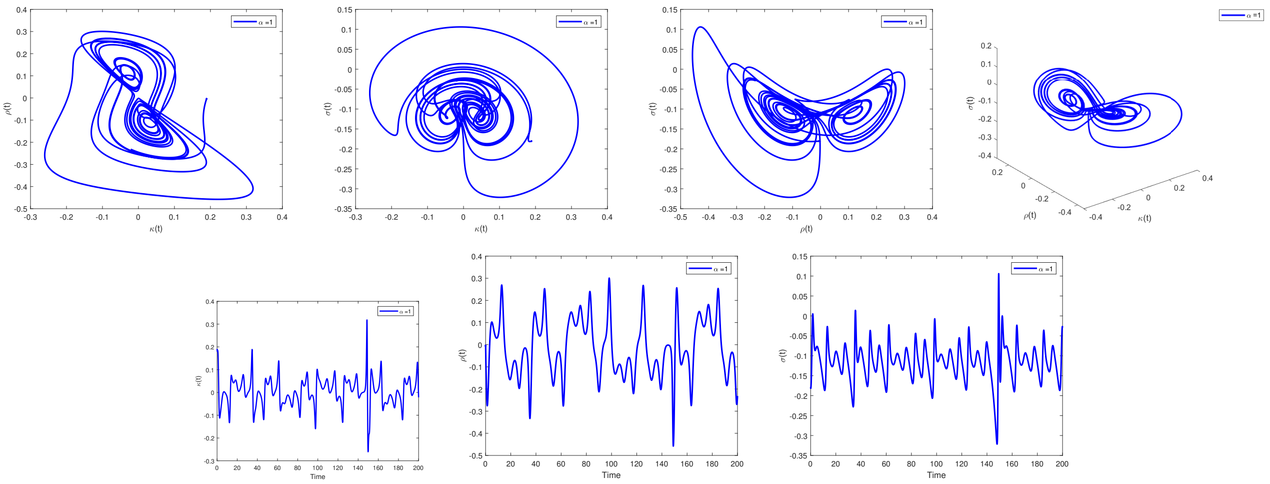

Figure 9. AS for (5.5) with $\kappa(t) = 1$.

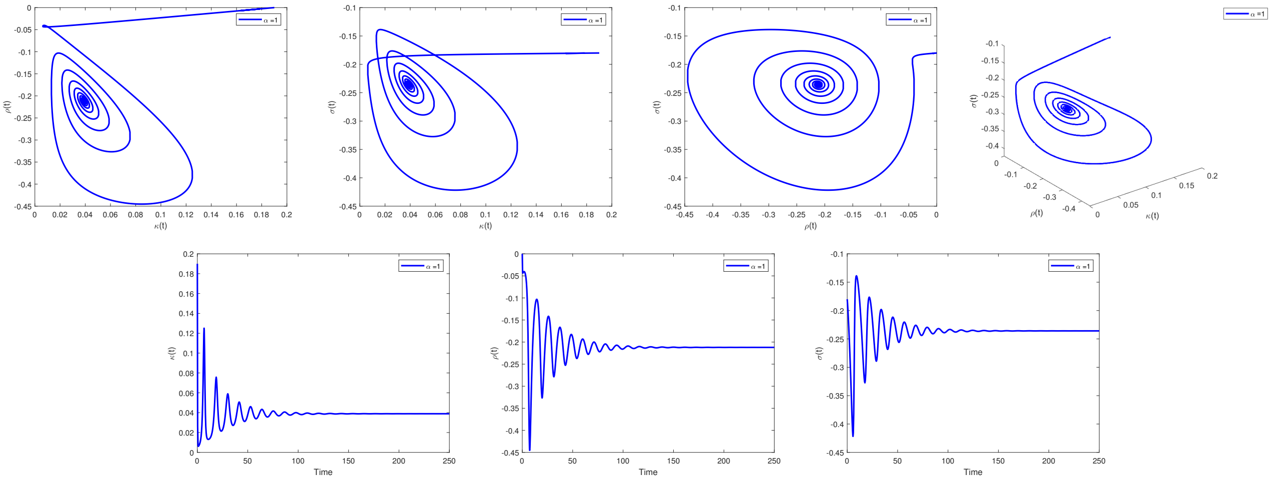

Figure 10. AS for (5.6) with $\kappa(t) = 1$.

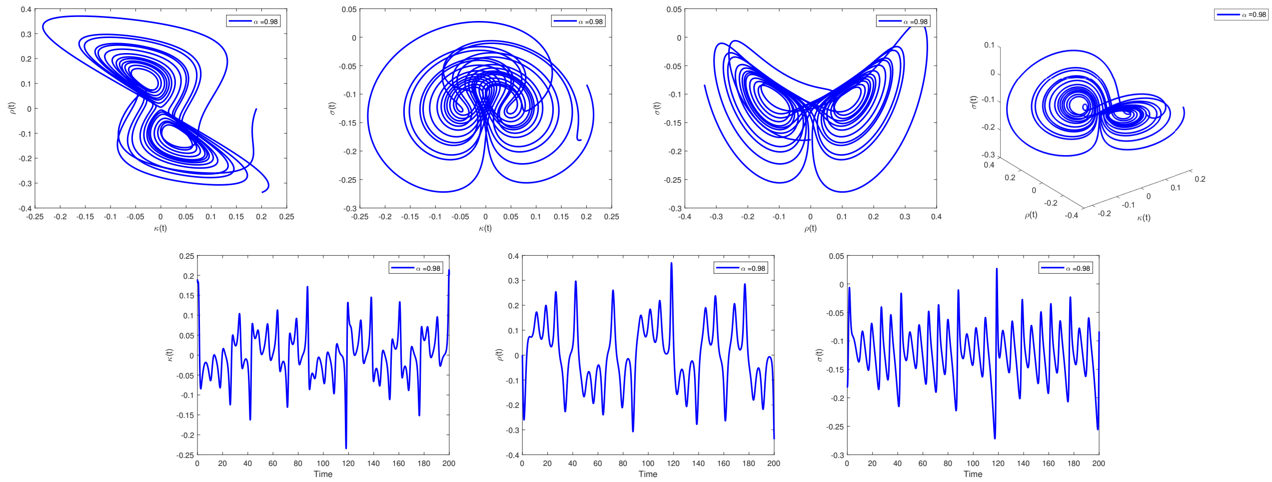

Figure 11. AS for (5.5) with $\kappa(t) = 0.98$.

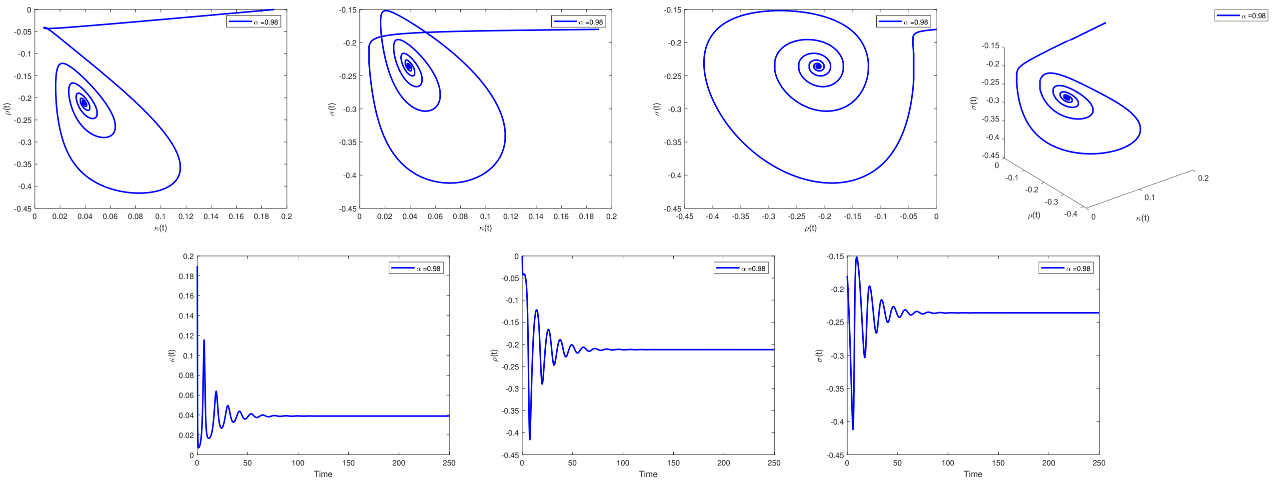

Figure 12. AS for (5.6) with $\kappa(t) = 0.98$.

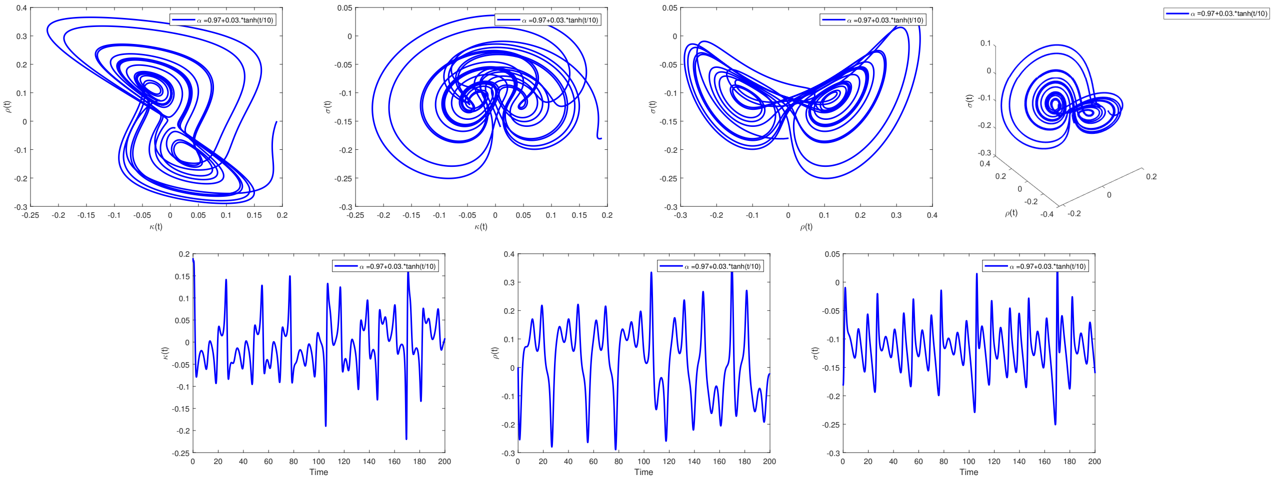

Figure 13. AS for (5.5) with $\kappa(t) = 0.97 + 0.03 \times tanh(t/10)$.

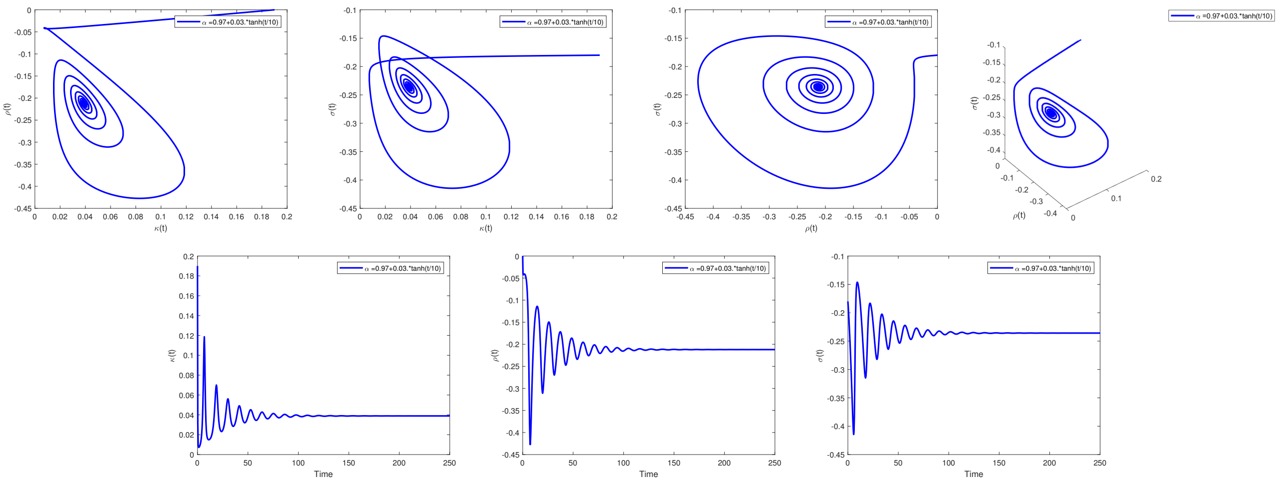

Figure 14. AS for (5.6) with $\kappa(t) = 0.97 + 0.03 \times tanh(t/10)$.

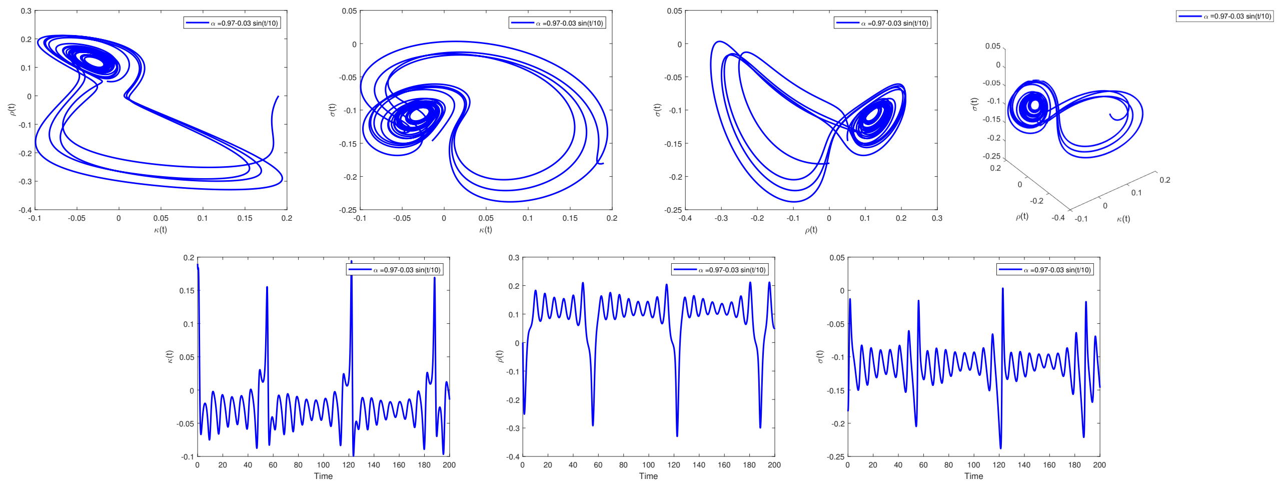

Figure 15. AS for (5.5) with $\kappa(t) = 0.97 - 0.03 \times sin(t/10)$.

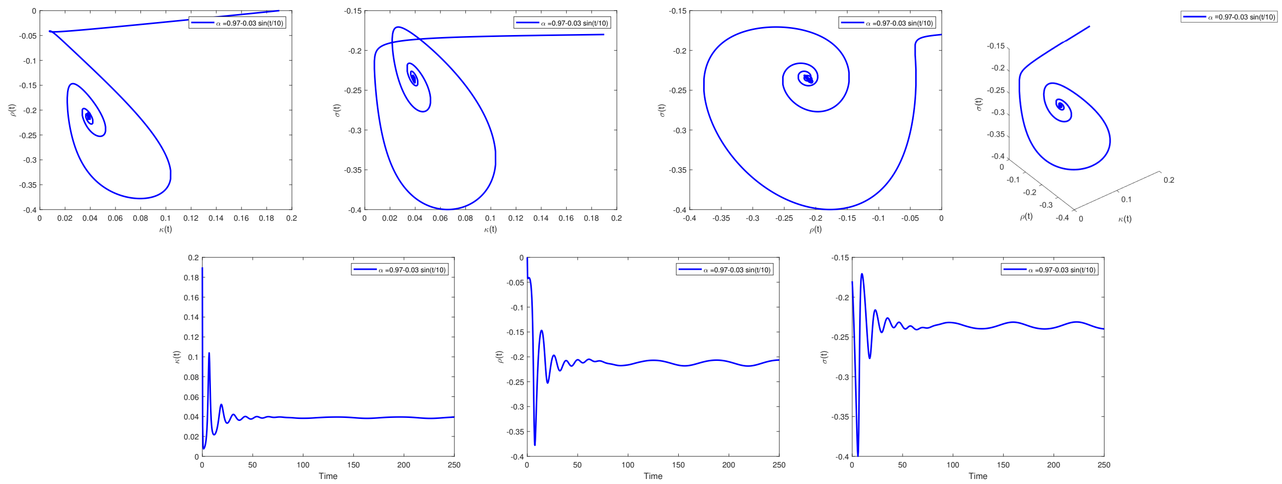

Figure 16. AS for (5.6) with $\kappa(t) = 0.97 - 0.03 \times sin(t/10)$.

6. Conclusions

Fractional differential operators have the ability to accurately replicate and reveal some chaos. However, because of their non-linearity, their analytical solutions are difficult to obtain and, in some circumstances, impossible to achieve due to their non-linearity. Researchers rely on numerical methods to understand physical behavior. We present a numerical method for chaotic problems. Using differential and integral operators in the sense of Atangana-Baleanu-Caputo, we investigated the Newton-Leipnik system of mathematical equations able to capture chaotic behavior. Solutions are obtained for the fractional-order Newton-Leipnik model using a fractional operator with a non-singular kernel. Uniqueness and boundedness for solutions are proved through the fixed point theory. Due to the high non-linearity of our problem, we used a suitable numerical scheme to solve this system of equations numerically. The presented scheme is applicable to many other systems, such as those in [36–48]. In future work, the existence and uniqueness of solutions reported for general component differential equations will be extended to multidimensional problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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