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Research article

On the high-th mean of one special character sums modulo a prime

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Abstract: Using the elementary method of the classical Gauss sums and the properties of character sums, we study a linear recurrence formula about the form $G(n) = 1 + \sum_{a=1}^{p-1}\left(\frac{a^2+n\bar{a}^2}{p}\right)$ and about the mean value of $G(n)$. This is a further exploration of Yuan and Zhang's research in 2022, which help us to better understand the character sums wide range application.

Keywords: quadratic character; quartic character; basic method; calculating formula; special character sums
Mathematics Subject Classification: 11L05, 11L40

1. Introduction

The key to solving the general quadratic congruence equation is to solve the equation of the form $x^2 \equiv a \mod p$, where a and p are integers, $p > 0$ and p is not divisible by a. For relatively large p, it is impractical to use the Euler criterion to distinguish whether the integer a with $(a, p) = 1$ is quadratic residue of modulo p. In order to study this issue, Legendre has proposed a new tool-Legendre's symbol.

Let p be an odd prime, the quadratic character modulo p is called the Legendre's symbol, which is defined as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p; \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p; \\ 0, & \text{if } p \mid a. \end{cases}$$

The Legendre's symbol makes it easy for us to calculate the level of quadratic residues. The basic properties of Legendre's symbol can be found in any book on elementary number theory, such as [1–3].

The properties of Legendre's symbol and quadratic residues play an important role in number theory. Many scholars have studied them and achieved some important results. For examples, see the [4–21].

One of the most representative properties of the Legendre's symbol is the quadratic reciprocal law: Let p and q be two distinct odd primes. Then, (see Theorem 9.8 in [1] or Theorems 4–6 in [3])

$$\left(\frac{p}{q}\right)\cdot\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

For any odd prime p with $p \equiv 1 \bmod 4$ there exist two non-zero integers $\alpha(p)$ and $\beta(p)$ such that

$$p = \alpha^2(p) + \beta^2(p). \tag{1}$$

In fact, the integers $\alpha(p)$ and $\beta(p)$ in the (1) can be expressed in terms of Legendre's symbol modulo p (see Theorems 4–11 in [3])

$$\alpha(p) = \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a^3+a}{p}\right) \quad \text{and} \quad \beta(p) = \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a^3+ra}{p}\right),$$

where r is any integer, and $(r, p) = 1$, $\left(\frac{r}{p}\right) = -1$, $\left(\frac{*}{p}\right) = \chi_2$ denote the Legendre's symbol modulo p.

Noting that Legendre's symbol is a special kind of character. For research on character, Han [7] studied the sum of a special character $\chi(ma + \bar{a})$, for any integer m with $(m, p) = 1$, then

$$\left|\sum_{a=1}^{p-1}\chi(ma+\bar{a})\right|^2 = 2p + \left(\frac{m}{p}\right)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{b(b-1)(a^2b-1)}{p}\right),$$

which is a special case of a general polynomial character sums $\sum_{a=N+1}^{N+M}\chi(f(a))$, where M and N are any positive integers, and $f(x)$ is a polynomial.

In [8], Du and Li introduced a special character sums $C(\chi, m, n, c; p)$ in the following form:

$$C(\chi, m, n, c; p) = \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi\left(a^2 + na - b^2 - nb + c\right)\cdot e\left(\frac{mb^2 - ma^2}{p}\right),$$

and studied the asymptotic properties of it. They obtained

$$\sum_{c=1}^{p-1}|C(\chi, m, n, c; p)|^{2k} = \begin{cases} p^{2k+1} + \frac{k^2-3k-2}{2}\cdot p^{2k} + O\left(p^{2k-1}\right), & \text{if } \chi \text{ is the Legendre symbol modulo } p; \\ p^{2k+1} + \frac{k^2-3k-2}{2}\cdot p^{2k} + O\left(p^{2k-1/2}\right), & \text{if } \chi \text{ is a complex character modulo } p. \end{cases}$$

Recently, Yuan and Zhang [12] researched the question about the estimation of the mean value of high-powers for a special character sum modulo a prime, let p be an odd prime with $p \equiv 1 \bmod 6$, then for any integer $k \geq 0$, they have the identity

$$S_k(p) = \frac{1}{3}\cdot\left[d^k + \left(\frac{-d+9b}{2}\right)^k + \left(\frac{-d-9b}{2}\right)^k\right],$$

where

$$S_k(p) = \frac{1}{p-1}\sum_{r=1}^{p-1}A^k(r),$$

$$A(r) = 1 + \sum_{a=1}^{p-1} \left(\frac{a^2 + r\bar{a}}{p} \right),$$

and for any integer r with $(r, p) = 1$.

More relevant research on special character sums will not be repeated. Inspired by these papers, we have the question: If we replace the special character sums with Legendre's symbol, can we get good results on $p \equiv 1 \bmod 4$?

We will convert $\beta(p)$ to another form based on the properties of complete residues

$$\beta(p) = \frac{1}{2} \sum_{a=1}^{p-1} \left(\frac{a + n\bar{a}}{p} \right),$$

where $\bar{a}$ is the inverse of a modulo p. That is, $\bar{a}$ satisfy the equation $x \cdot a \equiv 1 \mod p$ for any integer a with $(a, p) = 1$.

For any integer $k \geq 0$, $G(n)$ and $K_k(p)$ are defined as follows:

$$G(n) = 1 + \sum_{a=1}^{p-1} \left(\frac{a^2 + n\bar{a}^2}{p} \right) \quad \text{and} \quad K_k(p) = \frac{1}{p-1} \sum_{n=1}^{p-1} G^k(n).$$

In this paper, we will use the analytic methods and properties of the classical Gauss sums and Dirichlet character sums to study the computational problem of $K_k(p)$ for any positive integer k, and give a linear recurrence formulas for $K_k(p)$. That is, we will prove the following result.

Theorem 1. *Let p be an odd prime with* $p \equiv 1 \bmod 4$*, then we have*

$$K_k(p) = (4p + 2) \cdot K_{k-2}(p) - 8\left(2\alpha^2 - p\right) \cdot K_{k-3}(p) + \left(16\alpha^4 - 16p\alpha^2 + 4p - 1\right) \cdot K_{k-4}(p),$$

for all integer $k \geq 4$ *with*

$$K_0(p) = 1, \ K_1(p) = 0, \ K_2(p) = 2p + 1, \ K_3(p) = -3(4\alpha^2 - 2p),$$

where

$$\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + \bar{a}}{p} \right).$$

Applying the properties of the linear recurrence sequence, we may immediately deduce the following corollaries.

Corollary 1. *Let p be an odd prime with* $p \equiv 1 \bmod 4$*. Then we have*

$$\frac{1}{p-1} \sum_{n=1}^{p-1} \frac{1}{1 + \sum_{a=1}^{p-1} \left(\frac{a^2 + n\bar{a}^2}{p} \right)} = \frac{16\alpha^2 p - 28\alpha^2 - 8p^2 + 14p}{16\alpha^4 - 16\alpha^2 p + 4p - 1}.$$

Corollary 2. *Let p be an odd prime with* $p \equiv 1 \bmod 4$*. Then we have*

$$\frac{1}{p-1} \sum_{n=1}^{p-1} \sum_{m=0}^{p-1} \left(1 + \sum_{a=1}^{p-1} \left(\frac{a^2 + n\bar{a}^2}{p} \right) \right) \cdot e\left(\frac{nm^2}{p} \right) = -\sqrt{p}.$$

Corollary 3. *Let p be an odd prime with* $p \equiv 1 \bmod 4$. *Then we have*

$$\frac{1}{p-1}\sum_{n=1}^{p-1}\sum_{m=0}^{p-1}\left[1+\sum_{a=1}^{p-1}\left(\frac{a^2+n\bar{a}^2}{p}\right)\right]^2\cdot e\left(\frac{nm^2}{p}\right)=\left(4\alpha^2-2p\right)\cdot\sqrt{p}.$$

Corollary 4. *Let p be an odd prime with* $p \equiv 1 \bmod 8$. *Then we have*

$$\sum_{n=1}^{p-1}\left(1+\sum_{a=1}^{p-1}\left(\frac{a^2+n\bar{a}^2}{p}\right)\right)\cdot\sum_{m=0}^{p-1}e\left(\frac{nm^4}{p}\right)=\sqrt{p}\left(-1+B(1)\right)-p,$$

where

$$B(1)=\sum_{m=0}^{p-1}e\left(\frac{m^4}{p}\right).$$

If we consider such a sequence $F_k(p)$ as follows: Let p be a prime with $p \equiv 1 \bmod 8$, χ_4 be any fourth-order character modulo p. For any integer $k \geq 0$, we define the $F_k(p)$ as

$$F_k(p)=\sum_{n=1}^{p-1}\frac{1}{G^k(n)},$$

we have

$$\begin{aligned}F_k(p)=&\frac{1}{16\alpha^4-16\alpha^2p+4p-1}F_{k-4}(p)-\frac{(4p+2)}{16\alpha^4-16\alpha^2p+4p-1}F_{k-2}(p)\\&+\frac{4(4\alpha^2-2p)}{16\alpha^4-16\alpha^2p+4p-1}F_{k-1}(p).\end{aligned}$$

2. Some lemmas

Lemma 1. *Let p be an odd prime with* $p \equiv 1 \bmod 4$. *Then for any fourth-order character* $\chi_4 \bmod p$, *we have the identity*

$$\tau^2(\chi_4)+\tau^2(\bar{\chi}_4)=2\sqrt{p}\cdot\alpha,$$

where

$$\tau(\chi_4)=\sum_{a=1}^{p-1}\chi_4(a)e\left(\frac{a}{p}\right)$$

denotes the classical Gauss sums, $e(y)=e^{2\pi iy}$, $i^2=-1$, *and* α *is the same as in the Theorem 1.*
Proof. See Lemma 2.2 in [9].

Lemma 2. *Let p be an odd prime. Then for any non-principal character* ψ *modulo p, we have the identity*

$$\tau(\psi^2)=\frac{\psi^2(2)}{\tau(\chi_2)}\cdot\tau(\psi)\cdot\tau(\psi\chi_2),$$

where $\chi_2=\left(\frac{*}{p}\right)$ *denotes the Legendre's symbol modulo p.*

Proof. See Lemma 2 in [12].

Lemma 3. *Let p be a prime with $p \equiv 1 \bmod 4$, then for any integer n with $(n, p) = 1$ and fourth-order character $\chi_4 \bmod p$, we have the identity*

$$\sum_{a=1}^{p-1}\left(\frac{a^2+n\bar{a}^2}{p}\right) = -1-\chi_2(n)+\frac{1}{\sqrt{p}}\cdot\left(\chi_4(n)\cdot\tau^2(\bar{\chi_4})+\bar{\chi_4}(n)\cdot\tau^2(\chi_4)\right).$$

Proof. For any integer a with $(a, p) = 1$, we have the identity

$$1+\chi_4(a)+\chi_2(a)+\bar{\chi_4}(a)=4,$$

if a satisfies $a \equiv b^4 \bmod p$ for some integer b with $(b, p) = 1$ and

$$1+\chi_4(a)+\chi_2(a)+\bar{\chi_4}(a)=0,$$

otherwise. So from these and the properties of Gauss sums we have

$$\begin{aligned}\sum_{a=1}^{p-1}\left(\frac{a^2+n\bar{a}^2}{p}\right) &= \sum_{a=1}^{p-1}\left(\frac{a^2}{p}\right)\left(\frac{a^4+n}{p}\right)\\
&= \sum_{a=1}^{p-1}\chi_2\left(a^4\right)\chi_2\left(a^4+n\right)\\
&= \sum_{a=1}^{p-1}(1+\chi_4(a)+\chi_2(a)+\bar{\chi_4}(a))\cdot\chi_2(a)\cdot\chi_2(a+n)\\
&= \sum_{a=1}^{p-1}(1+\chi_4(na)+\chi_2(na)+\bar{\chi_4}(na))\cdot\chi_2(na)\cdot\chi_2(na+n)\\
&= \sum_{a=1}^{p-1}\chi_2(a)\chi_2(a+1)+\sum_{a=1}^{p-1}\chi_4(na)\chi_2(a)\chi_2(a+1)\\
&\quad+\sum_{a=1}^{p-1}\chi_2(na)\chi_2(a)\chi_2(a+1)+\sum_{a=1}^{p-1}\bar{\chi_4}(na)\chi_2(a)\chi_2(a+1)\\
&= \sum_{a=1}^{p-1}\chi_2(1+\bar{a})+\sum_{a=1}^{p-1}\chi_4(na)\chi_2(a)\chi_2(a+1)\\
&\quad+\sum_{a=1}^{p-1}\chi_2(n)\chi_2(a+1)+\sum_{a=1}^{p-1}\bar{\chi_4}(na)\chi_2(a)\chi_2(a+1).\end{aligned} \tag{2}$$

Noting that for any non-principal character χ,

$$\sum_{a=1}^{p-1}\chi(a)=0$$

and

$$\sum_{a=1}^{p-1}\chi(a)\chi(a+1)=\frac{1}{\tau(\bar{\chi})}\sum_{b=1}^{p-1}\sum_{a=1}^{p-1}\bar{\chi}(b)\chi(a)e\left(\frac{b(a+1)}{p}\right).$$

Then we have

$$\sum_{a=1}^{p-1}\chi_2(1+\bar{a})=-1,\quad \sum_{a=1}^{p-1}\chi_2(a+1)=-1,$$

$$\begin{aligned}\sum_{a=1}^{p-1}\chi_4(a)\chi_2(a)\chi_2(a+1)&=\frac{1}{\tau(\chi_2)}\sum_{b=1}^{p-1}\sum_{a=1}^{p-1}\chi_2(b)\chi_4(a)\chi_2(a)e\left(\frac{b(a+1)}{p}\right)\\&=\frac{1}{\tau(\chi_2)}\sum_{b=1}^{p-1}\bar{\chi_4}(b)e\left(\frac{b}{p}\right)\sum_{a=1}^{p-1}\chi_4(ab)\chi_2(ab)e\left(\frac{ab}{p}\right)\\&=\frac{1}{\tau(\chi_2)}\cdot\tau(\bar{\chi_4})\cdot\tau(\chi_4\chi_2).\end{aligned}\tag{3}$$

For any non-principal character ψ, from Lemma 2 we have

$$\tau(\psi^2)=\frac{\psi^2(2)}{\tau(\chi_2)}\cdot\tau(\psi)\cdot\tau(\psi\chi_2).\tag{4}$$

Taking $\psi=\chi_4$, note that

$$\tau(\chi_2)=\sqrt{p},\ \ \tau(\chi_4)\cdot\tau(\bar{\chi_4})=\chi_4(-1)\cdot p,$$

from (3) and (4), we have

$$\begin{aligned}\sum_{a=1}^{p-1}\chi_4(a)\chi_2(a)\chi_2(a+1)&=\frac{\bar{\chi_4}^2(2)\cdot\tau(\chi_4^2)\cdot\tau(\chi_2)\cdot\tau(\bar{\chi_4})}{\tau(\chi_2)\cdot\tau(\chi_4)}\\&=\frac{\chi_2(2)\cdot\tau(\chi_2)\cdot\tau^2(\bar{\chi_4})}{\tau(\chi_4)\cdot\tau(\bar{\chi_4})}\\&=\frac{\chi_2(2)\cdot\sqrt{p}\cdot\tau^2(\bar{\chi_4})}{\chi_4(-1)\cdot p}\\&=\frac{\chi_2(2)\cdot\tau^2(\bar{\chi_4})}{\chi_4(-1)\cdot\sqrt{p}}.\end{aligned}\tag{5}$$

Similarly, we also have

$$\sum_{a=1}^{p-1}\bar{\chi_4}(a)\chi_2(a)\chi_2(a+1)=\frac{\chi_2(2)\cdot\tau^2(\chi_4)}{\chi_4(-1)\cdot\sqrt{p}}.\tag{6}$$

Consider the quadratic character modulo p, we have

$$\left(\frac{2}{p}\right)=\chi_2(2)=\begin{cases}1, & \text{if } \ \mathrm{p}\equiv\pm1 \bmod 8;\\-1, & \text{if } \ \mathrm{p}\equiv\pm3 \bmod 8.\end{cases}\tag{7}$$

And when $p\equiv 1 \bmod 8$, we have $\chi_4(-1)=1$; when $p\equiv 5 \bmod 8$, we have $\chi_4(-1)=-1$.

Combining (2) and (5)–(7) we can deduce that

$$\sum_{a=1}^{p-1}\left(\frac{a^2+n\bar{a}^2}{p}\right) = -1-\chi_2(n)+\frac{1}{\sqrt{p}}\cdot\left(\chi_4(n)\cdot\tau^2(\bar{\chi}_4)+\bar{\chi}_4(n)\cdot\tau^2(\chi_4)\right).$$

This prove Lemma 3.

Lemma 4. *Let p be an odd prime with $p \equiv 1 \bmod 4$. Then for any integer $k \geq 4$ and n with $(n,p)=1$, we have the fourth-order linear recurrence formula*

$$G^k(n) = (4p+2)\cdot G^{k-2}(n) + 8(p-2\alpha^2)\cdot G^{k-3}(n) + [(4\alpha^2-2p)^2-(2p-1)^2]\cdot G^{k-4}(n),$$

where

$$\alpha = \alpha(p) = \frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a^3+a}{p}\right) = \sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\bar{a}}{p}\right),$$

$\left(\frac{*}{p}\right) = \chi_2$ *denotes the Legendre's symbol.*

Proof. For $p \equiv 1 \bmod 4$, any integer n with $(n,p)=1$, and fourth-order character χ_4 modulo p, we have the identity

$$\chi_4^4(n) = \bar{\chi}_4^{\,4}(n) = \chi_0(n), \;\; \chi_4^2(n) = \chi_2(n),$$

where χ_0 denotes the principal character modulo p.

According to Lemma 3,

$$\sum_{a=1}^{p-1}\left(\frac{a^2+n\bar{a}^2}{p}\right) = -1-\chi_2(n)+\frac{1}{\sqrt{p}}\cdot\left(\chi_4(n)\cdot\tau^2(\bar{\chi}_4)+\bar{\chi}_4(n)\cdot\tau^2(\chi_4)\right),$$

$$G(n) = 1+\sum_{a=1}^{p-1}\left(\frac{a^2+n\bar{a}^2}{p}\right).$$

We have

$$G(n) = -\chi_2(n)+\frac{1}{\sqrt{p}}\cdot\left(\chi_4(n)\cdot\tau^2(\bar{\chi}_4)+\bar{\chi}_4(n)\cdot\tau^2(\chi_4)\right), \tag{8}$$

$$\begin{aligned}G^2(n) =& [-\chi_2(n)+\frac{1}{\sqrt{p}}\cdot(\chi_4(n)\cdot\tau^2(\bar{\chi}_4)+\bar{\chi}_4(n)\cdot\tau^2(\chi_4))]^2\\ =&1-2\chi_2(n)\cdot\frac{1}{\sqrt{p}}\cdot\left(\chi_4(n)\cdot\tau^2(\bar{\chi}_4)+\bar{\chi}_4(n)\cdot\tau^2(\chi_4)\right)\\ &+\frac{1}{p}\cdot\left(\chi_2(n)\cdot\tau^4(\bar{\chi}_4)+\chi_2(n)\cdot\tau^4(\chi_4)+2p^2\right)\\ =&1-2\chi_2(n)\cdot\frac{1}{\sqrt{p}}\cdot\left(\chi_4(n)\cdot\tau^2(\bar{\chi}_4)+\bar{\chi}_4(n)\cdot\tau^2(\chi_4)\right)\\ &+\frac{1}{p}\cdot\left(\chi_2(n)\cdot(\tau^4(\bar{\chi}_4)+\tau^4(\chi_4))+2p^2\right).\end{aligned}$$

According to Lemma 1, we have

$$\left(\tau^2(\chi_4) + \tau^2(\bar{\chi_4})\right)^2 = \tau^4(\bar{\chi_4}) + \tau^4(\chi_4) + 2p^2 = 4p\alpha^2.$$

Therefore, we may immediately deduce

$$\begin{aligned} G^2(n) =& 1 - 2(\chi_2(n) \cdot (G(n) + \chi_2(n)) \\ &+ \frac{1}{p}\left(\chi_2(n) \cdot (\tau^4(\bar{\chi_4}) + \tau^4(\chi_4)) + 2p^2\right) \\ =& 1 - 2\chi_2(n) \cdot (G(n) + \chi_2(n)) \\ &+ \frac{1}{p} \cdot \left[\chi_2(n)((\tau^2(\bar{\chi_4}) + \tau^2(\chi_4))^2 - 2p^2) + 2p^2\right] \\ =& 2p - 1 - 2\chi_2(n) \cdot G(n) + \left(4\alpha^2 - 2p\right) \cdot \chi_2(n), \end{aligned} \tag{9}$$

$$\begin{aligned} G^3(n) =& [-\chi_2(n) + \frac{1}{\sqrt{p}} \cdot (\chi_4(n) \cdot \tau^2(\bar{\chi_4}) + \bar{\chi_4}(n) \cdot \tau^2(\chi_4))]^3 \\ =& \left(2p - 1 - 2\chi_2(n) \cdot G(n) + \left(4\alpha^2 - 2p\right) \cdot \chi_2(n)\right) \cdot G(n) \\ =& (4\alpha^2 - 2p)\chi_2(n) \cdot G(n) + (2p + 3)G(n) - (4p - 2)\chi_2(n) - 2(4\alpha^2 - 2p) \end{aligned} \tag{10}$$

and

$$\left[G^2(n) - (2p - 1)\right]^2 = \left[\chi_2(n) \cdot (4\alpha^2 - 2p) - 2\chi_2(n) \cdot G(n)\right]^2,$$

which implies that

$$G^4(n) = (4p + 2) \cdot G^2(n) + 8(p - 2\alpha^2) \cdot G(n) + [(4\alpha^2 - 2p)^2 - (2p - 1)^2]. \tag{11}$$

So for any integer $k \geq 4$, from (8)–(11), we have the fourth-order linear recurrence formula

$$\begin{aligned} G^k(n) &= G^{k-4}(n) \cdot G^4(n) \\ &= (4p + 2) \cdot G^{k-2}(n) + 8(p - 2\alpha^2) \cdot G^{k-3}(n) + [(4\alpha^2 - 2p)^2 - (2p - 1)^2] \cdot G^{k-4}(n). \end{aligned}$$

This proves Lemma 4.

3. Proof of the theorem

In this section, we will complete the proof of our theorem.

Let p be any prime with $p \equiv 1 \bmod 4$, then we have

$$K_0(p) = \frac{1}{p-1}\sum_{n=1}^{p-1} G^0(n) = \frac{p-1}{p-1} = 1. \tag{12}$$

$$K_1(p) = \frac{1}{p-1}\sum_{n=1}^{p-1} G^1(n)$$

$$= \frac{1}{p-1}\sum_{n=1}^{p-1}\left(-\chi_2(n) + \frac{1}{\sqrt{p}}\cdot(\chi_4(n)\tau^2(\bar{\chi_4}) + \bar{\chi_4}(n)\tau^2(\chi_4))\right) \tag{13}$$
$$= 0,$$

$$\begin{aligned} K_2(p) &= \frac{1}{p-1}\sum_{n=1}^{p-1} G^2(n) \\ &= \frac{1}{p-1}\sum_{n=1}^{p-1}\left(-\chi_2(n) + \frac{1}{\sqrt{p}}\cdot(\chi_4(n)\tau^2(\bar{\chi_4}) + \bar{\chi_4}(n)\tau^2(\chi_4))\right)^2 \qquad (14) \\ &= 2p+1, \end{aligned}$$

$$\begin{aligned} K_3(p) &= \frac{1}{p-1}\sum_{n=1}^{p-1} G^3(n) \\ &= \frac{1}{p-1}\sum_{n=1}^{p-1}\left(-\chi_2(n) + \frac{1}{\sqrt{p}}\cdot(\chi_4(n)\tau^2(\bar{\chi_4}) + \bar{\chi_4}(n)\tau^2(\chi_4))\right)^3 \qquad (15) \\ &= -3(4\alpha^2 - 2p). \end{aligned}$$

It is clear that from Lemma 4, if $k \geq 4$, we have

$$\begin{aligned} K_k(p) =& \frac{1}{p-1}\sum_{n=1}^{p-1} G^k(n) \\ =&(4p+2)\cdot K_{k-2}(p) - 8\left(2\alpha^2 - p\right)\cdot K_{k-3}(p) + (16\alpha^4 - 16p\alpha^2 + 4p - 1)\cdot K_{k-4}(p). \qquad (16) \end{aligned}$$

Now Theorem 1 follows (12)–(16). Obviously, using Theorem 1 to all negative integers, and that lead to Corollary 1.

This completes the proofs of our all results.

Some notes:

Note 1: In our theorem, know n is an integer, and $(n, p) = 1$. According to the properties of quadratic residual, $\chi_2(n) = \pm 1, \chi_4(n) = \pm 1$.

Note 2: In our theorem, we only discussed the case $p \equiv 1 \bmod 8$. If $p \equiv 3 \bmod 4$, then the result is trivial. In fact, in this case, for any integer n with $(n, p) = 1$, we have the identity

$$\begin{aligned} G(n) &= 1 + \sum_{a=1}^{p-1}\left(\frac{a^2 + n\bar{a}^2}{p}\right) = 1 + \sum_{a=1}^{p-1}\left(\frac{a^4}{p}\right)\cdot\left(\frac{a^4+n}{p}\right) \\ &= 1 + \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\cdot\left(\frac{a+n}{p}\right) = 1 + \sum_{a=1}^{p-1}\left(\frac{a^2+na}{p}\right) \\ &= 1 + \sum_{a=1}^{p-1}\left(\frac{1+n\bar{a}}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{1+na}{p}\right) = 0. \end{aligned}$$

Thus, for all prime p with $p \equiv 3 \bmod 4$ and $k \geq 1$, we have $K_k(p) = 0$.

4. Conclusions

The main result of this paper is Theorem 1. It gives an interesting computational formula for $K_k(p)$ with $p \equiv 1 \bmod 4$. That is, for any integer k, we have the identity

$$K_k(p) = (4p + 2) \cdot K_{k-2}(p) - 8\left(2\alpha^2 - p\right) \cdot K_{k-3}(p) + (16\alpha^4 - 16p\alpha^2 + 4p - 1) \cdot K_{k-4}(p).$$

Thus, the problems of calculating a linear recurrence formula of one kind special character sums modulo a prime are given.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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