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Research article

On a class of one-dimensional superlinear semipositone (p, q)-Laplacian problem

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Abstract: We study the existence of positive solutions for a class of one-dimensional superlinear (p, q)-Laplacian with Sturm-Liouville boundary conditions. We allow the reaction term to be singular at 0 with infinite semipositone behavior. Our approach depends on Amann's fixed point theorem.

Keywords: (p,q)-Laplacian; superlinear; positive solutions
Mathematics Subject Classification: Primary 34B15; Secondary 34B18

1. Introduction

In this paper, we investigate positive solutions for the one-dimensional BVP

$$\begin{cases} -(\phi_\varepsilon(u'))' = -\lambda g(u) + f(t,u), \ t \in (0,1), \\ au(0) - bu'(0) = 0, \ cu(1) + du'(1) = 0, \end{cases} \tag{1.1}$$

where $\varepsilon \geq 0$, $\phi_\varepsilon(s) = |s|^{p-2}s + \varepsilon|s|^{q-2}s, p > q > 1, a, b, c, d$ are nonnegative constants with $ac + ad + bc > 0$, $f : (0, 1) \times [0, \infty) \to \mathbb{R}$, $g : (0, \infty) \to [0, \infty)$, and λ is a nonnegative parameter.

For $\varepsilon = 0$, $-(\phi_\varepsilon(u'))'$ is the usual p-Laplacian while for $\varepsilon > 0$, the operator is referred to as the (p, q)-Laplacian. We are focusing on the case when $f(\cdot, u)$ is p-superlinear, and g is allowed to exhibit semipositone structure i.e., $-g(0^+) \in [-\infty, 0)$. For a rich literature on semipositone problems and their applications, see [9]. Using Amann's Fixed Point Theorem, we shall establish here the existence of a positive solution to (1.1) for $\lambda \geq 0$ small when $f(\cdot, u)$ is p-superlinear at 0 and ∞, and the superlinearity is involved with the first eigenvalue of the p-Laplacian operator when $\varepsilon = 0$. Our result in the p-Laplacian case improves previous ones in [3, 4, 8, 10, 12] (see Remark 1.1 below), while producing a new existence criteria in the (p,q)-Laplacian case. We refer to [6, 7, 13] and the references therein for related existence results to (1.1) in the superlinear/sublinear cases when $\varepsilon = 0$.

Let λ_1 be the principal eigenvalue of $-(\phi_0(u'))'$ on $(0, 1)$ with Sturm-Liouville boundary condition in (1.1), (see [2, 5]).

We consider the following hypotheses:

(A1) $g : (0, \infty) \to [0, \infty)$ is continuous, non-increasing, and integrable near 0.

(A2) $f : (0, 1) \times [0, \infty) \to \mathbb{R}$ is a Carathéodory function, and there exists $\gamma \in L^1(0, 1)$ such that

$$\inf_{z\in(0,\infty)} \frac{f(t, z)}{z^{p-1}} \geq -\gamma(t),$$

for a.e. $t \in (0, 1)$.

(A3) $\sup\limits_{z\in(0,c)} |f(t, z)|$ is integrable on $(0, 1)$ for all $c > 0$.

(A4) There exists a number $\sigma > 0$ such that

$$f(t, z) \leq \lambda_1 z^{p-1},$$

for $z \in (0, \sigma]$ and a.e. $t \in (0, 1)$, and in addition $f(t, z) \not\equiv \lambda_1 z^{p-1}$ on any subinterval of $[0, \sigma]$ if $\varepsilon = 0$.

(A5) $\lim\limits_{z\to\infty} \frac{f(t,z)}{z^{p-1}} = \infty$ if $\varepsilon > 0$, and $\liminf\limits_{z\to\infty} \frac{f(t,z)}{z^{p-1}} > \lambda_1$ if $\varepsilon = 0$, where the limits are uniform for a.e. $t \in (0, 1)$.

Let $p(t) = \min(t, 1 - t)$. By a positive solution of (1.1), we mean a function $u \in C^1[0, 1]$ with $\inf\limits_{(0,1)} \frac{u}{p} > 0$ and satisfying (1.1).

Our main result is

Theorem 1.1. *Let (A1)–(A5) hold. Then there exists a number* $\lambda_0 > 0$ *such that (1.1) has a positive solution for* $0 \leq \lambda < \lambda_0$.

Remark 1.1. *(i) When* $\varepsilon = 0$, *the existence of a positive solution to (1.1) was established in [3, 4], where* $g \equiv 0$ *with Sturm-Liouville condition in [3], and* $g(u) = u^{-\delta}$, $\delta \in (0, 1)$ *with Dirichlet boundary condition in [4], under the assumption*

$$\limsup_{z\to 0^+} \frac{f(t, z)}{z^{p-1}} < \lambda_1 < \liminf_{z\to\infty} \frac{f(t, z)}{z^{p-1}}.$$

The results in [3, 4] provided extensions of the work in [8, 10, 12]. Note that our condition (A4) allows the case $\limsup\limits_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} = \lambda_1$.

(ii) In the case $bd = 0$, *the proof of Theorem 1.1 shows that when* $\varepsilon = 0$, *(A4) can be replaced by the weaker condition* $f(t, z) \leq \lambda_1 z^{p-1}$ *and* $f(t, z) \not\equiv \lambda_1 z^{p-1}$ *on* $[0, \sigma]$ *for a.e.* $t \in (0, 1)$.

Example 1.1. *Let* $\delta, \nu \in (0, 1)$ *with* $\delta + \nu < 1$, *and* $r > p - 1$. *By Theorem 1.1, the following problems have a positive solution for* $\lambda \geq 0$ *small:*

(i)

$$\begin{cases} -(\phi_\varepsilon(u'))' = -\frac{\lambda}{u^\delta \ln^\nu(1+u)} + \lambda_1 u^{p-1} + u^r - u^s, \ t \in (0, 1), \\ au(0) - bu'(0) = 0, \ cu(1) + du'(1) = 0, \end{cases}$$

where $\varepsilon > 0$ *and* $r > s > p - 1$. *Indeed, here*

$$f(t, z) = \lambda_1 z^{p-1} + z^r - z^s \leq \lambda_1 z^{p-1} \ \textit{for} \ z \leq 1,$$

i.e., (A4) holds. Since

$$z^{1-p}f(t,z) = \lambda_1 + z^{r-(p-1)} - z^{s-(p-1)} \geq \lambda_1 - 1$$

for $z \in (0,\infty)$, *(A2) holds. Clearly (A1), (A3), and (A5) are satisfied.*

(ii)

$$\begin{cases} -(\phi_0(u'))' = -\frac{\lambda}{u^\delta \ln^\nu(1+u)} + \lambda_1 u^{p-1}e^{-u^\alpha} + u^r, \ t \in (0,1), \\ au(0) - bu'(0) = 0, \ cu(1) + du'(1) = 0, \end{cases}$$

where $\alpha \in (0, r-p+1)$. *Note that (A4) with* $\varepsilon = 0$ *is equivalent to*

$$\lambda_1(1 - e^{-z^\alpha}) \geq z^{r-(p-1)}$$

on $[0,\sigma]$ *and* $\lambda_1(1-e^{-z^\alpha}) \not\equiv z^{r-(p-1)}$ *on any subinterval of* $[0,\sigma]$ *for some* $\sigma > 0$. *This is true since* $\lim\limits_{z\to 0^+} \frac{1-e^{-z^\alpha}}{z^{r-p+1}} = \infty$. *Clearly the remaining conditions are satisfied.*

Note that $\limsup\limits_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} = \lambda_1$ *in both examples.*

2. Preliminaries

Let $0 \leq \alpha < \beta \leq 1$. In what follows, $\gamma \in L^1(\alpha,\beta)$ with $\gamma \geq 0$ and we shall denote the norm in $L^q(\alpha,\beta)$ and $C^1[\alpha,\beta]$ by $\|\cdot\|_q$ and $|\cdot|_1$ respectively.

Lemma 2.1. *Let* $u, v \in C^1[\alpha,\beta]$ *satisfy*

$$\begin{cases} -(\phi_\varepsilon(u'))' + \gamma(t)\phi_\varepsilon(u) \geq -(\phi_\varepsilon(v'))' + \gamma(t)\phi_\varepsilon(v) \quad a.e \ \ on \ (\alpha,\beta), \\ au(\alpha) - bu'(\alpha) \geq av(\alpha) - bv'(\alpha), \ cu(\beta) + du'(\beta) \geq cv(\beta) + dv'(\beta). \end{cases} \tag{2.1}$$

Then $u \geq v$ *on* $[\alpha,\beta]$.

Proof. Suppose $u(t_0) < v(t_0)$ for some $t_0 \in (\alpha,\beta)$. Let $I = (\alpha_0,\beta_0) \subset (\alpha,\beta)$ be the largest open interval containing t_0 such that $u < v$ on I. Then $u(\alpha_0) = v(\alpha_0)$ if $\alpha_0 > \alpha$ and $u(\beta_0) = v(\beta_0)$ if $\beta_0 < \beta$. Multiplying the inequation in (2.1) by $u - v$ and integrating on I gives

$$\int_I (\phi_\varepsilon(u') - \phi_\varepsilon(v'))(u' - v') \leq 0,$$

since $\gamma \geq 0$ and $-(\phi_\varepsilon(u') - \phi_\varepsilon(v')(u-v)|_{\alpha_0}^{\beta_0} \geq 0$ in view of the boundary conditions at α,β. Since ϕ_ε is increasing, it follows that $u' = v'$ on I and hence $u = v + \sigma$ on I, where σ is a negative constant. If $\alpha_0 > \alpha$ or $\beta_0 < \beta$ then $\sigma = 0$, a contradiction. On the other hand, if $\alpha_0 = \alpha$ and $\beta_0 = \beta$ then the boundary conditions in (2.1) gives $a\sigma, c\sigma \geq 0$ and thus $a = c = 0$, a contradiction and hence the result follows. □

Lemma 2.2. *Let* $k \in L^1(\alpha,\beta)$. *Then the problem*

$$\begin{cases} -(\phi_\varepsilon(z'))' + \gamma(t)\phi_\varepsilon(z) = k(t) \ \ on \ (\alpha,\beta), \\ az(\alpha) - bz'(\alpha) = 0, \ cz(\beta) + dz'(\beta) = 0 \end{cases} \tag{2.2}$$

has a unique solution $z \equiv T_\varepsilon k \in C^1[\alpha,\beta]$ *with*

$$|z|_1 \leq K\phi_\varepsilon^{-1}(\|k\|_1), \tag{2.3}$$

where the constant K is independent of $k, \alpha, \beta, z, \varepsilon$. *In addition, the map* $T_\varepsilon : L^1(\alpha,\beta) \to C[\alpha,\beta]$ *is completely continuous.*

Proof. Suppose first that $\gamma \equiv 0$.

By integrating, we see that the solution of (2.2) is given by

$$z(t) = C_1 - \int_\alpha^t \phi_\varepsilon^{-1}\left(C + \int_\alpha^s k\right) ds, \tag{2.4}$$

where the constants C, C_1 satisfy

$$\begin{cases} aC_1 + b\phi_\varepsilon^{-1}(C) = 0, \\ c\left(C_1 - \int_\alpha^\beta \phi_\varepsilon^{-1}\left(C + \int_\alpha^s k\right) ds\right) - d\phi_\varepsilon^{-1}\left(C + \int_\alpha^\beta k\right) = 0. \end{cases} \tag{2.5}$$

Note that (2.5) has a unique solution (C, C_1) since if $a = 0$ then $C = 0$ and

$$C_1 = \frac{d}{c}\phi_\varepsilon^{-1}\left(\int_\alpha^\beta k\right) + \int_\alpha^\beta \phi_\varepsilon^{-1}\left(\int_\alpha^s k\right) ds, \tag{2.6}$$

while if $a > 0$ then $C_1 = -\frac{b}{a}\phi_\varepsilon^{-1}(C)$, where C is the unique solution of

$$g_\varepsilon(C) \equiv bc\phi_\varepsilon^{-1}(C) + ac\int_\alpha^\beta \phi_\varepsilon^{-1}\left(C + \int_\alpha^s k\right) ds + ad\phi_\varepsilon^{-1}\left(C + \int_\alpha^\beta k\right) = 0. \tag{2.7}$$

Indeed, $g_\varepsilon(C) > 0$ for $C > \|k\|_1$ and $g_\varepsilon(C) < 0$ for $C < -\|k\|_1$. Thus (2.7) has a unique solution C with $|C| \le \|k\|_1$ since g_ε is continuous and increasing.

Using the inequality (see Proposition A(ii) in Appendix)

$$\phi_\varepsilon^{-1}(mx) \le m^{\frac{1}{q-1}}\phi_\varepsilon^{-1}(x)$$

for $m \ge 1$, $x \ge 0$, and (2.4)–(2.6), we get

$$|z(t)| + |z'(t)| \le |C_1| + 2\phi_\varepsilon^{-1}(2\|k\|_1) \le \left(c_0 + 2^{\frac{q}{q-1}}\right)\phi_\varepsilon^{-1}(\|k\|_1),$$

for all $t \in [\alpha, \beta]$, where $c_0 = (d/c + 1)$ if $a = 0$, $c_0 = b/a$ if $a > 0$, from which (2.3) follows.

Next, we consider the general case $\gamma \in L^1(\alpha, \beta)$ with $\gamma \ge 0$. In view of the above, there exist $z_1, z_2 \in C^1[\alpha, \beta]$ satisfying

$$-(\phi_\varepsilon(z_1'))' = -|k(t)| \text{ on } (\alpha, \beta), \quad -(\phi_\varepsilon(z_2'))' = |k(t)| \text{ on } (\alpha, \beta),$$

with Sturm-Liouville boundary conditions.

By Lemma 2.1, $z_1 \le 0 \le z_2$ on (α, β), which implies

$$-(\phi_\varepsilon(z_1'))' + \gamma(t)\phi_\varepsilon(z_1) \le -|k(t)| \le k(t) \text{ on } (\alpha, \beta)$$

and

$$-(\phi_\varepsilon(z_2'))' + \gamma(t)\phi_\varepsilon(z_2) \ge |k(t)| \ge k(t) \text{ on } (\alpha, \beta),$$

i.e., (z_1, z_2) is a pair of sub- and supersolution of (2.2) with $z_1 \le z_2$ on (α, β). Thus (2.2) has a solution $z \in C^1[\alpha, \beta]$ with $z_1 \le z \le z_2$ on (α, β). The solution is unique due to Lemma 2.1.

Since

$$-(\phi_\varepsilon(z'))' = k(t) - \gamma(t)\phi_\varepsilon(z) \text{ on } (\alpha,\beta)$$

and $\|z\|_\infty \le \max(\|z_1\|_\infty, \|z_2\|_\infty) \le K\phi_\varepsilon^{-1}(\|k\|_1)$ in view of (2.3) when $\gamma = 0$, it follows that

$$\|k(t) - \gamma(t)\phi_\varepsilon(z)\|_1 \le \|k\|_1 + \|\gamma\|_1\phi_\varepsilon(K_1\phi_\varepsilon^{-1}(\|k\|_1)) \le K_2\|k\|_1,$$

where $K_1 = \max(K, 1)$ and $K_2 = 1 + K_1^{p-1}\|\gamma\|_1$. Here we have used Proposition A(iii) in Appendix. Consequently, it is

$$|z|_1 \le K\phi_\varepsilon^{-1}(K_2\|k\|_1) \le KK_2^{\frac{1}{q-1}}\phi_\varepsilon^{-1}(\|k\|_1),$$

where we have used Proposition A(ii) in Appendix. Thus (2.3) holds. Next, we verify that T_ε is continuous. Let $(k_n) \subset L^1(\alpha,\beta)$ and $k \in L^1(\alpha,\beta)$ be such that $\|k_n - k\|_1 \to 0$. Let $u_n = T_\varepsilon k_n$ and $u = T_\varepsilon k$.

Multiplying the equation

$$-(\phi_\varepsilon(u_n') - \phi_\varepsilon(u'))' + \gamma(t)(\phi_\varepsilon(u_n) - \phi_\varepsilon(u)) = k_n - k \ \text{ on } (\alpha,\beta)$$

by $u_n - u$ and integrating between α and β, we obtain

$$c_n + \int_\alpha^\beta (\phi_\varepsilon(u_n') - \phi_\varepsilon(u'))(u_n' - u') \le \|k_n - k\|_1\|u_n - u\|_\infty, \tag{2.8}$$

where $c_n = -(\phi_\varepsilon(u_n') - \phi_\varepsilon(u')(u_n - u)|_\alpha^\beta \ge 0$. By [11, Lemma 30],

$$(\phi_\varepsilon(x) - \phi_\varepsilon(y))(x - y) \ge (|x|^{p-2}x - |y|^{p-2}y)(x - y) \ge c_0|x - y|^{\max(p,2)} \tag{2.9}$$

for all $x, y \in \mathbb{R}$ with $|x|+|y| \le 2M$, where $c_0 > 0$ is a constant depending only on p and M. Applying (2.9) with $x = u_n', y = u'$ and note that $|u_n|_1 + |u|_1 \le 2M$, where $M = K\max\left(\phi^{-1}(\|k_n\|_1), \phi^{-1}(\|k\|_1)\right)$, we obtain from (2.8) that

$$c_n + c_0\int_\alpha^\beta |u_n' - u'|^{\max(p,2)} \le 2M\|k_n - k\|_1 \to 0 \text{ as } n \to \infty. \tag{2.10}$$

If $b = 0$ then $(u_n - u)(\alpha) = 0$ and the Mean Value Theorem implies that

$$|u_n(t) - u(t)| \le \left|\int_\alpha^t |u_n' - u'|\right| \le \left(\int_\alpha^\beta |u_n' - u'|^{\max(p,2)}\right)^{\frac{1}{\max(p,2)}}$$

for $t \in [\alpha,\beta]$. Hence $\|u_n - u\|_\infty \to 0$ as $n \to \infty$ in view of (2.10).

If $b > 0$ then $u_n'(\alpha) = \frac{a}{b}u_n(\alpha), u'(\alpha) = \frac{a}{b}u(\alpha)$, and since (2.9) gives

$$\frac{bc_0}{a}\left(\frac{a}{b}\right)^{\max(p,2)}|u_n(\alpha) - u(\alpha)|^{\max(p,2)} \le \left(\phi_\varepsilon\left(\frac{a}{b}u_n(\alpha)\right) - \phi_\varepsilon\left(\frac{a}{b}u(\alpha)\right)\right)(u_n(\alpha) - u(\alpha)) \le c_n,$$

it follows from the Mean Value Theorem and (2.10) that

$$\|u_n - u\|_\infty \le |u_n(\alpha) - u(\alpha)| + \left(\int_\alpha^\beta |u_n' - u'|^{\max(p,2)}\right)^{\frac{1}{\max(p,2)}} \to 0$$

as $n \to \infty$. Hence T_ε is continuous. Since (u_n) is bounded in $C^1[\alpha,\beta]$, T_ε is completely continuous, which completes the proof. □

Lemma 2.3. *Let $k \in L^1(0,1)$ with $k \geq 0$, and $u \in C^1[0,1]$ satisfy*

$$\begin{cases} -(\phi_\varepsilon(u'))' + \gamma(t)\phi_\varepsilon(u) \geq -k(t) \;\; on\ (0,1), \\ au(0) - bu'(0) \geq 0,\;\; cu(1) + du'(1) \geq 0. \end{cases}$$

Then there exist constants $\kappa, C > 0$ independent of u, k, ε such that if $\|u\|_\infty \geq C\phi_\varepsilon^{-1}(\|k\|_1)$ then

$$u(t) \geq \kappa \|u\|_\infty p(t)$$

for $t \in [0,1]$.

Proof. Let $v \in C^1[0,1]$ satisfy

$$\begin{cases} -((\phi_\varepsilon(v'))' + \gamma(t)\phi_\varepsilon(v) = -k(t) \;\; \text{on } (0,1), \\ av(0) - bv'(0) = 0,\;\; cv(1) + dv'(1) = 0. \end{cases}$$

By Lemma 2.2, $|v|_1 \leq K\phi_\varepsilon^{-1}(\|k\|_1)$, where K is independent of k. By Lemma 2.1, $u \geq v$ on $[0,1]$. Suppose $\|u\|_\infty > K\phi_\varepsilon^{-1}(\|k\|_1)$, and $\|u\|_\infty = |u(\tau)|$ for some $\tau \in [0,1]$. Then $u(\tau) > 0$ because if $u(\tau) \leq 0$ then $\|u\|_\infty = -u(\tau) \leq -v(\tau) \leq K\phi_\varepsilon^{-1}(\|k\|_1)$, a contradiction. In what follows, we may increase K without mentioning if needed.

Suppose first that $\tau \in (0,1)$. Let $z \in C^1[0,\tau]$ satisfying

$$\begin{cases} -(\phi_\varepsilon(z'))' + \gamma(t)\phi_\varepsilon(z) = -k(t) \;\; \text{on } (0,\tau), \\ az(0) - bz'(0) = 0,\;\; z(\tau) = \|u\|_\infty. \end{cases} \tag{2.11}$$

Note that z_0 is a subsolution of (2.11) and $z_0 + \|u\|_\infty$ is a supersolution of (2.11), where z_0 satisfies

$$\begin{cases} -(\phi_\varepsilon(z_0'))' + \gamma(t)\phi_\varepsilon(z_0) = -k(t) \;\; \text{on } (0,\tau), \\ az_0(0) - bz_0'(0) = 0,\;\; z_0(\tau) = 0, \end{cases}$$

from which the existence of z follows. By Lemma 2.1, $u \geq z \geq v \geq -K\phi_\varepsilon^{-1}(\|k\|_1)$ on $[0,\tau]$. Define $z_1(t) = z(t) + K\phi_\varepsilon^{-1}(\|k\|_1)$. Then $z_1 \geq 0$ on $[0,1]$ and

$$z_1'(0) \geq -K_1\phi_\varepsilon^{-1}(\|k\|_1),$$

where $K_1 = K$ if $b = 0$ and $K_1 = K(1 + a/b)$ if $b > 0$. Indeed, if $b = 0$ then $z(0) = v(0) = 0$ and so $z_1'(0) = z'(0) \geq v'(0) \geq -K\phi_\varepsilon^{-1}(\|k\|_1)$, while if $b > 0$ then $z_1'(0) = (a/b)z(0) \geq -K(a/b)\phi_\varepsilon^{-1}(\|k\|_1)$.

Since $z \leq z_1$ on $(0,\tau)$ and $z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1) \geq 0$, it follows upon integrating the equation

$$(\phi_\varepsilon(z_1'))' = \gamma(t)\phi_\varepsilon(z) + k(t) \text{ on } (0,\tau)$$

that

$$\begin{aligned} z_1(t) &= z_1(0) + \int_0^t \phi_\varepsilon^{-1}\left(\phi_\varepsilon(z_1'(0)) + \int_0^s (\gamma(\xi)\phi_\varepsilon(z) + k(\xi))d\xi\right)ds \\ &\leq z_1(0) + \int_0^t \phi_\varepsilon^{-1}\left(\phi_\varepsilon(z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1)) + \int_0^s (\gamma(\xi)\phi_\varepsilon(z_1) + k(\xi))d\xi\right)ds \\ &\leq z_1(0) + \int_0^t \phi_\varepsilon^{-1}\left(\phi_\varepsilon(z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1)) + \int_0^t (\gamma(\xi)\phi_\varepsilon(z_1) + k(\xi))d\xi\right)ds \end{aligned}$$

$$\leq z_1(0) + \phi_\varepsilon^{-1}\left(\phi_\varepsilon(z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1)) + \int_0^t (\gamma(\xi)\phi_\varepsilon(z_1) + k(\xi))d\xi\right). \tag{2.12}$$

Applying ϕ_ε on both sides of (2.12) and using the inequality (see Proposition A(i) in Appendix)

$$\phi_\varepsilon(x+y) \leq M(\phi_\varepsilon(x) + \phi_\varepsilon(y)) \quad \forall x, y \geq 0,$$

where $M = 2^{\max(p-2,0)}$, we obtain

$$\phi_\varepsilon(z_1(t)) \leq M[\phi_\varepsilon(z_1(0)) + \phi_\varepsilon(z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1)) + \|k\|_1] + M\int_0^t \gamma(\xi)\phi_\varepsilon(z_1)d\xi.$$

By Gronwall's inequality,

$$\phi_\varepsilon(z_1(t)) \leq M[\phi_\varepsilon(z_1(0)) + \phi_\varepsilon(z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1)) + \|k\|_1]e^{M\|\gamma\|_1}$$

for $t \in [0, \tau]$. In particular when $t = \tau$, we obtain

$$\phi_\varepsilon(z_1(0)) + \phi_\varepsilon(z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1)) + \|k\|_1 \geq 2K_2\phi_\varepsilon(\|u\|_\infty),$$

where $K_2 = (2M)^{-1}e^{-M\|\gamma\|_1}$. Since $\phi_\varepsilon(x) + \phi_\varepsilon(y) \leq 2\phi_\varepsilon(x+y)$ for $x, y \geq 0$, this implies

$$\phi_\varepsilon(z_1(0) + z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1)) \geq K_2\phi_\varepsilon(\|u\|_\infty) - \frac{\|k\|_1}{2} \geq K_3\phi_\varepsilon(\|u\|_\infty) \geq \phi_\varepsilon\left(K_4\|u\|_\infty\right),$$

where $K_3 = K_2/2 < 1$ and $K_4 = K_3^{\frac{1}{q-1}}$, provided that $\phi_\varepsilon(\|u\|_\infty) \geq \|k\|_1/K_2$ which is true if $\|u\|_\infty \geq (1/K_2)^{\frac{1}{q-1}}\phi_\varepsilon^{-1}(\|k\|_1)$. Consequently,

$$z_1(0) + z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1) \geq K_4\|u\|_\infty,$$

which implies

$$z(0) + z'(0) \geq K_4\|u\|_\infty - (K + K_1)\phi_\varepsilon^{-1}(\|k\|_1) \geq K_5\|u\|_\infty, \tag{2.13}$$

where $K_5 = K_4/2$, provided that $\|u\|_\infty \geq \frac{2(K+K_1)}{K_4}\phi^{-1}(\|k\|_1)$. Since

$$(\phi_\varepsilon(z'))' = \gamma(t)\phi_\varepsilon(z) + k \;\; \geq -\gamma(t)\phi_\varepsilon(K\phi_\varepsilon^{-1}(\|k\|_1)) \;\; \geq -K^{\frac{1}{p-1}}\|k\|_1\gamma(t) \;\text{ on } (0, \tau),$$

it follows that

$$\phi_\varepsilon(z'(t)) \geq \phi_\varepsilon(z'(0)) - K^{\frac{1}{p-1}}\|k\|_1\|\gamma\|_1 \tag{2.14}$$

for $t \in [0, \tau]$. If $b = 0$ then $z(0) = 0$ and (2.13) becomes $z'(0) \geq K_5\|u\|_\infty$, from which (2.14) implies

$$\phi_\varepsilon(z'(t)) \geq \phi_\varepsilon(K_5\|u\|_\infty) - K^{\frac{1}{p-1}}\|\gamma\|_1\|k\|_1 \geq \frac{\phi_\varepsilon(K_5\|u\|_\infty)}{2} \geq \phi_\varepsilon(K_6\|u\|_\infty),$$

where $K_6 = 2^{1-q}K_5$, provided that $\phi_\varepsilon(K_5\|u\|_\infty) \geq 2K^{\frac{1}{p-1}}\|\gamma\|_1\|k\|_1$ which is true if $\|u\|_\infty \geq K_5^{-1}\left(2K^{\frac{1}{p-1}}\|\gamma\|_1\right)^{\frac{1}{q-1}}\phi_\varepsilon^{-1}(\|k\|_1)$. Consequently,

$$z'(t) \geq K_6\|u\|_\infty \text{ on } (0, \tau),$$

which implies upon integrating that

$$u(t) \geq z(t) \geq K_6\|u\|_\infty t \quad \text{for } t \in [0, \tau]. \tag{2.15}$$

If $b > 0$ then $z'(0) = (a/b)z(0)$ and (2.13) becomes

$$z(0) \geq \frac{K_5 b}{a+b}\|u\|_\infty. \tag{2.16}$$

Since $z'(0) \geq 0$, (2.14) gives

$$z'(t) \geq -\phi_\varepsilon^{-1}\left(K^{\frac{1}{p-1}}\|\gamma\|_1\|k\|_1\right) \geq -\tilde{K}\phi_\varepsilon^{-1}(\|k\|_1) \text{ on } (0, \tau),$$

where $\tilde{K} = \left(K^{\frac{1}{p-1}}\|\gamma\|_1\right)^{\frac{1}{q-1}}$. This, together with (2.16), implies

$$z(t) \geq z(0) - \tilde{K}\phi_\varepsilon^{-1}(\|k\|_1) \geq \frac{K_5 b}{a+b}\|u\|_\infty - \tilde{K}\phi_\varepsilon^{-1}(\|k\|_1).$$

Hence

$$u(t) \geq z(t) \geq K_7\|u\|_\infty \text{ for } t \in [0, \tau], \tag{2.17}$$

where $K_7 = \frac{K_5 b}{2(a+b)}$, provided that $\|u\|_\infty \geq \frac{2\tilde{K}(a+b)}{K_5 b}\phi_\varepsilon^{-1}(\|k\|_1)$.

Combining (2.15) and (2.17), we obtain

$$u(t) \geq \kappa_0\|u\|_\infty t, \quad \forall t \in [0, \tau], \tag{2.18}$$

where $\kappa_0 = \min(K_6, K_7)$.

Next, let $w \in C^1[\tau, 1]$ be the solution of

$$\begin{cases} -(\phi_\varepsilon(w'))' + \gamma(t)\phi_\varepsilon(w) = -k(t) \text{ on } (\tau, 1), \\ w(\tau) = \|u\|_\infty, \quad cw(1) + dw'(1) = 0. \end{cases}$$

Then $u \geq w$ on $[\tau, 1]$, and using similar arguments as above, we obtain

$$u(t) \geq \kappa_1\|u\|_\infty(1-t) \ \ \forall t \in [\tau, 1], \tag{2.19}$$

where $\kappa_1 > 0$ is a constant independent of k, provided that $\|u\|_\infty > C\phi_\varepsilon^{-1}(\|k\|)$ for some large constant C independent of u.

Combining (2.18) and (2.19), we see that Lemma 2.3 holds with $\kappa = \min(\kappa_0, \kappa_1)$. If $\tau = 0$ then (2.19) holds on $[0, 1]$, and if $\tau = 1$ then (2.17) holds on $[0, 1]$, which completes the proof. □

3. Proof of the main result

Let $E = C[0, 1]$ be with the usual sup-norm.

Proof of Theorem 1.1. Let C, κ be given by Lemma 2.3 and define $\sigma_0 = \kappa\sigma, h(t) = g(\sigma_0 p(t))$. For $v \in E$, $g(\max(v, \sigma_0 p)) \in L^1(0, 1)$ by (A1), and $0 \leq f(t, |v|) + \gamma(t)\phi_\varepsilon(|v|) \in L^1(0, 1)$ by (A2) and (A3). Let $\lambda \geq 0$ be small so that $C\phi_\varepsilon^{-1}(\lambda\|h\|_1) < \sigma$. Then the problem

$$\begin{cases} -(\phi_\varepsilon(u'))' + \gamma(t)\phi_\varepsilon(u) = -\lambda g(\max(v, \sigma_0 p)) + f(t, |v|) + \gamma(t)\phi_\varepsilon(|v|) \text{ on } (0, 1), \\ au(0) - bu'(0) = 0, \ cu(1) + du'(1) = 0 \end{cases}$$

has a unique solution $u = A_\varepsilon v \in C^1[0,1]$ in view of Lemma 2.2. Since the operator $S : E \to L^1(0,1)$ defined by $(Sv)(t) = -\lambda g(\max(v, \sigma_0 p)) + f(t, |v|) + \gamma(t)|v|^{p-1}$ is continuous, it follows from Lemma 2.2 that $A_\varepsilon : E \to E$ is completely continuous. We shall verify that
(i) $u = \theta A_\varepsilon u,\ \theta \in (0,1] \Longrightarrow \|u\|_\infty \neq \sigma$.

Let $u \in E$ satisfy $u = \theta A_\varepsilon u$ for some $\theta \in (0,1]$ with $\|u\|_\infty = \sigma$.

Suppose $\varepsilon > 0$. Then

$$-\left(\phi_\varepsilon\left(\frac{u'}{\theta}\right)\right)' + \gamma(t)\phi_\varepsilon\left(\frac{u}{\theta}\right) = -\lambda g(\max(u, \sigma_0 p(t))) + f(t, |u|) + \gamma(t)\phi_\varepsilon(|u|)$$

on $(0,1)$, which implies upon multiplying by θ^{p-1} that

$$\begin{aligned} -(\phi_{\varepsilon\theta^{p-q}}(u'))' + \gamma(t)\phi_{\varepsilon\theta^{p-q}}(u) &= \theta^{p-1}(-\lambda g(\max(u, \sigma_0 p(t))) + f(t, |u|) + \gamma(t)\phi_\varepsilon(|u|)) \\ &\geq -\lambda h(t) \quad \text{on } (0,1). \end{aligned} \tag{3.1}$$

Since $\|u\|_\infty > C\phi_\varepsilon^{-1}(\lambda\|h\|_1)$, Lemma 2.3 gives

$$u(t) \geq \kappa\|u\|_\infty p(t) \geq \sigma_0 p(t) > 0$$

for $t \in (0,1)$ (recall that $\kappa\sigma = \sigma_0$). Hence it follows from (3.1) and (A4) that

$$\begin{aligned} &-(\phi_{\varepsilon\theta^{p-q}}(u'))' = \theta^{p-1} f(t,u) - \lambda\theta^{p-1} g(u)) + \theta^{p-1}\gamma(t)\phi_\varepsilon(u) - \gamma(t)\phi_{\varepsilon\theta^{p-q}}(u) \\ &= \theta^{p-1} f(t,u) - \lambda\theta^{p-1} g(u) + \gamma(t)(\theta^{p-1} - 1)u^{p-1} + \varepsilon\gamma(t)(\theta^{p-1} - \theta^{p-q})u^{q-1} \\ &\leq \theta^{p-1} f(t,u) \leq \theta^{p-1}\lambda_1 u^{p-1} \end{aligned} \tag{3.2}$$

on $(0,1)$. Multiplying (3.2) by u and integrating gives

$$-\phi_{\varepsilon\theta^{p-q}}(u'(1))u(1) + \phi_{\varepsilon\theta^{p-q}}(u'(0))u(0) + \int_0^1 \phi_{\varepsilon\theta^{p-q}}(u')u' \leq \lambda_1 \int_0^1 u^p.$$

Since $au(0) - bu'(0) = 0 = cu(1) + du'(1)$ and $\varepsilon > 0$, this implies

$$-\phi_0(u'(1))u(1) + \phi_0(u'(0))u(0) + \int_0^1 |u'|^p < \lambda_1 \int_0^1 u^p, \tag{3.3}$$

Consequently,

$$\lambda_1 > \frac{-\phi_0(u'(1))u(1) + \phi_0(u'(0))u(0) + \int_0^1 |u'|^p}{\int_0^1 u^p}.$$

Since λ_1 is characterized by the Raleigh formula

$$\lambda_1 = \inf_{v \in V} \frac{-\phi_0(v'(1))v(1) + \phi_0(v'(0))v(0) + \int_0^1 |v'|^p}{\int_0^1 |v|^p}, \tag{3.4}$$

where $V = \{u \in C^1[0,1] : au(0) - bu'(0) = 0 = cu(1) + du'(1)\}$, we get a contradiction. Thus (i) holds.

Next, suppose $\varepsilon = 0$. Then the $<$ inequality in (3.3) is replaced by $\le$, which together with (3.4) imply

$$\lambda_1 = \frac{-\phi_0(u'(1))u(1) + \phi_0(u'(0))u(0) + \int_0^1 |u'|^p}{\int_0^1 |u|^p},$$

i.e., u is an eigenfunction corresponding to λ_1. Hence (3.2) gives

$$\lambda_1 u^{p-1} \le \theta^{p-1} f(t,u) \le \theta^{p-1}\lambda_1 u^{p-1} \le \lambda_1 u^{p-1} \text{ on } (0,1),$$

from which it follows that $f(t,u) = \lambda_1 u^{p-1}$ for a.e. $t \in (0,1)$. Since $\|u\|_\infty = \sigma$, we get a contradiction with (A4) with $\varepsilon = 0$. If $bd = 0$, then $u(0) = 0$ or $u(1) = 0$, and since $\|u\|_\infty = \sigma$, we have $u[0,1] = [0,\sigma]$, we get a contradiction if $f(t,z) \not\equiv \lambda_1 z^{p-1}$ on $[0,\sigma]$ for a.e. $t \in (0,1)$. Thus (i) holds.

Next, we verify that

(ii) There exists a constant $R > \sigma$ such that $u = A_\varepsilon u + \xi,\ \xi \ge 0 \Longrightarrow \|u\|_\infty \ne R$.

Let $u \in E$ satisfy $u = A_\varepsilon u + \xi$ for some $\xi \ge 0$. Then u satisfies

$$-(\phi_\varepsilon(u'))' + \gamma(t)\phi_\varepsilon(u-\xi) = -\lambda g(\max(u,\sigma_0 p(t))) + f(t,|u|) + \gamma(t)\phi_\varepsilon(|u|) \tag{3.5}$$

on $(0,1)$, which implies

$$-(\phi_\varepsilon(u'))' + \gamma(t)\phi_\varepsilon(u) \ge -\lambda h(t) \tag{3.6}$$

on $(0,1)$. Note that

$$au(0) - bu'(0) = a\xi \ge 0,\quad cu(1) + du'(1) = c\xi \ge 0. \tag{3.7}$$

Suppose $\|u\|_\infty = R > \sigma$. Then Lemma 2.3 gives

$$u(t) \ge \kappa\|u\|_\infty p(t) \ge \kappa R p(t) \ge \sigma_0 p(t) \tag{3.8}$$

for $t \in (0,1)$. Using (3.8) in (3.5), we get

$$-(\phi_\varepsilon(u'))' \ge -\lambda g(u) + f(t,u) \text{ on } (0,1). \tag{3.9}$$

Suppose $\varepsilon > 0$ and let $M > 0$. Since $\lim_{z\to\infty} \frac{f(t,z)-\lambda g(z)}{\phi_\varepsilon(z)} = \infty$ by (A1) and (A5), there exists a positive constant L such that

$$f(t,z) - \lambda g(z) \ge M\phi_\varepsilon(z) \tag{3.10}$$

for a.e. $t \in (0,1)$ and $z > L$. By (3.8),

$$u(t) \ge \frac{\kappa}{4}\|u\|_\infty = \frac{\kappa R}{4} > L \text{ for } t \in [1/4,3/4]$$

for R large, from which (3.9) and (3.10) imply

$$-(\phi_\varepsilon(u'))' \ge M\phi_\varepsilon(u) \ge M\phi_\varepsilon\left(\frac{\kappa\|u\|_\infty}{4}\right) \quad \text{on } [1/4,3/4].$$

Since $u(1/4)$ and $u(3/4)$ are positive, the comparison principle gives $u \ge \tilde{u}$ on $[1/4,3/4]$, where $\tilde{u}$ is the solution of

$$\begin{cases} -(\phi_\varepsilon(\tilde{u}'))' = M\phi_\varepsilon\left(\frac{\kappa\|u\|_\infty}{4}\right) \text{ on } (1/4,3/4), \\ \tilde{u}(1/4) = \tilde{u}(3/4) = 0. \end{cases}$$

Let $\|\tilde{u}\|_\infty = \tilde{u}(\tau)$ for some $\tau \in (1/4, 3/4)$. If $\tau \leq 1/2$ then we have

$$\|u\|_\infty \geq \tilde{u}(5/8)) = \int_{5/8}^{3/4} \phi_\varepsilon^{-1}\left(M\phi_\varepsilon\left(\frac{\kappa\|u\|_\infty}{4}\right)(s-\tau)\right)ds \geq \frac{1}{8}\phi_\varepsilon^{-1}\left(\frac{M}{8}\phi_\varepsilon\left(\frac{\kappa\|u\|_\infty}{4}\right)\right),$$

while if $\tau > 1/2$,

$$\|u\|_\infty \geq \tilde{u}(3/8) = \int_{1/4}^{3/8} \phi_\varepsilon^{-1}\left(M\phi_\varepsilon\left(\frac{\kappa\|u\|_\infty}{4}\right)(\tau-s)\right)ds \geq \frac{1}{8}\phi_\varepsilon^{-1}\left(\frac{M}{8}\phi_\varepsilon\left(\frac{\kappa\|u\|_\infty}{4}\right)\right).$$

Hence using Proposition A(iii) we see that in either case,

$$\phi_\varepsilon(8\|u\|_\infty) \geq \frac{M}{8}\phi_\varepsilon\left(\frac{\kappa\|u\|_\infty}{4}\right) \geq \phi_\varepsilon\left(\left(\frac{M}{8}\right)^{\frac{1}{p-1}}\frac{\kappa\|u\|_\infty}{4}\right)$$

i.e., $\|u\|_\infty \geq \frac{\kappa(M/8)^{\frac{1}{p-1}}\|u\|_\infty}{32}$, a contradiction if M is large enough, which proves (ii).

Suppose next that $\varepsilon = 0$. Since $\liminf_{z\to\infty} \frac{f(t,z)-\lambda g(z)}{z^{p-1}} > \lambda_1$ uniformly for a.e. $t \in (0,1)$, there exist positive constants $L_0, \tilde{\lambda}$ with $\tilde{\lambda} > \lambda_1$ such that

$$f(t,z) - \lambda g(z) \geq \tilde{\lambda} z^{p-1} \tag{3.11}$$

for a.e. $t \in (0,1)$ and all $z \geq L_0$. For $\delta_1 \in (0, 1/2)$, let λ_{1,δ_1} be the first eigenvalue of the problem

$$\begin{cases} -(\phi_0(v'))' = \lambda_{1,\delta_1}\phi_0(v) \quad \text{on } (\delta_1, \delta_2), \\ av(\delta_1) - bv'(\delta_1) = 0, \; cv(\delta_2) + dv'(\delta_2) = 0, \end{cases} \tag{3.12}$$

where $\delta_2 = 1 - \delta_1$. By the continuity of the first eigenvalue with respect to the domain, $\lambda_{1,\delta_1} \to \lambda_1$ as $\delta_1 \to 0$. Hence there exits $\delta > 0$ such that $\lambda_{1,\delta_1} < \tilde{\lambda}$ for $\delta_1 \leq \delta$.

Let $\delta_1 = \delta/2$, $\delta_2 = 1 - \delta/2$, and $\mu \in (\lambda_{1,\delta_1}, \tilde{\lambda})$. By decreasing δ if necessary, we have from (3.7) that

$$a\bar{u}(\delta_1) - b\bar{u}'(\delta_1) \geq 0 \text{ if } a > 0, \quad c\bar{u}(\delta_2) + d\bar{u}'(\delta_2) \geq 0 \text{ if } c > 0, \tag{3.13}$$

where $\bar{u} = u + 1$. By (3.8),

$$u(t) \geq \frac{\kappa R\delta}{4} \geq L_0 \tag{3.14}$$

for $t \in [\delta/4, 1-\delta/4]$ for R large. It follows from (3.9), (3.11) and (3.14) that

$$-(\phi_0(u'))' \geq -\lambda g(u) + f(t,u) \geq \tilde{\lambda} u^{p-1} \text{ on } [\delta/4, 1-\delta/4]. \tag{3.15}$$

By (3.6) and (3.15),

$$-(\phi_0(u'))' \geq -\lambda h(t) - \gamma(t)\phi_0(u) \geq -\gamma_L(t), \tag{3.16}$$

for a.e. $t \in (0,1)$, where $\gamma_L(t) = \lambda h(t) + \gamma(t)\phi_0(L) \geq 0$. We claim that the eigenvalue problem

$$\begin{cases} -(\phi_0(v'))' = \mu\phi_0(v) \quad \text{on } (\delta_1, \delta_2), \\ av(\delta_1) - bv'(\delta_1) = 0, \; cv(\delta_2) + dv'(\delta_2) = 0 \end{cases} \tag{3.17}$$

has a positive solution, provided that R is large enough.

Let ψ_1 be the positive solution of (3.12) with $\|\psi_1\|_\infty = 1$. Then clearly ψ_1 is a subsolution of (3.17). Since (3.14) implies

$$\frac{u}{u+1} \geq \frac{\kappa R\delta/4}{1+\kappa R\delta/4} \text{ on } [\delta/4, 1-\delta/4]$$

for R large and $\frac{\kappa R\delta/4}{1+\kappa R\delta/4} \to 1$ as $R \to \infty$, it follows from (3.15) that

$$-(\phi_0(\bar{u}'))' \geq \tilde{\lambda} u^{p-1} = \tilde{\lambda}\bar{u}^{p-1}\left(\frac{u}{u+1}\right)^{p-1} \geq \mu\bar{u}^{p-1} \text{ on } (\delta_1, \delta_2), \tag{3.18}$$

for R large.

Case 1. $a, c > 0$. *Then* $\bar{u}$ *is a supersolution of (3.17) in view of (3.13) and (3.18).*

Case 2. $ac = 0$. *If* $a = 0$ *then (3.7) gives* $u'(0) = 0$. *Combining (3.14)–(3.16), we deduce that for R large,*

$$-\phi_0(u'(\delta_1)) = -\int_0^{\delta_1}(\phi_0(u'))' \geq -\int_0^{\delta/4}\gamma_L + \tilde{\lambda}\int_{\delta/4}^{\delta/2} u^{p-1} > 0$$

i.e., $u'(\delta_1) < 0$. *Similarly if* $c = 0$ *then* $u'(1) = 0$, *and*

$$\phi_0(u'(\delta_2)) = -\int_{\delta_2}^{1}(\phi_0(u'))' \geq -\int_{1-\delta/4}^{1}\gamma_L + \tilde{\lambda}\int_{1-\delta/2}^{1-\delta/4} u^{p-1} > 0$$

i.e., $u'(\delta_2) > 0$. *Since* $a\bar{u}(\delta_1) - b\bar{u}'(\delta_1) > 0$ *and* $c\bar{u}(\delta_2) + d\bar{u}'(\delta_2) > 0$, *it follows from (3.18) that* $\bar{u}$ *is a supersolution of (3.17).*

Since $\psi_1 \leq 1 \leq \bar{u}$ on $[\delta_1, \delta_2]$, the existence of a solution v to (3.17) with $\psi_1 \leq v \leq \bar{u}$ on (δ_1, δ_2) follows, which is a contradiction. Thus (ii) holds. By Amann's fixed point theorem [1, Theorem 12.3] , A_ε has a fixed point $u \in E$ with $\|u\|_\infty > \sigma$. Using $\xi = 0$ in (ii) and (3.8), we obtain $u(t) \geq \sigma_0 p(t)$ for $t \in [0, 1]$ i.e., $g(\max(u, \sigma_0 p(t))) = g(u)$ and therefore u is a positive solution of (1.1), which completes the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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Appendix A

We provide here some inequalities regarding the operator ϕ_ε.

Proposition A.
(i) $\phi_\varepsilon(x+y) \le M(\phi_\varepsilon(x)+\phi_\varepsilon(y))$ *for* $x, y \ge 0$, *where* $M = 2^{\max(p-2,0)}$.
(ii) $\phi_\varepsilon^{-1}(mx) \le m^{\frac{1}{q-1}}\phi_\varepsilon^{-1}(x)$ *for* $m \ge 1, x \ge 0$.
(iii) $\phi_\varepsilon(cx) \le c^{p-1}\phi_\varepsilon(x)$ *for* $c \ge 1, x \ge 0$.

Proof. *(i)* Let $x, y \ge 0$. Since the function z^r is convex on $[0, \infty)$ for $r \ge 1$,

$$\left(\frac{x+y}{2}\right)^r \le \frac{x^r + y^r}{2}$$

i.e.,

$$(x+y)^r \le 2^{r-1}(x^r + y^r).$$

On the other hand if $0 < r < 1$, we have

$$(x+y)^r \le x^r + y^r.$$

Hence for $r > 0$,

$$(x+y)^r \le 2^{\max(r-1,0)}(x^r + y^r),$$

which implies

$$\begin{aligned}\phi_\varepsilon(x+y) &= (x+y)^{p-1} + \varepsilon(x+y)^{q-1}\\ &\le 2^{\max(p-2,0)}(x^{p-1}+y^{p-1}) + \varepsilon 2^{\max(q-2,0)}(x^{q-1}+y^{q-1})\\ &\le 2^{\max(p-2,0)}(\phi_\varepsilon(x)+\phi_\varepsilon(y))\end{aligned}$$

i.e., (i) holds.

(ii) Let $z \ge 0$ and $c \ge 1$. We claim that

$$\phi_\varepsilon(cz) \ge c^{q-1}\phi_\varepsilon(z). \tag{A.1}$$

Indeed,

$$\phi_\varepsilon(cz) = c^{p-1}z^{p-1} + \varepsilon c^{q-1}z^{q-1} \ge c^{q-1}\phi_\varepsilon(z)$$

i.e., (A.1) holds. Let $m \ge 1$, $x \ge 0$. Then by using (A.1) with $c = m^{\frac{1}{q-1}}$ and $z = \phi_\varepsilon^{-1}(x)$, we obtain

$$\phi_\varepsilon\left(m^{\frac{1}{q-1}}\phi_\varepsilon^{-1}(x)\right) \ge m\phi_\varepsilon(\phi_\varepsilon^{-1}(x)) = mx$$

i.e., (ii) holds.

(iii) Let $c \ge 1$ and $x \ge 0$. Then

$$\phi_\varepsilon(cx) = c^{p-1}x^{p-1} + \varepsilon c^{q-1}x^{q-1} \le c^{p-1}(x^{p-1} + \varepsilon x^{q-1})$$

i.e., (iii) holds. □

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