Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 25714–25728.
DOI: 10.3934/math.20231311
Received: 05 May 2023
Revised: 23 August 2023
Accepted: 27 August 2023
Published: 06 September 2023

Research article

Weighted estimates for commutators associated to singular integral operator satisfying a variant of Hörmander's condition

Jie Sun and Jiamei Chen*

Department of Mathematics, Mudanjiang Normal University, Mudanjiang 157011, Heilongjiang, China

* **Correspondence:** Email: chenjiamei256@163.com; Tel: +8615214549621.

Abstract: In this paper, we establish some boundedness for commutators generated by the singular integral operator satisfying a variant of Hörmander's condition and a weighted BMO function on weighted Hardy spaces and weighted Herz spaces. As an application, we obtain some classical results.

Keywords: a variant of Hörmander's condition; commutators; BMO space; weighted Hardy space; weighted Herz space
Mathematics Subject Classification: 42B20, 42B30

1. Introduction and results

In 1952, Calderón and Zygmund [1] introduced the concept of singular integral in the study of elliptic partial differential equations and proved the existence of singular integral. Later, Calderón and Zygmund [2] studied a class of singular integral of convolution type and proved the L^p-boundedness ($1 < p < \infty$).

In the classical Calderón-Zygmund theory, the Hörmander's condition [3]

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\mathrm{d}x \leq C, \quad \forall y \neq 0. \tag{1.1}$$

plays a foundational role in Harmonic analysis. As the development of singular integrals, the kernel K which does not satisfy the condition (1.1) has been extensively considered. In [4], Grubb and Moore introduced a variant of Hörmander's condition

$$\int_{|x|>2|y|} |K(x-y) - \sum_{j=1}^{m} \mathcal{B}_j(x)\phi_j(y)|\mathrm{d}x \leq C, \tag{1.2}$$

where $\mathcal{B}_j$ and ϕ_j are appropriate functions and proved the L^p-boundedness for the singular operator with kernel satisfying (1.2).

Later, Trujillo-González made the kernel's conditions stronger and established the weighted L^p-boundedness of singular integral operator as below.

Theorem A. [5] Let $K \in L^2(\mathbb{R}^n)$ satisfy

$(K1)$ $\|\hat{K}\|_\infty \le C_0$,

$(K2)$ $|K(x)| \le \frac{C_0}{|x|^n}$,

$(K3)$ there exist functions $\mathcal{B}_1 \cdots \mathcal{B}_m$ and $\Phi = \{\phi_1, \cdots \phi_m\} \subset L^\infty(\mathbb{R}^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(\mathbb{R}^{nm})$,

$(K4)$ for a fixed $\gamma > 0$ and any $|x| > 2|y| > 0$,

$$|K(x-y) - \sum_{j=1}^{m} \mathcal{B}_j(x)\phi_j(y)| \le C_0 \frac{|y|^\gamma}{|x-y|^{n+\gamma}}. \tag{1.3}$$

For any $f \in C_c^\infty(\mathbb{R}^n)$, we define the singular integral operator T related to the kernel K by:

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\mathrm{d}y.$$

Let $1 < p < \infty$, $\omega \in A_p$, then there exists a constant $C > 0$, such that

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega(x)\mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x.$$

Remark A. When $m = 1$, $\mathcal{B}_1(x) = K(x)$, $\phi_1(y) = 1$, then (1.2) is exactly Hörmander's condition and (1.3) is the classical Calderón-Zygmund kernel.

With the development of singular integral operators, their commutators have been well studied. In 1976, Coifman, Rochberg and Weiss [6] established the boundedness of commutators on some $L^p(\mathbb{R}^n)(1 < p < \infty)$. Let b be a locally integrable function on $\mathbb{R}^n$ and let T be a Calderón-Zygmund singular integral operator. Consider the commutator T_b defined for suitable functions f by

$$T_b f(x) = b(x)Tf(x) - T(bf)(x).$$

In 2004, Zhou [7] proved that the commutators generated by the singular integral operator and a BMO function are bounded on Hardy spaces. In 2009, Kong and Jiang [8] got the boundedness of commutators generated by the singular integral operator with a homogeneous kernel and a BMO function on weighted Hardy spaces and weighted Herz spaces. In 2010, Liu [9] showed that the commutators generated by the singular integral operator and a BMO function are bounded on Herz-Hardy spaces. For more information about this topic we refer to [10–13].

In 2012, Zhang [14] studied the commutators generated by the singular integral operator satisfying a variant of Hörmander's condition and a BMO function are bounded on Hardy spaces. In 2015, Xie [15] discussed the boundedness of multilinear operators from Lebesgue spaces to Orlicz spaces when the kernel K satisfies the conditions $(K1)$–$(K4)$. In 2017, Pan [16] obtained the boundedness of multilinear operators satisfying a variant of Hörmander's condition on Morrey spaces.

Motivated by these results, we will study the boundedness of commutators generated by the singular integral operator satisfying a variant of Hörmander's condition and a weighted BMO function on some function spaces in this paper. Now, we state our main results as follows.

Theorem 1.1. Let T be the singular integral operator with the kernel K satisfying $(K1)$–$(K4)$. Let γ be as in $K4$. Suppose that $\mu \in A_1$, $\frac{n}{n+\gamma} < p \leq 1$ and $b \in \mathrm{BMO}_\mu$. Then T_b is bounded from $\mathcal{H}^p_{\Phi,b}(\mu)$ to $L^p(\mu^{1-p})$.
Remark 1.1. When we take $m = 1, \mathcal{B}_1(x) = K(x), \phi_1(y) = 1$, then the commutator generated by the classical singular integral operator and a weighted BMO function is bounded from $\mathcal{H}^p_b(\mu)$ to $L^p(\mu^{1-p})$.
Theorem 1.2. Let T be the singular integral operator with the kernel K satisfying $(K1)$–$(K4)$. Suppose that $\mu \in A_1$ and $b \in \mathrm{BMO}_\mu$. Then T_b is bounded from $\mathcal{H}^1_\Phi(\mu)$ to weak $L^1(\mathbb{R}^n)$. That is to say, for any $\lambda > 0$, there exists $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |T_b f(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_{\mathcal{H}^1_\Phi(\mu)}.$$

Remark 1.2. When we take $m = 1, \mathcal{B}_1(x) = K(x), \phi_1(y) = 1$, then the commutator generated by the classical singular integral operator and a weighted BMO function is bounded from $\mathcal{H}^1(\mu)$ to weak $L^1(\mathbb{R}^n)$.
Theorem 1.3. Let T be the singular integral operator with the kernel K satisfying $(K1)$–$(K4)$. Suppose that $\mu \in A_1$, $0 < p \leq \infty, 1 < q < \infty$, $-\frac{n\delta}{q} < \eta < n\delta - \frac{n}{q}$ and $b \in \mathrm{BMO}_\mu$. Then T_b maps $\dot{K}^{\eta,p}_q(\mu,\mu)$ continuously into $\dot{K}^{\eta,p}_q(\mu,\mu^{1-q})$.
Remark 1.3. When we take $m = 1, \mathcal{B}_1(x) = K(x), \phi_1(y) = 1$, then the commutator generated by the classical singular integral operator and a weighted BMO function maps $\dot{K}^{\eta,p}_q(\mu,\mu)$ continuously into $\dot{K}^{\eta,p}_q(\mu,\mu^{1-q})$.
Theorem 1.4. Let T be the singular integral operator with the kernel K satisfying $(K1)$–$(K4)$. Suppose that $\mu \in A_1$, $0 < p \leq 1$, $1 < q < \infty$, $\eta = n\delta - \frac{n}{q}$, $q\delta \neq 1$ and $b \in \mathrm{BMO}_\mu$. Then T_b maps $\dot{K}^{\eta,p}_q(\mu,\mu)$ continuously into $W\dot{K}^{\eta,p}_q(\mu,\mu^{1-q})$.
Remark 1.4. When we take $m = 1, \mathcal{B}_1(x) = K(x), \phi_1(y) = 1, \eta = n\delta - \frac{n}{q}$, then the commutator generated by the classical singular integral operator and a weighted BMO function maps $\dot{K}^{\eta,p}_q(\mu,\mu)$ continuously into $W\dot{K}^{\eta,p}_q(\mu,\mu^{1-q})$.

Throughout this paper, we denote by p' the conjugate index of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$. The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same at each occurrence but is independent of the main parameters.

2. Preliminaries and lemmas

Firstly, let us introduce some important notations that will help us further.
Definition 2.1. [17] A non-negative locally integrable function is called a weight function. Let ω be a weight function, $1 < p < \infty$. If there is a constant $C > 0$, such that for any ball $B \subset \mathbb{R}^n$,

$$\Big(\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}\mathrm{d}x\Big)^{p-1} \leq C,$$

then we say $\omega \in A_p$. We say $\omega \in A_1$, if there is a constant $C > 0$, such that for any ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x \leq C\omega(x), a.e. x \in \mathbb{R}^n.$$

A weight function $\omega \in A_\infty$ if it satisfies the A_p condition for some $1 \leq p < \infty$. The smallest constant satisfying the fomulas above is called A_p constant of ω and we denote it by $[\omega]_{A_p}$.

For $1 \leq p < q < \infty$, we have $A_1 \subset A_p \subset A_q$ and $A_\infty = \cup_{p\geq 1} A_p$.

Definition 2.2. [5] Given a positive and locally integrable function g in $\mathbb{R}^n$, we say that g satisfies the reverse Hölder RH_∞ condition, in short, $g \in RH_\infty(\mathbb{R}^n)$, if for any cube Q centered at the origin we have

$$0 < \sup_{x\in Q} g(x) \leq C \frac{1}{|Q|} \int_Q g(x)\mathrm{d}x$$

with $C > 0$ being an absolute constant independent of Q.

Definition 2.3. [18] Suppose $\mu \in A_\infty$. We will say that a locally integrable function $b(x)$ belongs to the weighted BMO_μ , that is

$$\sup_B \frac{1}{\mu(B)} \int_B |b(x) - b_B|\mathrm{d}x \leq C < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, $\mu(B) = \int_B \mu(x)\mathrm{d}x$, the smallest constant C is denoted as $\|b\|_{*,\mu}$.

For $1 \leq p, q \leq \infty$, we have

$$C_1\|b\|_{*,\mu} \leq \sup_B \left(\frac{1}{\mu(B)} \int_B |b(x) - b_B|^p \mu(x)^{1-p}\mathrm{d}x\right)^{\frac{1}{p}} \leq C_2\|b\|_{*,\mu}.$$

Similar to the definition of the Hardy space related to Φ in [19] , we define the weighted Hardy space related to Φ.

Definition 2.4. For $0 < p \leq 1$, $\omega \in A_\infty$ and $\Phi = \{\phi_1, \cdots, \phi_m\} \subset L^\infty(\mathbb{R}^n)$, a function $a(x)$ is called an $\omega - (p, \infty, \Phi)$ atom centered at x_0, if

(1) $\mathrm{supp} a \subset B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$,

(2) $\|a\|_{L^\infty} \leq \omega(B(x_0, r))^{-\frac{1}{p}}$,

(3) $\int_{\mathbb{R}^n} a(x)\mathrm{d}x = \int_{\mathbb{R}^n} a(x)\phi_j(x - x_0)\mathrm{d}x = 0$, $j = 1, 2, \cdots, m$.

Definition 2.5. For $0 < p \leq 1$ and $\omega \in A_\infty$, we say that a distribution f on $\mathbb{R}^n$ belongs to the weighted Hardy space $\mathcal{H}^p_\Phi(\omega)$, if in distributional sense, it can be written as $f = \sum_{j=-N}^{N} \lambda_j a_j$, where each a_j is an $\omega - (p, \infty, \Phi)$ atom, $N \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$ and $\sum_{j=-N}^{N} |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{\mathcal{H}^p_\Phi(\omega)} = \inf_{\sum_{j=-N}^{N} \lambda_j a_j = f} \left(\sum_{j=-N}^{N} |\lambda_j|^p\right)^{\frac{1}{p}}$$

with the infimum taken over all decompositions of f.

Similar to the Definition 2.5, we define the weighted Hardy space related to Φ and b.

Definition 2.6. For $0 < p \leq 1$, $\omega \in A_\infty$, $\Phi = \{\phi_1, \cdots, \phi_m\} \subset L^\infty(\mathbb{R}^n)$ and b is a locally integrable function, a function $a(x)$ is called an $\omega - (p, \infty, \Phi, b)$ atom centered at x_0, if

(1) $\mathrm{supp} a \subset B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$,

(2) $\|a\|_{L^\infty} \leq \omega(B(x_0, r))^{-\frac{1}{p}}$,

(3) $\int_{\mathbb{R}^n} a(x)\mathrm{d}x = \int_{\mathbb{R}^n} a(x)\phi_j(x - x_0)\mathrm{d}x = \int_{\mathbb{R}^n} a(x)b(x)\phi_j(x - x_0)\mathrm{d}x = 0$, $j = 1, 2, \cdots, m$.

Definition 2.7. For $0 < p \leq 1$ and $\omega \in A_\infty$, we say that a distribution f on $\mathbb{R}^n$ belongs to the weighted Hardy space $\mathcal{H}^p_{\Phi,b}(\omega)$, if in distributional sense, it can be written as $f = \sum\limits_{j=-N}^{N} \lambda_j a_j$, where each a_j is an $\omega - (p, \infty, \Phi, b)$ atom, $N \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$ and $\sum\limits_{j=-N}^{N} |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{\mathcal{H}^p_{\Phi,b}(\omega)} = \inf_{\sum\limits_{j=-N}^{N} \lambda_j a_j = f} \Big(\sum_{j=-N}^{N} |\lambda_j|^p \Big)^{\frac{1}{p}}$$

with the infimum taken over all decompositions of f.

For $k \in \mathbb{Z}$, $B_k = B(0, 2^k)$ and $E_k = B_k \backslash B_{k-1}$. Let $\chi_k = \chi_{E_k}$ denote the characteristic function of the set E_k.

Definition 2.8. [20] Let $\eta \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$, ω_1 and ω_2 be non-negative weighted functions. The homogeneous Herz space $\dot{K}^{\eta,p}_q(\omega_1, \omega_2)$ is defined by

$$\dot{K}^{\eta,p}_q(\omega_1, \omega_2) = \{f \in L^q_{loc}(\mathbb{R}^n \backslash \{0\}, \omega_2) : \|f\|_{\dot{K}^{\eta,p}_q(\omega_1,\omega_2)} < \infty\},$$

where $\|f\|_{\dot{K}^{\eta,p}_q(\omega_1,\omega_2)} = \{ \sum\limits_{k=-\infty}^{\infty} \omega_1(B_k)^{\frac{\eta p}{n}} \|f\chi_k\|^p_{L^q_{\omega_2}} \}^{\frac{1}{p}}$ with the usual modification made when $p = \infty$.

Definition 2.9. [20] Let $\eta \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$, ω_1 and ω_2 be non-negative weighted functions. The measurable function $f(x)$ belongs to the weighted weak Herz space $W\dot{K}^{\eta,p}_q(\omega_1, \omega_2)$, if

$$\|f\|_{W\dot{K}^{\eta,p}_q(\omega_1,\omega_2)} = \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\frac{\eta p}{n}} m_k(\lambda, f)^{\frac{p}{q}} \Big\}^{\frac{1}{p}} < \infty,$$

where $m_k(\lambda, f) = \omega_2(\{x \in E_k : |f(x)| > \lambda\})$ with the usual modification made when $p = \infty$.

Lemma 2.1. [18] Suppose that $\omega \in A_1$. Then there exist constant $C_1, C_2 > 0$ and $0 < \delta < 1$ for each measurable subset of $B \subset \mathbb{R}^n$ such that

$$C_1 \frac{|E|}{|B|} \leq \frac{\omega(E)}{\omega(B)} \leq C_2 \Big(\frac{|E|}{|B|} \Big)^{\delta}.$$

Lemma 2.2. [8] Suppose that $\mu \in A_1$ and $b \in \mathrm{BMO}_\mu(\mathbb{R}^n)$. Then there exists a constant $C > 0$ for each measurable subset of $B \subset \mathbb{R}^n$ such that

$$|b_{2^j B} - b_B| \leq C \|b\|_{*,\mu} j \frac{\mu(B)}{|B|}.$$

Lemma 2.3. [21] Suppose that $\mu \in A_1$ and $1 < q < \infty$. Then there exists a constant $C > 0$ for each measurable subset of $B \subset \mathbb{R}^n$ such that

$$\int_{2^{j+1}B} \mu(x)^{1-q} \mathrm{d}x \leq \frac{|2^{j+1}B|^q}{\mu(2^{j+1}B)^{q-1}}.$$

According to [22], we have the following lemma :

Lemma 2.4. Let T be the singular integral operator with the kernel K satisfying $(K1)$–$(K4)$. Suppose that $1 < p < \infty$, $\mu \in A_1$ and $b \in \mathrm{BMO}_\mu$. Then T_b is bounded from $L^p(\mu)$ to $L^p(\mu^{1-p})$.

3. Proofs of theorems

Proof of Theorem 1.1. It suffices to prove that, for any $\mu-(p,\infty,\Phi,b)$ atom a, the following inequality holds

$$\|T_b a\|_{L^p(\mu^{1-p})} \le C. \tag{3.1}$$

In fact, if (3.1) holds, then for any $f = \sum_{j=-N}^{N} \lambda_j a_j \in \mathcal{H}^p_{\Phi,b}(\mu)$, where each a_j is a $\mu-(p,\infty,\Phi,b)$ atom, since $\frac{n}{n+\gamma} < p \le 1$, we have

$$\begin{aligned}
\|T_b f\|_{L^p(\mu^{1-p})} &= \Big(\int_{\mathbb{R}^n} \Big|T_b\Big(\sum_{j=-N}^{N}\lambda_j a_j\Big)\Big|^p \mu(y)^{1-p}\mathrm{d}y\Big)^{\frac{1}{p}}\\
&= \Big(\int_{\mathbb{R}^n} \Big|\sum_{j=-N}^{N}\lambda_j T_b a_j(y)\Big|^p \mu(y)^{1-p}\mathrm{d}y\Big)^{\frac{1}{p}}\\
&\le \Big(\sum_{j=-N}^{N}|\lambda_j|^p\|T_b a_j\|^p_{L^p(\mu^{1-p})}\Big)^{\frac{1}{p}}\\
&\le C\Big(\sum_{j=-N}^{N}|\lambda_j|^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

Then the proof can be completed by taking the infimum for all atomic decompositionns of f.
Now, we need to prove (3.1), suppose that supp$a \subset B = B(x_B, r)$. Write

$$\begin{aligned}
\|T_b a\|_{L^p(\mu^{1-p})} &\le \Big(\int_{2B}|T_b a(x)|^p\mu(x)^{1-p}\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{(2B)^c}|T_b a(x)|^p\mu(x)^{1-p}\mathrm{d}x\Big)^{\frac{1}{p}}\\
&:= I_1 + I_2.
\end{aligned}$$

By the Hölder inequality for $q > 1$ and Lemma 2.4, we have

$$\begin{aligned}
I_1 &\le \int_{2B}|T_b a(x)|\mathrm{d}x\mu(2B)^{\frac{1}{p}-1}\\
&= \Big(\int_{2B}|T_b a(x)|\mu(x)^{\frac{1}{q}-1}\mu(x)^{1-\frac{1}{q}}\mathrm{d}x\Big)\mu(2B)^{\frac{1}{p}-1}\\
&\le \Big(\int_{2B}|T_b a(x)|^q\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{2B}\mu(x)\mathrm{d}x\Big)^{\frac{1}{q'}}\mu(2B)^{\frac{1}{p}-1}\\
&\le \|T_b a\|_{L^q(\mu^{1-q})}\mu(2B)^{\frac{1}{q'}}\mu(2B)^{\frac{1}{p}-1}\\
&\le \|a\|_{L^q(\mu)}\mu(2B)^{\frac{1}{q'}}\mu(2B)^{\frac{1}{p}-1}\\
&\le C\|a\|_\infty\mu(B)^{\frac{1}{q}}\mu(2B)^{\frac{1}{q'}}\mu(2B)^{\frac{1}{p}-1}\\
&\le C\mu(B)^{-\frac{1}{p}}\mu(B)^{\frac{1}{q}}\mu(2B)^{\frac{1}{q'}}\mu(2B)^{\frac{1}{p}-1}\\
&\le C.
\end{aligned}$$

Let $C_k = 2^{k+1}B \backslash 2^k B$

$$I_2^p \leq \sum_{k=1}^{\infty} \int_{C_k} |(T_b a)(x)|^p \mu(x)^{1-p} \mathrm{d}x \leq \sum_{k=1}^{\infty} \Big(\int_{C_k} |(T_b a)(x)| \mathrm{d}x \Big)^p \Big(\int_{C_k} \mu(x) \mathrm{d}x \Big)^{1-p}.$$

Write

$$\begin{aligned} \int_{C_k} |(T_b a)(x)| \mathrm{d}x &\leq \int_{C_k} |b(x) - b_B| |Ta(x)| \mathrm{d}x + \int_{C_k} |T((b - b_B)a)(x)| \mathrm{d}x \\ &:= I_{21} + I_{22}. \end{aligned}$$

For I_{21}, when $y \in B$, $x \in 2^{k+1}B \backslash 2^k B$, we have $|y - x| \sim |x - x_B| \geq 2|y - x_B|$, by the vanishing condition of a, we obtain

$$\begin{aligned} I_{21} &= \int_{C_k} \Big|b(x) - b_B\Big| \Big| \int_B K(x-y) a(y) \mathrm{d}y - \sum_{j=1}^{m} \int_B a(y) \mathcal{B}_j(x - x_B) \phi_j(y - x_B) \mathrm{d}y \Big| \mathrm{d}x \\ &\leq \int_{C_k} \Big|b(x) - b_B\Big| \Big(\int_B \Big|K(x-y) - \sum_{j=1}^{m} \mathcal{B}_j(x - x_B) \phi_j(y - x_B)\Big| \Big|a(y)\Big| \mathrm{d}y \Big) \mathrm{d}x \\ &\leq C \int_{C_k} |b(x) - b_B| \int_B \frac{|y - x_B|^\gamma}{|x - x_B|^{n+\gamma}} |a(y)| \mathrm{d}y \mathrm{d}x \\ &\leq C \|a\|_\infty |B| 2^{-k\gamma} |2^{k+1}B|^{-1} \int_{2^{k+1}B} |b(x) - b_B| \mathrm{d}x \\ &\leq C \mu(B)^{-\frac{1}{p}} 2^{-k\gamma} \cdot 2^{-(k+1)n} \cdot (k+1) \cdot 2^{(k+1)n} \mu(B) \|b\|_{*,\mu} \\ &\leq C \|b\|_{*,\mu} (k+1) 2^{-k\gamma} \mu(B)^{1-\frac{1}{p}}. \end{aligned}$$

For I_{22}, we have

$$\begin{aligned} I_{22} &\leq \int_{C_k} \Big| \int_B K(x-y)[b(y) - b_B] a(y) \mathrm{d}y - \sum_{j=1}^{m} \int_B a(y)[b(y) - b_B] \mathcal{B}_j(x - x_B) \phi_j(y - x_B) \mathrm{d}y \Big| \mathrm{d}x \\ &\leq C \|a\|_\infty 2^{-k\gamma} |2^{k+1}B|^{-1} \int_B |b(y) - b_B| \mathrm{d}y \int_{2^{k+1}B} \mathrm{d}x \\ &\leq C \|a\|_\infty 2^{-k\gamma} \mu(B) \|b\|_{*,\mu} \\ &\leq C \|b\|_{*,\mu} 2^{-k\gamma} \mu(B)^{1-\frac{1}{p}}. \end{aligned}$$

Since $\frac{n}{n+\gamma} < p \leq 1$,

$$\begin{aligned} I_2 &\leq C \Big(\sum_{k=1}^{\infty} (k+1)^p \cdot 2^{-k\gamma p} \cdot \mu(B)^{p-1} \cdot \mu(2^{k+1}B)^{1-p} \Big)^{\frac{1}{p}} \\ &\leq C \Big(\sum_{k=1}^{\infty} (k+1)^p 2^{k(n-np-\gamma p)} \Big)^{\frac{1}{p}} \\ &\leq C. \end{aligned}$$

Combining the estimates for I_1 and I_2, we finish the proof. □

Proof of Theorem 1.2. We can write $f = \sum_{j=-N}^{N} \lambda_j a_j \in \mathcal{H}_{\Phi}^1(\mu)$ with each a_j being a $\mu-(1,\infty,\Phi)$ atom and $\sum_{j=-N}^{N} |\lambda_j| < \infty$. Suppose supp$a_j \subset B_j = B(x_j, r_j)$. Write

$$\begin{aligned}
|T_b f(x)| &\le \sum_{j=-N}^{N} \left|\lambda_j [b(x) - b_{2B_j}] T a_j(x) \chi_{2B_j}(x)\right| + \sum_{j=-N}^{N} \left|\lambda_j [b(x) - b_{2B_j}] T a_j(x) \chi_{(2B_j)^c}(x)\right| \\
&\quad + \left|T\Big(\sum_{j=-N}^{N} \lambda_j [b - b_{2B_j}] a_j\Big)(x)\right| \\
&:= J_1(x) + J_2(x) + J_3(x).
\end{aligned}$$

By the Hölder inequality and the Theorem A, we obtain

$$\begin{aligned}
&\|[b(\cdot) - b_{2B_j}] T a_j(\cdot) \chi_{2B_j}(\cdot)\|_{L^1} \\
\le &\int_{2B_j} |b(x) - b_{2B_j}| |T a_j(x)| \mathrm{d}x \\
\le &\|T a_j\|_{L^2(\mu)} \Big(\int_{2B_j} |b(x) - b_{2B_j}|^2 \mu(x)^{-1} \mathrm{d}x\Big)^{\frac{1}{2}} \\
\le &\|a_j\|_{L^2(\mu)} \mu(2B_j)^{\frac{1}{2}} \|b\|_{*,\mu} \\
\le &C,
\end{aligned}$$

and

$$\begin{aligned}
&\|(b(\cdot) - b_{2B_j}) T a_j(\cdot) \chi_{(2B_j)^c}(\cdot)\|_{L^1} \\
&\le \sum_{k=1}^{\infty} \int_{2^{k+1}B_j \backslash 2^k B_j} \left|b(x) - b_{2B_j}\right| \int_{B_j} \left|K(x-s) - \sum_{j=1}^{m} \mathcal{B}_j(x - x_j) \phi_j(s - x_j)\right| \left|a_j(s)\right| \mathrm{d}s \mathrm{d}x \\
&\le C \sum_{k=1}^{\infty} |B_j| 2^{-k\gamma} |2^{k+1} B_j|^{-1} \|a_j\|_{\infty} \int_{2^{k+1}B_j} |b(x) - b_{2B_j}| \mathrm{d}x \\
&\le C \sum_{k=1}^{\infty} 2^{-k\gamma} 2^{-kn-n} \|a_j\|_{\infty} k 2^{kn} \mu(2B_j) \|b\|_{*,\mu} \\
&\le C \sum_{k=1}^{\infty} k 2^{-k\gamma} \mu(B_j)^{-1} \mu(2B_j) \|b\|_{*,\mu} \\
&\le C \sum_{k=1}^{\infty} k 2^{-k\gamma} \|b\|_{*,\mu} \\
&\le C.
\end{aligned}$$

Thus, we have

$$|\{x \in \mathbb{R}^n : |J_i(x)| > \lambda/3\}| \le C\lambda^{-1} \|J_i\|_{L^1} \le C\lambda^{-1} \Big(\sum_{j=-N}^{N} |\lambda_j|\Big), \quad i = 1, 2.$$

Noting that

$$\int_{B_j} |b(y) - b_{2B_j}||a_j(y)|\mathrm{d}y \le C\|b\|_{*,\mu}\|a\|_\infty \mu(2B_j) \le C.$$

By the weak (1, 1) boundedness of T in [4], we obtain

$$|\{x \in \mathbb{R}^n : |J_3(x)| > \lambda/3\}| \le C\lambda^{-1}\Big\| \sum_{j=-N}^{N} \lambda_j(b - b_{2B_j})a_j\Big\|_{L^1} \le C\lambda^{-1} \sum_{j=-N}^{N} |\lambda_j|.$$

Therefore,

$$|\{x \in \mathbb{R}^n : |T_b f(x)| > \lambda\}| \le C \sum_{i=1}^{3} |\{x \in \mathbb{R}^n : |J_i(x)| > \lambda/3\}| \le C\lambda^{-1} \sum_{j=-N}^{N} |\lambda_j|.$$

Taking the infimum over all decompositions of f, the proof is finished. □

Proof of Theorem 1.3. We only consider the case $0 < p < \infty$. When $p = \infty$, we can make appropriate modifications.

$$\begin{aligned}
\|T_b f\|^p_{\dot{K}^{\eta,p}_q(\mu,\mu^{1-q})} &\le C \sum_{j=-\infty}^{\infty} \mu(B_j)^{\frac{\eta p}{n}} \Big(\sum_{k=-\infty}^{j-2} \|T_b(f\chi_k)\chi_j\|_{L^q(\mu^{1-q})}\Big)^p \\
&\quad + C \sum_{j=-\infty}^{\infty} \mu(B_j)^{\frac{\eta p}{n}} \Big(\sum_{k=j-1}^{j+1} \|T_b(f\chi_k)\chi_j\|_{L^q(\mu^{1-q})}\Big)^p \\
&\quad + C \sum_{j=-\infty}^{\infty} \mu(B_j)^{\frac{\eta p}{n}} \Big(\sum_{k=j+2}^{\infty} \|T_b(f\chi_k)\chi_j\|_{L^q(\mu^{1-q})}\Big)^p \\
&:= L_1 + L_2 + L_3.
\end{aligned}$$

Firstly, we estimate L_2. By Lemma 2.4

$$L_2 \le C \sum_{j=-\infty}^{\infty} \mu(B_j)^{\frac{\eta p}{n}} \sum_{k=j-1}^{j+1} \|f\chi_k\|^p_{L^q(\mu)} \le C\|f\|^p_{\dot{K}^{\eta,p}_q(\mu,\mu)}.$$

Obviously

$$\begin{aligned}
\|T_b(f\chi_k)\chi_j\|_{L^q(\mu^{1-q})} &\le C\Big(\int_{E_j} \Big|(b(x) - b_{B_k}) \int_{E_k} K(x-y)f(y)\chi_k(y)\mathrm{d}y\Big|^q \mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}} \\
&\quad + C\Big(\int_{E_j} \Big| \int_{E_k} K(x-y)(b(y) - b_{B_k})f(y)\chi_k(y)\mathrm{d}y\Big|^q \mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}} \\
&:= H_1 + H_2.
\end{aligned}$$

Now, let us estimate L_1, for $j \ge k + 2$, by Lemmas 2.1–2.3, we have

$$H_1 \le C2^{-jn}\|f\chi_k\|_{L^1}\Big(\int_{E_j} |b(x) - b_{B_k}|^q \mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\leq C2^{-jn}\|f\chi_k\|_{L^q(\mu)}\Big(\int_{B_k}\mu(x)^{1-q'}\mathrm{d}x\Big)^{\frac{1}{q'}}\Big(\int_{B_j}|b(x)-b_{B_k}|^q\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}}\\
&\leq C2^{-jn}\|f\chi_k\|_{L^q(\mu)}|B_k|\mu(B_k)^{-\frac{1}{q}}\mu(B_j)^{\frac{1}{q}}(j-k)2^{(j-k)n(1-\delta)}\\
&\leq C2^{-jn}\cdot 2^{kn}\cdot 2^{(j-k)\frac{n}{q}}\cdot(j-k)\cdot 2^{(j-k)n(1-\delta)}\|f\chi_k\|_{L^q(\mu)}\\
&\leq C(j-k)2^{(j-k)(\frac{n}{q}-n\delta)}\|f\chi_k\|_{L^q(\mu)},
\end{aligned}
$$

and

$$
\begin{aligned}
H_2&\leq C2^{-jn}\int_{E_k}\Big|(b(y)-b_{B_k})f\chi_k(y)\Big|\mathrm{d}y\Big(\int_{E_j}\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}}\\
&\leq C2^{-jn}\|f\chi_k\|_{L^q(\mu)}\Big(\int_{B_k}|b(y)-b_{B_k}|^{q'}\mu(y)^{1-q'}\mathrm{d}y\Big)^{\frac{1}{q'}}\Big(\int_{B_j}\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}}\\
&\leq C2^{-jn}\|f\chi_k\|_{L^q(\mu)}\mu(B_k)^{\frac{1}{q'}}|B_j|\mu(B_j)^{-\frac{1}{q'}}\\
&\leq C2^{\frac{n\delta}{q'}(k-j)}\|f\chi_k\|_{L^q(\mu)}\\
&= C2^{(j-k)n\delta(\frac{1}{q}-1)}\|f\chi_k\|_{L^q(\mu)}\\
&= C2^{(j-k)(\frac{n\delta}{q}-n\delta)}\|f\chi_k\|_{L^q(\mu)}.
\end{aligned}
$$

When $\eta>0$, we have

$$
\begin{aligned}
\|T_b(f\chi_k)\chi_j\|_{L^q(\mu^{1-q})}&\leq C2^{(k-j)\eta}(j-k)2^{(j-k)(\eta+\frac{n}{q}-n\delta)}\|f\chi_k\|_{L^q(\mu)}\\
&= C2^{(k-j)\eta}W(j,k)\|f\chi_k\|_{L^q(\mu)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
L_1&\leq C\sum_{j=-\infty}^{\infty}\mu(B_j)^{\frac{\eta p}{n}}\Big(\sum_{k=-\infty}^{j-2}2^{(k-j)\eta}W(j,k)\|f\chi_k\|_{L^q(\mu)}\Big)^p\\
&\leq C\begin{cases}\sum\limits_{j=-\infty}^{\infty}\sum\limits_{k=-\infty}^{j-2}\mu(B_k)^{\frac{\eta p}{n}}W(j,k)^p\|f\chi_k\|^p_{L^q(\mu)} & p\leq 1\\ \sum\limits_{j=-\infty}^{\infty}\Big(\sum\limits_{k=-\infty}^{j-2}\mu(B_k)^{\frac{\eta p}{n}}W(j,k)\|f\chi_k\|^p_{L^q(\mu)}\Big)\Big(\sum\limits_{k=-\infty}^{j-2}W(j,k)\Big)^{\frac{p}{p'}} & p>1\end{cases}\\
&\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{\eta p}{n}}\|f\chi_k\|^p_{L^q(\mu)}\sum_{j=k+2}^{\infty}W(j,k)^{\min(p,1)}\\
&\leq C\|f\|^p_{\dot{K}_q^{\eta,p}(\mu,\mu)}.
\end{aligned}
$$

When $\eta<0$, we have

$$
\begin{aligned}
\|T_b(f\chi_k)\chi_j\|_{L^q(\mu^{1-q})}&\leq C2^{(k-j)\eta\delta}(j-k)2^{(j-k)(\eta\delta+\frac{n}{q}-n\delta)}\|f\chi_k\|_{L^q(\mu)}\\
&= C2^{(k-j)\eta\delta}W_1(j,k)\|f\chi_k\|_{L^q(\mu)}.
\end{aligned}
$$

Thus

$$
L_1\leq C\sum_{j=-\infty}^{\infty}\mu(B_j)^{\frac{\eta p}{n}}\Big(\sum_{k=-\infty}^{j-2}2^{(k-j)\eta\delta}W_1(j,k)\|f\chi_k\|_{L^q(\mu)}\Big)^p
$$

$$\leq C\begin{cases}\sum\limits_{j=-\infty}^{\infty}\sum\limits_{k=-\infty}^{j-2}\mu(B_k)^{\frac{\eta p}{n}}W_1(j,k)^p\|f\chi_k\|_{L^q(\mu)}^p & p\leq 1\\ \sum\limits_{j=-\infty}^{\infty}\Big(\sum\limits_{k=-\infty}^{j-2}\mu(B_k)^{\frac{\eta p}{n}}W_1(j,k)\|f\chi_k\|_{L^q(\mu)}^p\Big)\Big(\sum\limits_{k=-\infty}^{j-2}W_1(j,k)\Big)^{\frac{p}{p'}} & p>1\end{cases}$$

$$\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{\eta p}{n}}\|f\chi_k\|_{L^q(\mu)}^p\sum_{j=k+2}^{\infty}W_1(j,k)^{\min(p,1)}$$

$$\leq C\|f\|_{\dot{K}_q^{\eta,p}(\mu,\mu)}^p.$$

Lastly, we estimate L_3, for $k\geq j+2$, by Lemmas 2.1–2.3, we have

$$\begin{aligned}H_1&\leq C2^{-kn}\|f\chi_k\|_{L^1}\Big(\int_{E_j}|b(x)-b_{B_k}|^q\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}}\\&\leq C2^{-kn}\|f\chi_k\|_{L^q(\mu)}\Big(\int_{B_k}\mu(x)^{1-q'}\mathrm{d}x\Big)^{\frac{1}{q'}}\Big(\int_{B_j}|b(x)-b_{B_k}|^q\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}}\\&\leq C2^{-kn}\|f\chi_k\|_{L^q(\mu)}|B_k|\mu(B_k)^{-\frac{1}{q}}(k-j)\mu(B_j)^{\frac{1}{q}}\|b\|_{*,\mu}\\&\leq C(k-j)2^{(j-k)\frac{n\delta}{q}}\|f\chi_k\|_{L^q(\mu)},\end{aligned}$$

and

$$\begin{aligned}H_2&\leq C2^{-kn}\Big(\int_{E_k}|(b(y)-b_{B_k})f\chi_k(y)|\mathrm{d}y\Big)\Big(\int_{E_j}\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{1}{q}}\\&\leq C2^{-kn}\|f\chi_k\|_{L^q(\mu)}\mu(B_k)^{\frac{1}{q'}}|B_j|\mu(B_j)^{-\frac{1}{q'}}\\&\leq C2^{jn-kn}\cdot 2^{(k-j)\frac{n}{q'}}\|f\chi_k\|_{L^q(\mu)}\\&\leq C2^{(j-k)n(1-\frac{1}{q'})}\|f\chi_k\|_{L^q(\mu)}\\&\leq C2^{(j-k)\frac{n}{q}}\|f\chi_k\|_{L^q(\mu)}.\end{aligned}$$

When $\eta>0$

$$\begin{aligned}\|T_b(f\chi_k)\chi_j\|_{L^q(\mu^{1-q})}&\leq C2^{(k-j)\eta\delta}(k-j)2^{(j-k)(\eta\delta+\frac{n\delta}{q})}\|f\chi_k\|_{L^q(\mu)}\\&=C2^{(k-j)\eta\delta}W_2(j,k)\|f\chi_k\|_{L^q(\mu)}.\end{aligned}$$

Thus

$$L_3\leq C\sum_{j=-\infty}^{\infty}\mu(B_j)^{\frac{\eta p}{n}}\Big(\sum_{k=j+2}^{\infty}2^{(k-j)\eta\delta}W_2(j,k)\|f\chi_k\|_{L^q(\mu)}\Big)^p$$

$$\leq C\begin{cases}\sum\limits_{j=-\infty}^{\infty}\sum\limits_{k=j+2}^{\infty}\mu(B_k)^{\frac{\eta p}{n}}W_2(j,k)^p\|f\chi_k\|_{L^q(\mu)}^p & p\leq 1\\ \sum\limits_{j=-\infty}^{\infty}\Big(\sum\limits_{k=j+2}^{\infty}\mu(B_k)^{\frac{\eta p}{n}}W_2(j,k)\|f\chi_k\|_{L^q(\mu)}^p\Big)\Big(\sum\limits_{k=j+2}^{\infty}W_2(j,k)\Big)^{\frac{p}{p'}} & p>1\end{cases}$$

$$\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{\eta p}{n}}\|f\chi_k\|_{L^q(\mu)}^p\sum_{j=-\infty}^{k-2}W_2(j,k)^{\min(p,1)}$$

$$\leq C\|f\|^p_{\dot{K}^{\eta,p}_q(\mu,\mu)}.$$

Similarly, when $\eta < 0$, we have

$$\begin{aligned}\|T_b(f\chi_k)\chi_j\|_{L^q(\mu^{1-q})} &\leq C2^{(k-j)\eta}(k-j)2^{(j-k)(\eta+\frac{n\delta}{q})}\|f\chi_k\|_{L^q(\mu)}\\ &= C2^{(k-j)\eta}W_3(j,k)\|f\chi_k\|_{L^q(\mu)}.\end{aligned}$$

$$\begin{aligned}L_3 &\leq C\sum_{j=-\infty}^{\infty}\mu(B_j)^{\frac{\eta p}{n}}\Big(\sum_{k=j+2}^{\infty}2^{(k-j)\eta}W_3(j,k)\|f\chi_k\|_{L^q(\mu)}\Big)^p\\ &\leq C\begin{cases}\sum\limits_{j=-\infty}^{\infty}\sum\limits_{k=j+2}^{\infty}\mu(B_k)^{\frac{\eta p}{n}}W_3(j,k)^p\|f\chi_k\|^p_{L^q(\mu)} & p\leq 1\\ \sum\limits_{j=-\infty}^{\infty}\Big(\sum\limits_{k=j+2}^{\infty}\mu(B_k)^{\frac{\eta p}{n}}W_3(j,k)\|f\chi_k\|^p_{L^q(\mu)}\Big)\Big(\sum\limits_{k=j+2}^{\infty}W_3(j,k)\Big)^{\frac{p}{p'}} & p>1\end{cases}\\ &\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{\eta p}{n}}\|f\chi_k\|^p_{L^q(\mu)}\sum_{j=-\infty}^{k-2}W_3(j,k)^{\min(p,1)}\\ &\leq C\|f\|^p_{\dot{K}^{\eta,p}_q(\mu,\mu)}.\end{aligned}$$

Then we finish the proof. □

Proof of Theorem 1.4. For $k\in\mathbb{Z}$, we can write f as

$$f(x)=\sum_{l=-\infty}^{k-2}f(x)\chi_l(x)+\sum_{l=k-1}^{\infty}f(x)\chi_l(x)=f_1(x)+f_2(x).$$

Thus

$$|T_bf(x)|\leq|T_bf_1(x)|+|T_bf_2(x)|,$$

and

$$\begin{aligned}&\|T_bf\|^p_{W\dot{K}^{\eta,p}_q(\mu,\mu^{1-q})}\\ &\leq\sup_{\lambda>0}\lambda^p\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{p\eta}{n}}\mu^{1-q}\Big(\Big\{x\in E_k:|(b(x)-b_{B_k})Tf_1(x)|>\frac{\lambda}{3}\Big\}\Big)^{\frac{p}{q}}\\ &\quad+\sup_{\lambda>0}\lambda^p\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{p\eta}{n}}\mu^{1-q}\Big(\Big\{x\in E_k:|T\big((b(x)-b_{B_k})f_1\big)(x)|>\frac{\lambda}{3}\Big\}\Big)^{\frac{p}{q}}\\ &\quad+\sup_{\lambda>0}\lambda^p\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{p\eta}{n}}\mu^{1-q}\Big(\Big\{x\in E_k:|T_bf_2(x)|>\frac{\lambda}{3}\Big\}\Big)^{\frac{p}{q}}\\ &:=G_1+G_2+G_3.\end{aligned}$$

By Lemma 2.4 and $0<p\leq 1$, we obtain

$$G_3\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{p\eta}{n}}\Big(\int_{\mathbb{R}^n}|T_bf_2(x)|^q\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{p}{q}}$$

$$
\begin{aligned}
&\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{p\eta}{n}}\Big(\int_{\mathbb{R}^n}|f_2(x)|^q\mu(x)\mathrm{d}x\Big)^{\frac{p}{q}}\\
&\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{p\eta}{n}}(\|f\chi_{k-1}\|^p_{L^q(\mu)}+\sum_{l=k}^{\infty}\|f\chi_l\|^p_{L^q(\mu)})\\
&\leq C\begin{cases}\sum\limits_{k=-\infty}^{\infty}\Big(\mu(B_{k-1})^{\frac{p\eta}{n}}2^{p\eta}\|f\chi_{k-1}\|^p_{L^q(\mu)}+\mu(B_l)^{\frac{p\eta}{n}}\sum\limits_{l=k}^{\infty}2^{(k-l)p\delta\eta}\|f\chi_l\|^p_{L^q(\mu)}\Big) & q\delta>1\\ \sum\limits_{k=-\infty}^{\infty}\Big(\mu(B_{k-1})^{\frac{p\eta}{n}}2^{p\delta\eta}\|f\chi_{k-1}\|^p_{L^q(\mu)}+\mu(B_l)^{\frac{p\eta}{n}}\sum\limits_{l=k}^{\infty}2^{(k-l)p\eta}\|f\chi_l\|^p_{L^q(\mu)}\Big) & q\delta<1\end{cases}\\
&\leq C\sum_{l=-\infty}^{\infty}\mu(B_l)^{\frac{p\eta}{n}}\|f\chi_l\|^p_{L^q(\mu)}\\
&\leq C\|f\|^p_{\dot{K}^{\eta,p}_q(\mu,\mu)}.
\end{aligned}
$$

For G_1, by the Hölder inequality and Lemmas 2.1–2.3, we have

$$
\begin{aligned}
G_1&\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{p\eta}{n}}\Big(\int_{E_k}\Big|(b(x)-b_{B_k})\int_{\mathbb{R}^n}K(x-y)f_1(y)\mathrm{d}y\Big|^q\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{p}{q}}\\
&\leq C\sum_{k=-\infty}^{\infty}2^{-knp}\mu(B_k)^{\frac{p\eta}{n}}\|f_1\|^p_{L_1}\Big(\int_{E_k}|b(x)-b_{B_k}|^q\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{p}{q}}\\
&\leq C\sum_{k=-\infty}^{\infty}2^{-knp}\mu(B_k)^{\frac{p\eta}{n}}\sum_{l=-\infty}^{k-2}\|f\chi_l\|^p_{L^q(\mu)}|B_l|^p\mu(B_l)^{-\frac{p}{q}}\mu(B_k)^{\frac{p}{q}}\\
&\leq C\sum_{l=-\infty}^{\infty}\mu(B_l)^{\frac{p\eta}{n}}\|f\chi_l\|^p_{L^q(\mu)}\sum_{k=l+2}^{\infty}\begin{cases}2^{(k-l)pn(\delta-1)} & q\delta>1\\ 2^{(k-l)pn[(\delta-1)(1+\frac{1}{q'})]} & q\delta<1\end{cases}\\
&\leq C\|f\|^p_{\dot{K}^{\eta,p}_q(\mu,\mu)}.
\end{aligned}
$$

For G_2, by Lemmas 2.1–2.3, we have

$$
\begin{aligned}
G_2&\leq C\sum_{k=-\infty}^{\infty}\mu(B_k)^{\frac{p\eta}{n}}\Big(\int_{E_k}\Big|\int_{\mathbb{R}^n}K(x-y)(b(y)-b_{B_k})f_1(y)\mathrm{d}y\Big|^q\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{p}{q}}\\
&\leq C\sum_{k=-\infty}^{\infty}2^{-knp}\mu(B_k)^{\frac{p\eta}{n}}\sum_{l=-\infty}^{k-2}\Big(\int_{E_l}|(b(y)-b_{B_k})f(y)\chi_l(y)|\mathrm{d}y\Big)^p\Big(\int_{B_k}\mu(x)^{1-q}\mathrm{d}x\Big)^{\frac{p}{q}}\\
&\leq C\sum_{k=-\infty}^{\infty}2^{-knp}\mu(B_k)^{\frac{p\eta}{n}}\sum_{l=-\infty}^{k-2}\|f\chi_l\|^p_{L^q(\mu)}(k-l)^p\mu(B_l)^{\frac{p}{q'}}|B_k|^p\mu(B_k)^{-\frac{p}{q'}}\\
&\leq C\sum_{l=-\infty}^{\infty}\mu(B_l)^{\frac{p\eta}{n}}\|f\chi_l\|^p_{L^q(\mu)}\sum_{k=l+2}^{\infty}(k-l)^p\begin{cases}2^{\frac{np}{q}(k-l)(\delta-1)} & q\delta>1\\ 2^{np\delta(k-l)(\delta-1)} & q\delta<1\end{cases}\\
&\leq C\|f\|^p_{\dot{K}^{\eta,p}_q(\mu,\mu)}.
\end{aligned}
$$

Then we finish the proof. □

4. Conclusions

Some new weighted inequalities for commutators related to singular integral operator satisfying a variant of Hörmander's condition are proved. Our results further generalize the corresponding results in [8] and include the unweighted case.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to thank the referee for his/her invaluable comments and suggestions.

This work was supported by a Research Project of the Basic Scientific Research Expenses for Heilongjiang Provincial Higher Institutions (No.1355ZD010), the Reform and Development Foundation for Local Colleges and Universities of the Central Government (Excellent Young Talents Project of Heilongjiang Province) (No.2020YQ07), the project of Mudanjiang Normal University (No.GP2019006), the Ideology and Politics Project of Postgraduate course (No.KCSZKC-2022026, KCSZAL-2022013) and the project of Scientific Research Team of Education Department of Heilongjiang Province (No.1354MSYTD006; 2019-KYYWF-0909).

Conflict of interest

All of authors in this article declare no conflict of interest.

References

1. A. P. Calderón, A. Y. Zygmund, On the existence of certain singular integrals, *Acta Math.*, **88** (1952), 85–139. https://doi.org/10.1007/978-94-009-1045-4_3
2. A. P. Calderón, A. Y. Zygmund, On singular integrals, *Am. J. Math.*, **78** (1956), 289–309. https://doi.org/10.1007/978-3-642-10918-8_3
3. L. Hörmander, Estimates for translation invariant operators in L^p spaces, *Acta Math.*, **104** (1960), 93–140. https://doi.org/10.1007/BF02547187
4. D. J. Grubb , C. N. Moore, A variant of Hörmander's condition for singular integrals, *Colloq. Math.*, **73** (1997), 165–172.
5. R. Trujillo-González, Weighted norm inequalities for singular integral operators satisfying a variant of Hörmander's condition, *Comment. Math. Univ. Ca.*, **44** (2003), 137–152.
6. R. R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. Math.*, **103** (1976), 611–635. https://doi.org/10.2307/1970954
7. W. J. Zhou, Multilinear commutators generated by singular integrals, fractional integrals and smooth functions, *Hunan Univ.*, 2004.

8. X. B. Kong, Y. S. Jiang, Weighted BMO estimates for commutators, *Acta Math. Sin. Chinese Ser.*, **52** (2009), 429–440.
9. Z. G. Liu, J. Liu, Boundedness of Calderón-Zygmund operators and commutators, *Acta Math. Sin. Chinese Ser.*, **53** (2010), 541–550.
10. B. T. Anh, X. T. Duong, On commutators of vector BMO functions and multilinear singular integrals with non-smooth kernels, *J. Math. Anal. Appl.*, **371** (2010), 80–94. https://doi.org/10.1016/j.jmaa.2010.03.058
11. I. Ekincioglu, C. Keskin, A. Serbetci, Multilinear commutators of Calderón-Zygmund operator on generalized variable exponent Morrey spaces, *Positivity*, **25** (2021), 1551–1567. https://doi.org/10.1007/s11117-021-00828-3
12. V. S. Guliyev, A. F. Ismayilova, Calderón-Zygmund operators with kernels of Dini's type and their multilinear commutators on generalized weighted Morrey spaces, *Twms J. Pure Appl. Math.*, **12** (2021), 265–277. https://doi.org/10.1007/s11117-021-00846-1
13. F. Deringoz, V. S. Guliyev, M. N. Omarova, M. A Ragusa, Calderón-Zygmund operators and their commutators on generalized weighted Orlicz-Morrey spaces, *B. Math. Sci.*, **13** (2023), 2250004. https://doi.org/10.48550/arXiv.2204.13898
14. D. Q. Zhang, Boundedness of commutators of singular integral operators satisfying a variant of Hörmander's condition, *Heilongjiang Univ.*, 2011.
15. Z. Q. Xie, Boundedness of multilinear singular integral operators satisfying a variant of Hörmander's condition, *Hunan Univ.*, 2017.
16. J. Pan, Boundedness on Morrey space of multilinear singular integral operators satisfing a variant of Hörmander's condition, *J. Math. Inequal.*, **11** (2017), 99–112. https://doi.org/10.7153/jmi-11-09
17. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Am. Math. Soc.*, **165** (1972), 207–226. https://doi.org/10.1090/S0002-9947-1972-0293384-6
18. J. Garciá-Cuerva, J. L. R. de Francia, Weighted norm inequalities and related topics, *Amsterdam: North Holland*, 1985. https://doi.org/10.1016/s0304-0208(08)x7154-3
19. D. Q. Zhang, J. Sun, The boundedness of commutataors satisfying a variant of Hörmander's condition on Hardy spaces, *J. Hebei Norm. Univ.*, **36** (2012), 230–233.
20. S. Z. Lu, D. C. Yang, The decomposition of weighted Herz space on $\mathbb{R}^n$ and its applications, *Sci. China*, **38A** (1995), 147–158.
21. Y. Lin, Z. G. Liu, M. M. Song, Lipschitz estimates for commutators of singular integral operators on weighted Herz spaces, *Jordan J. Math. Stat.*, **3** (2010), 53–64.
22. C. H. Wu, M. Zhang, Weighted sharp maximal function inequalities and boundedness of multilinear singular integral operator satisfying a variant of Hörmander's condition, *J. Inequal. Appl.*, **2014** (2014), 1–16. https://doi.org/10.1186/1029-242X-2014-57

AIMS Press