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Research article

Vector valued bilinear Calderón-Zygmund operators with kernels of Dini's type in variable exponents Herz-Morrey spaces

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Abstract: The main purpose of this paper is to establish the weighted boundedness result of vector valued bilinear $\varpi(t)$-type Calderón-Zygmund operators in variable exponents Herz-Morrey spaces, where ϖ being nondecreasing and $\varpi \in \text{Dini}(1)$.

Keywords: vector valued; $\varpi(t)$-type Calderón-Zygmund operator; Herz-Morrey space; variable exponent
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1. Introduction

1.1. Bilinear $\omega(t)$-type Calderón-Zygmund operators

In 1985, Yabuta [1] proposed the definitions of $\varpi(t)$-type Calderón-Zygmund operators, he introduced certain $\varpi(t)$-type Calderón-Zygmund operators to facilitate his study of certain classes of pseudodifferential operators. After that, Maldonado and Naibo [2] established the weighted norm inequalities for the bilinear Calderón-Zygmund operators of type $\varpi(t)$, and applied them to the study of para-products and bilinear pseudo-differential operators with mild regularity. In 2009, Lu and Zhang [3] established the a number of results concerning boundedness of multi-linear $\varpi(t)$-type Calderón-Zygmund operators. we recall the so-called $\varpi(t)$-type Calderón-Zygmund operators.

Let $\varpi(t)$: $[0, \infty) \to [0, \infty)$ be a nondecreasing function with $0 < \varpi(1) < \infty$. For $\alpha > 0$, we say that $\varpi \in$ Dini(a) if

$$|\varpi|_{\text{Dini}(\alpha)} = \int_0^1 \frac{\varpi^\alpha(t)}{t} \mathrm{d}t < \infty. \tag{1.1}$$

It is evident that for $0 < \alpha_1 < \alpha_2$, there is $\mathrm{Dini}(\alpha_1) < \mathrm{Dini}(\alpha_2)$. If $\varpi \in \mathrm{Dini}(1)$, then

$$\sum_0^\infty \varpi(2^{-j}) \approx \int_0^1 \frac{\varpi(t)}{t}\mathrm{d}t < \infty,$$

here and in what follows, for any quantities A and B, if there exists a constant $C > 0$ such that $A \leq CB$, we write $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$.

A measurable function $K(\cdot,\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \backslash \{(x, y_1, y_2) : x = y_1 = y_2\}$ is said to be a bilinear $\varpi(t)$-type Calderón-Zygmund kernel if it satisfies: for all $(x, y_1, y_2) \in \mathbb{R}^n$ with $x \neq y_i,\ i = 1, 2$, if there exists a constant $A > 0$ such that

$$|K(x, y_1, y_2)| \leq A\varpi(\sum_{i=1}^{2} |x - y_i|)^{-2n}, \tag{1.2}$$

and for $(x, y_1, y_2) \in (\mathbb{R}^n)^3$ with $x \neq y_1, y_2$, and

$$|K(x, y_1, y_2) - K(z, , y_1, y_2)| \leq A\omega\left(\frac{|x - z|}{\sum_{i=1}^{2} |x - y_i|}\right)\left[\sum_{i=1}^{2} |x - y_i|\right]^{-2n}. \tag{1.3}$$

whenever $2|x - z| < \max\{|x - y_1|, |x - y_2|\}$.

Definition 1.1. ([2]) Let $\varpi \in \mathrm{Dini}(1)$. One can say that T_ϖ is a bilinear $\varpi(t)$-type operator with the kernel K satisfying (1.2) and (1.3), for all f_1 ,$f_2 \in C_c^\infty(\mathbb{R}^n)$,

$$T_\varpi(f_1, f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2,\quad x \notin \mathrm{supp} f_1 \cap \mathrm{supp} f_2. \tag{1.4}$$

1.2. Products of weighted Herz-Morrey spaces with variable exponents

In the following, for each $k \in \mathbb{Z}$, we define $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $D_k = B_k \backslash B_{k-1}$, $\chi_k = \chi_{D_k}$, $m \geq 1$, $\widetilde{\chi}_0 = \chi_{B_0}$.

Given a function $p(x) \in \mathcal{P}(\mathbb{R}^n)$, the space $L^{p(x)}(\mathbb{R}^n)$ is now defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x \leq 1\right\}.$$

Denote $\mathcal{P}(\mathbb{R}^n)$ to be the set of the all measurable functions $p(x)$ with

$$p_- =: \operatorname{ess}\inf_{x \in \mathbb{R}^n} p(x) > 1$$

and

$$p_+ =: \operatorname{ess}\sup_{x \in \mathbb{R}^n} p(x) < \infty,$$

and $\mathcal{B}(\mathbb{R}^n)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that the Hardy-littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, $\mathcal{P}^0(\mathbb{R}^n)$ the set of all measurable functions $p(x)$ with $p_- > 0$ and $p_+ < \infty$.

The space $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) = \{f : f\chi_K \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) \text{ for all compact subsets K} \in \mathbb{R}^n\},$$

where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and ω be a nonnegative measurable function on $\mathbb{R}^n$. Then the weighted variable exponent Lebesgue space $L^{p(\cdot)}(\omega)$ is the set of all complex-valued measurable functions f such that $f\omega \in L^{p(\cdot)}$. The space $L^{p(\cdot)}(\omega)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\omega)} = \|f\omega\|_{L^{p(\cdot)}}.$$

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the standard Hardy-Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|} \int_B f(y)\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^n,$$

where the supremum is taken over all balls containing x in $\mathbb{R}^n$.

Definition 1.2. ([4]) Let $\alpha(\cdot)$ be a real-valued function on $\mathbb{R}^n$.

(i) For any $x, y \in \mathbb{R}^n$, $|x - y| < 1/2$, if

$$|\alpha(x) - \alpha(y)| \lesssim \frac{1}{\log(e + 1/|x - y|)},$$

then $\alpha(\cdot)$ is said local log-Hölder continuous on $\mathbb{R}^n$.

(ii) For all $x \in \mathbb{R}^n$, if

$$|\alpha(x) - \alpha(0)| \lesssim \frac{1}{\log(e + 1/|x|)},$$

then $\alpha(\cdot)$ is said log-Hölder continuous functions at origin, denote by $\mathcal{P}^{\log}_0(\mathbb{R}^n)$ the set of all log-Hölder continuous at origin.

(iii) If there exists $\alpha_\infty \in \mathbb{R}$, for $x \in \mathbb{R}^n$, if

$$|\alpha(x) - \alpha_\infty| \lesssim \frac{1}{\log(e + |x|)},$$

then $\alpha(\cdot)$ is said log-Hölder continuous at infinity, denote by $\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at infinity.

(iv) The function $\alpha(\cdot)$ is global log-Hölder continuous if $\alpha(\cdot)$ are both locally log-Hölder continuous and log-Hölder continuous at infinity. Denote by $\mathcal{P}^{\log}(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions.

Let ω be a weighted function on $\mathbb{R}^n$, that is, ω is real-valued, non-negative and locally integrable. ω is said to be a Muckenhoupt A_1 weight if

$$M\omega(x) \lesssim \omega(x) \quad a.e., x \in \mathbb{R}^n.$$

For $1 < p < \infty$, we say that ω is an A_p weight if

$$\sup_B \left(\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}\mathrm{d}x\right)^{p-1} < \infty.$$

Definition 1.3. ([5]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. For some constant C, a weight ω is said to be an $A_{p(\cdot)}$ weight, if for all balls B in $\mathbb{R}^n$ such that

$$\frac{1}{|B|}\|\omega\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\omega^{-1}\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 1.1. ([5]) If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $\omega \in A_{p(\cdot)}$, then for each $f \in L^{p(\cdot)}(\omega)$,

$$\|(Mf)\omega\|_{L^{p(\cdot)}} \lesssim \|f\omega\|_{L^{p(\cdot)}},$$

Before give the definitions of the weighted Herz space and Herz-Morrey space with variable exponents, we also need the notation of the variable mixed sequence space $\ell^q(L^{p(\cdot)})$, which was firstly defined in [6]. Let ω be a nonnegative measurable function. Given a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$, we define the modular

$$\rho_{\ell^q(L^{p(\cdot)}(\omega))}((f_j)_j) = \sum_{j\in\mathbb{Z}} \inf\Big\{\lambda_j : \int_{\mathbb{R}^n}\Big(\frac{|f_j(x)\omega(x)|}{\lambda_j^{\frac{1}{q(x)}}}\Big)^{p(x)}\mathrm{d}x \leq 1\Big\},$$

where $\lambda^{\frac{1}{\infty}} = 1$. If $q^+ < \infty$ or $q(\cdot) \leq p(\cdot)$, the above can be written as

$$\rho_{\ell^q(L^{p(\cdot)}(\omega))}((f_j)_j) = \sum_{j\in\mathbb{Z}} \left\| |f_j\omega|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

The norm is

$$\|(f_j)_j\|_{\rho_{\ell^q(L^{p(\cdot)}(\omega))}} = \inf\Big\{\mu > 0 : \rho_{\ell^q(L^{p(\cdot)}(\omega))}((\frac{f_j}{\mu})_j) \leq 1\Big\}.$$

Definition 1.4. ([7]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}^0(\mathbb{R}^n)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\omega)$ are defined by

$$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\omega) = \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\},\omega) : \|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\omega)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\omega)} = \|(2^{j\alpha(\cdot)}f\chi_j)_j\|_{\rho_{\ell^q(L^{p(\cdot)}(\omega))}}.$$

Lemma 1.2. ([7]) Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ and ω be a weight. If $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at the origin, then T

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\omega)} = \|f\|_{\dot{K}^{\alpha_\infty,q_\infty}_{p(\cdot)}(\omega)}.$$

Additionally, if $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at the origin, then

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\omega)} \approx \Big(\sum_{k\leq 0}\|2^{k\alpha(0)}f\chi_k\|^{q(0)}_{L^{p(\cdot)}}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k>0}\|2^{k\alpha_\infty}f\chi_k\|^{q(0)}_{L^{p(\cdot)}}\Big)^{\frac{1}{q_\infty}}.$$

Definition 1.5. ([8]) Let $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, $\lambda \in [0,1)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous weighted Herz-Morrey space $M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(\omega)$ are defined by

$$M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(\omega) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}, \omega) : \|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(\omega)} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(\omega)} = \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \|(2^{k\alpha(\cdot)k} f\chi_k)_{k\le L}\|_{\rho_{\ell^{q}(L^{p(\cdot)}(\omega))}}.$$

Lemma 1.3. ([8]) Let $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, ω be a weight, $\lambda \in [0,\infty)$ and $\alpha \in L^\infty(\mathbb{R}^n)$. If $\alpha(\cdot)$, $q(\cdot) \in \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, then for any $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}, \omega)$,

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(\omega)} \approx \max\Bigg\{ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \|(2^{k\alpha(0)} f\chi_k)_{k\le L}\|_{\ell^{q_0}(L^{p(\cdot)}(\omega))},$$
$$\sup_{L>0, L\in\mathbb{Z}} \Big[2^{-L\lambda} \|(2^{k\alpha(0)} f\chi_k)_{k\le L}\|_{\rho_{\ell^{q_0}(L^{p(\cdot)}(\omega))}} + 2^{-L\lambda} \|(2^{k\alpha_\infty} f\chi_k)^{L}_{k=0}\|_{\rho_{\ell^{q_0}(L^{p(\cdot)}(\omega))}} \Big] \Bigg\},$$

where and hereafter, $q_0 = q(0)$.

Lemma 1.4. ([8]) If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $\omega \in A_{p(\cdot)}$, then there exist constants $\delta_1, \delta_2 \in (0,1)$, such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\omega)}}{\|\chi_B\|_{L^{p(\cdot)}(\omega)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\omega^{-1})}}{\|\chi_B\|_{L^{p'(\cdot)}(\omega^{-1})}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

2. Background knowledges and notations

Before proving the main results, we need the following lemmas.

For $\delta > 0$, we denote $[M(|f|^\delta)]^{\frac{1}{\delta}}$ by M_δ. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then the sharp maximal function is defined by

$$M^\# f(x) = \sup_Q \frac{1}{Q} \int_Q |f(y) - f_Q| \text{dy},$$

where the supremum is taken over all the cubes Q containing the point x, and where as usual f_Q denotes the average of f on Q. we denote $[M^\#(|f|^\delta)]^{\frac{1}{\delta}}$ by $M^\#_\delta$.

Lemma 2.1. ([3]) Let T_ω be a bilinear $\omega(t)$-type Calderón-Zygmund operator with $\varpi \in$ Dini(1) and let $0 < \delta < \frac{1}{2}$. Then, for any vector function $\vec{f} = (f_1, f_2)$, where each component is smooth and with compact support, the following inequality holds

$$M^\#_\delta(T_\omega(f_1, f_2))(x) \lesssim M(f_1)(x) M(f_2)(x).$$

Lemma 2.2. ([9]) Let $0 < p,\ \delta < \infty$ and $\omega \in A_\infty$. There exists a positive constant C such that

$$\int_{\mathbb{R}^n} [M_\delta f(x)]^p \omega(x) \text{dx} \le \int_{\mathbb{R}^{\text{n}}} [\text{M}^\#_\delta \text{f(x)}]^{\text{p}} \omega(\text{x}) \text{dx}$$

for every function f such that the left hand side is finite.

Lemma 2.3. ([10]) Let $p(\cdot),\ p_1(\cdot),\ p_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ such that $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$. Then for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, there exists

$$\|fg\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot)}}\|g\|_{L^{p_2(\cdot)}}$$

If $p \in \mathcal{P}(\mathbb{R}^n)$, ω is a weight with $\omega = \omega_1 \times \omega_2$, there exists

$$\|fg\|_{L^{p(\cdot)}(\omega)} \lesssim \|f\|_{L^{p_1(\cdot)}(\omega_1)}\|g\|_{L^{p_2(\cdot)}(\omega_2)}.$$

Lemma 2.4. ([11]) Let $0 < p \leq \infty$, $\delta > 0$. Then for non-negative sequence $\{a_j\}_{j=-\infty}^{\infty}$, there exists

$$\Big(\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\infty} 2^{-|k-j|^{\delta}} a_k\Big)^p\Big)^{\frac{1}{p}} \lesssim (\sum_{j=-\infty}^{\infty} a_j^p)^{\frac{1}{p}},$$

when $p = \infty$, above inequality stands for

$$\sum_{k=-\infty}^{\infty} (2^{-|k-j|^{\delta}} a_k) \lesssim \sup_{j\in\mathbb{Z}} a_j.$$

Lemma 2.5. ([12]) Assume that for some $p_0 \in (0,\infty)$ and every $\omega_0 \in A_\infty$, let $\mathcal{F}$ be a family of pairs of non-negative functions such that

$$\int_{\mathbb{R}^n} f(x)^{p_0}\omega_0(x)\mathrm{d}x \lesssim \int_{\mathbb{R}^n} g_0(x)^{p_0}\omega_0(x)\mathrm{d}x,\ \ (f,g) \in \mathcal{F}. \tag{2.1}$$

Then for all $0 < p < \infty$ and $\omega_0 \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^{p}\omega_0(x)\mathrm{d}x \lesssim \int_{\mathbb{R}^n} g_0(x)^{p}\omega_0(x)\mathrm{d}x,\ \ (f,g) \in \mathcal{F}.$$

Furthermore, for every $p, q \in (0,\infty)$, $\omega_0 \in A_\infty$, and sequences $\{(f_j, g_j)\} \in \mathcal{F}$,

$$\Big\|\Big(\sum_{j=1}^{\infty}(f_j)\Big)^q\Big\|_{L^p(\omega_0)} \lesssim \Big\|\Big(\sum_{j=1}^{\infty}(g_j)\Big)^q\Big\|_{L^p(\omega_0)}. \tag{2.2}$$

Lemma 2.6. ([8]) Assume that for some p_0 and let F be a family of pairs of non-negative functions such that (2.1) holds. Let $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. If there exists $s \leq p_-$ such that $\omega^s \in A_{\frac{p(\cdot)}{s}}$ and M is bounded on $L^{(\frac{p(\cdot)}{s})'}(\omega^{-s})$. Then for every $q \in (1,\infty)$ and sequence $\{(f_j, g_j)\}_{j\in\mathbb{N}} \subset \mathcal{F}$

$$\Big\|\Big(\sum_{j=1}^{\infty}(f_j)\Big)^q\Big\|_{L^{p(\cdot)}(\omega)} \lesssim \Big\|\Big(\sum_{j=1}^{\infty}(g_j)\Big)^q\Big\|_{L^{p(\cdot)}(\omega)}.$$

Lemma 2.7. ([13]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and ω be a weight. If the maximal operator M is bounded both on $L^{p(\cdot)}(\omega)$ and $L^{p'(\cdot)}(\omega^{-1})$, $q \in (1.\infty)$, then

$$\Big\|\Big(\sum_{j=1}^{\infty}(Mf_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\omega)} \lesssim \Big\|\Big(\sum_{j=1}^{\infty}|f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\omega)}.$$

Lemma 2.8. Let T_ϖ be a bilinear Calderón-Zygmund operator with $\varpi \in \text{Dini}(1)$ and $p(\cdot) \in \mathcal{P}_0$ such that there exists $s \le p_-$ such that $\omega^s \in A_{\frac{p(\cdot)}{s}}$ and M is bounded on $L^{(\frac{p(\cdot)}{s})'}(\omega^{-s})$. Suppose that $\omega = \omega_1 \times \omega_2$ and $\omega_i \in A_{p_i(\cdot)},\ i = 1, 2$. If $p_i \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ $(i = 1, 2)$ satisfying

$$\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$$

for $x \in \mathbb{R}^n$. Then for compactly supported bounded functions $f_1^j, f_2^j \in L^{p_0}(\mathbb{R}^n)$, $j \in \mathbb{N}$ such that

$$\Big\|\Big(\sum_{j=1}^{\infty} |T_\varpi(f_1^j, f_2^j)|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\omega)} \lesssim \prod_{i=1}^{2} \Big\|\Big(\sum_{j=1}^{\infty} |f_i^j|^{q_i}\Big)^{\frac{1}{q_i}}\Big\|_{L^{p_i(\cdot)}(\omega_i)},$$

where $q_i \in (1, \infty)$ for $i = 1, 2$ and

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Proof of Lemma 2.8. Since f_1^j, f_2^j are bounded functions with compact support, $T_\varpi(f_1^j, f_2^j) \in L^p(\mathbb{R}^n)$ for every $0 < p < \infty$. With Lemmas 2.1 and 2.2, Lu and Zhang [3] showed that for all $\omega \in A_\infty$,

$$\int_{\mathbb{R}^n} |T_\varpi(f_1, f_2)(x)|^p \omega(x) \mathrm{d}x \lesssim \int_{\mathbb{R}^n} (Mf_1(x)Mf_2(x))^p \omega(x) \mathrm{d}x.$$

Therefore, by Lemmas 2.5 and 2.6, we have

$$\Big\|\Big(\sum_{j=1}^{\infty} |T_\varpi(f_1^j, f_2^j)|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\omega)} \lesssim \Big\|\Big(\sum_{j=1}^{\infty} |Mf_1^j(x)Mf_2^j(x)|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\omega)}.$$

Since

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

and $\omega = \omega_1\omega_2$, together with Hölders inequality, Lemmas 2.3 and 2.7, we have

$$\begin{aligned}
\Big\|\Big(\sum_{j=1}^{\infty} |Mf_1^j(x)Mf_2^j(x)|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(\omega)} &\lesssim \prod_{i=1}^{2} \Big\|\Big(\sum_{j=1}^{\infty} |Mf_i^j|^{q_i}\Big)^{\frac{1}{q_i}}\Big\|_{L^{p_i(\cdot)}(\omega_i)} \\
&\lesssim \prod_{i=1}^{2} \Big\|\Big(\sum_{j=1}^{\infty} |f_i^j|^{q_i}\Big)^{\frac{1}{q_i}}\Big\|_{L^{p_i(\cdot)}(\omega_i)}.
\end{aligned}$$

We complete the proof of Lemma 2.8.

3. Boundedness of the vector valued bilinear $\varpi(t)$-type Calderón-Zygmund operators

Theorem 3.1. Let T_ϖ be a bilinear ϖ-type Calderón-Zygmund operator with $\varpi \in \text{Dini}(1)$, p_1 and $p_2 \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}^{\log}(\mathbb{R}^n)$ santisfying

$$\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$$

and $p(\cdot)\in\mathcal{P}_0$ such that there exists $s\le p_-$ such that $\omega^s\in A_{\frac{p(\cdot)}{s}}$ and M is bounded on $L^{(\frac{p(\cdot)}{s})'}(\omega^{-s})$, where $\omega=\omega_1\omega_2$ and $\omega_i\in A_{p_i(\cdot)}$, $i=1,2$. Suppose that

$$\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n),\quad \alpha(0)=\alpha_1(0)+\alpha_2(0),$$

$$\alpha_\infty=\alpha_{1\infty}+\alpha_{2\infty},\quad q(\cdot)\in\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n),$$
$$\frac{1}{q(0)}=\frac{1}{q_1(0)}+\frac{1}{q_2(0)},\quad \frac{1}{q_\infty}=\frac{1}{q_{1\infty}}+\frac{1}{q_{2\infty}},$$
$$\lambda=\lambda_1+\lambda_2,\quad 0\le\lambda_i<\infty,\quad \delta_{i1},\delta_{i2}\in(0,1)$$

are the constants in Lemma 1.4 for exponents $p_i(\cdot)$ and weights $\omega_i(\ i=1,2)$. Let $r_i\in(1,\infty)$ and

$$\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}.$$

If $\lambda_i-n\delta_{i1}<\alpha_{i\infty}$, $\alpha_i(0)\le n\delta_{i2}$, then

$$\Big\|\Big(\sum_{j=1}^{\infty}|T_\varpi(f_1^j,f_2^j)|^r\Big)^{\frac{1}{r}}\Big\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(\omega)}\lesssim\prod_{i=1}^{2}\Big\|\Big(\sum_{j=1}^{\infty}|(f_i^j)|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(\omega_i)}$$

for all $f_i^j\in M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(\omega_i)$, $j\in\mathbb{N}$, $i=1,2$.

Proof of Theorem 3.1. We only consider bounded compact supported functions for the set of all bounded compactly supported functions is dense in weighted variable Lebesgue spaces (see [13]). Let f_1^v and f_2^v be bounded functions with compact support for $v\in\mathbb{N}$ and write

$$f_i^v=\sum_{l=-\infty}^{\infty}f_{il}^v\chi_l=\sum_{l=-\infty}^{\infty}f_{il}^v,\quad i=1,2,v\in\mathbb{N}.$$

By Lemma 1.3, we have

$$\begin{aligned}\Big\|\Big(\sum_{v=1}^{\infty}|T_\varpi(f_1^v,f_2^v)|^r\Big)^{\frac{1}{r}}\Big\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)}&\approx\max\Big\{\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\Big\|(2^{k\alpha(0)}(\sum_{v=1}^{\infty}|T_\varpi(f_1^v,f_2^v)|^r)^{\frac{1}{r}}\chi_k\Big)_{k\le L}\Big\|_{\ell^{q_0}(L^{p(\cdot)}(w))}\\&\sup_{L>0,L\in\mathbb{Z}}\Big[2^{-L\lambda}\Big\|\Big(2^{k\alpha(0)}(\sum_{v=1}^{\infty}|T_\varpi(f_1^v,f_2^v)|^r)^{\frac{1}{r}}\chi_k\Big)_{k<0}\Big\|_{\ell^{q_0}(L^{p(\cdot)}(w))}\\&+2^{-L\lambda}\Big\|\Big(2^{k\alpha_\infty}\Big(\sum_{v=1}^{\infty}\Big|T_\varpi(f_1^v,f_2^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big)_{k=0}^{L}\Big\|_{\ell^{q_\infty}(L^{p(\cdot)}(w))}\Big]\Big\}\\&=\max\{E,F\},\end{aligned}$$

where

$$\begin{aligned}E&=\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\Big\|\Big(2^{k\alpha(0)}\Big(\sum_{v=1}^{\infty}\Big|T_\varpi(f_1^v,f_2^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big)_{k\le L}\Big\|_{\ell^{q_0}(L^{p(\cdot)}(w))},\\F&=\sup_{L>0,L\in\mathbb{Z}}\{G+H\},\end{aligned}$$

$$
\begin{aligned}
G &= 2^{-L\lambda}\Big\|\Big(2^{k\alpha(0)}\Big(\sum_{v=1}^{\infty}\Big|T_{\varpi}(f_1^v,f_2^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big)_{k<0}\Big\|_{\ell^{q_0}(L^{p(\cdot)}(w))},\\
H &= 2^{-L\lambda}\Big\|\Big(2^{k\alpha_\infty}\Big(\sum_{v=1}^{\infty}\Big|T_{\varpi}(f_1^v,f_2^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big)_{k=0}^{L}\Big\|_{\ell^{q_\infty}(L^{p(\cdot)}(w))}.
\end{aligned}
$$

Since to estimate G is essentially similar to estimate E, it is suffice to obtain that E and H are bounded in Herz-Morrey space with variable exponents. It is easy to see that

$$
E \lesssim \sum_{i=i}^{9} E_i, \quad H \lesssim \sum_{i=i}^{9} H_i,
$$

where

$$
\begin{aligned}
E_1 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_2 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_3 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_4 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=-\infty}^{k-2} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_5 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_6 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_7 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}|\sum_{l=k+2}^{\infty}\sum_{j=-\infty}^{k-2} T_{\varpi}(f_{1l}^v,f_{2j}^v)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_8 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k-1}^{k+1} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_9 &= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
H_1 &= 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},\\
H_2 &= 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1} T_{\varpi}(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},
\end{aligned}
$$

$$H_3 = 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty}T_\varpi(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_4 = 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=-\infty}^{k-2}T_\varpi(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_5 = 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}T_\varpi(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_6 = 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty}T_\varpi(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_7 := 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=-\infty}^{k-2}T_\varpi(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_8 := 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k-1}^{k+1}T_\varpi(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_9 := 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty}T_\varpi(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}.$$

We will use the following estimates. If $l \le k-1$, by Hölder's inequality, Lemma 1.4 and Definition 1.3, we have

$$\begin{aligned}\Big\|2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}^v|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i)} &\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}w_i\chi_l\Big\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}\\ &\le C2^{-kn}|B_k|\|\chi_{B_k}\|_{L^{p_i'(\cdot)}(w_i^{-1})}^{-1}\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\\ &\le C2^{(l-k)n\delta_{2i}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}.\end{aligned}\tag{3.1}$$

If $l = k$, then

$$\begin{aligned}\Big\|2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}^v|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i} &\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}w_i\chi_l\Big\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}\\ &\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\\ &\le \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}.\end{aligned}\tag{3.2}$$

If $l \ge k+1$, then

$$\Big\|2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}^v|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i)} \le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}w_i\chi_l\Big\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}$$

$$
\begin{aligned}
&\leq C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|^{-1}_{L^{p_i(\cdot)}(w_i)}\\
&\quad\times\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\\
&\leq C2^{(l-k)n\left(1-\delta_{1i}\right)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}. \qquad (3.3)
\end{aligned}
$$

Reverse the order of f_1 and f_2, it is obviously that the estimates of E_2, E_3 and E_6 are similar to those of E_4, E_7 and E_8, respectively. Thus We just need to estimate E_1–E_3, E_5, E_6 and E_9.

For E_1, since $l,\ j \leq k-2$, then for $i = 1, 2$,

$$
|x-y_i| \geq |x|-|y_i| > 2^{k-1}-2^{\min\{l,j\}} \geq 2^{k-2}, \quad x\in D_k,\ y_1\in D_l,\ y_2\in D_j.
$$

Therefore, for $x \in D_k$, we have

$$
|K(x,y_1,y_2)| \leq C(|x-y_1|+|x-y_2|)^{-2n} \leq C2^{-2kn}.
$$

Thus, for any $x \in D_k$ and $\mathrm{l,j \leq k-2}$, we have

$$
\begin{aligned}
|T(f_{1l}^v, f_{2j}^v)(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_{1l}^v(y_1)||f_{2j}^v(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim 2^{-2kn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)||f_{2j}^v(y_2)|\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned}
$$

Hence, together with the Hölder's and Minkowski's inequality, we have

$$
\begin{aligned}
&\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}T_{\varpi}(f_{1l}^v, f_{2j}^v)\Big|^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
\lesssim\ &\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2)^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
\lesssim\ &\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\Big)^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\times\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2)^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\|_{L^{p_2(\cdot)}(w_2)}\\
\lesssim\ &\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\times\Big\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}. \qquad (3.4)
\end{aligned}
$$

Since

$$
\frac{1}{q_(0)}=\frac{1}{q_1(0)}+\frac{1}{q_2(0)} \quad \text{and} \quad \lambda=\lambda_1+\lambda_2,
$$

by Hölder's inequality, we have

$$\begin{aligned} E_1 \ &\lesssim \ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1})^{\frac{1}{r_1}}\,\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \\ &\quad \times\Big\|\sum_{j=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}} \\ &\lesssim \ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\times(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)}\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1})^{\frac{1}{r_1}}\,\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)})^{\frac{1}{q_1(0)}} \\ &\quad \times\sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\times(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\|\sum_{j=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2})^{\frac{1}{r_2}}\,\mathrm{d}y_2\chi_k\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)})^{\frac{1}{q_2(0)}} \\ &= \ E_{1,1}\times E_{1,2}. \end{aligned}$$

For convenience's sake, we write

$$E_{1,i} = \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i}\times\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha_i(0)q_i(0)}\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}(y_i)|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\Big\}^{\frac{1}{q_i(0)}}.$$

For $n\delta_{i2}-\alpha_i(0)>0$, by (3.1) and Lemma 2.4 we have

$$\begin{aligned} E_{1,i} \ &\lesssim \ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha_i(0)q_i(0)} Big.\times\Big(\sum_{l=-\infty}^{k-2} 2^{(l-k)n\delta_{i2}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\Big)^{q_i(0)}\Big\}^{\frac{1}{q_i(0)}} \\ &= \ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i}\times\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l=-\infty}^{k-2} 2^{l\alpha_i(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)\big(n\delta_{i2}-\alpha_i(0)\big)}\Big)^{q_i(0)}\Big\}^{\frac{1}{q_i(0)}} \\ &\lesssim \ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{L-2} 2^{l\alpha_i(0)q_i(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\Big)^{\frac{1}{q_i(0)}} \\ &\lesssim \ \Big\|\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}(w_i)}, \end{aligned}$$

where we write $2^{-|k-l|(n\delta_{i2}-\alpha_i(0))}=2^{-|k-l|\varepsilon_i}$ for $\varepsilon_i=n\delta_{i2}-\alpha_i(0)>0$, then we have

$$E_1 \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^{v}|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^{v}|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.$$

To estimate E_2, since $l\le k-2$, $k-1\le j\le k+1$, then we have

$$|x-y_2|\ge|x-y_1|\ge|x|-|y_1|\ge 2^{k-2},\quad x\in D_k,\ y_1\in D_l,\ y_2\in D_j.$$

Therefore, for $x\in D_k$, we have

$$|K(x,y_1,y_2)|\le C(|x-y_1|+|x-y_2|)^{-2n}\le C2^{-2kn}.$$

Thus, for any $x \in D_k, l \le k-2, k-1 \le j \le k+1$, we have

$$\begin{aligned} |T(f_{1l}^v, f_{2j}^v)(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_{1l}^v(y_1)||f_{2j}^v(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim 2^{-2kn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f_{1l}^v(y_1)||f_{2j}^v(y_2)| dy_1 dy_2. \end{aligned}$$

Combining the Hölder's with and Minkowski's inequality, hence we obtain

$$\begin{aligned} &\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1} T_{\varpi}(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\ \lesssim\ &\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|dy_1 \sum_{j=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|dy_2\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\ \lesssim\ &\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|dy_1\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\ &\times\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|dy_2\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)} \\ \lesssim\ &\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^r\Big)^{\frac{1}{r}}dy_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\ &\times\Big\|\sum_{j=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^r\Big)^{\frac{1}{r}}dy_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{3.5}$$

Since

$$\frac{1}{q(0)} = \frac{1}{q_1(0)} + \frac{1}{q_2(0)} \quad \text{and} \quad \lambda = \lambda_1 + \lambda_2,$$

by Hölder's inequality, we have

$$\begin{aligned} E_2 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^r\Big)^{\frac{1}{r}}dy_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \\ &\quad\times\Big\|\sum_{j=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^r\Big)^{\frac{1}{r}}dy_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}} \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\times\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^r\Big)^{\frac{1}{r}}dy_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}} \\ &\quad\times\sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}\Big|f_{2j}^v(y_2)\Big|^r\Big)^{\frac{1}{r}}dy_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\ &= E_{2,1}\times E_{2,2}. \end{aligned}$$

It is obvious that

$$E_{2,1} = E_{1,1} \lesssim \|(\sum_{v=1}^{\infty}|f_1^v|^{r_1})^{\frac{1}{r_1}}\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}.$$

Now we turn to estimate $E_{2,2}$. By (3.1)–(3.3), we have

$$\begin{aligned} E_{2,2} &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=k-1}^{k+1} 2^{(j-k)n}\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L+1} 2^{k\alpha_2(0)q_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\ &\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}, \end{aligned}$$

where we use $2^{-n\delta_{22}} < 1$ and $2^{(j-k)n(1-\delta_{12})} < 2^{(j-k)n} < 2^{2n}, j \in \{k-1, k, k+1\}$ for (3.1) and (3.3) respectively. Thus, we obtain

$$E_2 \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.$$

To estimate E_3, since $l \le k-2$ and $j \ge k+2$, we have

$$|x-y_1| \ge |x| - |y_1| \ge 2^{k-2}, \quad |x-y_2| \ge |y_2| - |x| > 2^{j-2}, \quad x \in D_k,\ y_1 \in D_l,\ y_2 \in D_j.$$

Therefore, for any $x \in D_k, l \le k-2, j \ge k+2$, we get

$$\begin{aligned} |T_\varpi(f_{1l}, f_{2j})(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_{1l}^v(y_1)||f_{2j}^v(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\ &\lesssim 2^{-kn}2^{-jn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)||f_{2j}^v(y_2)|\mathrm{d}y_1\mathrm{d}y_2. \end{aligned}$$

Thus, by Hölder's inequality and Minkowski's inequality, we have

$$\begin{aligned} &\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty} T(f_{1l}^v, f_{2j}^v)|^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\ \lesssim\ &\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\ \lesssim\ &\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\Big)^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\ &\times\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)} \\ \lesssim\ &\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\ &\times\Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{3.6}$$

Since

$$\frac{1}{q(0)}=\frac{1}{q_1(0)}+\frac{1}{q_2(0)}\quad \text{and}\quad \lambda=\lambda_1+\lambda_2,$$

by Hölder's inequality, we have

$$\begin{aligned}
E_3 &\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)}\\
&\quad\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}}\\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_1}\times\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}}\\
&\quad\times\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\times\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&= E_{3,1}\times E_{3,2}.
\end{aligned}$$

It is obvious that

$$E_{3,1}=E_{1,1}\lesssim\Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}.$$

Since $n\delta_{21}+\alpha_2(0)>0$, by (3.3), we obtain

$$\begin{aligned}
E_{3,2} &\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\times\Big(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_{21}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\times\Big(\sum_{k=-\infty}^{L}\Big(\sum_{j=k+2}^{L}2^{j\alpha_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)(n\delta_{21}+\alpha_2(0))}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\quad+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\times\Big(\sum_{k=-\infty}^{L}\Big(2^{k\alpha_2(0)}\sum_{j=L+1}^{0}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)n\delta_{21}}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\quad+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\times\Big(\sum_{k=-\infty}^{L}\Big(2^{k\alpha_2(0)}\sum_{j=1}^{\infty}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)n\delta_{21}}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&= I_1+I_2+I_3.
\end{aligned}$$

First, we consider I_1. By Lemma 2.4, we have

$$\begin{aligned}
I_1 &\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\times\Big(\sum_{k=-\infty}^{L}\Big(\sum_{j=k+2}^{L}2^{j\alpha_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)(n\delta_{21}+\alpha_2(0))}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{j=-\infty}^{L+2}2^{j\alpha_2(0)q_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}\big(w_2\big)},
\end{aligned}$$

where we write $2^{-|k-j|(n\delta_{21}+\alpha_2(0))} = 2^{-|k-j|\eta_2}$ for $\eta_2 = n\delta_{21} + \alpha_2(0) > 0$. Next, we consider I_2. Since $n\delta_{21} + \alpha_2(0) - \lambda_2 > 0$, we obtain

$$\begin{aligned}
I_2 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}\Big(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=L+1}^{0} 2^{j\alpha_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} \\
&\quad \times 2^{-j(n\delta_{21}+\alpha_2(0))}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \times 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}\Big(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=L+1}^{0} 2^{-j(n\delta_{21}+\alpha_2(0)-\lambda_2)}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}}\sup_{j\le 0} 2^{-j\lambda_2}2^{j\alpha_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{L\big(-n\delta_{21}-\alpha_2(0)\big)}\Big(\sum_{k=-\infty}^{L} 2^{k\big(n\delta_{21}+\alpha_2(0)\big)q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned}$$

Then, we consider I_3. Since $\delta_{21} + \alpha_2(0) - \lambda_2 > 0$, we obtain

$$\begin{aligned}
I_3 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}\Big(2^{k(n\delta_{21}+\alpha_2(0))}\times\sum_{j=1}^{\infty} 2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} 2^{-j(n\delta_{21}+\alpha_{2\infty})}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}}\sup_{j\ge 1} 2^{-j\lambda_2}2^{j\alpha_\infty}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} \\
&\quad \times 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}\Big(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=1}^{\infty} 2^{-j(n\delta_{21}+\alpha_{2\infty}-\lambda_2)}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L} 2^{k(n\delta_{21}+\alpha_2(0))q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)} \sup_{L\le 0, L\in\mathbb{Z}} 2^{L\big(-\lambda_2+n\delta_{21}+\alpha_2(0)\big)} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned}$$

Thus, we have

$$E_3 \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate E_5, using Hölder's inequality and Lemma 2.8, we have

$$E_5 \lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}\Big\|\Big(\sum_{v=1}^{\infty}\Big|T_\varpi(f_{1l},f_{2j})\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}}$$

$$
\begin{aligned}
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big(\Big\|\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{L^{p_1(\cdot)}(w_1)}\times\Big\|\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{L^{p_2(\cdot)}(w_2)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}}\\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_{1}^{v}|^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}}\\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_{2}^{v}|^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_{1}^{v}|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_{2}^{v}|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.
\end{aligned}
$$

To estimate E_6, since $k-1\le l\le k+1$ and $j\ge k+2$, we obtain

$$|x-y_1|>2^{k-2},\quad |x-y_2|>2^{j-2},\quad x\in D_k,\ y_1\in D_l,\ y_2\in D_j.$$

Thus, for any $x\in D_k, k-1\le l\le k+1$ and $j\ge k+2$, we obtain

$$
\begin{aligned}
|T(f_{1l}^{v},f_{2j}^{v})(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_{1l}^{v}(y_1)||f_{2j}^{v}(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim 2^{-kn}2^{-jn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_{1l}^{v}(y_1)||f_{2j}^{v}(y_2)|\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned}
$$

Therefore, by Hölder's inequality and Minkowski's inequality, we obtain

$$
\begin{aligned}
&\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty}T(f_{1l}^{v},f_{2j}^{v})\Big|^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^{v}(y_1)|\mathrm{d}y_1\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^{v}(y_2)|\mathrm{d}y_2\Big)^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^{v}(y_1)|\mathrm{d}y_1\Big)^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\quad\times\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^{v}(y_2)|\mathrm{d}y_2\Big)^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}\\
&\lesssim \Big\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\quad\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned}
\tag{3.7}
$$

Since

$$\frac{1}{q_(0)}=\frac{1}{q_1(0)}+\frac{1}{q_2(0)}\quad\text{and}\quad \lambda=\lambda_1+\lambda_2,$$

by Hölder's inequality, we have

$$E_6 \lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\Big\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)}$$

$$
\begin{aligned}
&\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}}\\
\lesssim\ & \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_1}\times\Big(\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}}\\
&\times\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_2}\times\Big(\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
=\ & E_{6,1}\times E_{6,2}.
\end{aligned}
$$

By the interchange of f_1 and f_2, we find the estimateof $E_{6,1}$ and $E_{2,2}$ are similar, and $E_{6,2} = E_{3,2}$. To estimate E_9, since $l, j \geq k+2$, we get

$$|x-y_i| > 2^{k-2}, \quad x\in D_k,\ y_1\in D_l,\ y_2\in D_j.$$

Therefore, for any $x\in D_k,\ l, j\geq k+2$, we have

$$
\begin{aligned}
|T_{\varpi}(f_{1l}^{v},f_{2j}^{v})(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_{1l}^{v}(y_1)||f_{2j}^{v}(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim 2^{-ln}2^{-jn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_{1l}^{v}(y_1)||f_{2j}^{v}(y_2)|\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned}
$$

Thus, by Hölder's inequality and Minkowski's inequality, we have

$$
\begin{aligned}
&\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty}T(f_{1l}^{v},f_{2j}^{v})\Big|^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
\lesssim\ &\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=k+2}^{\infty}2^{-ln}\int_{\mathbb{R}^n}|f_{1l}^{v}(y_1)|\mathrm{d}y_1\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^{v}(y_2)|\mathrm{d}y_2\Big)^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
\lesssim\ &\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=k+2}^{\infty}2^{-\ln}\int_{\mathbb{R}^n}|f_{1l}^{v}(y_1)|\mathrm{d}y_1\Big)^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\times\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^{v}(y_2)|\mathrm{d}y_2\Big)^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}\\
\lesssim\ &\Big\|\sum_{l=k+2}^{\infty}2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned}
\tag{3.8}
$$

Since

$$\frac{1}{q_(0)}=\frac{1}{q_1(0)}+\frac{1}{q_2(0)} \quad\text{and}\quad \lambda=\lambda_1+\lambda_2,$$

by Hölder's inequality, we have

$$E_9 \lesssim \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda}\Big(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\Big\|\sum_{l=k+2}^{\infty}2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)}$$

$$
\begin{aligned}
&\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}}\\
\lesssim\ &\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_1}\times\Big(\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{l=k+2}^{\infty}2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}}\\
&\times\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_2}\times\Big(\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
=\ &E_{9,1}\times E_{9,2}.
\end{aligned}
$$

Obviously, the estimates of $E_{9,i}$ are similar to those of $E_{3,2}(i=1,2)$.

All estimates for E_i $i=1,2,\cdots,9$ considered, we have

$$
E\lesssim\Big\|\Big(\sum_{v=1}^{\infty}|f_1^{v}|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^{v}|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.
$$

Finally, we estimate H. By the interchange of f_1 and f_2, we see that the estimates of H_2, H_3 and H_6 are similar to those of H_4, H_7 and H_8, respectively. Thus we just need to estimate H_1–H_3, H_5, H_6 and H_9.

For the subsequent proof process, we need following further preparation. If $l<0$, by Lemma 1.3, we have

$$
\begin{aligned}
\Big\|\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}&=2^{-l\alpha_i(0)}\Big(2^{l\alpha_i(0)q_i(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\Big)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{-l\alpha_i(0)}\Big(\sum_{t=-\infty}^{l}2^{t\alpha_i(0)q_i(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\chi_t\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\Big)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{l(\lambda-\alpha_i(0))}2^{-l\lambda}\Big(\sum_{t=-\infty}^{l}\Big\|2^{t\alpha_i(0)}\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\chi_t\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\Big)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{l(\lambda-\alpha_i(0))}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^{v}|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}(w_i)}.
\end{aligned}
\tag{3.9}
$$

To estimate H_1, since

$$
l,j\le k-2,\quad \frac{1}{q_\infty}=\frac{1}{q_{1\infty}}+\frac{1}{q_{2\infty}}
$$

and $\lambda=\lambda_1+\lambda_2$, by (3.4) and Hölder's inequality, we have

$$
\begin{aligned}
H_1\ &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty}\\
&\quad\times\Big\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\
&\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}}
\end{aligned}
$$

$$
\begin{aligned}
&\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}\\
= \;& H_{1,1}\times H_{1,2},
\end{aligned}
$$

where

$$
H_{1,i}=2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}(y_i)|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}.
$$

By (3.1), we obtain

$$
\begin{aligned}
H_{1,i} \;\lesssim\; & 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=-\infty}^{k-2}2^{(l-k)n\delta_{i2}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
\lesssim\; & 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\\
&+\sum_{l=0}^{k}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
\lesssim\; & 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&+2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=0}^{k}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
=\;& I_4+I_5.
\end{aligned}
$$

If $q_{i\infty}\geq 1$, since $n\delta_{i2}-\alpha_{i\infty}>0$ and $n\delta_{i2}-\alpha_i(0)>0$,by the Minkowski's inequality and (3.9), we obtain

$$
\begin{aligned}
I_4 \;=\;& 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\Big\{\sum_{k=0}^{L}\Big(2^{k\alpha_{i\infty}}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\sum_{l=-\infty}^{-1}2^{ln\delta_{i2}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\Big\{\sum_{k=0}^{L}2^{-k\left(n\delta_{i2}-\alpha_{i\infty}\right)q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}2^{-L\lambda_i}\sum_{l=-\infty}^{-1}2^{l\left(n\delta_{i2}+\lambda_i-\alpha_i(0)\right)}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}.
\end{aligned}
$$

If $q_{i\infty}<1$, since $n\delta_{i2}-\alpha_{i\infty}>0$ and $n\delta_{i2}-\alpha_i(0)>0$, by (3.9), we have

$$
I_4 \;\lesssim\; 2^{-L\lambda_i}\Big(\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}2^{(l-k)n\delta_{i2}q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}
$$

$$
\begin{aligned}
&= 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}2^{ln\delta_{i2}q_{i\infty}}\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}2^{-kn\delta_{i2}q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}\\
&= 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}2^{\ln\delta_{i2}q_{i\infty}}\sum_{k=0}^{L}2^{-k\big(n\delta_{i2}-\alpha_{i\infty}\big)q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}2^{\ln\delta_{i2}q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}(w_i)}2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{-1}2^{l\big(n\delta_{i2}+\lambda_i-\alpha_i(0)\big)q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}(w_i)}.
\end{aligned}
$$

We consider I_5. Since $n\delta_{i2}-\alpha_{i\infty}>0$, by Lemma 2.4, we have

$$
\begin{aligned}
I_5 &= 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=0}^{k}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&= 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}\Big(\sum_{l=0}^{k}2^{l\alpha_{i\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)(n\delta_{i2}-\alpha_{i\infty})}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\Big(\sum_{l=0}^{k}2^{l\alpha_{i\infty}q_{i\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}\big(w_i\big)},
\end{aligned}
$$

where we write $2^{-|k-l|(n\delta_{i2}-\alpha_{i\infty})}\lesssim 2^{-|k-l|\eta_i}$ for $\eta_i=n\delta_{i2}-\alpha_{i\infty}$.

Thus,we get

$$
H_1\lesssim\Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.
$$

To estimate H_2, since

$$
l\le k-2,\quad k-1\le j\le k+1,\quad \frac{1}{q_\infty}=\frac{1}{q_{1\infty}}+\frac{1}{q_{2\infty}}
$$

and $\lambda=\lambda_1+\lambda_2$, by (3.6) and Hölder's inequality, we have

$$
\begin{aligned}
H_2 &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v\big(y_1\big)|^r\Big)^{\frac{1}{r}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty}\\
&\quad\times\Big\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^r\Big)^{\frac{1}{r}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\
&\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^r\Big)^{\frac{1}{r}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}}
\end{aligned}
$$

$$\begin{aligned}
&\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r}\Big)^{\frac{1}{r}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}\\
= \ & H_{2,1}\times H_{2,2}.
\end{aligned}$$

It isobvious that

$$H_{2,1}=H_{1,1}\lesssim \|(\sum_{v=1}^{\infty}|f_1^v|^{r_1})^{\frac{1}{r_1}}\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}.$$

Now we estimate $H_{2,2}$. Combining (3.1)–(3.3), we have

$$\begin{aligned}
H_{2,2} &\lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\sum_{j=k-1}^{k+1}2^{(j-k)n}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}\\
&\lesssim 2^{-L\lambda_2}\Big(\sum_{k=-1}^{L+1}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)},
\end{aligned}$$

where we use $2^{-n\delta_{22}}<1$ and $2^{(j-k)n(1-\delta_{21})}<2^{(j-k)n}$ for (3.6) and (3.8), respectively. Thus, we obtain

$$H_2\lesssim \|(\sum_{v=1}^{\infty}|f_1^v|^{r_1})^{\frac{1}{r_1}}\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.$$

To estimate H_3, since

$$l\le k-2,\quad j\ge k+2,\quad \frac{1}{q_\infty}=\frac{1}{q_{1\infty}}+\frac{1}{q_{2\infty}}$$

and $\lambda=\lambda_1+\lambda_2$, together (3.6) with the Hölder's inequality, we have

$$\begin{aligned}
H_3 &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty}\\
&\quad\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{p\infty}}\Big)^{\frac{1}{q_\infty}}\\
&\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^{v}(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}}\\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}\\
&= H_{3,1}\times H_{3,2}.
\end{aligned}$$

It is easy to see that

$$H_{3,1}=H_{1,1}\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}.$$

Since $n\delta_{21}+\alpha_{2\infty}>0$, by (3.3), we obtain

$$\begin{aligned} H_{3,2} &\lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_{21}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}} \\ &\lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(\sum_{j=k+2}^{L+2}2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)\big(n\delta_{21}+\alpha_{2\infty}\big)}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}} \\ &\quad +2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(2^{k\alpha_{2\infty}}\sum_{j=L+3}^{\infty}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)n\delta_{21}}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}} \\ &= I_6+I_7. \end{aligned}$$

For I_6, by Lemma 2.4, we obtain

$$\begin{aligned} I_6 &\lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(\sum_{j=k+2}^{L+2}2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)\big(n\delta_{21}+\alpha_{2\infty}\big)}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}} \\ &\lesssim 2^{-L\lambda_2}\Big(\sum_{j=0}^{L+2}2^{j\alpha_{2\infty}q_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|^{q_{2\infty}}_{L^{p_2(\cdot)(w_2)}}\Big)^{\frac{1}{q_{2\infty}}} \\ &\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}\big(w_2\big)}, \end{aligned}$$

where we write $2^{-|k-j|(n\delta_{21}+\alpha_{2\infty})}=2^{-|k-j|\vartheta_2}$ for $\vartheta_2=n\delta_{21}+\alpha_{2\infty}>0$.

For I_7, since $n\delta_{21}+\alpha_{2\infty}-\lambda_2>0$, we have

$$\begin{aligned} I_7 &\lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(2^{k(n\delta_{21}+\alpha_{2\infty})}\sum_{j=L+3}^{\infty}2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}\times 2^{-j(n\delta_{21}+\alpha_{2\infty})}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2^\infty}}} \\ &\lesssim \sup_{j\geq 1}2^{-j\lambda_2}2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(2^{k(n\delta_{21}+\alpha_{2\infty})}\sum_{j=L+3}^{\infty}2^{-j(n\delta_{21}+\alpha_{2\infty}-\lambda_2)}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}} \\ &\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}2^{-L\lambda_2+\big(n\delta_{21}+\alpha_{2\infty}\big)L-L\big(n\delta_{21}+\alpha_{2\infty}-\lambda_2\big)} \\ &\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}. \end{aligned}$$

Thus, we get

$$H_3\lesssim\Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate H_5, using Hölder's inequality and Lemma 2.8, we have

$$H_5 \lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}\Big\|\Big(\sum_{v=1}^{\infty}\Big|T(f_{1l},f_{2j})\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|^{q_\infty}_{L^{p(\cdot)}(w)}\Big)^{\frac{1}{q_\infty}}$$

$$
\begin{aligned}
&\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big(\Big\|\Big(\sum_{v=1}^{\infty}|f_{1l}^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|\|_{L^{p_1(\cdot)}(w_1)}\times\Big\|\Big(\sum_{v=1}^{\infty}|f_{2j}^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{L^{p_2(\cdot)}(w_2)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\
&\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}}\\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned}
$$

To estimate H_6, since

$$
k-1\le l\le k+1,\quad j\ge k+2,\quad \frac{1}{q_\infty}=\frac{1}{q_{1\infty}}+\frac{1}{q_{2\infty}}
$$

and $\lambda=\lambda_1+\lambda_2$, by (3.7) and Hölder's sinequality, we have

$$
\begin{aligned}
H_6 &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty}\\
&\quad\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\
&\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}}\\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}\\
&= H_{6,1}\times H_{6,2}.
\end{aligned}
$$

By the interchange of f_1 and f_2,we see that that of $H_{6,1}$ is similar to the estimate of $H_{2,2}$ and $H_{6,2}=H_{3,2}$.

To estimate H_9, since

$$
l,j\ge k+2,\quad \frac{1}{q_\infty}=\frac{1}{q_{1\infty}}+\frac{1}{q_{2\infty}}
$$

and $\lambda=\lambda_1+\lambda_2$, by (3.8) and Hölder's inequality, we have

$$
\begin{aligned}
H_9 &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\sum_{l=k+2}^{\infty}2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty}\\
&\quad\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}\\
&\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=k+2}^{\infty}2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}}
\end{aligned}
$$

$$\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^{v}(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$= H_{9,1}\times H_{9,2}.$$

Obviously, the estimates of $H_{9,i}$ are similar to those of $H_{3,2}$ for $i = 1, 2$, respectively.

Taking all estimates for H_i together, $i = 1, 2, \cdots, 9$, we obtain

$$H \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

This completes the proof.

4. Conclusions

On the basis of vector valued bilinear Calderón-Zygmund operators with kernels of Dini's type are bounded on variable Lebesgue spaces, with the help of properties of the $\varpi(t)$ and space decomposition methods for variable exponents Herz-Morrey spaces. We establish the weighted boundedness result of vector valued bilinear $\varpi(t)$-type Calderón-Zygmund operators in variable exponents Herz-Morrey spaces, this is a new and meaningful result.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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