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Research article

Interpretation on nonlocal neutral functional differential equations with delay

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Abstract: This work deals with the existence and continuous dependence of an integral solution for neutral integro-differential equations with a nonlocal condition. This result is established by using an integrated resolvent operator under conditions of Lipschitz continuity and uniqueness via the Banach fixed point technique. We also study the existence of a strict solution on reflexive and general Banach spaces. In the last section, an example is provided related to this theory.

Keywords: integro-differential equations; integrated resolvent operator; Lipschitz condition; strict solution; fixed point theorem
Mathematics Subject Classification: 34B10, 35B65, 37L05, 45K05, 47H10, 47N20

1. Introduction

Neutral differential equations are studied by many authors with or without delay, to model many real situations in different fields like population studies, electronics, chemical kinetics and biological science. The below system is used to describe the heat conduction materials in [2].

$$\begin{cases} \frac{\partial}{\partial t}[z(t_1,x)+\int_{-\infty}^{t_1} e_1(t_1-\xi)z(\xi,x)d\xi] = d\Delta z(t_1)+\int_{-\infty}^{t_1} e_2(t_1-\xi)\Delta z(\xi,x)+f(t_1,z(\cdot,x)), t_1\geq 0,\\ z(t_1,x)=0, \qquad \text{for } x\in\partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is open, $(t_1, x) \in [0, \infty) \times \Omega$ and $z(t_1, x)$ denotes the heat in x at any time t_1. Let $d > 0; e_i : \mathbb{R} \to \mathbb{R}$ represents the internal energy of fading memory materials. In [7, 8] Ezzinbi et al. proved the existence and regularity of solutions of neutral equations by using resolvent operator theory and fixed point theorems. In [3, 15, 21] the authors proved the existence solution of neutral integro-differential equations by using fractional powers of operators and the Schauder fixed point theorem. Also, in [1,28], the authors proved the existence of solutions of differential equations by using fractional powers of operators under the condition of Krasnoselskii's fixed point theorems. In [24] Murugesu and Suguna proved the existence solution for neutral functional integro-differential equations by using fractional powers of operators and Sadovskii's fixed point theorem. The existence result for integro-differential equations in [9, 22, 27] was proved by using resolvent operator theory and Monch-Krasnoselskii's and Sadovskii's fixed point theorems. In [4, 23] the authors established the existence of a mild solution for neutral differential equations by using Schaefer fixed-point theorem.

The nonlocal initial conditions are more effective, realistic and accurate in the solutions and uniqueness than the classical one proved by many researchers see [6, 20]. Recently published [18, 19] proves the existence and uniqueness of solutions of functional integro-differential equations with nonlocal conditions; the authors also proved the existence of a strict solution by using an integrated resolvent operator. The main tool for proving the uniqueness and existence of solutions of differential equations by using the Banach fixed point theorem has been established in [12–14]. In [26] the authors proved that the mild solution, strong solution and classical solutions obtained by using the semigroup theory of evolution equations also explained the uniqueness of the solution. The semigroup and resolvent operator theories are important methods to find the solutions of integro-differential equations in Banach space(BS), and the authors established integrated semigroup theory in [16]. In recent years, many differential equations have been reformed as integral equations and scholars have proved that the existence of solutions can be obtained via appropriate fixed-point theorems, which is the common technique for proving the existence of solutions of the integral equations. In [11, 17], proved the existence solutions of integro-differential equations through the use of resolvent operators with finite delay furthermore, the authors used the integrated resolvent operator in [11].

In the recently published article [29] the authors established the following system

$$\begin{cases} \frac{d}{dt}[x(t) - F(t, x(h_1(t)))] = Ax(t) + \int_0^t B(t-s)x(s)ds + G(t, x(h_2(t))), \quad \text{for } t \in [0, a], \\ x(0) + g(x) = x_0. \end{cases}$$

For this problem they proved the existence of the solution of nondensely defined neutral equations via the integrated resolvent operator technique. They also proved continuous dependence and differentiability. They assumed that A is a closed linear operator on X and its domain does not equals to X. Motivated by this above-mentioned article, we established the theory for the neutral integro-differential equations with nonlocal and finite delay. This theory contains the integrated resolvent operator in the proof of the existence of the solution and assumptions of Lipschitz continuity; we also prove the uniqueness by applying Banach fixed-point theory and verified its differentiability.

Regarding this, we have to show the existence of the integral solution of the below system:

$$\begin{cases} \frac{d}{dt}[\omega(t) - q(t, \omega_t)] = \mathcal{D}\omega(t) + \int_0^t \mathcal{H}(t-\zeta)\omega(\zeta)d\zeta \\ \qquad\qquad +\varphi\Big(t, \omega_t, \int_0^t h(t, \zeta, \omega_\zeta)d\zeta\Big) \qquad \text{for } t \in [0, a] = I, \\ \omega(0) = \phi + g(\omega) \in C([-r, 0]; \mathcal{E}) = C. \end{cases} \tag{1.1}$$

In this article, $\mathcal{E}$ denotes the BS and $\mathcal{D}$ is the closed linear operator on $\mathcal{E}$; its domain $\overline{D(\mathcal{D})} \neq \mathcal{E}$, which satisfies the Hille-Yosida theorem. Let $\mathcal{H}(t)$ be the set of bounded linear operators in $\mathcal{E}$ with $D(\mathcal{D}) \subset D(B(t))$, $t \geq 0$ from $D(\mathcal{D}) = Y$ into $\mathcal{E}$. The functions $q : I \times C \to \mathcal{E}$, $h : I \times I \times C \to \mathcal{E}$ and $\varphi : I \times C \times \mathcal{E} \to \mathcal{E}$ are continuous as specified later. Let $C = C([-r, 0]; X)$ be a set of continuous functions on $[-r, 0]$ in $\mathcal{E}$ and ϕ, g be continuous functions defined on C.

Note that ω belongs to the continuous function $C([-r, \infty); \mathcal{E})$, $t \geq 0$; the function $\omega_t \in C$ given that $\omega_t(\sigma) = \omega(t + \sigma)$ for $\sigma \in [-r, 0]$. The general form of (1.1) is an abstract formation of a large number of partial integro-differntial equations, particularly for applications such as electronic circuits, economics, biological sciences, medicine and more. In this article we use the Banach theorem to prove the existence of a solution to the nonlocal system given by Eq (1.1). The existence and uniqueness of the abstract form given by Eq (1.1) have been established in previous articles and by using different approaches this is particularly true for the existence of solutions and valid properties of differential equations which have been established by applying the resolvent operator technique in [5, 7, 10].

This paper is summarized as follows. In Section 2, we provide the preliminary results and definitions regarding integrated resolvent operator theory. In Section 3, we discuss the existence and uniqueness of the solution and continuous dependence. In Section 4, we prove the differentiability of the solution; in Section 5, we provide an example related to our basic results.

2. Basic results and definitions

Here, this section includes some basic results and definitions regarding integrated resolvent operators. Let $\mathcal{E}$ ba a BS and $\mathcal{D}$ be a closed linear operator; $0 \in \rho(\mathcal{D})$ then $\mathcal{D}^{-1}$ exists. Let Y be the BS $(D(\mathcal{D}), \|\cdot\|)$ with a graph of the norm $\|v\| = \|\mathcal{D}v\| + \|v\|$, $\forall v \in D(\mathcal{D})$. Let $\mathcal{L}(Y, \mathcal{E})$ be the bounded linear operator $Y \to \mathcal{E}$ with the norm $\|\cdot\|$ and it is $\mathcal{L}(\mathcal{E})$ when $Y = \mathcal{E}$. Let $C([-r, 0]; \mathcal{E})$ represent the functions on $[-r, 0]$ denoted by C that are continuous in X and have the sup-norm $\|\cdot\|_C$.

Next we recall a few definitions and results about the integrated resolvent operators established in [25] for linear nondensely defined integro-differential equations.

Consider the below homogeneous linear integro-differential system:

$$\begin{cases} v'(t) = \mathcal{D}v(t) + \int_0^t \mathcal{H}(t - \zeta)v(\zeta)d\zeta \quad \text{for } t \in [0, a] \\ v(0) = v_0 \in \mathcal{E}. \end{cases} \tag{2.1}$$

Here, the operators $\mathcal{D}$ and $\mathcal{H}(\cdot)$ are defined already in Eq (2.1). Then the integrated resolvent operator for Eq (2.1) is as follows:

Definition 2.1. *[25] A set of operators $(Q(t))_{t\geq 0}$ in $\mathcal{L}(\mathcal{E})$ constitute an integrated resolvent operator for Eq (2.1) if it satisfies the following:*

$(R1)$ $\forall v \in \mathcal{E}$, $Q(\cdot)v \in C([0, +\infty); \mathcal{E})$.

$(R2)$ $\forall v \in \mathcal{E}$, $\int_0^{\cdot} Q(\zeta) \in Y$.

$(R3)$ $Q(t)v - tv = \mathcal{D} \int_0^t Q(\zeta)vd\zeta + \int_0^t \mathcal{H}(t - \zeta) \int_0^\zeta Q(r)vdrd\zeta$, $\forall v \in \mathcal{E}$, $t \geq 0$.

$(R4)$ $Q(t)v - tv = \int_0^t Q(\zeta)\mathcal{D}vd\zeta + \int_0^t \int_0^\zeta Q(\zeta - r)\mathcal{H}(r)vdrd\zeta$, $\forall v \in D(\mathcal{D})$, $t \geq 0$.

Definition 2.2. *([25]) The operator* $(\mathcal{Q}(t))_{t\geq 0}$ *defined in the above definition is locally Lipschitz continuous (LLC) if* $\forall a > 0$*and* $\exists K_a = K(a) > 0$ *implies the following*

$$\|\mathcal{Q}(\xi_1) - \mathcal{Q}(\xi_2)\| \leq K_a|\xi_1 - \xi_2|, \text{ where } \xi_1, \xi_2 \in [0, a].$$

Consider the following non homogeneous integro-differential system:

$$\begin{cases} v'(t) = \mathcal{D}v(t) + \int_0^t \mathcal{H}(t-\zeta)v(\zeta)d\zeta + f(t) & \text{for } t \in [0, a]. \\ v(0) = v_0 \in \mathcal{E}. \end{cases} \tag{2.2}$$

We follow a previous article see [25] to write the integral solution and strict solution of Eq (2.2) as follows:

Definition 2.3. *For* $f \in L^1([0,\infty); \mathcal{E})$ *and* $v_0 \in \mathcal{E}$*, a function* $v : [0, a] \to \mathcal{E}$ *is called an integral solution of Eq (2.2) if*

(1) $v \in C([0, a]; \mathcal{E})$,

(2) $\int_0^{\cdot} v(\zeta)d\zeta \in C([0, a]; Y)$,

(3) $v(t) = v_0 + \mathcal{D}\int_0^t v(\zeta)d\zeta + \int_0^t \mathcal{H}(t-\zeta)\int_0^{\zeta} v(\xi)d\xi d\zeta + \int_0^t f(\zeta)d\zeta$, $t \in [0, a]$.

Lemma 2.4. *([25]) Assume that* $(\mathcal{Q}(t))_{t\geq 0}$ *is an LLC integrated resolvent operator of Eq (2.2) with* $\rho(\mathcal{D}) \neq \emptyset$ *then, we have the following:*

(i) *If* $v_0 \in \overline{D(\mathcal{D})}$ *and* $f \in L^1([0,+\infty); \mathcal{E})$ *then* $\exists$ *a unique integral solution* $v(\cdot)$ *of problem Eq (2.2); then,*

$$v(t) = \mathcal{Q}'(t)v_0 + \frac{d}{dt}\int_0^t \mathcal{Q}(t-\zeta)f(\zeta)d\zeta, \quad t \in [0, a]. \tag{2.3}$$

Further,

$$\|v(t)\| \leq C\Big(\|v_0\| + \int_0^t \|f(\zeta)\|d\zeta\Big), \quad t \in [0, a]. \tag{2.4}$$

(ii) *Suppose that* $v_0 \in D(\mathcal{D})$, $f \in W^{1,1}([0, a]; \mathcal{E})$ *and* $\mathcal{D}v_0 + f(0) \in \overline{D(\mathcal{D})}$*;* $\exists$ *a unique strict solution* $v(\cdot)$ *for Eq (2.2) and*

$$\|v'(t)\| \leq C\Big(\|\mathcal{D}v_0 + f(0)\| + \int_0^t \|\mathcal{H}(\zeta)v_0 + f'(\zeta)\|d\zeta\Big), \quad t \in [0, a].$$

Here $C_1 > 0$ *is a constant.*

Remark 2.5. *Suppose that* $(\mathcal{Q}(t))_{t\geq 0}$ *is an LLC integrated resolvent operator; from [25, Theorem 2.7], for* $v \in \overline{D(\mathcal{D})}$*,* $t \mapsto \mathcal{Q}(t)v$ *is differentiable on* $[0, a]$.

Lemma 2.6. *([25, Theorem 2.6]) The set of* $(\mathcal{G}(t))_{t\geq 0} \in \mathcal{L}(\mathcal{E})$ *is LLC with* $\mathcal{G}(0) = 0$*; then, we have the following:*

(*i*) *If* $f \in L^1([0,a];\mathcal{E})$*, then* $\int_0^{\cdot} \mathcal{G}(\cdot - \zeta) f(\zeta) d\zeta \in C^1([0,a];\mathcal{E})$ *and*

$$\|\mathcal{K}(t)\| \leq K_T \int_0^t \|f(\zeta)\| d\zeta \text{ for } t \in [0,a]. \tag{2.5}$$

Here $\mathcal{K} = \frac{d}{dt}\int_0^t \mathcal{G}(t-\zeta)f(\zeta)d\zeta$, $t \in [0,a]$ *and* $K_T > 0$ *is Lipschitzian of* $\{\mathcal{G}(t) : t \in [0,a]\}$ *. Further if* $\|f(t)\| \leq K_0$ *,* $\exists K_0 > 0$*,* $t \in [0,a]$ *and*

$$\|\mathcal{K}(t_1 + \zeta) - \mathcal{K}(t_1)\| \leq K_0 K_T \zeta + K_T \int_0^{t_1} \|f(\zeta + \xi) - f(\xi)\| d\xi \text{ for } t_1, \zeta, t + \zeta \in [0,a]. \tag{2.6}$$

(*ii*) *Suppose that* $f : [0,a] \to \mathcal{E}$ *is a strongly bounded variation; then,* $\mathcal{K}(\cdot)$ *is Lipschitz continuous on* $[0,a]$.

3. Existence result

Here, we have to show the existence of the solution of Eq (1.1). Due to Lemma 2.4, the integral solution of Eq (1.1) with the nonlocal condition is as follows:

Definition 3.1. *Let* $\omega_0 \in \overline{D(\mathcal{D})}$*. A function* $\omega \in ([-r,+\infty);\mathcal{E})$ *is an integral solution of the system given by Eq (1.1) if it satisfies the following:*

$$\begin{cases} \omega(t) = q(t,\omega_t) + Q'(t)[\phi(0) + g(\omega)(0) - q(0,\omega_0)] + \frac{d}{dt}\int_0^t Q(t-s) \\ \qquad \times\Big[\mathcal{D}q(s,\omega_s) + \int_0^s q(s,\omega_s)\mathcal{H}(\xi)d\xi + \varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)\Big]ds, t \geq 0. \\ \phi(t) + g(\omega)(t) \text{ for } t \in [-r,0]. \end{cases} \tag{3.1}$$

Remark 3.2. *From the above definition, if* $\omega(\cdot)$ *is an integral solution of Eq (1.1) on* $[0,a]$ *then, for each* $t \in [0,a], \omega(t) - q(t,\omega_t) \in \overline{D(\mathcal{D})}$*. Further* $\omega(0) - q(0,\omega_0) \in \overline{D(\mathcal{D})}$*.*

To establish the solution of the existence of Eq (1.1), we need the support of the below assumptions:

($H1$) The function $q : I \times C \to D(\mathcal{D})$ is Lipschitz continuous; there exists a constant $L_1 > 0$; then,

$$\|\mathcal{D}q(\xi,x_1) - \mathcal{D}q(\xi,y_1)\| \leq L_1\|x_1 - y_1\| \text{ and } \|\mathcal{D}q(\xi,x_1)\| \leq L_1(\|x_1\| + 1)$$

for any $0 \leq \xi \leq a, x_1, y_1 \in C$.

($H2$) The function $\varphi : I \times C \times \mathcal{E} \to \mathcal{E}$ is Lipschitz continuous; $\exists L_2 > 0$ so that

$$\|\varphi(t,\xi_1,\nu_1) - \varphi(t,\xi_2,\nu_2)\| \leq L_2(\|\xi_1 - \xi_2\| + \|\nu_1 - \nu_2\|)$$

and

$$\|\varphi(t,\xi_1,\nu_1)\| \leq L_2(\|\xi_1\|_C + \|\nu_1\|)$$

for every $\xi_1, \xi_2 \in C, \nu_1, \nu_2 \in \mathcal{E}$.

$(H3)$ The map $g : C([0,a];\mathcal{E}) \to C$ is Lipschitz continuous and $\exists L_3 > 0$; then,

$$\|g(v_1) - g(v_2)\| \le L_3\|v_1 - v_2\|_C \text{ and } \|g(u)\| \le L_3\|u\|_C$$

for each $v_1, v_2 \in ([0,a];\mathcal{E})$ and for $u \in ([0,a];\mathcal{E})$.

$(H4)$ The map $h : I \times I \times C \to \mathcal{E}$ is Lipschitz continuous; there exists a constant $L_h > 0$; then,

$$\|h(\xi_1, \xi_2, \phi) - h(\xi_1, \xi_2, \psi)\| \le L_h\|\phi - \psi\| \text{ and } \|h(\xi_1, \xi_2, \phi)\| \le L_h\|\phi\|$$

for each $\xi_1, \xi_2 \in I$, $\phi, \psi \in C$.

Theorem 3.3. *Let $\omega_0 \in \overline{D(\mathcal{D})}$, $0 \in \rho(\mathcal{D})$ satisfy (H1–H4) and $\forall \phi \in C$; then, the system given by Eq (1.1) has at least one mild solution on $[-r, +\infty)$ provided that*

$$M_1L_1 + C\left[L_3 + (M_1 + a + M_2a^2)L_1 + L_2a(1 + L_h)\right] < 1. \tag{3.2}$$

Here C is from Lemma 2.4, $M_1 = \|\mathcal{D}^{-1}\|$ and $M_2 = \sup_{t\in I}\|\mathcal{H}(t)\|$.

Proof. Let $a > 0$ and $C([0,a];\mathcal{E})$ is a set of continuous maps from $[0,a]$ into $\mathcal{E}$ with the uniform norm topology. We prove this existence by using the Banach fixed point theorem.

The operator Γ on $C([0,a];\mathcal{E})$ is defined by

$$(\Gamma\omega)(t) = \begin{cases} \omega(t) = q(t,\omega_t) + Q'(t)[\phi(0) + g(\omega)(0) - q(0,\omega_0)] + \frac{d}{dt}\int_0^t Q(t-s) \\ \quad \times\Big[\mathcal{D}q(s,\omega_s) + \int_0^s q(s,\omega_s)\mathcal{H}(\xi)d\xi \\ \quad +\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)\Big]ds, \text{ for } t \ge 0. \\ \phi(t) + g(\omega)(t) \text{ for } t \in [-r,0]. \end{cases} \tag{3.3}$$

We prove that this operator Γ has a fixed point in the closed ball $B_r = \{\omega \in C([-r,a];\mathcal{E}), \|\omega\| \le r\}$. Before we prove that Γ is a map on B_r, for each $\omega \in \overline{B_r}$ and $t \in [-r,0]$, we take $\Gamma_1 = \phi(t) + g(\omega)(t)$; we have

$$\begin{aligned} \|(\Gamma_1\omega)(t)\| &= \|\phi(t) + g(\omega)(t)\| \\ &\le \|\phi(t)\| + \|g(\omega)(t)\| \\ &\le \|\phi\| + L_3\|\omega\| \\ &\le \|\phi\| + L_3r. \end{aligned}$$

Next if $t \in [0,a]$, let $\Gamma_2 = \omega(t)$; then,

$$\begin{aligned} \|(\Gamma_2\omega)(t)\| &= \Big\|q(t,\omega_t) + Q'(t)\left[\phi(0) + g(\omega)(0) - q(0,\omega_0)\right] \\ &+ \frac{d}{dt}\int_0^t Q(t-s)\Big[\mathcal{D}q(s,\omega_s) + \int_0^s q(s,\omega_s)\mathcal{H}(\xi)d\xi \\ &+ \varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)\Big]ds\Big\| \end{aligned}$$

$$
\begin{aligned}
&\leq \|q(t,\omega_t)\| + \|Q'(t)\left[\phi(0)+g(\omega)(0)-q(0,\omega_0)\right]\| \\
&+ \left\|\frac{d}{dt}\int_0^t Q(t-s)\Big[\mathcal{D}q(s,\omega_s)+\int_0^s q(s,\omega_s)\mathcal{H}(\xi)d\xi \\
&+ \varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)\Big]ds\right\|.
\end{aligned}
$$

Using the hypotheses, we have

$$
\begin{aligned}
\|(\Gamma_2\omega)(t)\| &\leq M_1L_1(\|\omega_t\|+1)+C\left(\|\phi\|+L_3\|\omega\|+M_1L_1(\|\omega_0\|+1)\right) \\
&+ C\int_0^t\Big[\|\mathcal{D}q(s,\omega_s)\|+\int_0^s \|q(s,\omega_s)\mathcal{H}(\xi)\|d\xi \\
&+ \|\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)\|\Big]ds \\
&\leq M_1L_1(r+1)+C\left[\|\phi\|+L_3r+M_1L_1(r+1)\right] \\
&+ Ca\left(L_1(r+1)+aM_2L_1(r+1)+L_2(r+L_hr)\right) \\
&\leq M_1L_1(r+1)+C\Big[\|\phi\|+L_3r+(M_1+a+M_2a^2)L_1(r+1)+L_2(r+L_hr)\Big].
\end{aligned}
$$

It follows from the above two cases that

$$
\|(\Gamma\omega)(t)\| \leq M_1L_1(r+1)+C\Big[\|\phi\|+L_3r+(M_1+a+M_2a^2)L_1(r+1)+L_2(r+L_hr)\Big]\leq r.
$$

Hence the operator Γ is well defined in B_r; next, we show that Γ is a contractive map on B_r. The map Γ is defined on B_r as

$$
\begin{aligned}
(\Gamma\omega)(t) &= q(t,\omega_t)+Q'(t)[\phi(0)+g(\omega)(0)-q(0,\omega_0)]+\frac{d}{dt}\int_0^t Q(t-s) \\
&\times\Big[\mathcal{D}q(s,\omega_s)+\int_0^s q(s,\omega_s)\mathcal{H}(\xi)d\xi+\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)\Big]ds, \text{ for } t\geq 0.
\end{aligned}
$$

The extension $\tilde{\omega}:[-r,0]\to\mathcal{E}$ is as follows

$$
\tilde{\omega}(t)=\begin{cases}(\omega)(t), & \text{for } t\in[0,a],\\ \phi(t)+g(\omega)(t), & \text{for } t\in[-r,0].\end{cases}
$$

Let $\sigma(t),\tau(t)\in B_r$ represent the solution of Eq (1.1); for $t\in[0,a]$ we have

$$
\begin{aligned}
\|(\Gamma\sigma)(t)-(\Gamma\tau)(t)\| &\leq \|q(t,\tilde{\sigma}_t)-q(t,\tilde{\tau}_t)\| \\
&+ \|Q'(t)[\phi(0)+g(\tilde{\sigma})(0)-q(0,\tilde{\sigma}_0)-\phi(0)-g(\tilde{\tau})(0)+q(0,\tilde{\tau}_0)]\| \\
&+ \frac{d}{dt}\int_0^t Q(t-s)\Big[\Big\|\mathcal{D}q(s,\tilde{\sigma}_s)+\int_0^s q(s,\tilde{\sigma}_s)\mathcal{H}(\xi)d\xi \\
&+ \varphi\Big(s,\tilde{\sigma}_s,\int_0^s h(s,\xi,\tilde{\sigma}_\xi)d\xi\Big)-\mathcal{D}q(s,\tilde{\tau}_s)-\int_0^s q(s,\tilde{\tau}_s)\mathcal{H}(\xi)d\xi \\
&- \varphi\Big(s,\tilde{\tau}_s,\int_0^s h(s,\xi,\tilde{\tau}_\xi)d\xi\Big)\Big\|\Big]ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|(\Gamma\sigma)(t)-(\Gamma\tau)(t)\| &\le \|q(t,\tilde{\sigma}_t)-q(t,\tilde{\tau}_t)\| \\
&+ \|Q'(t)[(g(\tilde{\omega})(0)-g(\tilde{\tau})(0))-(q(0,\tilde{\sigma}_0)-q(0,\tilde{\tau}_0))]\| \\
&+ \frac{d}{dt}\int_0^t Q(t-s)\Big[\|\mathcal{D}q(s,\tilde{\sigma}_s)-\mathcal{D}q(s,\tilde{\tau}_s)\| \\
&+ \int_0^s \|\mathcal{H}(\xi)q(s,\tilde{\sigma}_s)-q(s,\tilde{\tau}_s)\|d\xi \\
&+ \Big\|\varphi\Big(s,\tilde{\sigma}_s,\int_0^s h(s,\xi,\tilde{\sigma}_\xi)d\xi\Big)-\varphi\Big(s,\tilde{\tau}_s,\int_0^s h(s,\xi,\tilde{\tau}_\xi)d\xi\Big)\Big\|\Big]ds.
\end{aligned}
$$

By using the hypotheses

$$
\begin{aligned}
\|(\Gamma\sigma)(t)-(\Gamma\tau)(t)\| &\le \|M_1L_1\|\tilde{\sigma}_t-\tilde{\tau}_t\|_C + C(L_3\tilde{\sigma}_t(0)-\tilde{\tau}_t(0)\| + M_1L_1\|\tilde{\sigma}_0-\tilde{\tau}_0) \\
&+ C\int_0^t \Big[L_1\|\tilde{\sigma}_s-\tilde{\tau}_s\| + \int_0^s M_2L_1\|\tilde{\sigma}_s-\tilde{\tau}_s\|d\xi \\
&+ L_2(\|\tilde{\sigma}_s-\tilde{\tau}_s\| + L_h\|\tilde{\sigma}_s-\tilde{\tau}_s\|)\Big] \\
&\le \Big[M_1L_1 + C(L_3+M_1L_1+aL_1+M_2a^2L_1+aL_2+aL_2L_h)\Big]\|\tilde{\sigma}_s-\tilde{\tau}_s\| \\
&\le \Big[M_1L_1 + C\Big(L_3+(M_1+a+M_2a^2)L_1+L_2a(1+L_h)\Big)\Big]\|\sigma-\tau\|.
\end{aligned}
$$

Thus from Eq (3.2),

$$
\|(\Gamma\sigma)(t)-(\Gamma\tau)(t)\| \le k_0\|\sigma(t)-\tau(t)\|,
$$

where

$$
k_0 = \Big[M_1L_1 + C\Big(L_3+(M_1+a+M_2a^2)L_1+L_2a(1+L_h)\Big)\Big] < 1.
$$

Hence Γ has a fixed point $\omega(\cdot)$ and is a unique integral solution of Eq (1.1) on $[0,a]$.

Next we consider the continuous dependence of the solution for Eq (1.1) in the sense of the below theorem:

Theorem 3.4. *Suppose that the axioms of Theorem 3.3 hold and let $u(\cdot)$, $v(\cdot)$ be solutions of Eq (1.1) with the initial conditions $u_0, v_0 \in \overline{D(\mathcal{D})}$ respectively; then, the solution of Eq (1.1) has continuous dependence upon initial values, provided that*

$$
\|u(t)-v(t)\| \le \frac{(CL_3+CM_1L_1)e^{Ca[L_1+M_2L_1a+L_2+L_2L_h]}}{1-M_1L_1e^{Ca[L_1+M_2L_1a+L_2+L_2L_h]}}\|u_0-v_0\|
$$

$$
\text{and } M_1L_1e^{Ca[L_1+M_2L_1a+L_2+L_2L_h]} < 1. \tag{3.4}
$$

Proof. Let $u = u(\cdot), v = v(\cdot)$ be two solutions of Eq (1.1). For $t \in [0,a]$,

$$
\begin{aligned}
u(t)\| = q(t,u_t) &+ \mathcal{Q}'(t)\,[\phi(0)+g(u)(0)-q(0,u_0)]\\
&+ \frac{d}{dt}\int_0^t \mathcal{Q}(t-s)\Big[\mathcal{D}q(s,u_s)+\int_0^s q(s,u_s)\mathcal{H}(\xi)d\xi\\
&+ \varphi\Big(s,u_s,\int_0^s h(s,\xi,u_\xi)d\xi\Big)\Big]ds.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\|u(t)-v(t)\| &\leq \|q(t,u_t)-q(t,v_t)\|\\
&+ \|\mathcal{Q}'(t)\Big([g(u)(0)-g(v)(0)]+[q(0,u_0)-q(0,v_0)]\Big)\|\\
&+ \Big\|\frac{d}{dt}\int_0^t \mathcal{Q}(t-s)\Big[\mathcal{D}q(s,u_s)-\mathcal{D}q(s,v_s)+\int_0^s \mathcal{H}[q(s,u_s)-q(s,v_s)]d\xi\\
&+ \varphi\Big(s,u_s,\int_0^s h(s,\xi,u_\xi)d\xi\Big)-\varphi\Big(s,v_s,\int_0^s h(s,\xi,v_\xi)d\xi\Big)\Big]ds\Big\|.
\end{aligned}
$$

$$
\begin{aligned}
\|u(t)-v(t)\| &\leq \|q(t,u_t)-q(t,v_t)\|\\
&+ \|\mathcal{Q}'(t)\Big([g(u)(0)-g(v)(0)]+[q(0,u_0)-q(0,v_0)]\Big)\|\\
&+ C\int_0^t \Big[\|\mathcal{D}q(s,u_s)-\mathcal{D}q(s,v_s)\|+\int_0^s \|\mathcal{H}[q(s,u_s)-q(s,v_s)]\|d\xi\\
&+ \Big\|\varphi\Big(s,u_s,\int_0^s h(s,\xi,u_\xi)d\xi\Big)-\varphi\Big(s,v_s,\int_0^s h(s,\xi,v_\xi)d\xi\Big)ds\Big\|\Big]ds\\
&\leq M_1L_1\|u_t-v_t\|+C\Big(L_3\|u(0)-v(0)\|+M_1L_1\|u_0-v_0\|\Big)\\
&+ C\int_0^t \Big(L_1\|u_s-v_s\|+M_2L_1a\|u_s-v_s\|+L_2(\|u_s-v_s\|+L_h\|u_\xi-v_\xi\|)\Big)\\
&\leq M_1L_1\|u_t-v_t\|+C\Big[L_3\|u(0)-v(0)\|+M_1L_1\|u_0-v_0\|\\
&+ (L_1+M_2L_1a+L_2+L_2L_h)\int_0^t \sup_{0\leq\xi\leq s}\|u_s-v_s\|ds\Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|u(t)-v(t)\| &\leq \Big[M_1L_1\|u_t-v_t\|+(CL_3+CM_1L_1)\|u_0-v_0\|\Big]\\
&+ C(L_1+M_2L_1a+L_2+L_2L_h)\int_0^t \sup_{0\leq\xi\leq s}\|u_s-v_s\|ds.
\end{aligned}
$$

Hence by Gronwall's lemma,

$$
\sup_{0\leq s\leq t}\|u(s)-v(s)\| \leq \Big[M_1L_1\|u_t-v_t\|+(CL_3+CM_1L_1)\|u_0-v_0\|\Big]e^{\int_0^a [C(L_1+M_2L_1a+L_2+L_2L_h)]ds}.
$$

$$\begin{aligned}\|u(t)-v(t)\| &- M_1L_1\|u_t-v_t\|e^{[Ca(L_1+M_2L_1a+L_2+L_2L_h)]}\\ &\le (CL_3+CM_1L_1)\|u_0-v_0\|e^{[Ca(L_1+M_2L_1a+L_2+L_2L_h)]}.\end{aligned}$$

$$\begin{aligned}\|u(t)-v(t)\|(1-M_1L_1e^{[Ca(L_1+M_2L_1a+L_2+L_2L_h)]}) &\le (CL_3+CM_1L_1)\\ &\times e^{[Ca(L_1+M_2L_1a+L_2+L_2L_h)]}\|u_0-v_0\|.\end{aligned}$$

$$\|u(t)-v(t)\| \le \frac{(CL_3+CM_1L_1)e^{[Ca(L_1+M_2L_1a+L_2+L_2L_h)]}}{1-M_1L_1e^{[Ca(L_1+M_2L_1a+L_2+L_2L_h)]}}\|u_0-v_0\|.$$

Thus from Eq (3.4), the integral solution of Eq (1.1) has continuous dependence on the initial conditions.

4. Existence of a strict solution

Here, we study the strict solution of the problem given by Eq (1.1), by using the integrated resolvent operator theory and under some considerations.

Definition 4.1. *A function $\omega(\cdot) : [-r,+\infty) \to \mathcal{E}$ is a strict solution of Eq (1.1), if $\omega(t) - q(t,\omega_t) \in C^1([0,+\infty);\mathcal{E}) \cap C([0,+\infty);Y)$ and ω holds as in Eq (1.1) on $[-r,+\infty)$.*

First we prove this in reflexive BS in the sense of the below theorem:

Theorem 4.2. *Assume that the hypotheses of Theorem 3.3 hold, and that the following conditions are satisfied:*

$(H5)$ $\phi(0) + g(\omega)(0) \in D(\mathcal{D}), \mathcal{D}[\phi(0) + g(\omega)(0) - q(0,\omega_0)] + \varphi(0,\omega_0,0) \in \overline{D(\mathcal{D})}$.

$(H6)$ *It holds that*

$$((M_1 + K_a a + K_a M_2 a^2)L_1 + (1 + L_h)K_a a L_2) < 1. \tag{4.1}$$

Then Eq (1.1) has a strict solution on $[-r,a]$.

Proof. Let the operator Γ on $C([-r,a];\mathcal{E})$ be as given in Theorem 3.3. Let the closed ball $B_{r_0} = \{\omega \in C([-r,a];\mathcal{E} : \|\omega\| \le r_0, \|\omega(t_2) - \omega(t_1)\| \le L^*|t_2 - t_1|, t_1, t_2 \in [-r,a]\}$. Here $L^* > 0$ is a constant and we prove that Γ has a fixed point on B_{r_0}. By Theorem 3.3, $\Gamma(B_{r_0}) \subset B_{r_0}$; it suffices to show that

$$\|(\Gamma\omega)(t_2) - (\Gamma\omega)(t_1)\| \le L^*|t_2 - t_1| \text{ for } \omega \in B_{r_0}, t_1, t_2 \in [-r,a]. \tag{4.2}$$

The extension of the operator solution $\Gamma(\tilde{\omega}(t))$ is defined by

$$\Gamma(\tilde{\omega}(t)) = \begin{cases}(\tilde{\omega})(t), & t \in [0,a],\\ \phi(t) + g(\tilde{\omega})(t), & t \in [-r,0].\end{cases}$$

Now,

$$\begin{aligned}\|(\Gamma\omega)(t_2) - (\Gamma\omega)(t_1)\| &\le \|f(t_2,\tilde{\omega}_{t_2}) - q(t_1,\tilde{\omega}_{t_1})\|\\ &+ \|[Q'(t_2) - Q'(t_1)](\phi(0) + g(\tilde{\omega})(0) - q(0,\tilde{\omega}_0))\|\end{aligned}$$

$$
\begin{aligned}
&+ \quad \Big\| \frac{d}{dt_2} \int_0^{t_2} \mathcal{Q}(t_2 - s) \Big[\mathcal{D} q(s, \tilde{\omega}_s) + \int_0^s q(s, \tilde{\omega}_s) \mathcal{H}(\xi) d\xi \\
&+ \quad \varphi\Big(s, \tilde{\omega}_s, \int_0^s h(s, \xi, \tilde{\omega}_\xi) d\xi\Big)\Big] ds \\
&- \quad \frac{d}{dt_1} \int_0^{t_1} \mathcal{Q}(t_1 - s) \Big[\mathcal{D} q(s, \tilde{\omega}_s) + \int_0^s q(s, \tilde{\omega}_s) \mathcal{H}(\xi) d\xi \\
&+ \quad \varphi\Big(s, \tilde{\omega}_s, \int_0^s h(s, \xi, \tilde{\omega}_\xi) d\xi\Big)\Big] ds \Big\| \\
&=: \quad I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
I_1 = \|q(t_2, \tilde{\omega}_{t_2}) - q(t_1, \tilde{\omega}_{t_1})\|,
$$

$$
I_2 = \|[\mathcal{Q}'(t_2) - \mathcal{Q}'(t_1)](\phi(0) + g(\tilde{\omega})(0) - q(0, \tilde{\omega}_0))\|,
$$

$$
\begin{aligned}
I_3 \quad = \quad & \Big\| \frac{d}{dt_2} \int_0^{t_2} \mathcal{Q}(t_2 - s) \Big[\mathcal{D} q(s, \tilde{\omega}_s) + \int_0^s q(s, \tilde{\omega}_s) \mathcal{H}(\xi) d\xi \\
+ \quad & \varphi\Big(s, \tilde{\omega}_s, \int_0^s h(s, \xi, \tilde{\omega}_\xi) d\xi\Big)\Big] ds \\
- \quad & \frac{d}{dt_1} \int_0^{t_1} \mathcal{Q}(t_1 - s) \Big[\mathcal{D} q(s, \tilde{\omega}_s) + \int_0^s q(s, \tilde{\omega}_s) \mathcal{H}(\xi) d\xi \\
+ \quad & \varphi\Big(s, \tilde{\omega}_s, \int_0^s h(s, \xi, \tilde{\omega}_\xi) d\xi\Big)\Big] ds \Big\|.
\end{aligned}
$$

Now take I_1:

$$
\begin{aligned}
I_1 \quad &\leq \quad M_1 L_1 (|t_2 - t_1| + \|\tilde{\omega}_{t_2} - \tilde{\omega}_{t_1}\|) \\
&\leq \quad M_1 L_1 (|t_2 - t_1| + L^* |t_2 - t_1|) \\
&\leq \quad (M_1 L_1 + M_1 L_1 L^*) |t_2 - t_1|.
\end{aligned}
$$

From $(R4)$,

$$
\begin{aligned}
\mathcal{Q}'(t)\omega - \omega \quad &= \quad \mathcal{Q}\mathcal{D}\omega + \int_0^t \mathcal{Q}(t - s)\mathcal{H}(s)\omega ds. \\
I_2 \quad &\leq \quad \|\mathcal{Q}(t_2) - \mathcal{Q}(t_1)\| \|\mathcal{D}[\phi(0) + g(\tilde{\omega})(0) - q(0, \tilde{\omega}_0)]\| \\
&+ \quad \int_0^{t_1} \|\mathcal{Q}(t_2 - s) - \mathcal{Q}(t_1 - s)\| \|\mathcal{H}(\xi)[\phi(0) + g(\tilde{\omega})(0) - q(0, \tilde{\omega}_0)]\| ds \\
&+ \quad \int_{t_1}^{t_2} \|\mathcal{Q}(t_2 - s)\| \mathcal{H}(\xi)[\phi(0) + g(\tilde{\omega})(0) - q(0, \tilde{\omega}_0)]\| ds \\
&\leq \quad K_a \|\mathcal{D}[\phi(0) + g(\tilde{\omega})(0) - q(0, \tilde{\omega}_0)]\| |t_2 - t_1| \\
&+ \quad K_a M_2 \|\mathcal{D}[\phi(0) + g(\tilde{\omega})(0) - q(0, \tilde{\omega}_0)]\| a |t_2 - t_1|
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{t\in[0,a]} \|Q(t)\|M_2\mathcal{D}[\phi(0)+g(\tilde{\omega})(0)-q(0,\tilde{\omega_0})]\|\|t_2-t_1| \\
&\leq (K_a+K_aM_2a+MM_2)\,M_1\,(\|\phi\|+L_3r_0-L_1(r_0+1))\,|t_2-t_1| \\
&\leq [K_aM_1(1+aM_2)+MM_1M_2]\,(\|\phi\|+L_3r_0)|t_2-t_1|.
\end{aligned}
$$

From I_3, we note that

$$
\begin{aligned}
\|\mathcal{D}q(s,\tilde{\omega_s})+\int_0^s \mathcal{H}(\xi)q(\xi,\tilde{\omega_\xi})d\xi+\varphi\left(s,\tilde{\omega_s},\int_0^s h(s,\xi,\tilde{\omega_\xi})d\xi\right)\| \\
\leq [L_1+M_2L_1a+L_2+L_2L_h]\,r_0.
\end{aligned}
$$

Now

$$
\begin{aligned}
I_3 \leq\ & K_a[L_1+M_2L_1a+L_2+L_2L_h]r_0|t_2-t_1| \\
+\ & K_a\int_0^{t_1}\Big[\|\mathcal{D}q(t_2-t_1+s,\tilde{\omega}_{t_2-t_1+s})-\mathcal{D}q(s,\tilde{\omega_s})\| \\
+\ & \int_0^s \|\mathcal{H}(s-\xi)\|\|q(t_2-t_1+\xi,\tilde{\omega}_{t_2-t_1+\xi})-q(\xi,\tilde{\omega_\xi})\|d\xi \\
+\ & \int_{-(t_2-t_1)}^0 \|\mathcal{H}(s-\xi)q(t_2-t_1+\xi,\tilde{\omega}_{t_2-t_1+\xi})\|d\xi \\
+\ & \Big\|\varphi\Big(t_2-t_1+s,\tilde{\omega}_{t_2-t_1+s},\int_0^{t_2-t_1+s} h(t_2-t_1+s,t_2-t_1+\xi,\tilde{\omega}_{t_2-t_1+\xi})d\xi\Big) \\
-\ & \varphi\left(s,\tilde{\omega}_s,\int_0^s h(s,\xi,\tilde{\omega_\xi})d\xi\right)\Big\|\Big]ds,
\end{aligned}
$$

$$
\begin{aligned}
I_3 \leq\ & K_a[L_1+M_2L_1a+L_2+L_2L_h](r_0+1)|t_2-t_1| \\
+\ & K_aa\Big[L_1(1+L^*)+(M_2L_1a(1+L^*))+M_2L_1(r_0+1)+L_2(1+L^*+L_hL^*)\Big]|t_2-t_1|.
\end{aligned}
$$

From the estimates of I_1, I_2, I_3, we have

$$
\begin{aligned}
\|(\Gamma\omega)(t_2)-(\Gamma\omega)(t_1)\| \leq\ & \Big[(M_1L_1+M_1L_1L^*)+[K_aM_1(1+aM_2)+MM_1M_2][\|\phi\|+L_3r_0] \\
+\ & K_a(L_1+M_2L_1a+L_2+L_2L_h)(r_0+1) \\
+\ & K_aa[L_1+L_1L^*+M_2L_1a+M_2L_1L^*a+M_2L_1(r_0+1) \\
+\ & (L_2+L_2L^*+L_2L_hL^*)]\Big]|t_2-t_1| \\
\leq\ & \Big[C^*+\Big((M_1+K_aa+K_aM_2a^2)L_1+(1+L_h)K_aaL_2\Big)L^*\Big]|t_2-t_1|,
\end{aligned}
$$

where $C^*\in\mathbb{R}$, and different from L^*, $\omega\in B_{r_0}$; thus, from $(H6)$

$$
\|(\Gamma\omega)(t_2)-(\Gamma\omega)(t_1)\|\leq L^*|t_2-t_1|,
$$

where we assume that L^* is large enough $\left(\geq \frac{C^*}{1-((M_1+K_aa+K_aM_2a^2)L_1+(1+L_h)K_aaL_2)}\right)$.

Thus Γ has a unique fixed point $\omega(\cdot)$ and is an integral solution of Eq (1.1). Further $\omega(t)$ is Lipschitz-continuous on $[0, a]$; moreover,

$$s \to \mathcal{D}q(s, \omega_s) + \int_0^s \mathcal{H}(s-\xi)q(\xi, \omega_\xi)d\xi + \varphi\Big(s, \omega_s, \int_0^s h(s, \xi, \omega_\xi)d\xi\Big)$$

is also Lipschitz continuous on $[0, a]$ and $\mathcal{E}$ is a reflexive BS; hence, by the Radon-Nikodym property,

$$\mathcal{D}q(s, \omega_s) + \int_0^s \mathcal{H}(s-\xi)q(\xi, \omega_\xi)d\xi + \varphi\Big(s, \omega_s, \int_0^s h(s, \xi, \omega_\xi)d\xi\Big) \in W^{1,1}([0, a]; \mathcal{E}).$$

By Lemma 2.4, we have that $\omega(t)-q(t, \omega_t)$ is differentiable on $[0, a]$ and also a strict solution of Eq (1.1) on $[0, a]$. Next we consider that $\mathcal{E}$ is a general BS; further, we assume the following:

$(H7)$ The function $q \in C^1(\mathbb{R}^+ \times \mathcal{E}; Y)$ and the partial derivatives $D_1q(\cdot, \cdot), D_2q(\cdot, \cdot)$ are Lipschitz-continuous with respect to the second variable; $\exists L_q^i > 0$; then,

$$\|D_iq(t_1, s_1) - D_iq(t_1, s_2)\| \leq L_q^i\|s_1 - s_2\|$$

for $t_1 \in [0, a], s_1, s_2 \in C, i = 1, 2$.

$(H8)$ The function $\varphi \in C^1(\mathbb{R}^+ \times C \times \mathcal{E}; \mathcal{E})$ and the partial derivatives $D_1\varphi(\cdot, \cdot, \cdot), D_2\varphi(\cdot, \cdot, \cdot)$ are Lipschitz-continuous function with respect to second variable; $\exists L_\varphi^i > 0$; then,

$$\|D_i\varphi(t_1, r_1, s_1) - D_i\varphi(t_1, r_2, s_2)\| \leq L_\varphi^i(\|r_1 - r_2\| + \|s_1 - s_2\|)$$

for any $t_1 \in [0, a], r_1, r_2 \in C, s_1, s_2 \in \mathcal{E}$.

Theorem 4.3. *Suppose that (H1)–(H4),* $(H7)$ *and* $(H8)$ *are true with* $M_1L_1 < 1$. *If* $\omega(\cdot)$ *is an integral solution of Eq (1.1),* $\phi + g(\omega) \in D(\mathcal{D})$ *and* $\mathcal{D}[\phi(0) + g(\omega)(0) - q(0, \omega_0)] + \varphi(0, \omega_0, 0) \in \overline{D(\mathcal{D})}$*; then,* $\omega(\cdot)$ *is a strict solution of Eq (1.1).*

Proof. Let $\omega(\cdot)$ be an integral solution of Eq (1.1); see the following system

$$\begin{aligned} y(t) = {} & D_1q(t, \omega_t) + D_2q(t, \omega_t)y_t + Q'(t)\mathcal{D}[\phi(0) + g(\omega)(0) - q(0, \omega_0)] + Q'(t)\mathcal{D}q(0, \omega_0) \\ & + \frac{d}{dt}\int_0^t Q(t-s)\mathcal{H}(s)q(0, \omega_0)ds + Q'(t)\varphi(0, \omega_0, 0) \\ & + \frac{d}{dt}\int_0^t Q(t-s)\mathcal{H}(s)(\phi(0) + g(\omega)(0) - q(0, \omega_0))ds \\ & + \frac{d}{dt}\int_0^t Q(t-s)\Big[\mathcal{D}D_1q(s, \omega_s) + \mathcal{D}D_2q(s, \omega_s)y_s \\ & + \int_0^s \mathcal{H}(s-\xi)[D_1q(\xi, \omega_\xi) + D_2q(\xi, \omega_\xi)y_\xi]d\xi \\ & + D_1\varphi\Big(s, \omega_s, \int_0^s h(s, \xi, \omega_\xi)\Big) + D_2\varphi\Big(s, \omega_s, \int_0^s h(s, \xi, \omega_\xi)\Big)y_s\Big]ds. \end{aligned} \tag{4.3}$$

From the Banach principle, there exists a unique solution $y(\cdot) \in C([0, a]; \mathcal{E})$ to Eq (4.3). Let the map $z(t)$ be defined by

$$z(t) = \phi(0) + g(\omega)(0) + \int_0^t y(s)ds \text{ for } t \in [0, a].$$

We shall prove that $\omega(\cdot) = z(\cdot)$ on $[0, a]$.

$$
\begin{aligned}
z(t) = \phi(0) + g(\omega)(0) + \int_0^t [D_1 q(s,\omega_s) + D_2 q(s,\omega_s) y_s] ds \\
+ \mathcal{Q}\mathcal{D}[\phi(0) + g(\omega)(0) - q(0,\omega_0)] + \mathcal{Q}\mathcal{D}(q(0,\omega_0)) \\
+ \int_0^t \mathcal{Q}(t-s)\mathcal{H}(s) q(0,\omega_0) ds + \mathcal{Q}(t)(\varphi(0,\omega_0,0)) \\
+ \int_0^t \mathcal{Q}(t-s)\mathcal{H}(s)(\phi(0) + g(\omega)(0) - q(0,\omega_0)) ds \\
+ \int_0^t \mathcal{Q}(t-s)\Big[\mathcal{D}D_1 q(s,\omega_s) + \mathcal{D}D_2 q(s,\omega_s) y_s \\
+ \int_0^s \mathcal{H}(s-\xi)[D_1 q(\xi,\omega_\xi) + D_2 q(\xi,\omega_\xi) y_\xi] d\xi \\
+ D_1\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)\Big) + D_2\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)\Big) y_s\Big] ds.
\end{aligned} \tag{4.4}
$$

From $(R4)$,

$$
\begin{aligned}
\mathcal{Q}\mathcal{D}[\phi(0) + g(\omega)(0) - q(0,\omega_0)] = \mathcal{Q}'(t)[\phi(0) + g(\omega)(0) - q(0,\omega_0)] \\
- (\phi(0) + g(\omega)(0) - q(0,\omega_0)) \\
- \int_0^t \mathcal{Q}(t-s)\mathcal{H}(s)[\phi(0) + g(\omega)(0) - q(0,\omega_0)] ds.
\end{aligned} \tag{4.5}
$$

Consequently,

$$
q(0, z_0) = q(t, z_t) - \int_0^t [D_1 q(s, z_s) + D_2 q(s, z_s) y_s] ds. \tag{4.6}
$$

Further, we obtain

$$
\begin{aligned}
& \mathcal{Q}\mathcal{D}q(0,z_0) + \int_0^t \mathcal{Q}(t-s)\mathcal{H}(s) q(0,z_0) ds + \mathcal{Q}\varphi(0,z_0,0) \\
& = \frac{d}{dt}\int_0^t \mathcal{Q}(t-s)\Big[\mathcal{D}q(s,z_s) + \int_0^s \mathcal{H}(s-\xi) q(\xi,z_\xi) d\xi \\
& + \varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi) d\xi\Big)\Big] ds - \int_0^t \mathcal{Q}(t-s)\Big[\mathcal{D}D_1 q(s,z_s) + \mathcal{D}D_2 q(s,z_s) y_s \\
& + \int_0^s \mathcal{H}(s-\xi)[D_1 q(\xi,z_\xi) + D_2 q(\xi,z_\xi) y_\xi] d\xi \\
& + D_1\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi) d\xi\Big) + D_2\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi) d\xi\Big) y_s\Big] ds.
\end{aligned} \tag{4.7}
$$

Since $\omega_0 = z_0$ putting Eqs (4.5)–(4.7) in Eq (4.4), we have

$$
\begin{aligned}
z(t) &= q(t,z_t) + \int_0^t [D_1 q(s,\omega_s) + D_2 q(s,\omega_s) y_s] ds - \int_0^t [D_1 q(s,z_s) + D_2 q(s,z_s) y_s] ds \\
&+ \mathcal{Q}'(t)[\phi(0) + g(\omega)(0) - q(0,\omega_0)]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{d}{dt}\int_0^t Q(t-s)\Big[\mathcal{D}q(s,z_s) + \int_0^s \mathcal{H}(s-\xi)q(\xi,z_\xi)d\xi + \varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big)\Big]ds \\
&- \quad \int_0^t Q(t-s)\Big[[\mathcal{D}D_1q(s,z_s) + \mathcal{D}D_2q(s,z_s)y_s] + \int_0^s \mathcal{H}(s-\xi)[D_1q(\xi,z_\xi) + D_2q(\xi,z_\xi)y_\xi]d\xi \\
&+ \quad D_1\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big) + D_2\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big)y_s\Big]ds \\
&+ \quad \int_0^t Q(t-s)\Big[\mathcal{D}D_1q(s,\omega_s) + \mathcal{D}D_2q(s,\omega_s)y_s + \int_0^s \mathcal{H}(s-\xi)[D_1q(\xi,\omega_\xi) + D_2q(\xi,\omega_\xi)y_\xi]d\xi \\
&+ \quad D_1\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big) + D_2\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)y_s\Big]ds.
\end{aligned}
$$

Now

$$
\begin{aligned}
z(t)-\omega(t) &= q(t,z_t) - q(t,\omega_t) + \frac{d}{dt}\int_0^t Q(t-s)[\mathcal{D}q(s,z_s) - \mathcal{D}q(s,\omega_s)]ds \\
&+ \frac{d}{dt}\int_0^t Q(t-s)\int_0^s \mathcal{H}(s-\xi)[q(\xi,z_\xi) - q(\xi,\omega_\xi)]d\xi ds \\
&+ \frac{d}{dt}\int_0^t Q(t-s)\Big[\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big) - \varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)\Big]ds \\
&+ \int_0^t [D_1q(s,\omega_s) - D_1q(s,z_s)]ds + \int_0^t [D_2q(s,\omega_s) - D_2q(s,z_s)]y_s ds \\
&+ \int_0^t Q(t-s)[\mathcal{D}D_1q(s,\omega_s) - \mathcal{D}D_1q(s,z_s)]ds \\
&+ \int_0^t Q(t-s)[\mathcal{D}D_2q(s,\omega_s) - \mathcal{D}D_2q(s,z_s)]y_s ds \\
&+ \int_0^t Q(t-s)\int_0^s \mathcal{H}(s-\xi)[D_1q(\xi,\omega_\xi) - D_1q(\xi,z_\xi)]d\xi ds \\
&+ \int_0^t Q(t-s)\int_0^s \mathcal{H}(s-\xi)[D_2q(\xi,\omega_\xi) - D_2q(\xi,z_\xi)]y_\xi d\xi ds \\
&+ \int_0^t Q(t-s)\Big[D_1\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big) - D_1\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big)\Big]ds \\
&+ \int_0^t Q(t-s)\Big[D_2\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big) - D_2\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big)\Big]y_s ds \\
&:= J_1 + J_2 + J_3
\end{aligned}
$$

where

$$
\begin{aligned}
\|J_1\| &\leq \|q(t,z_t) - q(t,\omega_t)\| + \frac{d}{dt}\int_0^t Q(t-s)\|[\mathcal{D}q(s,z_s) - \mathcal{D}q(s,\omega_s)]\|ds \\
&+ \frac{d}{dt}\int_0^t Q(t-s)\int_0^s \|\mathcal{H}(s-\xi)[q(\xi,z_\xi) - q(\xi,\omega_\xi)]\|d\xi ds \\
&+ \frac{d}{dt}\int_0^t Q(t-s)\Big\|\Big[\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big) - \varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big)\Big]\Big\|ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq M_1 L_1 \sup_{0\leq s\leq t} \|\omega_s - z_s\| \\
&+ K_a\Big(L_1 + M_2 L_1 a + L_2(\|z_s - w_s\| + L_h\|z_\xi - \omega_\xi\|)\Big)\int_0^t \sup_{0\leq \xi\leq s}\|\omega_\xi - z_\xi\|ds \\
&\leq M_1 L_1 \sup_{0\leq s\leq t}\|\omega_s - z_s\| + K_a(L_1 + M_2 L_1 a + L_2 + L_2 L_h)\int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds,
\end{aligned}
$$

$$
\begin{aligned}
\|J_2\| &\leq \int_0^t \|[D_1 q(s,\omega_s) - D_1 q(s,z_s)]\|ds + \int_0^t \|[D_2 q(s,\omega_s) - D_2 q(s,z_s)]\|\|y_s\|ds \\
&+ \int_0^t \|Q(t-s)\|\|[\mathcal{D}D_1 q(s,\omega_s) - \mathcal{D}D_1 q(s,z_s)]\|ds \\
&+ \int_0^t \|Q(t-s)\|\|[\mathcal{D}D_2 q(s,\omega_s) - \mathcal{D}D_2 q(s,z_s)]\|\|y_s\|ds \\
&+ \int_0^t \|Q(t-s)\| \int_0^s \|\mathcal{H}(s-\xi)[D_1 q(\xi,\omega_\xi) - D_1 q(\xi,z_\xi)]\|d\xi ds \\
&+ \int_0^t \|Q(t-s)\| \int_0^s \|\mathcal{H}(s-\xi)[D_2 q(\xi,\omega_\xi) - D_2 q(\xi,z_\xi)]\|\|y_\xi\|d\xi ds \\
&\leq M_1 L_q^1 \int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds + M_1 M_3 L_q^2 \int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds \\
&+ \sup_{0\leq s\leq a}\|Q(s)\|L_q^1 \int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds + \sup_{0\leq s\leq a}\|Q(s)\|M_3 L_q^2 \int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds \\
&+ \sup_{0\leq s\leq a}\|Q(s)\|M_2 L_q^1 a \int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds \\
&+ \sup_{0\leq s\leq a}\|Q(s)\|M_2 M_3 L_q^2 a \int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds \\
&\leq \Big[M_1(L_q^1 + M_3 L_q^2) + \sup_{0\leq s\leq a}\|Q(s)\|(L_q^1 + M_3 L_q^2 + M_2 L_q^1 a + M_2 M_3 L_q^2 a)\Big] \\
&\times \int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds,
\end{aligned}
$$

where $M_3 = \sup_{0\leq s\leq a}\|y_s\|$,

$$
\begin{aligned}
\|J_3\| &\leq \int_0^t \|Q(t-s)\|\Big\|\Big[D_1\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big) - D_1\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big)\Big]\Big\|ds \\
&+ \int_0^t \|Q(t-s)\|\Big\|\Big[D_2\varphi\Big(s,\omega_s,\int_0^s h(s,\xi,\omega_\xi)d\xi\Big) - D_2\varphi\Big(s,z_s,\int_0^s h(s,\xi,z_\xi)d\xi\Big)\Big]\Big\|\|y_s\|ds \\
&\leq \sup_{0\leq s\leq a}\|Q(s)\|\Big(L_\varphi^1(\|\omega_s - z_s\| + L_h\|\omega_\xi - z_\xi\|)\Big)\int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds \\
&+ \sup_{0\leq s\leq a}\|Q(s)\|\Big(L_\varphi^1 + L_\varphi^1 L_h + M_3 L_\varphi^2 + M_3 L_\varphi^2 L_h\Big)\int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds
\end{aligned}
$$

$$\leq \sup_{0\leq s\leq a}\|Q(s)\|\Big(L_\varphi^1(1+L_h)+M_3L_\varphi^2(1+L_h)\Big)\int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds$$
$$\leq \sup_{0\leq s\leq a}\|Q(s)\|(1+L_h)(L_\varphi^1+M_3L_\varphi^2)\int_0^t \sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds.$$

By the values of J_1, J_2, J_3 we have

$$\begin{aligned}\sup_{0\leq s\leq t}\|\omega_s - z_s\| \leq\ & M_1L_1\sup_{0\leq s\leq t}\|\omega_s - z_s\| + \Big[K_a(L_1+M_2L_1a+L_2+L_2L_h)+M_1(L_q^1+M_3L_q^2)\\ &+ \sup_{0\leq s\leq a}\|Q(s)\|\Big(L_q^1+M_3L_q^2+M_2L_q^1a+M_2M_3L_q^2a\\ &+ (1+L_h)(L_\varphi^1+M_3L_\varphi^2)\Big)\Big]\int_0^t\sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds.\end{aligned}$$

Since $M_1L_1 < 1$, we obtain that

$$\sup_{0\leq s\leq t}\|\omega_s - z_s\| \leq \frac{N}{1-M_1L_1}\int_0^t\sup_{0\leq\xi\leq s}\|\omega_\xi - z_\xi\|ds,$$

where

$$\begin{aligned}N =& K_a(L_1+M_2L_1a+L_2+L_2L_h)+M_1(L_q^1+M_3L_q^2)\\ &+\sup_{0\leq s\leq a}\|Q(s)\|\Big(L_q^1+M_3L_q^2+M_2L_q^1a+M_2M_3L_q^2a+(1+L_h)(L_\varphi^1+M_3L_\varphi^2)\Big).\end{aligned}$$

Then by the Gronwall lemma, it follows that $\omega_t = z_t$ for all $t \in [-r, a]$, which shows that $\omega(t)$ is continuously differentiable on $[-r, a]$; consequently, $\omega(\cdot)$ is a strict solution of Eq (1.1).

5. Example

The application of this theory, we consider the following system:

$$\begin{cases}\frac{\partial}{\partial t}[\omega(t,x)-\int_{-r}^0 g(t,\omega(t+\theta,x))d\theta] = \frac{\partial^2}{\partial x^2}\omega(t,x)\\ \quad + \int_0^t p(t-s)\frac{\partial^2}{\partial x^2}\omega(s,x)ds + \int_0^t\int_{-r}^0 c(t,\omega(t+\theta,x))\omega(t,x)d\theta ds \text{ for } t\geq 0, x\in[0,1],\\ \omega(t,0)=\omega(t,1) \text{ for } t\in[0,1],\\ \omega(0,x)+\sum_{i=1}^p\int_0^1\eta_i(x,y)\omega(t_i,y)dy = \omega_0(x) \text{ for } t\in[-r,0], x,y\in[0,1],\end{cases} \tag{5.1}$$

where the function p is a locally bounded variation from $\mathbb{R}^+$ to $\mathbb{R}$, $0 < t_1 < t_2 < .... < t_p < 1$, $\omega_0(x)$ is an initial function on the BS $\mathcal{E} = C([0,1];\mathbb{R})$ and $\omega_0 : [-r,0]\times[0,1] \to \mathbb{R}$, $g : \mathbb{R}^+\times\mathbb{R} \to \mathbb{R}$ is continuous; we further assume the following

$(A1)$ We have the function $g \in C^2([-r,0]\times[0,1];\mathbb{R})$ with $g(\cdot,0) = g(\cdot,1) = 0$ and $\exists q_1(\cdot,\cdot) \in L^1([0,1]\times[0,1];\mathbb{R})$ then,

$$\left|\frac{\partial^2}{\partial x^2}g(x_1,y_1) - \frac{\partial^2}{\partial x^2}g(x_2,y_2)\right| < q_1(|x_1-x_2|+|y_1-y_2|)$$

and

$$\left|\frac{\partial^2}{\partial x^2} g(x_1, y_1)\right| < q_1(|x_1| + 1) \text{ for } x_1, x_2 \in [0, 1], y_1, y_2 \in \mathbb{R}.$$

$(A2)$ The map $c : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous $\exists L_c > 0$ and

$$|c(t_1, v_1) - c(t_2 - v_2)| \leq L_c(|t_1 - t_2| + |v_1 - v_2|) \text{ for } 0 \leq t_1, t_2 \leq 1, v_1, v_2 \in \mathbb{R}$$

and

$$|c(t, v)| \in L_c(|v| + 1) \text{ for } (t, v) \in [0, 1] \times \mathbb{R}.$$

$(A3)$ The functions $\eta_i : [0, 1] \times [0, 1] \to \mathbb{R}$ are continuous with $\eta_i(0, \cdot) = \eta_i(1, \cdot) = 0$ and $k_i = \sup |\eta_i(x, y)|; 0 \leq x \leq 1, 0 \leq y \leq 1 < 1, i = 1, 2, ...p$.

Now, we write Eq (5.1) as an abstract form of Eq (1.1) in $\mathcal{E}$. Let $\mathcal{D}$ be the operator by $\mathcal{D}v = v''$ and the domain

$$D(\mathcal{D}) = v \in \mathcal{E}; v'' \in \mathcal{E} \text{ and } v(0) = v(1) = 0.$$

Let the operators $\mathcal{H}(t) : D(\mathcal{D}) \subset \mathcal{E} \to \mathcal{E}, t \geq 0$ be defined by

$$\mathcal{H}(t)v = p(t)v'', \text{ and } D(\mathcal{H}(t)) = D(\mathcal{D}).$$

Hence $(\mathcal{H}(t)) \subset \mathcal{L}(Y, \mathcal{E})$ and $\mathcal{H}(\cdot)v \in BV_{loc}(\mathbb{R}; \mathcal{E})$, $v \in D(\mathcal{D})$. Here Y is already explained in Section 2, so there exists an LLC integrated resolvent operator $Q_{t\geq 0}$ related to Eq (5.1).

Further let $\omega(t)(x) = \omega(t, x)$, $q : [0, 1] \times C \to \mathcal{E}, \varphi : [0, 1] \times C \times \mathcal{E} \to \mathcal{E}, g : C([-r, 0]; \mathcal{E}) \to \overline{D(\mathcal{D})}$ be respectively,

$$\begin{aligned}
q(t, v)(x) &= \int_{-r}^{0} g(t, v(\theta))d\theta, v \in \mathcal{E},\\
\varphi(t, u, v)(x) &= \int_0^t \int_{-r}^{0} c(t, \omega(t + \theta, x))\omega(t, x)d\theta ds,\\
g(\omega(t)) &= \sum_{i=1}^{p} \int_{-r}^{0} \eta_i(x, y)\omega(t_i, y)dy, \omega \in C([-r, 0]; \mathcal{E}).
\end{aligned}$$

Under the above assumptions Eq (5.1) is rewritten in the form (1.1) and conditions of Theorems 3.3 and 3.4 are fulfilled. Also the functions q, φ satisfy the Lipschitz-continuous conditions in (H1) and (H2) respectively. In fact for $t_1, t_2 \in [0, 1]$ and $v_1, v_2 \in \mathcal{E}$ we have

$$\begin{aligned}
\|\mathcal{D}q(t_1, v_1) - \mathcal{D}q(t_2, v_2)\| &\leq \sup_{x\in[0,1]} \int_{-r}^{0} \left|\frac{\partial^2}{\partial x^2} g(t_1, v_1(\theta)) - \frac{\partial^2}{\partial x^2} g(t_2, v_2(\theta))\right| d\theta\\
&\leq \sup_{x\in[0,1]} \int_{-r}^{0} q_1(x, y)(|t_1 - t_2| + |v_1(\theta) - v_2(\theta))d\theta\\
&\leq L_{q_1}(|t_1 - t_2| + \|v_1 - v_2\|),
\end{aligned}$$

where $L_{q_1} > 0$ is a Lipschitz constant of q that is Lipschitz continuous on its domain C^2; also clearly, φ is satisfies (H2) with $L_2 = L_c$. On the other hand, under the condition of (A3), for $\omega_1, \omega_2 \in C([-r, 0]; \mathcal{E})$,

$$\begin{aligned}\|g(\omega_1(t)) - g(\omega_2(t))\| &\leq \sum_{i=1}^{p} \sup_{x\in[0,1]} \int_{-r}^{0} |\eta_i(x,y)| \|\omega_1(t_i)(y) - \omega_2(t_i)(y)\| dy \\ &\leq \sum_{i=1}^{p} k_i \|\omega_1(t_i) - \omega_2(t_i)\| \\ &\leq L_\eta \|\omega_1 - \omega_2\|_C\end{aligned}$$

which shows that g satisfies (H3). According to Theorems 3.3 and 3.4, we state that the following:

(1) If

$$M_1 L_{q_1} + C[L_\eta + (M_1 + 1 + M_2)L_{q_1} + L_c] < 1$$

from Theorem 3.3, for the initial map $\omega_0(x) \in \mathcal{E}$ and $\omega_0(0) = \omega_0(1) = 0$, $\exists$ a unique integral solution of Eq (5.1) on [0,1]. Further from Theorem 3.4, if

$$M_1 L_{q_1} e^{C[L_{q_1} + M_2 L_{q_1} + L_c]} < 1$$

then the estimates of Eq (3.4) holds such that the solution of Eq (5.1) has continuous dependence upon the initial data.

(2) Further assume that $\omega_0(\cdot) \in C^2([0,1] \times \mathbb{R})$ with $\omega_0(0) = \omega_0(1) = 0, c(0, \omega(0,0)) = c(0, \omega(0,1))$ the function $\omega(\cdot,\cdot) \in C([0,1] \times [0,1]; \mathbb{R})$ and $\eta_i \in C^2([-r,0] \times [0,1]; \mathbb{R})$. Thus

$$\mathcal{D}[\phi(0) + g(\omega)(0) - q(0, \omega_0)] + \varphi(0, \omega_0, 0) \in \overline{D(\mathcal{D})}.$$

Moreover $c \in C^2([0,1] \times \mathbb{R}; \mathbb{R})$; then, it is clear that (H7) and (H8) are hold. If $M_1 L_{q_1} < 1$ then by Theorem 4.3, the integral solution of Eq (5.1) becomes a strict solution.

6. Conclusions

In this work, we obtained the existence results for the system of neutral integro-differential equations given by (1.1) with the nonlocal condition in finite delay situations by using the Banach fixed point theorem. Also, we verified that the integral solution of the system given by (1.1) has continuous dependence with respect to the initial data, and we proved the existence of a strict solution by using integrated resolvent operator theory and Gronwall's lemma. We considered most of the functions in Eq (1.1) to be Lipschitz continuous and then obtained the results. The future work will consider the partial neutral functional integro-differential equations with the initial conditions and we will apply the integrated resolvent operator technique to this system.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declares that they have no conflicts of interest.

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