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Research article

Some novel refinements of Hermite-Hadamard and Pachpatte type integral inequalities involving a generalized preinvex function pertaining to Caputo-Fabrizio fractional integral operator

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Abstract: In this article, we aim to introduce and explore a new class of preinvex functions called $\mathfrak{n}$-polynomial m-preinvex functions, while also presenting algebraic properties to enhance their numerical significance. We investigate novel variations of Pachpatte and Hermite-Hadamard integral inequalities pertaining to the concept of preinvex functions within the framework of the Caputo-Fabrizio fractional integral operator. By utilizing this direction, we establish a novel fractional integral identity that relates to preinvex functions for differentiable mappings of first-order. Furthermore, we derive some novel refinements for Hermite-Hadamard type inequalities for functions whose first-order derivatives are polynomial preinvex in the Caputo-Fabrizio fractional sense. To demonstrate the practical utility of our findings, we present several inequalities using specific real number means. Overall, our investigation sheds light on convex analysis within the context of fractional calculus.

Keywords: convex function; preinvex function; Hermite-Hadamard inequality; Pachpatte inequality; Caputo-Fabrizio operator
Mathematics Subject Classification: 26A33, 26A51, 26D07, 26D10, 26D15

Abbreviations

C-FFIO	Caputo-Fibrizio fractional integral operator
H–H	Hermite-Hadamard
R–L	Riemann-Liouville
w.r.t.	with respect to
$G -_m PF$	Generalized *m* preinvex function means *n*-polynomial *m*-preinvex function

1. Introduction

The term convex functions have been propagated intensively in the renowned book, namely "Inequalities", which was written by John Edenser Littlewood, G. H. Hardy, and George Polya [1]. Over the last century, convexity and related premises have developed into an exciting domain of applied sciences. Numerous scholars have assisted and provided their skills and insights into this topic by providing updated versions of specific inequalities involving convex functions. It is obvious that the outstanding point of view on the notion of convexity always provides new ideas and important relevant uses in any discipline of applied sciences. Numerous paragons of mathematics continually make an effort to utilize and enjoy the new concepts for the beautification and improvement of convexity theory. The widespread use of the concept of convexity in applications, uses convex optimization is the primary one. The term inequality and the idea of generalized convexity are frequently employed in optimization [2]. The term convexity with the help of the concept of optimization has a magnificent impact on many field of applied sciences, including finance [3], estimation and signal processing [4], control systems [5], data analysis and computer science [6] and mathematical optimization for modeling [7, 8]. In economics [9], this concept performs a fundamental influence in duality theory and equilibrium. There are some renowned books devoted to variant directions of the theory of convex and optimization. Among them, "Convex analysis" by Rockafellar [10], "Convex optimization" by Vandenberghe and Boyd [11], "Introductory lectures on convex optimization" by Nesterov [12], "Convex analysis and minimization algorithms" by Lemarechal and Hiriart-Urruty [13] and "Convex analysis and nonlinear optimization" by Borwein and Lewis [14] are some books that kept the interest of researchers. This theory provides an excellent work plan for starting and creating numerical techniques for solving and researching challenging mathematical issues.

The study of integral inequalities along with convex analysis offers a fascinating and stimulating area of study in the realm of mathematical perception. Because of their widespread perspectives and applications, the tactics and literature of convexity and inequalities have recently become the topic of substantial research in both contemporary and historical times. The Hermite-Hadamard-type inequalities are the most frequently employed among all the inequalities. These convex function-based inequalities are crucial and basic in practical math. Thus, convexity and inequalities have been recommended as an engrossing area for researchers due to their vital role and fruitful importance in numerous branches of science. Integral inequalities have remarkable uses in probability, optimization theory, information technology, stochastic processes, statistics, integral operator theory, and numerical integration. For the applications, see the references [15–22].

Fractional calculus, focusing on fractional integration over complex domains, has recently gained attention due to its practical applications and has attracted mathematicians' interest. Sarikaya et al. [23] introduced a fractional Hermite-Hadamard inequality. Fractional operators play a crucial role in the development of fractional calculus, and the study of fractional integral inequality has been motivated by the exploration of well-known inequalities like Ostrowski, Simpson, and Hadamard. Fractional calculus finds application in various fields, such as transform theory, engineering, modeling, finance, mathematical biology, fluid flow, natural phenomena prediction, healthcare, and image processing. The references [24–27] provide further insights into this topic.

In [28] a comprehensive and up-to-date review on Hermite-Hadamard-type inequalities for different kinds of convexities and different kinds of fractional integral operators is presented. In this manuscript, we aim to examine the fractional Hermite-Hadamard and fractional Pachpatte-type inequalities related to polynomial preinvexity pertaining to Caputo-Fabrizio fractional integral operator. The Caputo-Fabrizio fractional operator introduced by Caputo and Fabrizio in [29], features a nonsingular kernel in its fractional derivatives and does not rely on the Gamma function. Notably, this operator allows any real power to be transformed into an integer order using Laplace transformation, enabling solutions to various related problems. This operator has found applications in modeling COVID-19 [30, 31], Hepatitis B epidemic model [32], financial [33], ground water flow [34], and more.

In light of the preceding topic and as motivated by the significant research initiatives, the conceptualization of this work is organized in the following ways. First of all, in Section 2, we add some recognized definitions, theorems and remarks because all these are required in upcoming subsequent sections. In Section 3, we introduce the new definition namely ɲ-polynomial m-preinvex function. Further, we add in this section its algebraic properties. In Section 4, we prove the Hermite-Hadamard inequalities via newly introduced concept via Caputo-Fabrizio fractional operator. In Section 5, we examine a new lemma and utilizing this newly introduced lemma with the help of Hölder and power mean inequality, we show some new variants of Hermite-Hadamard type inequality via newly introduced definition utilizing Caputo-Fabrizio fractional operator. In Section 6, employing the above equality with the help of improved version Hölder and power mean inequality, we attain some new variants of Hermite-Hadamard type inequality via newly introduced definition utilizing Caputo-Fabrizio fractional operator. In Section 7, we examine a new sort of Pachpatte type inequality via Caputo-Fabrizio fractional operator via newly introduced definition. Next, in Section 8, we offer some applications in the manner of constructed results. Finally, in Section 9, we give a conclusion.

2. Preliminaries

In this section, it would be appropriate to examine and concentrate on a few theorems, remarks and definitions, for the benefit of the readers' attention, clarity, relevance, and quality. This section's main objective is to research and discuss specific related ideas and concepts that are pertinent to our analysis in later sections of this paper. In the context of classical calculus, the subject convexity and the generalized form of the H-H inequality are initially studied and investigated. Additionally, we reviewed the m-preinvex function, Sobolev space, extended condition C and C-FFIO. We finish this part by reviewing the importance of the $G -_m PF$ for our research.

Jensen first time introduced the term convexity in the following manner:

Definition 2.1. *[35, 36] A function $\mathfrak{D} : [\mathfrak{d}_1, \mathfrak{d}_2] \to \mathbb{R}$ is called convex, if*

$$\mathfrak{D}(\mathfrak{s}x + (1-\mathfrak{s})y) \leq \mathfrak{s}\mathfrak{D}(x) + (1-\mathfrak{s})\mathfrak{D}(y),$$

holds true for every $x, y \in [\mathfrak{d}_1, \mathfrak{d}_2]$ and $\mathfrak{s} \in [0, 1]$.

Theorem 2.1. *(see [37]) Let $\mathfrak{D} : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $\mathfrak{d}_1 < \mathfrak{d}_2$ and $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{I}$. Then the following inequality holds true:*

$$\mathfrak{D}\left(\frac{\mathfrak{d}_1 + \mathfrak{d}_2}{2}\right) \leq \frac{1}{\mathfrak{d}_2 - \mathfrak{d}_1} \int_{\mathfrak{d}_1}^{\mathfrak{d}_2} \mathfrak{D}(x)dx \leq \frac{\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{D}(\mathfrak{d}_2)}{2}. \tag{2.1}$$

For some recent estimations, see [38–40].

Definition 2.2. *([41]) Let $\mathbb{I} \subseteq \mathbb{R}^n$. Then $\mathbb{I}$ is m-invex w.r.t $\mathcal{N} : \mathbb{I} \times \mathbb{I} \times (0, 1] \to \mathbb{R}$, if*

$$m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) \in \mathbb{I},$$

holds for every $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{I}$, $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$.

Example 2.1. *([41]) Suppose $m = \frac{1}{4}$, $\mathbb{I} = [\frac{-\pi}{2}, 0) \bigcup (0, \frac{1}{2}]$ and*

$$\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \begin{cases} m\cos(\mathfrak{d}_2 - \mathfrak{d}_1) & if\ \mathfrak{d}_1 \in (0, \frac{\pi}{2}], \mathfrak{d}_2 \in (0, \frac{\pi}{2}]; \\ -m\cos(\mathfrak{d}_2 - \mathfrak{d}_1) & if\ \mathfrak{d}_1 \in [\frac{-\pi}{2}, 0), \mathfrak{d}_2 \in [\frac{-\pi}{2}, 0); \\ m\cos(\mathfrak{d}_1) & if\ \mathfrak{d}_1 \in (0, \frac{\pi}{2}], \mathfrak{d}_2 \in [\frac{-\pi}{2}, 0); \\ -m\cos(\mathfrak{d}_1) & if\ \mathfrak{d}_1 \in [-\frac{\pi}{2}, 0), \mathfrak{d}_2 \in (0, \frac{\pi}{2}]. \end{cases}$$

The above example is valid only for m-invex set not for convex set.

Recently, Deng [42] introduced generalized m-preinvex function, which is defined as:

Definition 2.3. *Let $\mathfrak{D} : \mathbb{I} \to \mathbb{R}$. Then $\mathfrak{D}$ is generalized m-preinvex w.r.t. $\mathcal{N} : \mathbb{I} \times \mathbb{I} \times (0, 1] \to \mathbb{R}$ for $m \in (0, 1]$, if*

$$\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq m(1-\mathfrak{s})\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{s}\mathfrak{D}(\mathfrak{d}_2), \tag{2.2}$$

holds for every $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{I}$, $\mathfrak{s} \in [0, 1]$.

The following generalized Condition C first time introduced by Ting Song Du [43] in the aspect of m-preinvex.

Extended Condition-C: Suppose $\mathbb{A} \subset \mathbb{R}^n$ is an open m-invex subset w.r.t. $\mathcal{N} : \mathbb{A} \times \mathbb{A} \times (0, 1] \to \mathbb{R}$, for any $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{A}$, $\mathfrak{s} \in [0, 1]$. We then have

$$\begin{aligned} \mathcal{N}(\mathfrak{d}_1, m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m), m) &= -\mathfrak{s}\,\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m), \\ \mathcal{N}(\mathfrak{d}_2, m\mathfrak{d}_1 + \mathfrak{s}\,\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m), m) &= (1-\mathfrak{s})\,\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m), \\ \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) &= -\mathcal{N}(\mathfrak{d}_1, \mathfrak{d}_2, m). \end{aligned}$$

We provide a few essential definitions from fractional calculus theory that will be utilised in the results that follow.

Definition 2.4. *([44]) Suppose* $p \in [1, \infty)$ *and* $(\mathfrak{d}_1, \mathfrak{d}_2) \subseteq \mathbb{R}$, *then the Sobolev space* $H^p(\mathfrak{d}_1, \mathfrak{d}_2)$ *is defined by*

$$H^p(\mathfrak{d}_1, \mathfrak{d}_2) = \{\mathfrak{D} \in L^2(\mathfrak{d}_1, \mathfrak{d}_2) : \Delta^u \mathfrak{D} \in L^2(\mathfrak{d}_1, \mathfrak{d}_2), \forall \ |u| \leq p\}.$$

Definition 2.5. *([45]) Let* $\mathfrak{D} \in H^1(\mathfrak{d}_1, \mathfrak{d}_2)$, $\omega \in [0, 1]$ *and* $\mathfrak{d}_1 < \mathfrak{d}_2$. *The C-FFIO in the left sense is defined as*

$$\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(\mathfrak{s}) = \frac{(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(\mathfrak{s}) + \frac{\omega}{\mathfrak{B}(\omega)}\int_{\mathfrak{d}_1}^{\mathfrak{s}} \mathfrak{D}(x)dx.$$

Similarly, the right C-FFIO is stated as

$$\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(\mathfrak{s}) = \frac{(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(\mathfrak{s}) + \frac{\omega}{\mathfrak{B}(\omega)}\int_{\mathfrak{s}}^{\mathfrak{d}_2} \mathfrak{D}(x)dx,$$

where $\mathfrak{B}(\omega) > 0$ is a normalization function satisfying $\mathfrak{B}(0) = \mathfrak{B}(1) = 1$.

Recent extensions and estimations of fractional inequalities can exist in the literature (see for example [46–50]).

In this manuscript, we extend the equality utilizing C-FFIO, which is introduced by Dragomir [51].

Theorem 2.2. *([45]) Let* $\mathfrak{D} : \mathbb{I} \to \mathbb{R}$ *be a convex function on* $\mathbb{I}$. *If* $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{I}$ *with* $\mathfrak{d}_1 < \mathfrak{d}_2$ *and* $\mathfrak{D} \in \mathcal{L}[\mathfrak{d}_1, \mathfrak{d}_2]$, *then*

$$\begin{aligned}\mathfrak{D}\left(\frac{\mathfrak{d}_1+\mathfrak{d}_2}{2}\right) &\leq \frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k) - \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\right] \\ &\leq \frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}, \end{aligned} \tag{2.3}$$

where $\omega \in [0, 1]$ *and* $k \in [\mathfrak{d}_1, \mathfrak{d}_2]$.

Definition 2.6. *([52]) Let* $\mathfrak{D}$ *be a non-negative and real-valued function, Then* $\mathfrak{D}$ *is* $\mathfrak{n}$*-polynomial preinvex, if*

$$\mathfrak{D}(\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1)) \leq \frac{1}{\mathfrak{n}}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[1-\mathfrak{s}^{\mathfrak{u}}\right]\mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[1-(1-\mathfrak{s})^{\mathfrak{u}}\right]\mathfrak{D}(\mathfrak{d}_2), \tag{2.4}$$

for all $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{I}$, $\mathfrak{n} \in \mathbb{N}$ *and* $\mathfrak{s} \in [0, 1]$.

3. Generalized preinvex functions and its algebraic properties

Convexity has emerged as a captivating and practical field of study in both applied and pure sciences. The investigation of convex functions has led to innovative methods and calculations, providing a meaningful structure for addressing complex mathematical problems. The relationship between convexity and inequalities has garnered significant interest, leading to the exploration of various versions of classical inequalities.

Here, we are going to address the $G -_m PF$, an intriguing idea for preinvex functions, and look at some of its algebraic features and examples.

Definition 3.1. *Let a non empty subset $\mathbb{A}$ of $\mathbb{R}$ be m-invex set w.r.t $\mathcal{N} : \mathbb{A} \times \mathbb{A} \times (0,1] \to \mathbb{R}$. Then, $\mathfrak{D} : \mathbb{A} \to \mathbb{R}$ is $G -_m PF$ if*

$$\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_2), \tag{3.1}$$

holds for every $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{A}$, $m \in (0,1]$, $\mathfrak{s} \in [0,1]$ and $\mathfrak{n} \in \mathbb{N}$.

Remark 3.1. (*i*) *Choosing $\mathfrak{n} = 1$, then*

$$\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq m(1 - \mathfrak{s})\, \mathfrak{D}(\mathfrak{d}_1) + (1 - (1 - \mathfrak{s}))\, \mathfrak{D}(\mathfrak{d}_2). \tag{3.2}$$

(*ii*) *Choosing $\mathfrak{n} = m = 1$, then Definition 3.1 is collapses to the concept of preinvex which was introduced by Weir [53].*
(*iii*) *Choosing $\mathfrak{n} = m = 1$ and $\mathcal{N}(\mathfrak{d}_1, \mathfrak{d}_2, m) = \mathfrak{d}_1 - m\mathfrak{d}_2$, then Definition 3.1 collapses to the concept of convex function which was discussed by Niculescu [54].*

Lemma 3.1. *For all $\mathfrak{s} \in [0,1]$, $m \in (0,1]$ and $\mathfrak{n} \in \mathbb{N}$, the following inequalities*

$$\mathfrak{s} \leq \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \quad \text{and} \quad m(1 - \mathfrak{s}) \leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})$$

are hold.

Proof. The proof is obvious.

Proposition 3.1. *Every non-negative m-preinvex function is $G -_m PF$.*

Proof. Employing to the definition of m-preinvexity and Lemma 3.1, we have

$$\begin{aligned}\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) &\leq m(1 - \mathfrak{s})\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{s}\mathfrak{D}(\mathfrak{d}_2) \\ &\leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})\, \mathfrak{D}(\mathfrak{d}_2).\end{aligned}$$

Proposition 3.2. *Every $G -_m PF$ is an (h, m)-preinvex function with $h(\mathfrak{s}) = \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})$.*

Proof. Utilizing the property of $G -_m PF$ and given condition, we have

$$\begin{aligned}\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) &\leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})\, \mathfrak{D}(\mathfrak{d}_2) \\ &\leq mh(1 - \mathfrak{s})\mathfrak{D}(\mathfrak{d}_1) + h(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_2).\end{aligned}$$

From the preceding proposition, it seems obvious that the newly developed preinvexity is very large in comparison to already published functions, such as convex and preinvex. This is the most appealing feature of the intended new Definition 3.1.

Next, we show some examples in manner of the newly introduce idea.

Example 3.1. $\mathfrak{D}(\mathfrak{d}) = |\mathfrak{d}| \quad \forall \mathfrak{d} \geq 0$ *is a convex function, implies that the given function is a preinvex function (see [55]). Further, implies that, it is m-preinvex function if* $m = 1$. *By utilizing Proposition 3.1, it is an* $G -_m PF$.

Example 3.2. $\mathfrak{D}(\mathfrak{d}) = e^{\mathfrak{d}} \quad \forall \mathfrak{d} \geq 0$ *is a convex function, implies that the given function is preinvex function (see [55]). Further, implies that, it is m-preinvex function if* $m = 1$. *By utilizing Proposition 3.1, it is an* $G -_m PF$.

Now we will investigate and expound on several examples of the newly introduced notion. We can see that $\mathfrak{D}(x) = -|x|$ is preinvex but not convex.

Example 3.3. *Suppose* $\mathfrak{D} : \mathbb{R}^+ \to \mathbb{R}^+$ *is defined by*

$$\mathfrak{D}(\mathfrak{d}) = \begin{cases} \mathfrak{d}, & 0 \leq \mathfrak{d} \leq 1, \\ 1, & \mathfrak{d} > 1, \end{cases}$$

and

$$\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \begin{cases} \mathfrak{d}_2 - m\mathfrak{d}_1 & \mathfrak{d}_2 \leq 0, \mathfrak{d}_1 \leq 0, \\ \mathfrak{d}_2 - m\mathfrak{d}_1, & 0 \leq \mathfrak{d}_2 \leq 1, \mathfrak{d}_1 \leq 1, \\ -2 - m\mathfrak{d}_1, & \mathfrak{d}_2 \leq 0, 0 \leq \mathfrak{d}_1 \leq 1, \\ 2 - m\mathfrak{d}_1, & 0 \leq \mathfrak{d}_2 \leq 1, \mathfrak{d}_1 \leq 0. \end{cases}$$

The non-negative function $\mathfrak{D}(\mathfrak{d})$ *is m-preinvex but not convex. Employing to Proposition 3.1, it is an* $G -_m PF$ *w.r.t.* $\mathcal{N}$ *on* $X = \mathbb{R}^+ \times \mathbb{R}^+ \times (0, 1)$ *if* $m = 1$.

Example 3.4. *Let* $\mathfrak{D} : \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by*

$$\mathfrak{D}(\mathfrak{d}) = \begin{cases} \mathfrak{d} + 1, & 0 \leq \mathfrak{d} \leq 1, \\ 1, & \mathfrak{d} > 1, \end{cases}$$

and

$$\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \begin{cases} \mathfrak{d}_2 + m\mathfrak{d}_1, & \mathfrak{d}_2 \leq \mathfrak{d}_1, \\ 2(\mathfrak{d}_2 + m\mathfrak{d}_1), & \mathfrak{d}_2 > \mathfrak{d}_1, \end{cases}$$

$\forall \mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{R}^+ = [0, +\infty)$. *The above example is valid for m-preinvex but not convex. Employing to Proposition 3.1, it is an* $G -_m PF$ *if* $m = 1$.

Theorem 3.1. *Let* $\mathfrak{D}_g, \mathfrak{D}_y : \mathbb{X} = [\mathfrak{d}_1, \mathfrak{d}_2] \to \mathbb{R}$. *If* $\mathfrak{D}_g, \mathfrak{D}_y$ *are two* $G -_m PF$, *then* $\left(\mathfrak{D}_g + \mathfrak{D}_y\right)$ *is also an* $G -_m PF$.

Proof. Since given that $\mathfrak{D}_g$ and $\mathfrak{D}_g$ are two $G -_m PF$, then

$$\begin{aligned}
&\left(\mathfrak{D}_g + \mathfrak{D}_y\right)(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \\
&= \mathfrak{D}_g(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) + \mathfrak{D}_y(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \\
&\leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \mathfrak{D}_g(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}_g(\mathfrak{d}_2) \\
&+ \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \mathfrak{D}_y(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}_y(\mathfrak{d}_2) \\
&= \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \left[\mathfrak{D}_g(\mathfrak{d}_1) + \mathfrak{D}_y(\mathfrak{d}_1)\right] + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \left[\mathfrak{D}_g(\mathfrak{d}_2) + \mathfrak{D}_y(\mathfrak{d}_2)\right]
\end{aligned}$$

$$= \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) (\mathfrak{D}_g + \mathfrak{D}_y)(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})(\mathfrak{D}_g + \mathfrak{D}_y)(\mathfrak{d}_2).$$

This is the required proof.

Theorem 3.2. *Let* $\mathfrak{D} : \mathbb{X} = [\mathfrak{d}_1, \mathfrak{d}_2] \to \mathbb{R}$. *If* $\mathfrak{D}$ *is* $G -_m PF$, *then* $(c\mathfrak{D})$ *is also an* $G -_m PF$.

Proof. Since given that $\mathfrak{D}$ is $G -_m PF$ and c is any constant number, then

$$\begin{aligned}(c\mathfrak{D})(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) &\leq c\Big[\frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_2)\Big] \\ &= \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) c\mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) c\mathfrak{D}(\mathfrak{d}_2) \\ &= \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) (c\mathfrak{D})(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) (c\mathfrak{D})(\mathfrak{d}_2).\end{aligned}$$

This completes the proof.

Theorem 3.3. *Let* $\mathfrak{D}_g : \mathbb{X} \to \mathbb{Y}$ *and* $\mathfrak{D}_y : \mathbb{Y} \to \mathbb{R}$ *be* $G -_m PF$ *and increasing functions respectively. Then,* $\left(\mathfrak{D}_y \circ \mathfrak{D}_g\right)$ *is also an* $G -_m PF$ *w.r.t same* $\mathcal{N}$, *where dot product* $\circ$ *represent the composition operation of* $\mathfrak{D}_y$ *and* $\mathfrak{D}_g$.

Proof.

$$\begin{aligned}&\left(\mathfrak{D}_y \circ \mathfrak{D}_g\right)(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \\ &= \mathfrak{D}_y(\mathfrak{D}_g(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))) \\ &\leq \mathfrak{D}_y\Big[\frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \mathfrak{D}_g(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}_g(\mathfrak{d}_2)\Big] \\ &\leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \mathfrak{D}_y(\mathfrak{D}_g(\mathfrak{d}_1)) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}_y(\mathfrak{D}_g(\mathfrak{d}_2)) \\ &= \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) (\mathfrak{D}_y \circ \mathfrak{D}_g)(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) (\mathfrak{D}_y \circ \mathfrak{D}_g)(\mathfrak{d}_2).\end{aligned}$$

Theorem 3.4. *Let* $0 < \mathfrak{d}_1 < \mathfrak{d}_2$, $\mathfrak{D}_j : \mathbb{X} = [\mathfrak{d}_1, \mathfrak{d}_2] \to [0, +\infty)$ *be a family of* $G -_m PF$ *w.r.t.* $\mathcal{N}$ *and* $\mathfrak{D}(u) = \sup_j \mathfrak{D}_j(u)$. *Then,* $\mathfrak{D}$ *is an* $G -_m PF$ *w.r.t.* $\mathcal{N}$ *for* $m \in (0, 1]$, $\mathfrak{s} \in [0, 1]$, *and* $U = \{\mathfrak{d} \in [\mathfrak{d}_1, \mathfrak{d}_2] : \mathfrak{D}(\mathfrak{d}_{\mathfrak{u}}) < \infty\}$ *is an interval.*

Proof. Let $\mathfrak{d}_1, \mathfrak{d}_2 \in U$, $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$, then

$$\begin{aligned}\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) &= \sup_j \mathfrak{D}_j(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \\ &\leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \sup_j \mathfrak{D}_j(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \sup_j \mathfrak{D}_j(\mathfrak{d}_2)\end{aligned}$$

$$= \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_2) < \infty.$$

Theorem 3.5. *If $\mathfrak{D}_{\mathfrak{u}} : \mathbb{R}^{\mathfrak{n}} \to \mathbb{R}$ is an $G -_m$ PF w.r.t. $\mathcal{N}$ for $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$, then $\mathbb{M} = \{\mathfrak{d} \in \mathbb{R} : \mathfrak{D}_{\mathfrak{u}}(\mathfrak{d}) \leq 0, \mathfrak{u} = 1, 2, 3, \dots, \mathfrak{n}\}$ is an m-invex set.*

Proof. Given that $\mathfrak{D}_{\mathfrak{u}}(\mathfrak{d})$, $(\mathfrak{u} = 1, 2, 3, \dots, \mathfrak{n})$ are $G -_m$ *PF* w.r.t. $\mathcal{N}$ for $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$, then for all $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{R}^{\mathfrak{n}}$,

$$\mathfrak{D}_{\mathfrak{u}}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_2)$$

holds.

When $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{M}$, we know $\mathfrak{D}_{\mathfrak{u}}(\mathfrak{d}_1) \leq 0$ and $\mathfrak{D}_{\mathfrak{u}}(\mathfrak{d}_2) \leq 0$, then above inequality implies that

$$\mathfrak{D}_{\mathfrak{u}}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq 0, \quad \mathfrak{u} = 1, 2, 3, \dots, \mathfrak{n}.$$

That is, $m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) \in \mathbb{M}$. $\Rightarrow \mathbb{M}$ is an m-invex set.

Theorem 3.6. *If $\mathfrak{D} : \mathbb{A} \subseteq \mathbb{R}^{\mathfrak{n}} \to \mathbb{R}$ is an $G -_m$ PF on m-invex set $\mathbb{A}$ w.r.t $\mathcal{N}$ for $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$, then $\mathfrak{D}$ is also generalized quasi m-preinvex function.*

Proof. Given that $\mathfrak{D}$ is an $G -_m$ *PF* w.r.t $\mathcal{N}$ for $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$, and assuming that $\mathfrak{D}(\mathfrak{d}_1) \leq \mathfrak{D}(\mathfrak{d}_2)$, for all $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{X}$, we have

$$\begin{aligned} \mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) &\leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_2) \\ &\leq \left[\frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}}) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \right] \mathfrak{D}(\mathfrak{d}_2) \\ &\leq \mathfrak{D}(\mathfrak{d}_2). \end{aligned}$$

In the same manner, if we let $\mathfrak{D}(\mathfrak{d}_2) \leq \mathfrak{D}(\mathfrak{d}_1)$ for all $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{X}$, we can also get

$$\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq \mathfrak{D}(\mathfrak{d}_1).$$

Consequently,

$$\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq \max\{\mathfrak{D}(\mathfrak{d}_1), \mathfrak{D}(\mathfrak{d}_2)\}.$$

That is, $\mathfrak{D} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ is generalized quasi m-preinvex function on m-invex set $\mathbb{X}$ with respect to $\mathcal{N}$.

Theorem 3.7. *If $\mathfrak{D}_{\mathfrak{u}} : \mathbb{A} \subseteq \mathbb{R}^{\mathfrak{n}} \to \mathbb{R}$ $(\mathfrak{u} = 1, 2, \dots, \mathfrak{n})$ are $G -_m$ PF on $\mathbb{A}$ w.r.t $\mathcal{N} : \mathbb{A} \times \mathbb{A} \times (0, 1] \to \mathbb{R}$ for $m \in (0, 1]$, then the function*

$$\mathfrak{D} = \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \mathfrak{d}_{\mathfrak{u}} \mathfrak{D}_{\mathfrak{u}}, \quad \mathfrak{d}_{\mathfrak{u}} \geq 0, \quad (\mathfrak{u} = 1, 2, 3, \dots, \mathfrak{n})$$

is also an $G -_m$ PF on $\mathbb{A}$ w.r.t $\mathcal{N}$ for $m \in (0, 1]$, where $\mathfrak{d}_{\mathfrak{u}}$ is constant.

Proof. The proof is obvious.

Theorem 3.8. *Let $\mathfrak{D} : \mathbb{R}_0 \to \mathbb{R}_0 = [0, \infty)$ be $G -_m$ PF w.r.t $\mathcal{N} : \mathbb{R}_0 \times \mathbb{R}_0 \times (0, 1] \to \mathbb{R}_0$ for $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$. Assume that $\mathfrak{D}$ is monotonic decreasing, $\mathcal{N}$ is monotonic increasing regarding m for fixed $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{R}_0$ and $m_1 \le m_2 \quad (m_1, m_2 \in (0, 1])$. If $\mathfrak{D}$ is $G -_{m_1}$ PF on $\mathbb{R}_0$ with respect to $\mathcal{N}$, then $\mathfrak{D}$ is $G -_{m_2}$ PF on $\mathbb{R}_0$ with respect to $\mathcal{N}$.*

Proof. Given that $\mathfrak{D}$ is $G -_{m_1}$ *PF*, for all $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{R}_0$,

$$\mathfrak{D}(m_1\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m_1)) \le \frac{m_1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_2).$$

Combining the monotone decreasing of the function $\mathfrak{D}$ with the monotone increasing of the mapping $\mathcal{N}$ regarding m for fixed $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{R}_0$ and $m_1 \le m_2$, it follows that

$$\begin{aligned}
&\mathfrak{D}(m_2\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m_2)) \le \mathfrak{D}(m_1\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m_1)) \\
&\Rightarrow \mathfrak{D}(m_2\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m_2)) \le \frac{m_1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_2) \\
&\Rightarrow \mathfrak{D}(m_2\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m_2)) \le \frac{m_2}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}(\mathfrak{d}_2).
\end{aligned}$$

This completes the proof.

Theorem 3.9. *Suppose $\mathfrak{D}_g, \mathfrak{D}_y : \mathbb{X} = [\mathfrak{d}_1, \mathfrak{d}_2] \to \mathbb{R}$. If both functions are $G -_m$ PF and similarly ordered functions for $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$, then the product of these functions is $G -_m$ PF .*

Proof. For $m \in (0, 1]$ and $\mathfrak{s} \in [0, 1]$, then

$$\begin{aligned}
&\mathfrak{D}_g(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))\mathfrak{D}_y(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \\
&\le \left[\frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}_g(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}_g(\mathfrak{d}_2)\right] \\
&\times \left[\frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}_y(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) \mathfrak{D}_y(\mathfrak{d}_2)\right] \\
&\le \frac{m^2}{\mathfrak{n}^2} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})^2\, \mathfrak{D}_g(\mathfrak{d}_1)\mathfrak{D}_y(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}^2} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})^2 \mathfrak{D}_g(\mathfrak{d}_2)\mathfrak{D}_y(\mathfrak{d}_2) \\
&+ \frac{m}{\mathfrak{n}^2} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})(1 - \mathfrak{s}^{\mathfrak{u}}) [\mathfrak{D}_g(\mathfrak{d}_1)\mathfrak{D}_y(\mathfrak{d}_2) + \mathfrak{D}_g(\mathfrak{d}_2)\mathfrak{D}_y(\mathfrak{d}_1)] \\
&\le \frac{m^2}{\mathfrak{n}^2} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})^2\, \mathfrak{D}_g(\mathfrak{d}_1)\mathfrak{D}_y(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}^2} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})^2 \mathfrak{D}_g(\mathfrak{d}_2)\mathfrak{D}_y(\mathfrak{d}_2) \\
&+ \frac{m}{\mathfrak{n}^2} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})(1 - \mathfrak{s}^{\mathfrak{u}}) [\mathfrak{D}_g(\mathfrak{d}_1)\mathfrak{D}_y(\mathfrak{d}_1) + \mathfrak{D}_g(\mathfrak{d}_2)\mathfrak{D}_y(\mathfrak{d}_2)]
\end{aligned}$$

$$= \left[\frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}_g(\mathfrak{d}_1)\mathfrak{D}_y(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})\mathfrak{D}_g(\mathfrak{d}_2)\mathfrak{D}_y(\mathfrak{d}_2)\right]$$
$$\times \left(\frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}}) + \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\right)$$
$$\leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - \mathfrak{s}^{\mathfrak{u}})\, \mathfrak{D}_g(\mathfrak{d}_1)\mathfrak{D}_y(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} (1 - (1 - \mathfrak{s})^{\mathfrak{u}})\, \mathfrak{D}_g(\mathfrak{d}_2)\mathfrak{D}_y(\mathfrak{d}_2).$$

4. H-H inequality involving $\mathfrak{n}$-polynomial m-preinvex function via Caputo-Fabrizio operator

Since the notion of convex function was first introduced more than a century ago, an enormous number of outstanding inequalities have been proven in the domian of the convex theory. The most widely recognized and frequently utilized inequality in the field of convex theory is the H-H inequality. Hermite and Hadamard were the ones who first suggested this inequality. Many mathematicians were motivated by the idea of this inequality to investigate and analyze the classical inequalities utilizing the many convexity senses. For instance, Xi [56], Mehreen [57], and Kirmaci [58] presented several kind of this inequality via convex functions. Hudzik [59], Dragomir [60] and Ozcan [61] addressed the notion of s-convex function and proved a novel variant of this inequality. Butt [62] and Rashid [63] proved this inequality in the polynomial version involving a new class of convexity.

The principal goal of this section is to use the $G -_m PF$ and the C-FFIO to derive and demonstrate the H-H inequality.

Theorem 4.1. *Let $\mathbb{A}^\circ \subseteq \mathbb{R}$ ($\mathbb{A}^\circ$ the interior of $\mathbb{A}$) be an open m-invex subset w.r.t. $\mathcal{N} : \mathbb{A}^\circ \times \mathbb{A}^\circ \times (0, 1] \to \mathbb{R}$ for some fixed $m \in (0, 1]$. Let $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{A}^\circ$, $\mathfrak{d}_1 < \mathfrak{d}_2$ with $m\mathfrak{d}_1 \leq m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)$. Suppose $\mathfrak{D} : [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)] \to \mathbb{R}$ is $G -_m PF$ and satisfies extended condition-C. Then*

$$\frac{1}{2}\left(\frac{\mathfrak{n}}{\mathfrak{n} + 2^{-\mathfrak{n}} - 1}\right)\mathfrak{D}\left(\mathfrak{d}_1 + \frac{1}{2}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)\right) \leq \frac{1}{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \mathfrak{D}(x)dx$$
$$\leq \frac{m\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{D}(\mathfrak{d}_2)}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \frac{\mathfrak{u}}{\mathfrak{u} + 1}. \tag{4.1}$$

Proof. Employing the property of $G -_m PF$ of $\mathfrak{D}$, we have that

$$\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq \frac{m}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} [1 - \mathfrak{s}^{\mathfrak{u}}]\mathfrak{D}(\mathfrak{d}_1) + \frac{1}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} [1 - (1 - \mathfrak{s})^{\mathfrak{u}}]\mathfrak{D}(\mathfrak{d}_2),$$

$$\int_0^1 \mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))d\mathfrak{s} \leq \frac{m\mathfrak{D}(\mathfrak{d}_1)}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \int_0^1 [1 - \mathfrak{s}^{\mathfrak{u}}]d\mathfrak{s} + \frac{\mathfrak{D}(\mathfrak{d}_2)}{\mathfrak{n}} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \int_0^1 [1 - (1 - \mathfrak{s})^{\mathfrak{u}}]d\mathfrak{s}.$$

But

$$\int_0^1 \mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))d\mathfrak{s} = \frac{1}{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \mathfrak{D}(x)dx,$$

so

$$\frac{1}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(x)dx\le\frac{m\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{\mathfrak{n}}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}.$$

This completes the right side inequality.

For left side, utilizing the definition of $G -_m PF$ and extended condition C for $\mathcal{N}$ and integrating over $[0, 1]$ we get

$$\begin{aligned}
&\mathfrak{D}\Big(m\mathfrak{d}_1+\frac{1}{2}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)\Big)\\
&=\mathfrak{D}(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))+\frac{1}{2}\mathcal{N}(m\mathfrak{d}_1+(1-\mathfrak{s})\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m),m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))\\
&\le\frac{1}{\mathfrak{n}}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\Big[1-\Big(\frac{1}{2}\Big)^{\mathfrak{u}}\Big]\Big[\int_0^1\mathfrak{D}(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))d\mathfrak{s}+\int_0^1\mathfrak{D}(m\mathfrak{d}_1+(1-\mathfrak{s})\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))d\mathfrak{s}\Big]\\
&\le\frac{1}{\mathfrak{n}}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\Big[1-\Big(\frac{1}{2}\Big)^{\mathfrak{u}}\Big]\frac{2}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(x)dx\\
&\le\Big[\frac{\mathfrak{n}+2^{-\mathfrak{n}}-1}{\mathfrak{n}}\Big]\frac{2}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(x)dx.
\end{aligned}$$

This completes the proof.

Corollary 4.1. *Choosing $m = 1$, then*

$$\begin{aligned}
\frac{1}{2}\Big(\frac{\mathfrak{n}}{\mathfrak{n}+2^{-\mathfrak{n}}-1}\Big)\mathfrak{D}\Big(\mathfrak{d}_1+\frac{1}{2}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)\Big)&\le\frac{1}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\int_{\mathfrak{d}_1}^{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(x)dx\\
&\le\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{\mathfrak{n}}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}.
\end{aligned}$$

Corollary 4.2. *Choosing $n = 1$, then*

$$\begin{aligned}
\mathfrak{D}\Big(\mathfrak{d}_1+\frac{1}{2}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)\Big)&\le\frac{1}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(x)dx\\
&\le\frac{m\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}.
\end{aligned}$$

Corollary 4.3. *Choosing $n = m = 1$, then*

$$\mathfrak{D}\Big(\mathfrak{d}_1+\frac{1}{2}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)\Big)\le\frac{1}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\int_{\mathfrak{d}_1}^{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(x)dx\le\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}.$$

Corollary 4.4. *Choosing $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$, then*

$$\begin{aligned}
\frac{1}{2}\Big(\frac{\mathfrak{n}}{\mathfrak{n}+2^{-\mathfrak{n}}-1}\Big)\mathfrak{D}\left(\frac{m\mathfrak{d}_1+\mathfrak{d}_2}{2}\right)&\le\frac{1}{(\mathfrak{d}_2-m\mathfrak{d}_1)}\int_{m\mathfrak{d}_1}^{\mathfrak{d}_2}\mathfrak{D}(x)dx\\
&\le\frac{m\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{\mathfrak{n}}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}.
\end{aligned}$$

Corollary 4.5. *Choosing $n = 1$ and $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$, then*

$$\mathfrak{D}\left(\frac{(m\mathfrak{d}_1 + \mathfrak{d}_2)}{2}\right) \leq \frac{1}{\mathfrak{d}_2 - m\mathfrak{d}_1}\int_{m\mathfrak{d}_1}^{\mathfrak{d}_2} \mathfrak{D}(x)dx \leq \frac{m\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{D}(\mathfrak{d}_2)}{2}.$$

Remark 4.1. *If we put $m = 1$ and $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$ in Theorem 4.1, then we get (h, m)-preinvex (3.1) in [64].*

Remark 4.2. *If we put $\mathfrak{n} = m = 1$ and $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$ in Theorem 4.1, then we attain the H-H inequality in [37].*

Theorem 4.2. *Let $\mathbb{A}^\circ \subseteq \mathbb{R}$ be an open m-invex subset w.r.t. $\mathcal{N} : \mathbb{A}^\circ \times \mathbb{A}^\circ \times (0, 1] \to \mathbb{R}$ for some fixed $m \in (0, 1]$. Let $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathbb{A}^\circ$, $\mathfrak{d}_1 < \mathfrak{d}_2$ with $m\mathfrak{d}_1 \leq m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)$. Suppose $\mathfrak{D} : [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)] \to \mathbb{R}$ is $G -_m PF$, then the following H-H type inequalities hold:*

$$\begin{aligned}
&\frac{2^{-1}n}{n + 2^{-n} - 1}\mathfrak{D}\left(\frac{2m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2}\right)\\
&\leq \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}\right)(k) - \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\right]\\
&\leq \frac{m\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{D}(\mathfrak{d}_2)}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u} + 1},
\end{aligned}$$

where $\omega \in [0, 1]$ and $k \in [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$.

Proof. Employing the definition of $G -_m PF$, it follows from the Inequality (4.1) that

$$\begin{aligned}
&\frac{2^{-1}n}{n + 2^{-n} - 1}\mathfrak{D}\left(\frac{2m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2}\right)\\
&\leq \frac{2}{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \mathfrak{D}(x)dx\\
&= \frac{2}{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\left(\int_{m\mathfrak{d}_1}^{k} \mathfrak{D}(x)dx + \int_{k}^{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \mathfrak{D}(x)dx\right). \qquad (4.2)
\end{aligned}$$

Multiplying both sides of (4.2) by $\frac{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2\mathfrak{B}(\omega)}$ gives

$$\begin{aligned}
&\frac{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2\mathfrak{B}(\omega)}\frac{2^{-1}n}{n + 2^{-n} - 1}\mathfrak{D}\left(\frac{2m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2}\right)\\
&\leq \frac{\omega}{\mathfrak{B}(\omega)}\left(\int_{m\mathfrak{d}_1}^{k} \mathfrak{D}(x)dx + \int_{k}^{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \mathfrak{D}(x)dx\right). \qquad (4.3)
\end{aligned}$$

Adding $\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)$ to both sides of (4.3)

$$\begin{aligned}
&\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k) + \frac{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2\mathfrak{B}(\omega)}\frac{2^{-1}n}{n + 2^{-n} - 1}\mathfrak{D}\left(\frac{2m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2}\right)\\
&\leq \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k) + \frac{\omega}{\mathfrak{B}(\omega)}\left(\int_{m\mathfrak{d}_1}^{k} \mathfrak{D}(x)dx + \int_{k}^{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \mathfrak{D}(x)dx\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{(1-\omega)}{\mathfrak{B}(\omega)} \mathfrak{D}(k) + \frac{\omega}{\mathfrak{B}(\omega)} \int_{m\mathfrak{d}_1}^{k} \mathfrak{D}(x)dx \right) \\
&+ \left(\frac{(1-\omega)}{\mathfrak{B}(\omega)} \mathfrak{D}(k) + \frac{\omega}{\mathfrak{B}(\omega)} \int_{k}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)} \mathfrak{D}(x)dx \right) \\
&= \left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D} \right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D} \right)(k).
\end{aligned} \tag{4.4}
$$

On the other hand, from the Inequality (4.1), we have

$$
\frac{2}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)} \int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)} \mathfrak{D}(x)dx \leq \frac{m\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{n} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \frac{2\mathfrak{u}}{\mathfrak{u}+1}. \tag{4.5}
$$

If we multiply (4.5) by $\frac{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2\mathfrak{B}(\omega)}$ and add $\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)$ to the resulting inequality, we obtain

$$
\begin{aligned}
&\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D} \right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D} \right)(k) \\
&\leq \frac{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{\mathfrak{B}(\omega)} \frac{m\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{n} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \frac{\mathfrak{u}}{\mathfrak{u}+1} + \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k).
\end{aligned} \tag{4.6}
$$

Combining the above Inequalities (4.4) and (4.6), we obtain the desired result.

Corollary 4.6. *Choosing* $\mathfrak{n}=1$*, Theorem 4.2 becomes,*

$$
\begin{aligned}
&\mathfrak{D}\left(\frac{2m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2} \right) \\
&\leq \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)} \left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D} \right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D} \right)(k) - \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k) \right] \\
&\leq \frac{m\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}.
\end{aligned}
$$

Corollary 4.7. *Assuming that* $m=1$*, Theorem 4.2 becomes,*

$$
\begin{aligned}
&\frac{2^{-1}n}{n+2^{-n}-1}\mathfrak{D}\left(\frac{2\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2} \right) \\
&\leq \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)} \left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D} \right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D} \right)(k) - \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k) \right] \\
&\leq \frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{n} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \frac{\mathfrak{u}}{\mathfrak{u}+1}.
\end{aligned}
$$

Corollary 4.8. *Assuming that* $\mathfrak{n}=m=1$*, Theorem 4.2 becomes,*

$$
\begin{aligned}
&\mathfrak{D}\left(\frac{2\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2} \right) \\
&\leq \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)} \left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D} \right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D} \right)(k) - \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k) \right] \\
&\leq \frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}.
\end{aligned}
$$

Corollary 4.9. *Assume that* $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$*. Then Theorem 4.2 becomes,*

$$\begin{aligned}
&\frac{2^{-1}n}{n + 2^{-n} - 1}\mathfrak{D}\left(\frac{m\mathfrak{d}_1 + \mathfrak{d}_2}{2}\right) \\
&\leq \frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2 - m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k) - \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\right] \\
&\leq \frac{m\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{D}(\mathfrak{d}_2)}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}.
\end{aligned}$$

Corollary 4.10. *Assume that* $\mathfrak{n} = 1$ *and* $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$*. Then Theorem 4.2 becomes,*

$$\begin{aligned}
&\mathfrak{D}\left(\frac{m\mathfrak{d}_1 + \mathfrak{d}_2}{2}\right) \\
&\leq \frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2 - m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k) - \frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\right] \\
&\leq \frac{m\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{D}(\mathfrak{d}_2)}{2}.
\end{aligned}$$

Remark 4.3. *If we put* $m = 1$ *and* $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$ *in Theorem 4.2, it collapses to (Theorem 4, [46]).*

Remark 4.4. *Choosing* $\mathfrak{n} = m = 1$ *and* $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$ *in Theorem 4.2, it collapses to Theorem 2.2.*

5. Refinements of H-H type inequalities involving power mean and Hölder inequality via Caputo-Fabrizo operator

In the topic of convex analysis, a number of researchers have lately contributed on unique approaches to this challenge from various viewpoints. Recent research on H-H inequalities for convex functions has led to a wide range of innovations and improvements. Noor [65] utilized the notion of preinvex function and demostrated a new sort of H-H inequality. After Noor's published idea, many mathematicains set up new estimations of this inequality in the aspect of numerous kind of preinvexity. For example, Barani et al. [66] first time derived some refinements and estimations of this inequality for functions whose derivative of absolute values are preinvex. Noor [67] proved this inequality and its refinements pertaining to h-preinvexity. For the identical and similar preinvexity notions, we discuss to Wu et al. [68], Park [69], Sarikaya et al. [70], and Wang and Liu [71].

First, in this section, we look into the preinvex function lemma. We will include the results with the help of Hölder and power-mean inequality based on the recently investigated lemma. We add a few corollaries and observations in this section to increase the importance and caliber of this section.

Lemma 5.1. *Let* $\mathfrak{D} : \mathbb{I} = [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)] \to \mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^{\circ}$, $m\mathfrak{d}_1 < m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)$*. If* $\mathfrak{D}' \in \mathcal{L}[m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$*, then*

$$\frac{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2}\int_0^1 (1 - 2\mathfrak{s})\mathfrak{D}'(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))d\mathfrak{s} + \frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}(k)$$

$$= \frac{-\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{2} + \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k) + {}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\right],$$

where $\omega \in [0, 1]$ *and* $k \in [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$.

Proof. It is easy to see that

$$\begin{aligned}
&\int_0^1 (1 - 2\mathfrak{s})\mathfrak{D}'(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))d\mathfrak{s} \\
&= -\frac{\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} + \frac{2}{(\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))^2} \\
&\times \left(\int_{m\mathfrak{d}_1}^k \mathfrak{D}(x)dx + \int_k^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)} \mathfrak{D}(x)dx\right).
\end{aligned}$$

Multiplying both sides with $\frac{\omega(\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))^2}{2\mathfrak{B}(\omega)}$ and adding $\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)$, we have

$$\begin{aligned}
&\frac{\omega(\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))^2}{2\mathfrak{B}(\omega)} \int_0^1 (1 - 2\mathfrak{s})\mathfrak{D}'(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))d\mathfrak{s} + \frac{2(1 - \omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k) \\
&= -\frac{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{\mathfrak{B}(\omega)} \frac{\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{2} + \left(\frac{(1 - \omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k) + \frac{\omega}{\mathfrak{B}(\omega)} \int_{m\mathfrak{d}_1}^k \mathfrak{D}(x)dx\right) \\
&+ \left(\frac{(1 - \omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k) + \frac{\omega}{\mathfrak{B}(\omega)} \int_k^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)} \mathfrak{D}(x)dx\right) \\
&= \frac{-\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{2} \\
&+ \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k) + {}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\right].
\end{aligned}$$

This completes the proof.

Theorem 5.1. *Let* $\mathfrak{D} : \mathbb{I} = [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)] \to \mathbb{R}$ *be a differentiable function on* $\mathbb{I}^\circ$, $m\mathfrak{d}_1 < m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)$ *and* $|\mathfrak{D}'|$ *is* $G-_m PF$ *on* $[m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$. *If* $\mathfrak{D}' \in \mathcal{L}[m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$, *then*

$$\begin{aligned}
&\Bigg| -\frac{\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{2} - \frac{2(1 - \omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}(k) \\
&+ \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg| \\
&\leq \frac{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{n} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \left[\frac{(\mathfrak{u}^2 + \mathfrak{u} + 2)2^{\mathfrak{u}}\mathfrak{u} - 2}{(\mathfrak{u} + 1)(\mathfrak{u} + 2)2^{\mathfrak{u}+1}}\right] \left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)| + |\mathfrak{D}'(\mathfrak{d}_2)|}{2}\right),
\end{aligned} \tag{5.1}$$

where $\omega \in [0, 1]$, $m \in (0, 1]$ *and* $k \in [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$.

Proof. Employing the Lemma 5.1, we have

$$\Bigg| -\frac{\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{2} - \frac{2(1 - \omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}(k)$$

$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\int_0^1|1-2\mathfrak{s}||\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))|d\mathfrak{s}.$$

Employing the property of $G-_m PF$, we have

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\int_0^1|1-2\mathfrak{s}|\left(\frac{m}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}(1-\mathfrak{s}^{\mathfrak{u}})|\mathfrak{D}'(\mathfrak{d}_1)|+\frac{1}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}(1-(1-\mathfrak{s})^{\mathfrak{u}})|\mathfrak{D}'(\mathfrak{d}_2)|\right)d\mathfrak{s}$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2n}\left[m|\mathfrak{D}'(\mathfrak{d}_1)|\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1|1-2\mathfrak{s}|(1-\mathfrak{s}^{\mathfrak{u}})d\mathfrak{s}+|\mathfrak{D}'(\mathfrak{d}_2)|\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1|1-2\mathfrak{s}|(1-(1-\mathfrak{s})^{\mathfrak{u}})d\mathfrak{s}\right]$$
$$=\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2n}\left[m|\mathfrak{D}'(\mathfrak{d}_1)|\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right]+|\mathfrak{D}'(\mathfrak{d}_2)|\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right]\right]$$
$$=\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right]\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|+|\mathfrak{D}'(\mathfrak{d}_2)|}{2}\right).$$

This completes the proof.

Corollary 5.1. *Choosing* $\mathfrak{n}=1$*, then we have*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)\,(m|\mathfrak{D}'(\mathfrak{d}_1)|+|\mathfrak{D}'(\mathfrak{d}_2)|)}{8}.$$

Corollary 5.2. *Assume that* $m=1$*. Then we have*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}\mathfrak{u}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right]\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|+|\mathfrak{D}'(\mathfrak{d}_2)|}{2}\right),$$

Corollary 5.3. *Choosing* $\mathfrak{n}=m=1$*, then we have*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$

$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)\left(|\mathfrak{D}'(\mathfrak{d}_1)|+|\mathfrak{D}'(\mathfrak{d}_2)|\right)}{8}.$$

Corollary 5.4. *If we put $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$, then*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right]\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|+|\mathfrak{D}'(\mathfrak{d}_2)|}{2}\right).$$

Corollary 5.5. *If we put $m=1$ and $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$, then*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathfrak{d}_2-\mathfrak{d}_1}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right]\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|+|\mathfrak{D}'(\mathfrak{d}_2)|}{2}\right).$$

Corollary 5.6. *Assume that $\mathfrak{n}=1$ and $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$. Then we have*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{(\mathfrak{d}_2-m\mathfrak{d}_1)\left(m|\mathfrak{D}'(\mathfrak{d}_1)|+|\mathfrak{D}'(\mathfrak{d}_2)|\right)}{8}.$$

Remark 5.1. *If we put $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ and $\mathfrak{n}=m=1$, then we get the Theorem 5 in published article [45].*

Theorem 5.2. *Assume that $\mathfrak{D}:\mathbb{I}=[m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)]\to\mathbb{R}$ is a differentiable function on $\mathbb{I}^{\circ}$, $m\mathfrak{d}_1<m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)$ and $|\mathfrak{D}'|^q$ is $G-_m$ PF on $[m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)]$. If $\mathfrak{D}'\in\mathcal{L}[m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)]$, then*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}\right)^{\frac{1}{q}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}},\tag{5.2}$$

where $\omega\in[0,1]$, $m\in(0,1]$ and $k\in[m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)]$.

Proof. Employing Lemma 5.1, we have

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$

$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\int_0^1|1-2\mathfrak{s}||\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))|d\mathfrak{s}.$$

Employing Hölder inequality, we have

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\int_0^1|1-2\mathfrak{s}|^p d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1|\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))|^q d\mathfrak{s}\right)^{\frac{1}{q}}.$$

Employing the property of $G-_m PF$ of $|\mathfrak{D}'|^q$, we have

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\int_0^1(1-\mathfrak{s}^{\mathfrak{u}})d\mathfrak{s}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\int_0^1(1-(1-\mathfrak{s})^{\mathfrak{u}})d\mathfrak{s}\right]^{\frac{1}{q}}$$
$$=\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}\right]^{\frac{1}{q}}$$
$$=\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}\right)^{\frac{1}{q}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

This is the required proof.

Corollary 5.7. *Choosing* $\mathfrak{n}=1$*, then we obtain*

$$\Bigg|-\frac{\omega(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{q}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.8. *Assume that* $m=1$*. Then we obtain*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|$$

$$\leq \frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.9. *Assume that* $\mathfrak{n}=m=1$. *Then we obtain*

$$\Bigg|-\frac{\omega(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq \frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.10. *Choosing* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$, *then*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq \frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}\right)^{\frac{1}{q}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.11. *Choosing* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ *and* $m=1$, *then*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq \frac{\mathfrak{d}_2-\mathfrak{d}_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{\mathfrak{u}+1}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.12. *Choosing* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ *and* $\mathfrak{n}=1$, *then*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq \frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Remark 5.2. *Choosing* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)=\mathfrak{d}_2-\mathfrak{d}_1$ *and* $\mathfrak{n}=m=1$, *then we get the Theorem 6 in [45].*

Theorem 5.3. *Let* $\mathfrak{D}:\mathbb{I}=[m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)]\to\mathbb{R}$ *be a differentiable function on* $\mathbb{I}^{\circ}$ *and* $\mathfrak{d}_1,\mathfrak{d}_2\in\mathbb{I}$ *with* $m\mathfrak{d}_1<m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)$ *and assume that* $\mathfrak{D}'\in\mathcal{L}[m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)]$. *If* $|\mathfrak{D}'|^q$ *is* $G-_m$ *PF on* $[m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)]$, *then we obtain*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq \frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right)^{\frac{1}{q}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}, \quad (5.3)$$

where $\omega\in[0,1]$, $m\in(0,1]$ *and* $k\in[m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)]$.

Proof. Employing Lemma 5.1, we have

$$\begin{aligned}&\Bigg|\frac{-\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\\&+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\,\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\right]\Bigg|\\&\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\int_0^1|1-2\mathfrak{s}|\,|\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))|\,d\mathfrak{s}.\end{aligned}$$

Employing power mean inequality, we have

$$\begin{aligned}&\Bigg|\frac{-\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\\&+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\,\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\right]\Bigg|\\&\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\int_0^1|1-2\mathfrak{s}|d\mathfrak{s}\right)^{1-\frac{1}{q}}\left(|1-2\mathfrak{s}|\,|\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))|^q\,d\mathfrak{s}\right)^{\frac{1}{q}}.\end{aligned}$$

Employing the property of $G-_m PF$ of $|\mathfrak{D}'|^q$, we have

$$\begin{aligned}&\Bigg|\frac{-\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\\&+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\,\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\right]\Bigg|\\&\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\&\times\left(\int_0^1|1-2\mathfrak{s}|\left[\frac{m}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}[1-\mathfrak{s}^{\mathfrak{u}}]|\mathfrak{D}'(\mathfrak{d}_1)|^q+\frac{1}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}[1-(1-\mathfrak{s})^{\mathfrak{u}}]|\mathfrak{D}'(\mathfrak{d}_2)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\\&\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\&\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1|1-2\mathfrak{s}|[1-\mathfrak{s}^{\mathfrak{u}}]d\mathfrak{s}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1|1-2\mathfrak{s}|[1-(1-\mathfrak{s})^{\mathfrak{u}}]d\mathfrak{s}\right)^{\frac{1}{q}}\\&\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\&\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right)^{\frac{1}{q}}.\end{aligned}$$

By simplifying we obtain the desired result.

Corollary 5.13. *Choosing* $\mathfrak{n}=1$*, then*

$$\Bigg|\frac{-\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$

$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.14. *Choosing $m=1$, then*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.15. *Choosing $\mathfrak{n}=m=1$, then*

$$\Bigg|\frac{-\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[{}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.16. *Choosing $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$, then*

$$\left|\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}(k)\right]\right|$$
$$\leq\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}(\frac{1}{2})^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right)^{\frac{1}{q}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.17. *Choosing $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ and $m=1$, then*

$$\left|\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\left[{}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}(k)\right]\right|$$
$$\leq\frac{\mathfrak{d}_2-\mathfrak{d}_1}{2}(\frac{1}{2})^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.18. *Choosing $\mathfrak{n}=1$ and $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$, then*

$$\left|\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}(k)\right]\right|$$
$$\leq\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 5.19. *Choosing* $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$ *and* $\mathfrak{n} = m = 1$*, then*

$$\left|\frac{\mathfrak{D}(\mathfrak{d}_1) + \mathfrak{D}(\mathfrak{d}_2)}{2} - \frac{2(1-\omega)}{\omega(\mathfrak{d}_2 - \mathfrak{d}_1)}\mathfrak{D}(k) + \frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2 - \mathfrak{d}_1)}\left[{}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k) + {}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}(k)\right]\right|$$
$$\leq \frac{\mathfrak{d}_2 - \mathfrak{d}_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q + |\mathfrak{D}'(\mathfrak{d}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

6. Refinements of Hermite-Hadamard type inequalities using Hölder Iscan and improved power mean inequality via Caputo-Fabrizo operator

In the area of integral inequalities, several mathematicians have lately collaborated on novel strategies to this subject from numerous viewpoints and multiple views. In 2019, the Hölder Iscan integral inequality was studied by İşcan [72] for the first time as estimations of Hölder inequality. The refined version of power mean inequality is called the improved power mean integral inequality. This inequality was first time explored by Kadakal [73] in 2019.

The major goal of this aspect is to achieve fresh findings using Hölder-Iscan inequality and improved power mean inequality. Here, we add some remarks and corollaries for value and worth.

Theorem 6.1. *Let* $\mathfrak{D} : \mathbb{I} = [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)] \to \mathbb{R}$ *be a differentiable function on* $\mathbb{I}^{\circ}$*, with* $m\mathfrak{d}_1 < m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)$*,* $q > 1$*,* $p^{-1} + q^{-1} = 1$ *and assume that* $\mathfrak{D}' \in \mathcal{L}[m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$*. If* $|\mathfrak{D}'|^q$ *is* $G -_m$ *PF on* $[m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$*, then*

$$\left| -\frac{\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{2} - \frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}(k) \right.$$
$$\left. + \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}\right)(k)\right]\right|$$
$$\leq \frac{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)} + \frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}$$
$$+ \frac{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)} + \frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}},$$

where $\omega \in [0, 1]$*,* $m \in (0, 1]$ *and* $k \in [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$.

Proof. Employing Lemma 5.1, we have

$$\left| -\frac{\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{2} - \frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}(k) \right.$$
$$\left. + \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}\right)(k)\right]\right|$$
$$\leq \frac{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{2}\int_0^1 |1 - 2\mathfrak{s}||\mathfrak{D}'(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))|d\mathfrak{s}.$$

Employing Hölder-İşcan inequality, we have

$$\left| -\frac{\mathfrak{D}(m\mathfrak{d}_1) + \mathfrak{D}(m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))}{2} - \frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}\mathfrak{D}(k) \right.$$

$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\int_0^1(1-\mathfrak{s})|1-2\mathfrak{s}|^p d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1(1-\mathfrak{s})|\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_1,\mathfrak{d}_2,m))|^q d\mathfrak{s}\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\int_0^1\mathfrak{s}|1-2\mathfrak{s}|^p d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1\mathfrak{s}|\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_1,\mathfrak{d}_2,m))|^q d\mathfrak{s}\right)^{\frac{1}{q}}.$$

Employing $G-_m PF$ of $|\mathfrak{D}'|^q$, we have

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1(1-\mathfrak{s})(1-\mathfrak{s}^{\mathfrak{u}})d\mathfrak{s}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1(1-\mathfrak{s})(1-(1-\mathfrak{s})^{\mathfrak{u}})d\mathfrak{s}\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1\mathfrak{s}(1-\mathfrak{s}^{\mathfrak{u}})d\mathfrak{s}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1\mathfrak{s}(1-(1-\mathfrak{s})^{\mathfrak{u}})d\mathfrak{s}\right)^{\frac{1}{q}}$$
$$=\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)}+\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}.$$

This is the required proof.

Corollary 6.1. *Choosing* $\mathfrak{n}=1$ *we obtain*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{2m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+2|\mathfrak{D}'(\mathfrak{d}_2)|^q}{6}\right)^{\frac{1}{q}}\right\}.$$

Corollary 6.2. *Assume that* $m=1$. *Then we obtain*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$

$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)}+\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}.$$

Corollary 6.3. *Assume that* $\mathfrak{n}=m=1$. *Then we obtain*

$$\Bigg|-\frac{\omega(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$
$$+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{2|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+2|\mathfrak{D}'(\mathfrak{d}_2)|^q}{6}\right)^{\frac{1}{q}}\right\}.$$

Corollary 6.4. *If we put* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$, *then*

$$\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}$$
$$+\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)}+\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}.$$

Corollary 6.5. *If we put* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ *and* $m=1$, *then*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq\frac{\mathfrak{d}_2-\mathfrak{d}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}$$
$$+\frac{\mathfrak{d}_2-\mathfrak{d}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}(\mathfrak{u}+3)}{2(\mathfrak{u}+1)(\mathfrak{u}+2)}+\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{\mathfrak{u}}{2(\mathfrak{u}+2)}\right)^{\frac{1}{q}}.$$

Corollary 6.6. *If we put* $\mathfrak{n}=1$ *and* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$, *then*

$$\left|\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[{}^{CF}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}(k)\right]\right|$$
$$\leq\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{2m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+2|\mathfrak{D}'(\mathfrak{d}_2)|^q}{6}\right)^{\frac{1}{q}}\right\}.$$

Corollary 6.7. *If we put* $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) = \mathfrak{d}_2 - m\mathfrak{d}_1$ *and* $\mathfrak{n} = m = 1$, *then*

$$\left|\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2} - \frac{2(1-\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\mathfrak{D}(k) + \frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\left[{}^{CF}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k) + {}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}(k)\right]\right|$$
$$\leq \frac{\mathfrak{d}_2-\mathfrak{d}_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{2|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+2|\mathfrak{D}'(\mathfrak{d}_2)|^q}{6}\right)^{\frac{1}{q}}\right\}.$$

Theorem 6.2. *Let* $\mathfrak{D} : \mathbb{I} = [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)] \to \mathbb{R}$ *be a differentiable function on* $\mathbb{I}^\circ$, *with* $m\mathfrak{d}_1 < m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)$, $q \geq 1$ *and assume that* $\mathfrak{D}' \in \mathcal{L}[m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$. *If* $|\mathfrak{D}'|^q$ *is* $G -_m$ *PF on* $[m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$, *then*

$$\Bigg| - \frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2} - \frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+ \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq \frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}$$
$$\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)} + \frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}$$
$$+ \frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}$$
$$\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)} + \frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}},$$

where $\omega \in [0, 1]$, $m \in (0, 1]$ *and* $k \in [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$.

Proof. First, assume that $q > 1$. According to the Lemma 5.1, we have

$$\Bigg| - \frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2} - \frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+ \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq \frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\int_0^1 |1-2\mathfrak{s}||\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))|d\mathfrak{s}.$$

Utilizing the statement of the improved power mean inequality, we have

$$\Bigg| - \frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2} - \frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)$$
$$+ \frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k) + \left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|$$
$$\leq \frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\int_0^1(1-\mathfrak{s})|1-2\mathfrak{s}|d\mathfrak{s}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\mathfrak{s})|1-2\mathfrak{s}||\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_1,\mathfrak{d}_2,m))|^q d\mathfrak{s}\right)^{\frac{1}{q}}$$

$$+\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\int_0^1 \mathfrak{s}|1-2\mathfrak{s}|d\mathfrak{s}\right)^{1-\frac{1}{q}}\left(\int_0^1 \mathfrak{s}|1-2\mathfrak{s}||\mathfrak{D}'(m\mathfrak{d}_1+\mathfrak{s}\mathcal{N}(\mathfrak{d}_1,\mathfrak{d}_2,m))|^q d\mathfrak{s}\right)^{\frac{1}{q}}.$$

Employing $G-_m PF$ of $|\mathfrak{D}'|^q$, we have

$$\begin{aligned}&\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\\
&+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|\\
&\le\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1(1-\mathfrak{s})|1-2\mathfrak{s}|(1-\mathfrak{s}^{\mathfrak{u}})d\mathfrak{s}\right.\\
&\left.+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1(1-\mathfrak{s})|1-2\mathfrak{s}|(1-(1-\mathfrak{s})^{\mathfrak{u}})d\mathfrak{s}\right)^{\frac{1}{q}}\\
&+\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1\mathfrak{s}|1-2\mathfrak{s}|(1-\mathfrak{s}^{\mathfrak{u}})d\mathfrak{s}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\int_0^1\mathfrak{s}|1-2\mathfrak{s}|(1-(1-\mathfrak{s})^{\mathfrak{u}})d\mathfrak{s}\right)^{\frac{1}{q}}\\
&\le\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\\
&\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}\\
&+\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\\
&\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}.\end{aligned}$$

This is the required proof.

Corollary 6.8. *Choosing* $\mathfrak{n}=1$ *in the Inequality* (5.2)*, we attain*

$$\begin{aligned}&\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\\
&+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\right)(k)\right]\Bigg|\\
&\le\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left\{\left(\frac{3m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+3|\mathfrak{D}'(\mathfrak{d}_2)|^q}{16}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 6.9. *Choosing* $m=1$ *in the Inequality* (5.2)*, we attain*

$$\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)$$

$$
\begin{aligned}
&+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{C}\mathfrak{F}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{C}\mathfrak{F}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|\\
&\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\\
&\times\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}\\
&+\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\\
&\times\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 6.10. *Choosing* $\mathfrak{n}=m=1$ *in the Inequality* (5.2)*, we attain*

$$
\begin{aligned}
&\Bigg|-\frac{\omega(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1))}{2}-\frac{2(1-\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}(k)\\
&+\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{CF}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{CF}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\right)(k)\right]\Bigg|\\
&\leq\frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left\{\left(\frac{3|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+3|\mathfrak{D}'(\mathfrak{d}_2)|^q}{16}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Corollary 6.11. *If we put* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ *in the Inequality* (5.2)*, then*

$$
\begin{aligned}
&\Bigg|-\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{C}\mathfrak{F}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{C}\mathfrak{F}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|\\
&\leq\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\\
&\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}\\
&+\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\\
&\times\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 6.12. *If we put* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ *and* $m=1$ *in the Inequality* (5.2)*, then*

$$
\begin{aligned}
&\Bigg|-\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\left[\left({}^{\mathfrak{C}\mathfrak{F}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\right)(k)+\left({}^{\mathfrak{C}\mathfrak{F}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\right)(k)\right]\Bigg|\\
&\leq\frac{\mathfrak{d}_2-\mathfrak{d}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}
\end{aligned}
$$

$$\times\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}$$
$$+\frac{\mathfrak{d}_2-\mathfrak{d}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}$$
$$\times\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{2^{\mathfrak{u}+2}(\mathfrak{u}+2)(\mathfrak{u}+3)}+\frac{|\mathfrak{D}'(\mathfrak{d}_2)|^q}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\frac{(\mathfrak{u}+5)[(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2]}{2^{\mathfrak{u}+2}(\mathfrak{u}+1)(\mathfrak{u}+2)(\mathfrak{u}+3)}\right)^{\frac{1}{q}}.$$

Corollary 6.13. *Assume that* $\mathfrak{n}=1$ *and* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ *in above Theorem. Then*

$$\left|\frac{\mathfrak{D}(m\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}(k)\right]\right|$$
$$\leq\frac{\mathfrak{d}_2-m\mathfrak{d}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left\{\left(\frac{3m|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{m|\mathfrak{D}'(\mathfrak{d}_1)|^q+3|\mathfrak{D}'(\mathfrak{d}_2)|^q}{16}\right)^{\frac{1}{q}}\right\}.$$

Corollary 6.14. *Assume that* $\mathfrak{n}=m=1$ *and* $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ *in above Theorem. Then*

$$\left|\frac{\mathfrak{D}(\mathfrak{d}_1)+\mathfrak{D}(\mathfrak{d}_2)}{2}-\frac{2(1-\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\mathfrak{D}(k)+\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-\mathfrak{d}_1)}\left[{}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}(k)\right]\right|$$
$$\leq\frac{\mathfrak{d}_2-\mathfrak{d}_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left\{\left(\frac{3|\mathfrak{D}'(\mathfrak{d}_1)|^q+|\mathfrak{D}'(\mathfrak{d}_2)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{D}'(\mathfrak{d}_1)|^q+3|\mathfrak{D}'(\mathfrak{d}_2)|^q}{16}\right)^{\frac{1}{q}}\right\}.$$

7. Pachpatte type (h,m)-preinvex via $\mathfrak{n}$-polynomial m-preinvex function pertaining to Caputo-Fabrizio fractional integral operator

The term "convexity" has drawn substantial interest in the past couple of decades due to its relevance and recognition of the concept of inequality. Due to developments of the convexity in applied sciences, many inequalities have been proposed and identified in the realm of convex analysis. Preinvexity has also been discussed by a large number of mathematicians, and numerous books have been written that provide new estimates and generalizations. With the help of these studies and research, the amazing Pachpatte-type inequality in the aspect of preinvexity is greatly enhanced. Preinvexity is an essential notion in the formation of extended convex programming. Nian Li [74] utilized the concept of time scales in 2009 and attained the novel variants of Pachpatte-type integral inequalities. In 2021, Butt [75] first time addressed fractional Pachpatte-Mercer-type inequalities in the sense of harmonic convexity. In 2022, Sahoo [76] utilized the idea of interval analysis and attain the novel sorts of Pachpatte-type integral inequalities in the aspect of center-radius order via preinvexity. This inequality in the mode of fractional operator associated with concept of exponential kernel is addressed by Sahoo [77] in 2022. Tariq et. al. [78] employed the non-conformable operator and attained a new kind of Pachpatte-type inequality via generalized preinvexity.

In light of the foregoing literature, we will investigate and research the Pachpatte-type inequality for the C-FFIO. A corollary and many remarks are added to heighten the relevance and worth of this section.

Theorem 7.1. *Assume that* $\omega \in [0,1]$ *and* $k \in [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$. *Let* $\mathfrak{D}, \mathcal{G} : \mathbb{I} = [m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)] \to \mathbb{R}$ *be two functions such that* $m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) \in \mathbb{I}$ *with* $\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) > 0$ *and* $\mathfrak{D} \in \mathcal{L}[m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)]$. *If* $\mathfrak{D}$ *and* $\mathcal{G}$ *are* $G -_m PF$ *, then*

$$\frac{\mathfrak{B}(\omega)}{\omega \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \left[\left({}^{\mathfrak{C}\mathfrak{F}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega} \mathfrak{D}\mathcal{G} \right)(k) + \left({}^{\mathfrak{C}\mathfrak{F}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)} \mathfrak{D}\mathcal{G} \right)(k) - \frac{2(1-\omega)}{\mathfrak{B}(\omega)} \mathfrak{D}(k)\mathcal{G}(k) \right]$$
$$\leq \mathbb{M}(\mathfrak{d}_1, \mathfrak{d}_2) + \mathbb{N}(\mathfrak{d}_1, \mathfrak{d}_2),$$

where

$$\mathbb{M}(\mathfrak{d}_1, \mathfrak{d}_2) = m^2 \Delta_1(\mathfrak{s}) \mathfrak{D}(\mathfrak{d}_1) \mathcal{G}(\mathfrak{d}_1) + \Delta_4(\mathfrak{s}) \mathfrak{D}(\mathfrak{d}_2) \mathcal{G}(\mathfrak{d}_2),$$
$$\mathbb{N}(\mathfrak{d}_1, \mathfrak{d}_2) = m \Delta_2(\mathfrak{s}) \mathfrak{D}(\mathfrak{d}_2) \mathcal{G}(\mathfrak{d}_1) + m \Delta_3(\mathfrak{s}) \mathfrak{D}(\mathfrak{d}_1) \mathcal{G}(\mathfrak{d}_2),$$

and

$$\Delta_1 = \frac{1}{n_1 n_2} \sum_{\mathfrak{u}=1}^{n_1} [1 - \mathfrak{s}^{\mathfrak{u}}] \sum_{\mathfrak{u}=1}^{n_2} [1 - \mathfrak{s}^{\mathfrak{u}}], \quad \Delta_2 = \frac{1}{n_1 n_2} \sum_{\mathfrak{u}=1}^{n_1} [1 - \mathfrak{s}^{\mathfrak{u}}] \sum_{\mathfrak{u}=1}^{n_2} [1 - (1-\mathfrak{s})^{\mathfrak{u}}],$$
$$\Delta_3 = \frac{1}{n_1 n_2} \sum_{\mathfrak{u}=1}^{n_1} [1 - (1-\mathfrak{s})^{\mathfrak{u}}] \sum_{\mathfrak{u}=1}^{n_2} [1 - \mathfrak{s}^{\mathfrak{u}}], \quad \Delta_4 = \frac{1}{n_1 n_2} \sum_{\mathfrak{u}=1}^{n_1} [1 - (1-\mathfrak{s})^{\mathfrak{u}}] \sum_{\mathfrak{u}=1}^{n_2} [1 - (1-\mathfrak{s})^{\mathfrak{u}}].$$

Proof. Let $\mathfrak{D}$ and $\mathcal{G}$ be the $G -_m PF$, then for $\mathfrak{s} \in [0,1]$,

$$\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq \frac{m}{n_1} \sum_{\mathfrak{u}=1}^{n_1} [1 - \mathfrak{s}^{\mathfrak{u}}] \mathfrak{D}(\mathfrak{d}_1) + \frac{1}{n_1} \sum_{\mathfrak{u}=1}^{n_1} [1 - (1-\mathfrak{s})^{\mathfrak{u}}] \mathfrak{D}(\mathfrak{d}_2). \tag{7.1}$$
$$\mathcal{G}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \leq \frac{m}{n_2} \sum_{\mathfrak{u}=1}^{n_2} [1 - \mathfrak{s}^{\mathfrak{u}}] \mathcal{G}(\mathfrak{d}_1) + \frac{1}{n_2} \sum_{\mathfrak{u}=1}^{n_2} [1 - (1-\mathfrak{s})^{\mathfrak{u}}] \mathcal{G}(\mathfrak{d}_2). \tag{7.2}$$

Multiplying the above Inequalities (7.1) and (7.2), gives

$$\mathfrak{D}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))\mathcal{G}(m\mathfrak{d}_1 + \mathfrak{s}\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))$$
$$\leq \frac{m^2}{n_1 n_2} \sum_{\mathfrak{u}=1}^{n_1} [1 - \mathfrak{s}^{\mathfrak{u}}] \sum_{\mathfrak{u}=1}^{n_2} [1 - \mathfrak{s}^{\mathfrak{u}}] \mathfrak{D}(\mathfrak{d}_1)\mathcal{G}(\mathfrak{d}_1)$$
$$+ \frac{m}{n_1 n_2} \sum_{\mathfrak{u}=1}^{n_1} [1 - \mathfrak{s}^{\mathfrak{u}}] \sum_{\mathfrak{u}=1}^{n_2} [1 - (1-\mathfrak{s})^{\mathfrak{u}}] \mathfrak{D}(\mathfrak{d}_1)\mathcal{G}(\mathfrak{d}_2)$$
$$+ \frac{m}{n_1 n_2} \sum_{\mathfrak{u}=1}^{n_1} [1 - (1-\mathfrak{s})^{\mathfrak{u}}] \sum_{\mathfrak{u}=1}^{n_2} [1 - \mathfrak{s}^{\mathfrak{u}}] \mathfrak{D}(\mathfrak{d}_2)\mathcal{G}(\mathfrak{d}_1)$$
$$+ \frac{1}{n_1 n_2} \sum_{\mathfrak{u}=1}^{n_1} [1 - (1-\mathfrak{s})^{\mathfrak{u}}] \sum_{\mathfrak{u}=1}^{n_2} [1 - (1-\mathfrak{s})^{\mathfrak{u}}] \mathfrak{D}(\mathfrak{d}_2)\mathcal{G}(\mathfrak{d}_2)$$
$$= m^2 \Delta_1(\mathfrak{s}) \mathfrak{D}(\mathfrak{d}_1)\mathcal{G}(\mathfrak{d}_1) + m\Delta_2(\mathfrak{s}) \mathfrak{D}(\mathfrak{d}_2)\mathcal{G}(\mathfrak{d}_1)$$
$$+ m\Delta_3(\mathfrak{s}) \mathfrak{D}(\mathfrak{d}_1)\mathcal{G}(\mathfrak{d}_2) + \Delta_4(\mathfrak{s}) \mathfrak{D}(\mathfrak{d}_2)\mathcal{G}(\mathfrak{d}_2). \tag{7.3}$$

Taking integration both sides of the above Inequality (7.3) over [0, 1], we have

$$\begin{aligned}&\frac{1}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\int_{m\mathfrak{d}_1}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(x)\mathcal{G}(x)dx\\&\leq\int_0^1\Big[m^2\Delta_1(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_1)\mathcal{G}(\mathfrak{d}_1)+m\Delta_2(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_2)\mathcal{G}(\mathfrak{d}_1)+m\Delta_3(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_1)\mathcal{G}(\mathfrak{d}_2)+\Delta_4(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_2)\mathcal{G}(\mathfrak{d}_2)\Big]d\mathfrak{s}\\&=\mathbb{M}(\mathfrak{d}_1,\mathfrak{d}_2)+\mathbb{N}(\mathfrak{d}_1,\mathfrak{d}_2).\end{aligned}$$

That is,

$$\frac{1}{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\int_{m\mathfrak{d}_1}^{k}\mathfrak{D}(x)\mathcal{G}(x)dx+\int_{k}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(x)\mathcal{G}(x)dx\right]\leq\mathbb{M}(\mathfrak{d}_1,\mathfrak{d}_2)+\mathbb{N}(\mathfrak{d}_1,\mathfrak{d}_2). \tag{7.4}$$

Now multiplying (7.4) by $\frac{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{\mathfrak{B}(\omega)}$ and add $\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\mathcal{G}(k)$ to the resulting inequality , we obtain

$$\begin{aligned}&\frac{\omega}{\mathfrak{B}(\omega)}\left[\int_{m\mathfrak{d}_1}^{k}\mathfrak{D}(x)\mathcal{G}(x)\mathrm{dx}+\int_{k}^{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(x)\mathcal{G}(x)\mathrm{dx}\right]+\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\mathcal{G}(k)\\&\qquad\leq\frac{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{\mathfrak{B}(\omega)}\left[\mathbb{M}(\mathfrak{d}_1,\mathfrak{d}_2)+N(\mathfrak{d}_1,\mathfrak{d}_2)\right]+\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\mathcal{G}(k).\end{aligned}$$

Hence,

$$\begin{aligned}&{}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}(k)\mathcal{G}(k)+{}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}(k)\mathcal{G}(k)\\&\leq\frac{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{\mathfrak{B}(\omega)}\left[\mathbb{M}(\mathfrak{d}_1,\mathfrak{d}_2)+N(\mathfrak{d}_1,\mathfrak{d}_2)\right]+\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\mathcal{G}(k).\end{aligned}$$

This is the required.

Corollary 7.1. *Assume that $n_1 = n_2 = 1$ in Theorem 7.1. Then*

$$\begin{aligned}&\frac{2\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\mathcal{G}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}\mathfrak{D}\mathcal{G}\right)(k)-\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\mathcal{G}(k)\right]\\&\leq\frac{2}{3}M(\mathfrak{d}_1,\mathfrak{d}_2)+\frac{1}{3}N(\mathfrak{d}_1,\mathfrak{d}_2).\end{aligned}$$

Corollary 7.2. *Assume that $m = 1$ in Theorem 7.1. Then*

$$\begin{aligned}&\frac{\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\mathcal{G}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\mathcal{G}\right)(k)-\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\mathcal{G}(k)\right]\\&\leq\mathbb{M}(\mathfrak{d}_1,\mathfrak{d}_2)+\mathbb{N}(\mathfrak{d}_1,\mathfrak{d}_2).\end{aligned}$$

where

$$\mathbb{M}(\mathfrak{d}_1,\mathfrak{d}_2)=\Delta_1(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_1)\mathcal{G}(\mathfrak{d}_1)+\Delta_4(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_2)\mathcal{G}(\mathfrak{d}_2),$$

$$\mathbb{N}(\mathfrak{d}_1,\mathfrak{d}_2)=\Delta_2(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_2)\mathcal{G}(\mathfrak{d}_1)+\Delta_3(\mathfrak{s})\mathfrak{D}(\mathfrak{d}_1)\mathcal{G}(\mathfrak{d}_2).$$

Corollary 7.3. *Assume that $n_1 = n_2 = m = 1$ in Theorem 7.1. Then*

$$\frac{2\mathfrak{B}(\omega)}{\omega\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\mathcal{G}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1)}\mathfrak{D}\mathcal{G}\right)(k)-\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\mathcal{G}(k)\right]$$
$$\leq \frac{2}{3}\mathbb{M}(\mathfrak{d}_1,\mathfrak{d}_2)+\frac{1}{3}\mathbb{N}(\mathfrak{d}_1,\mathfrak{d}_2).$$

Corollary 7.4. *If we put $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ in Theorem 7.1, then*

$$\frac{\mathfrak{B}(\omega)}{\omega(\mathfrak{d}_2-m\mathfrak{d}_1)}\left[\left({}^{\mathfrak{CF}}_{m\mathfrak{d}_1}\mathrm{I}^{\omega}\mathfrak{D}\mathcal{G}\right)(k)+\left({}^{\mathfrak{CF}}\mathrm{I}^{\omega}_{\mathfrak{d}_2}\mathfrak{D}\mathcal{G}\right)(k)-\frac{2(1-\omega)}{\mathfrak{B}(\omega)}\mathfrak{D}(k)\mathcal{G}(k)\right]$$
$$\leq \mathbb{M}(\mathfrak{d}_1,\mathfrak{d}_2)+\mathbb{N}(\mathfrak{d}_1,\mathfrak{d}_2).$$

Remark 7.1. *If we put $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ and $m = 1$ in Theorem 7.1, it reduces to the Theorem 5 in [46].*

Remark 7.2. *If we put $\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)=\mathfrak{d}_2-m\mathfrak{d}_1$ and $n_1 = n_2 = m = 1$ in Theorem 7.1, it collapses to the Theorem 3 in [45].*

8. Applications to means

The subject convex analysis and fractional mathematics are both utilized in applied sciences. The literature makes it clear that these ideas have a broad spectrum of potential uses in multiple fields of research, from fluid dynamics to optimization. In order to be more precise, we are going to apply certain mean-type inequalities, such as arithmetic, geometric, and harmonic means inequalities, to the H-H inequality associated with the C-FFIO via $G -_m PF$. The following mean-type inequalities have remarkable utilization in the domains of probabilities, statistics, circuit theory, stochastic processes, engineering, numerical approximations, and machine learning. In this section, we investigate the means as applications for two positive number $\mathfrak{d}_1,\mathfrak{d}_2$ with $\mathfrak{d}_1<\mathfrak{d}_2$, which are given as:

(1) The arithmetic mean
$$\mathcal{A}=\mathcal{A}(\mathfrak{d}_1,\mathfrak{d}_2)=\frac{\mathfrak{d}_1+\mathfrak{d}_2}{2},\quad \mathfrak{d}_1,\mathfrak{d}_2\in\mathbb{R}.$$

(2) The generalized logarithmic mean
$$\mathcal{L}=\mathcal{L}_r^r(\mathfrak{d}_1,\mathfrak{d}_2)=\frac{\mathfrak{d}_2^{r+1}-\mathfrak{d}_1^{r+1}}{(r+1)(\mathfrak{d}_2-\mathfrak{d}_1)}.$$

Now employing the results in part 5, we investigate several inequalities involving special means. So here, we take $\mathfrak{B}(\omega)=\mathfrak{B}(1)=1$.

Proposition 8.1. *Assume that $m\mathfrak{d}_1, m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)\in\mathbb{R}^+$ and $m\mathfrak{d}_1<m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)$, then*

$$\left|-\mathcal{A}(m\mathfrak{d}_1^2,(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))^2)+\mathcal{L}_2^2(m\mathfrak{d}_1,m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))\right|$$
$$\leq \frac{\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m)}{n}\sum_{\mathfrak{u}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{u}^2+\mathfrak{u}+2)2^{\mathfrak{u}}-2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}}\right][|\mathfrak{d}_1|+|\mathfrak{d}_2|]. \tag{8.1}$$

Proof. If $\mathfrak{D}(z) = z^2$ with $\omega = 1$ and $\mathfrak{B}(\omega) = \mathfrak{B}(1) = 1$ in Theorem 5.1, then we attain the above proposition 8.1.

Proposition 8.2. *Assume that* $m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) \in \mathbb{R}^+$ *and* $m\mathfrak{d}_1 < m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)$*, then*

$$\left| -\mathcal{A}(e^{m\mathfrak{d}_1}, e^{(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}) + \mathcal{L}(e^{m\mathfrak{d}_1}, e^{(m\mathfrak{d}_1+\mathcal{N}(\mathfrak{d}_2,\mathfrak{d}_1,m))}) \right| \leq \frac{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{n} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \left[\frac{(\mathfrak{u}^2 + \mathfrak{u} + 2)2^{\mathfrak{u}} - 2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}} \right] \left[\frac{e_1^{\mathfrak{d}} + e_2^{\mathfrak{d}}}{2} \right]. \tag{8.2}$$

Proof. If $\mathfrak{D}(z) = e^z$ with $\omega = 1$ and $\mathfrak{B}(\omega) = \mathfrak{B}(1) = 1$ in Theorem 5.1, then we attain the above proposition 8.2.

Proposition 8.3. *Assume that* $m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m) \in \mathbb{R}^+, m\mathfrak{d}_1 < m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)$*, then*

$$\left| -\mathcal{A}((m\mathfrak{d}_1)^n, (m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m))^n) - \mathcal{L}_n^n(m\mathfrak{d}_1, m\mathfrak{d}_1 + \mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)) \right| \leq \frac{\mathcal{N}(\mathfrak{d}_2, \mathfrak{d}_1, m)}{n} \sum_{\mathfrak{u}=1}^{\mathfrak{n}} \left[\frac{(\mathfrak{u}^2 + \mathfrak{u} + 2)2^{\mathfrak{u}} - 2}{(\mathfrak{u}+1)(\mathfrak{u}+2)2^{\mathfrak{u}+1}} \right] \left[n\frac{|\mathfrak{d}_1^{n-1}| + |\mathfrak{d}_2^{n-1}|}{2} \right]. \tag{8.3}$$

Proof. If $\mathfrak{D}(z) = z^n$ with $\omega = 1$ and $\mathfrak{B}(\omega) = \mathfrak{B}(1) = 1$ in Theorem 5.1, then we attain the above Proposition 8.3.

9. Conclusions

Fractional calculus has a greater influence and provides more precise results when examining computer models. Fractional calculus is widely utilized in applied mathematics, mathematical biology, engineering, simulation, and inequality theory. Numerous researchers across multiple scientific domains have expressed a keen interest in fractional calculus. In this paper:

1) First, we explored a novel type of preinvex function namely $G -_m PF$. Further, we added some algebraic properties regarding this newly introduced definition.
2) We proved some refinements of the H-H and Pachpatte type inequalities via $G -_m PF$ in the aspect of the C-FFIO.
3) In addition, a novel integral identity is presented and several results in the sense of C-FFIO are attained via a newly introduced concept.
4) To improve the reader's interest and overall quality, we showed the refinements of H-H inequality regarding the newly introduced lemma with the help of improved power mean and Hölder Iscan inequality.
5) Some corollaries and remarks are added.
6) Finally, some meaningful applications regarding newly introduced ideas are explored.

This paper's new notion can be extended to numerous inequalities employing the fractional H-H. Further generalizations can be made using some fresh ideas, such as interval-valued R-L convexities, center-radius order convexities and fuzzy interval convexities. To observe the behavior of this inequality, excited and inspired mathematicians can also use interval-valued functions and quantum calculus, etc. The investigation of fractional versions of inequalities using various new convex function types will be very intriguing to watch.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they do not have conflict of interest regarding this manuscript.

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