Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 25550–25563.
DOI: 10.3934/math.20231304
Received: 06 July 2023
Revised: 07 August 2023
Accepted: 10 August 2023
Published: 04 September 2023

Research article

Positive solutions to integral boundary value problems for singular delay fractional differential equations

Xiulin Hu* **and Lei Wang**

School of Artificial Intelligence and Big Data, Hefei University, Hefei, China

* **Correspondence:** Email: huxlxy12@163.com; huxlxy@hfuu.edu.cn.

Abstract: Delay fractional differential equations play very important roles in mathematical modeling of real-life problems in a wide variety of scientific and engineering applications. The objective of this manuscript is to study the existence and uniqueness of positive solutions for singular delay fractional differential equations with integral boundary data. To investigate the described system, we construct a u_0-positive operator first. New research technique of by constructing u_0-positive operator is used to overcome the difficulties caused by both the delays and the boundary value conditions. Then the sufficient conditions for the existence and uniqueness of positive solutions of a class of the singular delay fractional differential equations with integral boundary is proved by using the fixed point theorem in cone.

Keywords: integral boundary value problems; singular delay fractional differential equations; u_0-positive operators; fixed point theorem
Mathematics Subject Classification: 34A08, 34B15, 34B17

1. Introduction

The fractional differential equations and delays arise naturally in a wide range of real-world phenomena and processes. Theory and applications of fractional differential system in different areas were considered by many researchers. For more details one can refer the books [1–3]. Some real-world models by fractional derivatives in engineering systems are presented in the book [4]. HIV/AIDS transmission models are investigated in [5]. Accelerated mass-spring systems are studied in [6]. Biochemical reaction models are studied in [7]. Chemical graph theory is given in [8,9]. During the last few decades, a lot of papers have been devoted to investigate the positive solutions of boundary value problems for fractional differential equations, such as [10–18]. The systems they studied are delay-free and most of them investigated the existence of positive solutions by using the classical fixed pointed methods. However, to our knowledge, on account of the need for resolving the difficulties caused by both the delays and the boundary value conditions, few results on the boundary value problems for fractional differential equations with time delays are appeared. For example, Qiao and Zhou [18] studied a class of boundary value problems for a fractional differential equation with integral boundary conditions but without time delays

$$\begin{cases} D_{0^+}^p x(t) + p(t)f(t, x(t)) + q(t) = 0,\ t \in (0,1), \\ x(0) = x'(0) = 0, x(1) = \int_0^1 l(s)x(s)ds, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to (0, +\infty)$ is continuous, $q(t), l(t) \in C((0,1), [0, +\infty))$. Liao and Ye [19] investigated the existence and uniqueness of positive solutions for a class of nonlinear delay fractional differential equations

$$\begin{cases} L(D)[x(t) - x(0)] = f(t, x_t),\ t \in (0, T], \\ x(t) = \phi(t),\ t \in [-r, 0], \end{cases} \tag{1.2}$$

where $f : [0,1] \times C \to \mathbb{R}^+$ is continuous, in which $\mathbb{R}^+ = [0, +\infty), C = C([-r, 0], \mathbb{R}^+)$ is the space of continuous functions from $[-r, 0]$ to $\mathbb{R}^+, r > 0$, $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - \cdots - a_1 D^{s_1}$,$0 < s_j < s_{j+1} < 1, a_j > 0, j = 1, 2, \cdots, n-1$. D^{s_j} denotes the standard Riemann-Liouville fractional derivative. Using Krasnosel'skii fixed point theorem, Su in [20] examined the positive solutions to the singular delay fractional differential equations with easy boundary data

$$\begin{cases} D^d x(t) + f(t, x(t-\tau)) = 0,\ t \in (0,1)\backslash\{\tau\}, \\ x(t) = \eta(t),\ t \in [-\tau, 0], \\ x(1) = 0, \end{cases} \tag{1.3}$$

where $f : (0,1) \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and may be singular at $t = 0, t = 1, x = 0$, $1 < d \leq 2$ is a real number, D^d is the Riemann-Liouville fractional derivative, $\mathbb{R}^+ = [0, +\infty)$. Li et al. [21] and Agarwal and Hristova [22] studied boundary value problems of some fractional functional differential equations involving the Caputo fractional derivative. However, the boundary value conditions in the above mentioned results for delay fractional differential equations are not concerned with the integral data. Despite many excellent works on integral boundary value problems for ordinary differential equations are available, there are only relatively scare results on the integral boundary value problems for delay fractional differential equations.

Inspired by the works mentioned above, the present paper is related to studying the existence and uniqueness of positive solutions for the following delay fractional differential equations

$$D_{0^+}^d x(t) + f(t, x(t-\tau)) = 0,\ t \in (0,1)\backslash\{\tau\} \tag{1.4}$$

with the more complicated integral boundary value conditions

$$\begin{cases} x(t) = p(t),\ t \in [-\tau, 0], \\ x'(0) = 0,\ x(1) = \int_0^1 q(s)x(s)ds, \end{cases} \tag{1.5}$$

where $D_{0^+}^d$ is a standard Riemann-Liouville fractional derivative and d is a real number with $2 < d \leq 3$. The time delay τ is a constant which admits $0 < \tau < 1$. Throughout the present paper, the integral boundary value problems (IBVPs) (1.4) and (1.5) refers to the Eq (1.4) with boundary data (1.5). The functions f, p, q involved in IBVPs (1.4) and (1.5) are assumed to satisfy the following conditions:

(H_1) $f : (0, 1) \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, may be singular at $t = 0, t = 1$ and $x = 0$.

(H_2) $p(t) \in C([-\tau, 0]), p(t) > 0$ for $t \in [-\tau, 0)$, and $p(0) = 0, p'_-(0) = 0$, where $p'_-(0)$ denotes the left derivative of p at $t = 0$.

(H_3) $q : (0, 1) \to \mathbb{R}^+$ is continuous, and satisfies

$$0 \le Q := \int_0^1 t^{d-1} q(t)dt < 1.$$

A function x is said to be a positive solution of IBVPs (1.4) and (1.5) if $x(t)$ is nonnegative on $[-\tau, 1]$, $x(t) > 0$ for $t \in [-\tau, 1]\backslash\{0\}$ and it admits the Eq (1.4).

The novelty of the present paper is twofold. First, IBVPs (1.4) and (1.5) under consideration involve not only the past time delay but also the fractional derivative with the order $2 < d \le 3$. Second, the technique used in this paper is to construct a u_0-positive operator as to overcome the difficulties caused by the singularity of the function f. Based on a fixed point theorem, some new existence and uniqueness criteria of positive solutions are established.

The rest of this study is organized as follows. In Section 2, some definitions and lemmas are reviewed. In Section 3, we construct a u_0-positive operator to demonstrate our main results. Then, the criteria to existence and uniqueness of positive solutions for IBVPs (1.4) and (1.5) can be established. We make a conclusion in Section 4.

2. Preliminaries

In this section, we resume with several necessary definitions and lemmas from fractional calculus theory.

Definition 2.1. *(e.g., [1–3]) The Riemann-Liouville fractional integral of a function $u : (0, +\infty) \to \mathbb{R}$ with order $d > 0$ is given by*

$$I_{0^+}^d u(t) = \frac{1}{\Gamma(d)} \int_0^t (t - s)^{d-1} u(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

In this section, we resume with several necessary definitions and lemmas from fractional calculus theory.

Definition 2.2. *(e.g., [1–3]) The Riemann-Liouville fractional integral of a function $u : (0, +\infty) \to \mathbb{R}$ with order $d > 0$ is given by*

$$I_{0^+}^d u(t) = \frac{1}{\Gamma(d)} \int_0^t (t - s)^{d-1} u(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.3. *(e.g., [1–3]) The Riemann-Liouville fractional derivative of a continuous function u : $(0, +\infty) \to \mathbb{R}$ with order $d > 0$ is given by*

$$D_{0^+}^d u(t) = \frac{1}{\Gamma(n - d)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-d-1} u(s)ds,$$

where $n - 1 \le d < n$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

The definition of a u_0-positive operator is given as follows.

Definition 2.4. *(e.g., [23]) Let X be a Banach space, and P a cone in X. A bounded linear operator $S : X \to X$ is said to be a u_0-positive operator on the cone P if there exists $u_0 \in P\backslash\{\theta\}$ such that for every $u \in P\backslash\{\theta\}$, there exists a natural number n and positive constants $\alpha(x)$,$\beta(x)$ such that the following symmetric inequality is satisfied*

$$\alpha(x)u_0 \leq S^n u \leq \beta(x)u_0.$$

Lemma 2.5. *([24]) Let $d > 0$ and $u(t)$ be an integrable function. Then,*

$$I_{0^+}^d D_{0^+}^d u(t) = u(t) + c_1 t^{d-1} + c_2 t^{d-2} + \cdots + c_n t^{d-n},$$

where $c_i \in \mathbb{R}(i = 1, 2, \cdots, n)$, and n is the smallest integer greater than or equal to d.

The following is an existence and uniqueness result of solutions for a linear boundary value problem, which is paramount for us in the following analysis.

Lemma 2.6. *Assume that $\rho \in C(0,1) \cap L(0,1), 2 < d \leq 3$. Then, the unique solution of the following BVPs*

$$\begin{cases} D_{0^+}^d x(t) + \rho(t) = 0, t \in (0,1), \\ x(0) = x'(0) = 0, x(1) = \int_0^1 q(s)x(s)ds, \end{cases} \tag{2.1}$$

is described by

$$x(t) = \int_0^1 G(t,s)\rho(s)ds - \frac{\int_0^1 (\int_s^1 q(t)(t-s)^{d-1}\rho(s)dt)ds}{(1-Q)\Gamma(d)} t^{d-1}, \tag{2.2}$$

in which the constant

$$Q := \int_0^1 t^{d-1} q(t)dt \in [0,1),$$

and

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(d)}[\frac{t^{d-1}(1-s)^{d-1}}{1-Q} - (t-s)^{d-1}], \ 0 \leq s \leq t \leq 1 \\ \\ \frac{1}{\Gamma(d)} \frac{t^{d-1}(1-s)^{d-1}}{1-Q}, \ 0 \leq t \leq s \leq 1 \end{cases} \tag{2.3}$$

is called the Green function of the BVPs (2.1).

Proof. Deduced from Lemma 2.5, we have

$$x(t) = -I_{0^+}^d \rho(t) + c_1 t^{d-1} + c_2 t^{d-2} + c_3 t^{d-3}.$$

So, the solution of Eq (2.1) is

$$x(t) = -\frac{1}{\Gamma(d)} \int_0^t (t-s)^{d-1}\rho(s)ds + c_1 t^{d-1} + c_2 t^{d-2} + c_3 t^{d-3},$$

where $2 < d \leq 3$. By the conditions $x(0) = x'(0) = 0$, we have that $c_2 = c_3 = 0$. On the other hand, the condition $x(1) = \int_0^1 q(s)x(s)ds$ yields

$$-\frac{1}{\Gamma(d)}\int_0^1 (1-s)^{d-1}\rho(s)ds + c_1 = \int_0^1 q(s)[-\frac{1}{\Gamma(d)}\int_0^s (s-\tau)^{d-1}\rho(\tau)d\tau + c_1 s^{d-1}]ds.$$

By swapping the upper and lower limits, we have

$$c_1 = \frac{\int_0^1 (1-s)^{d-1}\rho(s)ds - \int_0^1 [\int_t^1 q(s)(s-t)^{d-1}\rho(t)ds]dt}{(1-Q)\Gamma(d)},$$

where $Q := \int_0^1 t^{d-1}q(t)dt \in [0, 1)$.

Therefore, the solution of BVPs (2.1) is

$$\begin{aligned}
x(t) &= -\frac{1}{\Gamma(d)}\int_0^t (t-s)^{d-1}\rho(s)ds + \frac{t^{d-1}}{(1-Q)\Gamma(d)}\int_0^1 (1-s)^{d-1}\rho(s)ds \\
&\quad - \frac{\int_0^1 [\int_t^1 q(s)(s-t)^{d-1}\rho(t)ds]dt}{(1-Q)\Gamma(d)} t^{d-1} \\
&= \int_0^t \frac{1}{\Gamma(d)}[\frac{t^{d-1}(1-s)^{d-1}}{(1-Q)} - (t-s)^{d-1}]\rho(s)ds + \int_t^t \frac{1}{\Gamma(d)}\frac{t^{d-1}(1-s)^{d-1}}{(1-Q)}\rho(s)ds \\
&\quad - \frac{\int_0^1 [\int_t^1 q(s)(s-t)^{d-1}\rho(t)ds]dt}{(1-Q)\Gamma(d)} t^{d-1} \\
&= \int_0^1 G(t,s)\rho(s)ds - \frac{\int_0^1 [\int_s^1 q(t)(t-s)^{d-1}\rho(s)dt]ds}{(1-Q)\Gamma(d)} t^{d-1}.
\end{aligned}$$

This completes the proof. ■

Setting

$$L(s) = \frac{\int_s^1 q(t)(t-s)^{d-1}dt}{(1-Q)\Gamma(d)}. \tag{2.4}$$

Then, the solution of the BVPs (2.1) can be written as

$$x(t) = \int_0^1 [G(t,s) - L(s)t^{d-1}]\rho(s)ds.$$

We enjoy the following Lemma.

Lemma 2.7. *The Green function $G(t, s)$ defined by (2.3) admits the following inequality*

$$G(t,s) - L(s)t^{d-1} \geq 0, \ for\ t, s \in (0, 1).$$

Proof. From (2.3) and (2.4), for $0 \leq s \leq t \leq 1$, one can calculate directly that

$$G(t,s) - L(s)t^{d-1} = \frac{1}{\Gamma(d)}[\frac{t^{d-1}(1-s)^{d-1}}{1-Q} - (t-s)^{d-1}] - \frac{\int_s^1 q(t)(t-s)^{d-1}dt}{(1-Q)\Gamma(d)} t^{d-1}$$

$$
\begin{aligned}
&\geq \frac{1}{\Gamma(d)}[\frac{t^{d-1}(1-s)^{d-1}}{1-Q} - (t-ts)^{d-1}] - \frac{\int_s^1 q(t)(t-s)^{d-1}dt}{(1-Q)\Gamma(d)} t^{d-1} \\
&= \frac{t^{d-1}}{(1-Q)\Gamma(d)}[(1-s)^{d-1}Q - \int_s^1 q(t)(t-s)^{d-1}dt] \\
&\geq \frac{t^{d-1}}{(1-Q)\Gamma(d)}[\int_0^1 q(t)t^{d-1}(1-s)^{d-1}dt - \int_0^1 q(t)(t-s)^{d-1}dt] \\
&\geq 0.
\end{aligned}
$$

On the other hand, for $0 \leq t \leq s \leq 1$, noticing that $0 \leq Q < 1$, one can deduce that

$$
\begin{aligned}
G(t,s) - L(s)t^{d-1} &= \frac{1}{\Gamma(d)} \frac{t^{d-1}(1-s)^{d-1}}{1-Q} - \frac{\int_s^1 q(t)(t-s)^{d-1}dt}{(1-Q)\Gamma(d)} t^{d-1} \\
&> \frac{t^{d-1}}{(1-Q)\Gamma(d)}[(1-s)^{d-1}Q - \int_s^1 q(t)(t-s)^{d-1}dt] \\
&\geq \frac{t^{d-1}}{(1-Q)\Gamma(d)}[\int_0^1 q(t)t^{d-1}(1-s)^{d-1}dt - \int_0^1 q(t)(t-s)^{d-1}dt] \\
&\geq 0.
\end{aligned}
$$

Hence, for any $t, s \in (0,1)$, we have $G(t,s) - L(s)t^{d-1} \geq 0$, which is the desired results. ■

To proceed, we recall a result in [23].

Lemma 2.8. *(e.g., [23]) Let X be a Banach space, and P a cone in X. Suppose that $S : X \to X$ is a completely continuous linear operator and $S(P) \subset P$. If there exists $\psi \in X\backslash(-P)$ and a constant $c > 0$ such that $cS\psi \geq \psi$, then the spectral radius $r(S) \neq 0$, and S has a positive eigenfunction φ corresponding to its first eigenvalue $\lambda_1 = \frac{1}{r(S)}$, that is, $\varphi = \lambda_1 S\varphi$.*

3. Main results

In this section, we begin with constructing a u_0-positive operator; followed by the obtained results, the existence and uniqueness of positive solutions for IBVPs (1.4) and (1.5) is discussed in the succeeding subsection.

3.1. Construction of u_0-positive operators

We also need to define a space

$$E = \{x(t) : x \in C([-\tau, 1], \mathbb{R}^+), x(t) = 0 \; for \; t \in [-\tau, 0]\}, \tag{3.1}$$

with the norm

$$\| x \| = \sup_{t\in[-\tau,1]} | x(t) | = \sup_{t\in[0,1]} | x(t) | .$$

Then, it is not difficult to find that $(E, \| \cdot \|)$ is a Banach space. A cone in the space E can be described as

$$P = \{x \in E : x(t) \geq 0 \; for \; t \in [-\tau, 1]\}.$$

Let $\rho \in C(0,1) \cap L(0,1)$ be a nonnegative function. We define the functions

$$\overline{p}(t) = \begin{cases} p(t), t \in [-\tau, 0], \\ \\ 0, \ t \in [0,1], \end{cases}$$

$$\nu(t) = \begin{cases} 0, \ t \in [-\tau, 0], \\ \\ \int_0^1 [G(t,s) - L(s)t^{d-1}]\rho(s)ds, t \in [0,1], \end{cases}$$

and, for any $x \in P$,

$$x^*(t) = \max\{x(t) + \overline{p}(t) - \nu(t), 0\} = \begin{cases} p(t), \ t \in [-\tau, 0], \\ \\ \max\{x(t) - \nu(t), 0\}, t \in [0,1]. \end{cases}$$

The following is naturally followed by Lemma 2.6.

Remark 3.1. *The restriction of the function* ν *on* $[0,1]$

$$\nu\,|_{[0,1]} = \int_0^1 [G(t,s) - L(s)t^{d-1}]\rho(s)ds$$

is exactly the solution of the BVPs (2.1).

To proceed, define an operator A in P as

$$(Ax)(t) = \begin{cases} 0, \ t \in [-\tau, 0], \\ \\ \int_0^1 [G(t,s) - L(s)t^{d-1}][f(s, x^*(s-\tau)) + \rho(s)]ds, t \in (0,1]. \end{cases} \tag{3.2}$$

Lemma 2.6 means that if $\tilde{x}$ is a fixed point of the operator A, then $\tilde{x}$ is the solution of the following integral BVPs

$$\begin{cases} D_{0^+}^d \tilde{x}(t) + f(t, \tilde{x}^*(t-\tau)) + \rho(t) = 0, t \in (0,1) \backslash \{\tau\}, \\ \tilde{x}(t) = 0, \ t \in [-\tau, 0], \\ \tilde{x}'(0) = 0, \tilde{x}(1) = \int_0^1 q(s)\tilde{x}(s)ds. \end{cases}$$

Hence, if the following inequality $\tilde{x}(t-\tau) + \overline{p}(t-\tau) - \nu(t-\tau) \geq 0$ fulfills for $t \in [0,1]$, then

$$\tilde{x}^*(t-\tau) = \tilde{x}(t-\tau) + \overline{p}(t-\tau) - \nu(t-\tau).$$

Let

$$x(t) = \tilde{x}(t) + \overline{p}(t) - \nu(t), \tag{3.3}$$

where $\tilde{x}(t)$ is a fixed point of the operator A.

By the definitions of the functions $\overline{p}(t), \nu(t)$, it is easy to conclude that $x(t) = \tilde{x}(t) + \overline{p}(t) - \nu(t) = p(t)$, for any $t \in [-\tau, 0]$. By Remark 3.1, $\nu(1) = \int_0^1 q(s)\nu(s)ds$. Then, from the variable substitution (3.3) we have $x(1) = \tilde{x}(1) + \overline{p}(1) - v(1) = \int_0^1 q(s)\tilde{x}(s)ds - \int_0^1 q(s)v(s)ds = \int_0^1 q(s)x(s)ds$. Hence, by the condition $p'_-(0) = 0$ in (H_2), we conclude that the function x defined in (3.3) is the solution of IBVPs (1.4) and (1.5). As a result, in what follows one can just need to focus our study on finding the fixed points of the operator A defined by (3.2).

Define another operator T be defined in P by

$$(Ty)(t) = \begin{cases} 0, \ t \in [-\tau, 0], \\ \\ \int_0^1 [G(t,s) - L(s)t^{d-1}]y(s-\tau)ds, t \in (0,1]. \end{cases} \tag{3.4}$$

By Lemma 2.7, it is not difficult to see that $T : E \to E$ is a linear completely continuous operator and $T(P) \subset P$.

The u_0-positive operator of the operator T defined by (3.4) can be constructed in the following theory.

Theorem 3.2. *The operator T defined by (3.4) is a u_0-positive operator with $u_0(t) = t^{d-1}$.*

Proof. First, by the definition of the constant Q and the function $L(s)$ defined in Lemma 2.6, we easily possess

$$\begin{aligned} \frac{Q(1-s)^{d-1}}{(1-Q)\Gamma(d)} - L(s) &= \frac{(1-s)^{d-1}}{(1-Q)\Gamma(d)} \int_0^1 t^{d-1}q(t)dt - \frac{\int_s^1 (t-s)^{d-1}q(t)dt}{(1-Q)\Gamma(d)} \\ &= \frac{1}{(1-Q)\Gamma(d)}[\int_0^1 t^{d-1}(1-s)^{d-1}q(t)dt - \int_s^1 (t-s)^{d-1}q(t)dt] \\ &\geq \frac{1}{(1-Q)\Gamma(d)}[\int_0^1 (t-s)^{d-1}q(t)dt - \int_s^1 (t-s)^{d-1}q(t)dt] \\ &\geq 0, \end{aligned}$$

for $t, s \in (0,1)$. Notice that $0 \leq Q < 1$. Thus, the following inequality

$$\frac{(1-s)^{d-1}}{(1-Q)\Gamma(d)} - L(s) \geq \frac{Q(1-s)^{d-1}}{(1-Q)\Gamma(d)} - L(s) \geq 0$$

holds for $t, s \in (0,1)$.

For any $y \in P \setminus \{\theta\}$, by (3.4) and (2.3), one can calculate that

$$\begin{aligned} (Ty)(t) &= \int_0^t \{\frac{1}{\Gamma(d)}[\frac{t^{d-1}(1-s)^{d-1}}{1-Q} - (t-s)^{d-1}] - L(s)t^{d-1}\}y(s-\tau)ds \\ &\quad + \int_t^1 [\frac{1}{\Gamma(d)}\frac{t^{d-1}(1-s)^{d-1}}{1-Q} - L(s)t^{d-1}]y(s-\tau)ds \\ &= \int_0^t \frac{t^{d-1}(1-s)^{d-1}}{(1-Q)\Gamma(d)}y(s-\tau)ds - \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)}y(s-\tau)ds \\ &\quad - \int_0^t L(s)t^{d-1}y(s-\tau)ds + \int_t^1 \frac{t^{d-1}(1-s)^{d-1}}{(1-Q)\Gamma(d)}y(s-\tau)ds \end{aligned}$$

$$
\begin{aligned}
&-\int_t^1 L(s)t^{d-1}y(s-\tau)ds\\
&\leq \int_0^1 \frac{t^{d-1}(1-s)^{d-1}}{(1-Q)\Gamma(d)}y(s-\tau)ds-\int_0^1 L(s)t^{d-1}y(s-\tau)ds\\
&= \int_0^1 [\frac{(1-s)^{d-1}}{(1-Q)\Gamma(d)}-L(s)]y(s-\tau)ds\cdot t^{d-1},\ t,s\in(0,1).
\end{aligned}
$$

On the other hand, for $0 \leq Q < 1$, one can deduce that

$$
\begin{aligned}
(Ty)(t) &= \int_0^t \{\frac{1}{\Gamma(d)}[\frac{t^{d-1}(1-s)^{d-1}}{1-Q}-(t-s)^{d-1}]-L(s)t^{d-1}\}y(s-\tau)ds\\
&+\int_t^1 [\frac{1}{\Gamma(d)}\frac{t^{d-1}(1-s)^{d-1}}{1-Q}-L(s)t^{d-1}]y(s-\tau)ds\\
&\geq \int_0^t \{\frac{1}{\Gamma(d)}[\frac{t^{d-1}(1-s)^{d-1}}{1-Q}-(t-ts)^{d-1}]-L(s)t^{d-1}\}y(s-\tau)ds\\
&+\int_t^1 [\frac{1}{\Gamma(d)}\frac{t^{d-1}(1-s)^{d-1}}{1-Q}-L(s)t^{d-1}]y(s-\tau)ds\\
&= \int_0^t [\frac{Qt^{d-1}(1-s)^{d-1}}{(1-Q)\Gamma(d)}-L(s)t^{d-1}]y(s-\tau)ds\\
&+\int_t^1 [\frac{t^{d-1}(1-s)^{d-1}}{(1-Q)\Gamma(d)}-L(s)t^{d-1}]\cdot y(s-\tau)ds\\
&\geq \int_0^1 \frac{Qt^{d-1}(1-s)^{d-1}}{(1-Q)\Gamma(d)}y(s-\tau)ds-\int_0^1 L(s)t^{d-1}y(s-\tau)ds\\
&= \int_0^1 [\frac{Q(1-s)^{d-1}}{(1-Q)\Gamma(d)}-L(s)]y(s-\tau)ds\cdot t^{d-1},\ t,s\in(0,1).
\end{aligned}
$$

Therefore, for any $y \in P \setminus \{\theta\}$, one can deduce that

$$\alpha(y)\cdot u_0 \leq (Ty)(t) \leq \beta(y)\cdot u_0,$$

where

$$
\begin{aligned}
\alpha(y) &= \int_0^1 [\frac{Q(1-s)^{d-1}}{\Gamma(d)(1-Q)}-L(s)]y(s-\tau)ds,\\
\beta(y) &= \int_0^1 [\frac{(1-s)^{d-1}}{\Gamma(d)(1-Q)}-L(s)]y(s-\tau)ds.
\end{aligned}
$$

This implies that the operator T is a u_0-operator with $u_0(t) = t^{d-1}$. ∎

By the proof of Theorem 3.2 and Lemma 2.8, we have the following lemma.

Lemma 3.3. *The spectral radius of the operator T admits $r(T) \neq 0$ and T has a positive eigenfunction $\varphi^*(t)$ corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Proof. Let

$$
\psi(t) = \begin{cases} 0,\ t\in[-\tau,0],\\ t^{d-1}, t\in[0,1], \end{cases}
$$

and a constant

$$c = \{\int_0^1 [\frac{Q(1-s)^{d-1}}{\Gamma(d)(1-Q)} - L(s)]\psi(s-\tau)ds\}^{-1} > 0.$$

Then, from the proof of Theorem 3.2, we find that

$$c(T\psi)(t) \geq \psi(t).$$

Thus, by Lemma 2.8, the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction $\varphi^*(t)$ corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$, that is $\varphi^*(t) = \lambda_1(T\varphi^*)(t)$. ■

The following result can be used in the proof of the main result in this paper.

Remark 3.4. *Let $\varphi^*(t)$ be the positive eigenfunction of operator T corresponding to λ_1, that is,*

$$\lambda_1(T\varphi^*)(t) = \varphi^*(t).$$

Then, by Theorem 3.2 and Definition 2.4, there exists $k_1(\varphi^*), k_2(\varphi^*)$ such that

$$k_1(\varphi^*) \cdot u_0 \leq T\varphi^* = \frac{1}{\lambda_1}\varphi^* \leq k_2(\varphi^*) \cdot u_0, \ \varphi^* \in P \setminus \{\theta\}.$$

Hence, we obtain that T defined by (3.4) is a u_0-positive operator with $u_0(t) = \varphi^*(t)$.

3.2. Existence and uniqueness of positive solutions

In this subsection, based on a fixed point theorem, we study the existence and uniqueness of positive solutions for IBVPs (1.4) and (1.5) by using the u_0-positive operators. The following theorem is the main result in this paper.

Theorem 3.5. *Assume that conditions (H_1)–(H_3) hold and there exists a constant $k \in [0, 1)$ such that*

$$|\, f(t,u) - f(t,v) \,| \leq k\lambda_1 \,|\, u - v \,|, \ for\ any\ t \in [0,1], \ u, v \in \mathbb{R}, \tag{3.5}$$

where λ_1 is the first eigenvalue of the operator T defined by (3.4). Then, IBVPs (1.4) and (1.5) has a unique positive solution x^*. Moreover, for any $x_0 \in P$, the iterative sequence $x_n = Ax_{n-1}(n = 1, 2, \ldots)$ converges to x^*.

Proof. Owing to the continuity of f and the fact that T is a linear completely continuous operator, it is not difficult to verify that the operator $A : E \to E$ defined by (3.2) is completely continuous and satisfies $A(P) \subset P$.

For any given $x_0 \in P$, define the iterative sequence $x_n = Ax_{n-1}(n = 1, 2, \ldots)$. Since $A(P) \subset P$, it follows that $\{x_n\} \subset P$.

Since λ_1 is the first eigenvalue of T, that is $T(\varphi^*(t)) = \frac{1}{\lambda_1}\varphi^*(t)$, by the linearity of the operator T, stepwise recursive yields

$$T^{n-1}(\varphi^*(t)) = T^{n-2}(\frac{1}{\lambda_1}\varphi^*(t)) = \frac{1}{\lambda_1}T^{n-2}(\varphi^*(t)) = \ldots = \frac{1}{\lambda_1^{n-1}}\varphi^*(t).$$

Thus, for $n \in \mathbb{N}^+$, by (3.5), one can deduce that

$$|\, x_{n+1}(t) - x_n(t) \,| =|\, (Ax_n)(t) - (Ax_{n-1})(t) \,|$$

$$
\begin{aligned}
&=\left| \int_0^1 [G(t,s) - L(s)t^{d-1}][f(s, x_n^*(s-\tau)) + \rho(s)]ds \right. \\
&\left. - \int_0^1 [G(t,s) - L(s)t^{d-1}][f(s, x_{n-1}^*(s-\tau)) + \rho(s)]ds \right| \\
&\le \int_\tau^1 [G(t,s) - L(s)t^{d-1}] \mid f(s, x_n(s-\tau) - \nu(s-\tau)) \\
&- f(s, x_{n-1}(s-\tau) - \nu(s-\tau)) \mid ds \\
&\le k\lambda_1 \int_\tau^1 [G(t,s) - L(s)t^{\alpha-1}] \mid x_n(s-\tau) - x_{n-1}(s-\tau) \mid ds \\
&\le k\lambda_1 T(\mid x_n - x_{n-1} \mid)(t) \le \ldots \le k^n \lambda_1^n T^n(\mid x_1 - x_0 \mid)(t).
\end{aligned}
$$

By Theorem 3.2 and Remark 3.4, there is a constant $\delta_1 = \delta_1(\mid x_1 - x_0 \mid) > 0$ such that

$$T(\mid x_1 - x_0 \mid)(t) \le \delta_1 \varphi^*(t),\ t \in [0, 1],$$

where $\varphi^*(t)$ is the positive eigenfunction of operator T corresponding to λ_1. Then, for $n \in \mathbb{N}^+$, we have

$$
\begin{aligned}
\mid x_{n+1}(t) - x_n(t) \mid &\le k^n \lambda_1^n T^{n-1}(\delta_1 \varphi^*(t)) = \delta_1 k^n \lambda_1^n T^{n-1}(\varphi^*(t)) \\
&= \delta_1 k^n \lambda_1^n \cdot \frac{1}{\lambda_1^{n-1}} \varphi^*(t) = \delta_1 \lambda_1 k^n \varphi^*(t).
\end{aligned}
$$

It follows that for any $m \in N^+$

$$
\begin{aligned}
\mid x_{n+m}(t) - x_n(t) \mid &\le \mid x_{n+m}(t) - x_{n+m-1}(t) \mid + \ldots + \mid x_{n+1}(t) - x_n(t) \mid \\
&\le \delta_1 \lambda_1 (k^{n+m-1} + \ldots + k^n) \varphi^*(t) \\
&= \delta_1 \lambda_1 \frac{k^n(1-k^m)}{1-k} \varphi^*(t) \\
&\le \delta_1 \lambda_1 \frac{k^n}{1-k} \varphi^*(t),
\end{aligned}
$$

which means that

$$\| x_{n+m} - x_n \| \le \delta_1 \lambda_1 \frac{k^n}{1-k} \|\varphi^*\|.$$

Note that $\lim\limits_{n\to\infty} \beta_1 \lambda_1 \frac{k^n}{1-k} \varphi^*(t) = 0$. Thus, $\{x_n\}$ is a Cauchy sequence. Therefore, from the completeness of the space E and the closeness of the operator P, there exists $x^* \in P$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Since the operator A is continuous, taking the limit into $x_n = Ax_{n-1}$ demonstrate that x^* is a fixed point of A in P.

Next, we demonstrate that A has at most one fixed point in P. Suppose that there exist two elements $x, y \in X$ with $x = Ax$ and $y = Ay$. Then, by the condition (3.5), for any $n \in N^+$, one can calculate

$$\mid x(t) - y(t) \mid = \mid (A^n x)(t) - (A^n y)(t) \mid$$

$$
\begin{aligned}
&=\mid [A(A^{n-1}x)](t) - [A(A^{n-1}y)](t) \mid \\
&=\mid \int_0^1 [G(t,s) - L(s)t^{d-1}][f(s,(A^{n-1}x)^*(s-\tau)) + \rho(s)]ds \\
&- \int_0^1 [G(t,s) - L(s)t^{d-1}][f(s,(A^{n-1}y)^*(s-\tau)) + \rho(s)]ds \mid \\
&\le \int_\tau^1 [G(t,s) - L(s)t^{d-1}] \mid f(s,(A^{n-1}x)(s-\tau) - \nu(s-\tau)) \\
&- f(s,(A^{n-1}y)(s-\tau) - \nu(s-\tau)) \mid ds \\
&\le k\lambda_1 \int_\tau^1 [G(t,s) - L(s)t^{d-1}] \mid (A^{n-1}x)(s-\tau) - (A^{n-1}y)(s-\tau) \mid ds \\
&\le k\lambda_1 T(\mid A^{n-1}x - A^{n-1}y \mid)(t) \le \ldots \le k^n\lambda_1^n T^n(\mid x-y \mid)(t).
\end{aligned}
$$

Invoking again Theorem 3.2 and Corollary 3.4, there exists a constant $\delta_2 = \delta_2(\mid x-y \mid) > 0$ such that

$$T(\mid x-y \mid)(t) \le \delta_2\varphi^*(t),\ t \in [0,1].$$

Hence, we obtain

$$
\begin{aligned}
\mid x(t) - y(t) \mid \le k^n\lambda_1^n T^n(\mid x-y \mid)(t) &\le k^n\lambda_1^n T^{n-1}(\delta_2\varphi^*(t)) \\
&= \delta_2 k^n\lambda_1^n \cdot \frac{1}{\lambda_1^{n-1}}\varphi^*(t) = \delta_2\lambda_1 k^n\varphi^*(t), \\
T(\mid x_1 - x_0 \mid)(t) &\le \delta_1\varphi^*(t),\ t \in [0,1],
\end{aligned}
$$

where $\varphi^*(t)$ is the positive eigenfunction of operator T corresponding to the first eigenvalueλ_1. It follows that

$$\| x-y \| \le \delta_2\lambda_1 k^n\|\varphi^*\|.$$

Observing that $k \in [0,1)$, we have

$$\lim_{n\to+\infty} \delta_2\lambda_1 k^n\|\varphi^*\| = 0,$$

so $\| x-y \| \le 0$, and thus $x = y$.

Based on the above analysis, x^* is a unique fixed point of A in P, i.e., x^* is the unique positive solution of IBVP (1.4) and (1.5). ■

4. Conclusions

In this paper, a novel technique of u_0-positive operator is invoked to establish the existence and uniqueness of positive solutions for a class of the singular delay fractional differential equations with integral boundary, that is, IBVPs (1.4) and (1.5), which involves not only the past time delay but also the fractional derivative with the order $2 < d \le 3$. We first get the corresponding Green's function. Consequently, the u_0-positive operator T is derived by the equivalent integral equation of IBVP (1.4) and (1.5). Hence, the sufficient conditions for the existence and uniqueness of positive solutions of the problem is proved by using the fixed point theorem in cone.

On open questions for further research, it would be interesting to see what happen when the equation includes this term $x(t)$ in the function f, i.e., $f(t, x(t), x(t-\tau))$. Another potentially interesting research direction would be to take $d \in (1,2]$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by the University Science Research Key Project of Anhui Province under Grant No. KJ2021A1000 and No. KJ2021A0996, Talent Research Fund of Hefei University under Grant No. 20RC26.

Conflict of interest

The authors declare no conflicts of interest regarding this article.

References

1. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, B. V. Amsterdam, 2006.
2. I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.
3. A. A. Kilbas, O. I. Marichev, S. G. Samko, *Fractional integral and derivatives: Theory and applications*, Gordon and Breach, Yverdon, 1993.
4. R. L. Magin, *Fractional calculus in bioengineering*, Begell House Publishers, Redding, CT, USA, 2006.
5. D. Baleanu, M. Hasanabadi, A. M. Vaziri, A. Jajarmi, A new intervention strategy for an HIV/AIDs transmission by a general fractional modeling and an optimal control approach, *Chaos Soliton. Fract.*, **167** (2023), 113078. https://doi.org/10.1016/j.chaos.2022.113078
6. O. Defterli, D. Baleanu, A. Jajarmi, S. S. Sajjadi, N. Alshaikh, J. H. Asad, Fractional treatment: An accelerated mass-spring system, *Romanian Rep. Phys.*, **74** (2022), 122.
7. A. Akgül, S. A. Khoshnaw, Application of fractional derivative on non-linear biochemical reaction models, *Int. J. Intell. Networks*, **1** (2020), 52–58. https://doi.org/10.1016/j.ijin.2020.05.001
8. A. Turab, Z. D. Mitrović, A. Savić, Existence of solutions for a class of nonlinear boundary value problems on the hexasilinane graph, *Adv. Differ. Equ.*, **2021** (2021), 494. https://doi.org/10.1186/s13662-021-03653-w
9. M. M. Moutamal, C. Joseph, Optimal control of fractional Sturm-Liouville wave equations on a star graph, *Optimization*, 2022. http://dx.doi.org/10.1080/02331934.2022.2088370
10. N. Bouteraa, S. Benaicha, A study of existence and multiplicity of positive solutions for nonlinear fractional differential equations with non-local boundary conditions, *Stud. Univ. Babes-Bol. Math.*, **66** (2021), 361–380. http://dx.doi.org/10.24193/subbmath. 2021.2.1
11. Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, *Appl. Math. Lett.*, **51** (2016), 48–54. https://doi.org/10.1016/j.aml.2015.07.002

12. R. Agarwal, Y. Liu, D. O'Regan, C. Tian, Positive solutions of two-point boundary value problems for fractional singular differential equations, *Differ. Equ.*, **48** (2012), 619–629.
13. I. Bachar, H. Mâagli, F. Toumi, Z. Z. Abidine, Existence and global asymptotic behavior of positive solutions for sublinear and superlinear fractional boundary value problems, *Chinese Ann. Math. B*, **37** (2016), 1–28. https://doi.org/10.1007/s11401-015-0943-3
14. Y. Xu, Fractional boundary value problems with integral and anti-periodic boundary conditions, *B. Malays. Math. Sci. So.*, **39** (2016), 571–587. https://doi.org/10.1007/s40840-015-0126-0
15. H. Feng, C. Zhai, Existence and uniqueness of positive solutions for a class of fractional differential equation with integral boundary conditions, *Nonlinear Anal.-Model.*, **22** (2017), 160–172. https://doi.org/10.15388/NA.2017.2.2
16. Y. Cui, W. Ma, Q. Sun, X. Su, New uniqueness results for boundary value problem of fractional differential equation, *Nonlinear Anal.-Model.*, **23** (2018), 31–39. https://doi.org/10.15388/NA.2018.1.3
17. W. Liu, L. Liu, Existence and stability analysis of solutions for a new kind of boundary value problems of nonlinear fractional differential equations, *Nonlinear Anal.-Model.*, **27** (2022), 1068–1087. https://doi.org/10.15388/namc.2022.27.29420
18. Y. Qiao, Z. Zhou, Existence and uniqueness of positive solutions for a fractional differential equation with integral boundary conditions, *Adv. Differ. Equ.*, **2016** (2016), 106–113. https://doi.org/10.1186/s13662-016-0772-z
19. C. Liao, H. Ye, Existence of positive solutions of nonlinear fractional delay differential equations, *Positivity*, **13** (2009), 601–609. https://doi.org/10.1007/s11117-008-2251-6
20. X. Su, Positive solutions to singular boundary value problems for fractional functional differential equations with changing sign nonlinearity, *Comput. Math. Appl.*, **64** (2012), 3425–3435. https://doi.org/10.1016/j.camwa.2012.02.043
21. Y. Li, S. Sun, D. Yang, Z. Han, Three-point boundary value problems of fractional functional differential equations with delay, *Bound. Value Probl.*, **38** (2013), 1–15. https://doi.org/10.1186/1687-2770-2013-38
22. R. P. Agarwal, S. Hristova, Boundary value problem for multi-term nonlinear delay generalized proportional Caputo fractional differential equations, *Fractal Fract.*, **6** (2022), 691. https://doi.org/10.3390/fractalfract6120691
23. M. A. Krasnosel'skii, *Positive solutions of operator equations*, Noordhoff, Groningen, The Netherlands, 1964.
24. D. Jiang, C. Yuan, The positive properties of the green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its applications, *Nonlinear Anal.-Theor.*, **72** (2010), 710–719. https://doi.org/10.1016/j.na.2009.07.012