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Research article

Region of variablity for Bazilevic functions

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Abstract: Let $\mathcal{H}$ be the family of analytic functions defined in an open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and

$$\mathcal{A} = \left\{f \in \mathcal{H} : f(0) = f^{'}(0) - 1 = 0, \qquad (z \in \mathbb{U})\right\}.$$

For $A \in \mathbb{C}$, $B \in [-1, 0)$ and $\gamma \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, a function $h \in \mathcal{P}_\gamma[\xi, A; B]$ can be written as:

$$h(z) = \cos\gamma \frac{1 + A\omega(z)}{1 + B\omega(z)} + i \sin\gamma, \qquad (\omega(0) = 0, |\omega(z)| < 1, z \in \mathbb{U}),$$

where $\xi = \omega'(0) \in \overline{\mathbb{U}}$. The family $\mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$ contains analytic functions f in $\mathbb{U}$ such that

$$\frac{e^{i\gamma} z f'(z)}{[f(z)]^{1-\beta}[\psi(z)]^\beta} \in \mathcal{P}_\gamma[\xi, A; B],$$

where ψ is a starlike function. In this research, we find the region of variability denoted by $\mathcal{V}_\gamma[\psi, z_0, \xi, A; B]$ for $f(z_0)$, where f is ranging over the family $\mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$ for any fixed $z_0 \in \mathbb{U}$ and $\xi \in \overline{\mathbb{U}}$.

Keywords: spirallike and Bazilevič functions; Schwarz lemma
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1. Introduction and definitions

For a fixed point ξ in $\mathbb{U}$, the set of values of $\log\left(\frac{f(\xi)}{\xi}\right)$ as f ranging on the family of injective or univalent mappings is always a closed disk. This useful and important fact was proved by Grunsky [6]. The region of variability for various subfamilies of univalent functions became a fascinating area of the current research. Several authors have studied about such regions for certain subfamilies of analytic functions. These regions for the functions of bounded derivative were discussed by Yanagihara in 2005 as seen in [21]. Functions in such a subfamily satisfy the conditions $|f'(z)| \leq 1$ and $\mathrm{Re}(f'(z)) > 0$. Later on, Ponnusamy discussed the region of variability for the Kaplan family of functions $\mathcal{K}$ as described in [9]. Moreover, Yanagihara found the range of values for a subfamily of convex function, as seen in [23]. Ponnusamy discussed these problems for subfamilies of $\mathcal{S}^*$ and $\mathcal{K}$, as found in [10]. In 2008, see [15], Ponnusamy again considered these aspects for the spirallike functions. Vasudevarao discussed similar results when f is ranging over the functions with positive real parts, for detail see [11].

Most of the authors have introduced new subfamilies and studied the region of variability for these subfamilies of analytic and univalent functions. In 2010, Chen and Aiwu discussed these problems for functions with bounded Mocanu variations as seen in [2]. In 2011, Ponnusamy et al. [12] investigated these regions for the families of exponentially convex functions.

More useful and interesting results on these regions are also discussed by many authors. In 2014, Sunil Varma et al. [18], also worked out these issues for the related subfamilies along with Bappaditya who utilized the idea of subordination in his work as found in [1]. Also some related findings are seen in [10, 14, 16, 19, 20].

Let $\mathcal{H}$ be the family of analytic functions defined in $\mathbb{U} = \{z : |z| < 1\}$ the open unit disk with the center at origin O included in the z-plane and $\mathcal{A} \subset \mathcal{H}$. Then any function $f \in \mathcal{A}$ takes the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathbb{U}. \tag{1.1}$$

From (1.1), we note that $f(0) = f'(0) - 1 = 0$. The subfamily of univalent or injective mappings is represented by $\mathcal{S}$ and obviously $\mathcal{S} \subset \mathcal{A}$. A holomorphic or analytic mapping is called univalent as it never takes the same value twice. For example, in the unit disk $\mathbb{U}$, $f(z) = 1 + 2z + z^2 \in \mathcal{S}$. Formally, we have:

A mapping $f \in \mathcal{A}$ is univalent or injective in $\mathbb{U}$, if from $f(z_1) = f(z_2)$, we have $z_1 = z_2$. On the other hand, for a mapping $f \in \mathcal{S}$ in $\mathbb{U}$, from $z_1 \neq z_2$ we have $f(z_1) \neq f(z_2)$. The Koebe function $k(z)$ as defined by

$$k(z) = z/(1-z)^2,\ z \in \mathbb{U}, \tag{1.2}$$

and the mapping $\mathcal{M}(z) = \dfrac{\eta z + \beta}{\nu z + \mu}, \eta, \nu, \beta, \mu \in \mathbb{C} : \nu\beta - \eta\mu \neq 0$, are univalent. The family $\mathcal{S}$ is preserved under basic transformations, for reference, see [5]. Furthermore, from the condition $f'(z_0) \neq 0$, we have local univalence of f at z_0.

Let $\mathcal{P}$ be the family of holomorphic mappings $p : p(\mathbb{U})$ is the right half ω-plane having series form

$$p(z) = 1 + \sum_{j=1}^{\infty} \sigma_j z^j, z \in \mathbb{U}.$$

Obviously $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. This and other related families have a significant role in the recent development of the subject. Many subfamilies of $\mathcal{S}$ are connected with the class $\mathcal{P}$. For $p \in \mathcal{P}$, it may not be necessary that $p \in \mathcal{S}$. For example $p(z) = 1 + z^j \in \mathcal{P}$, but $p(z) \notin \mathcal{S}$ for $j \geq 2$. Obviously, the mapping $L_0(z)$ so that

$$L_0(z) = \frac{1+z}{1-z} \in \mathcal{P}.$$

We take $\mathcal{B}$ as a family of analytic function $\omega : \mathbb{U} \longrightarrow \mathbb{U}$ with $|\omega(z)| < 1$ and $\omega(0) = 0$. A function f is subordinate to a function F and we write $f \prec F$, if there exists $\omega \in \mathcal{B}$ such that $f(z) = F(\omega(z))$. Particularly, if $F \in \mathcal{S}$, then $f \prec F$ can be equivalently reformulated as $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$. For $A \in \mathbb{C}$, $B \in [-1, 0)$, the function $p \in \mathcal{P}_\gamma[A; B]$ can be written as:

$$p(z) = \cos\gamma \frac{1 + A\omega(z)}{1 + B\omega(z)} + i \sin\gamma \qquad (z \in \mathbb{U}), \tag{1.3}$$

where $\omega \in \mathcal{B}$. Moreover, we can defined the family $\mathcal{P}_\gamma[\xi, A; B]$ of analytic functions as:

$$\mathcal{P}_\gamma[\xi, A; B] = \left\{ p \in \mathcal{P}_\gamma[A; B] : p'(0) = (A - B)\xi \cos\gamma \right\}, \tag{1.4}$$

where $p \in \mathcal{P}_\gamma[A, B]$ defined above by (1.3), $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), A \in \mathbb{C}, B \in [-1, 0), \xi = \omega'(0) \in \overline{\mathbb{U}}$ and $z \in \mathbb{U}$.

The family $\mathcal{S}^*$ contains starlike functions f such that $f(\mathbb{U})$ is starlike about O in the w-plane. This family has been extensively studied in the literature, as seen in [5].

Spacek [17] extended the family $\mathcal{S}^*$ by using the logarithmic spirals besides the line segments. Let $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The curve $\vartheta_\gamma : \mathbb{R} \to \mathbb{C} : \vartheta_\gamma(t) = te^{i\gamma}, t \in \mathbb{R}$ and its rotation $e^{i\theta}\vartheta_\gamma(t), \theta \in \mathbb{R}$ are called γ-spirals. A domain $\mathbb{D} \subset \mathbb{C}$ is known as γ-spirallike about the origin, if the spiral has initial point at the origin and terminal of the spiral is any other point of $\mathbb{D}$. A function $f \in \mathcal{A}$ is spirallike, if $f(\mathbb{U})$ is spirallike about the origin. The family of spirallike functions represented by $\mathcal{S}_\gamma$ is defined by

$$\mathcal{S}_\gamma = \left\{ f \in \mathcal{H} : \operatorname{Re}\left(e^{i\gamma} \frac{zf'(z)}{f(z)} \right) > 0, \ \ z \in \mathbb{U} \right\}.$$

For details, we refer [5]. A normalized analytic or holomorphic function $f \in C_\gamma$ iff $zf' \in \mathcal{S}_\gamma^*$. For details see [16]. For β with $\operatorname{Re}\beta > 0$, let $\mathcal{B}(\psi, \beta)$ be the family of functions f so that

$$\operatorname{Re}\left(\frac{zf'(z)}{[f(z)]^{1-\beta} [\psi(z)]^\beta} \right) > 0, \psi \in \mathcal{S}^* \text{ and } z \in \mathbb{U}.$$

These Bazilevic functions are obviously univalent in $\mathbb{U}$. Let $\mathcal{B}_\gamma(\psi, \beta)$ be the family of holomorphic functions f in $\mathbb{U}$ such that

$$\operatorname{Re}\left\{ e^{i\gamma} \left(\frac{zf'(z)}{[f(z)]^{1-\beta} [\psi(z)]^\beta} \right) \right\} > 0, \ \psi \in \mathcal{S}^* \text{ and } z \in \mathbb{U}.$$

These functions are called spirallike Bazilevic functions. Let $A \in \mathbb{C}$, $B \in [-1, 0)$, $\beta > 0$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f \in \mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$ if

$$\frac{e^{i\gamma} zf'(z)}{[f(z)]^{1-\beta} [\psi(z)]^\beta} \prec \cos\gamma \frac{1 + Az}{1 + Bz} + i \sin\gamma,$$

where ψ is a starlike function.

In view of Herglotz form of Janowski functions, we write $f \in \mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$ as

$$\frac{e^{i\gamma} z f'(z)}{[f(z)]^{1-\beta} [\psi(z)]^{\beta}} = \cos\gamma \int_{-\pi}^{\pi} \left(\frac{1 + A\omega(z)e^{-it}}{1 + B\omega(z)e^{-it}} \right) d\mu(t) + i \sin\gamma$$

or we note that

$$\frac{z f'(z)}{[f(z)]^{1-\beta} [\psi(z)]^{\beta}} = e^{-i\gamma} \left\{ \cos\gamma \int_{-\pi}^{\pi} \left(\frac{A\omega(z)e^{-it} + 1}{B\omega(z)e^{-it} + 1} \right) d\mu(t) + i \sin\gamma \right\},$$

which can further take the form

$$\frac{z f'(z)}{[f(z)]^{1-\beta}} = \frac{1}{z} [\psi(z)]^{\beta} e^{-i\gamma} \left\{ \cos\gamma \int_{-\pi}^{\pi} \left(\frac{A\omega(z)e^{-it} + 1}{B\omega(z)e^{-it} + 1} \right) d\mu(t) + i \sin\gamma \right\}.$$

Integrating on both sides to have

$$\frac{[f(z)]^{\beta}}{\beta} = \int_{0}^{z} \left[\frac{1}{z} [\psi(z)]^{\beta} e^{-i\gamma} \left\{ \cos\gamma \int_{-\pi}^{\pi} \left(\frac{A\omega(z)e^{-it} + 1}{B\omega(z)e^{-it} + 1} \right) d\mu(t) + i \sin\gamma \right\} \right] dz$$

which leads to

$$f(z) = \left(\beta \int_{0}^{z} \left[\frac{1}{z} [\psi(z)]^{\beta} e^{-i\gamma} \left\{ \cos\gamma \int_{-\pi}^{\pi} \left(\frac{A\omega(z)e^{-it} + 1}{B\omega(z)e^{-it} + 1} \right) d\mu(t) + i \sin\gamma \right\} \right] dz \right)^{\frac{1}{\beta}}. \tag{1.5}$$

Suppose that

$$p(z) = e^{i\gamma} \left(\frac{z f'(z)}{[f(z)]^{1-\beta} [\psi(z)]^{\beta}} \right), \tag{1.6}$$

where

$$p \in \mathcal{P}_\gamma[A; B] : p'(0) = (A - B)\xi \cos\gamma, \xi \in \overline{\mathbb{U}},$$

then for $\omega \in \mathcal{B}$, we see that

$$p(z) = \cos\gamma \left(\frac{A\omega(z) + 1}{B\omega(z) + 1} \right) + i \sin\gamma = \frac{\cos\gamma\,(1 + A\omega(z)) + i \sin\gamma (B\omega(z) + 1)}{1 + B\omega(z)}.$$

We can also write

$$p(z) = \frac{(A\cos\gamma + iB\sin\gamma)\,\omega(z) + e^{i\gamma}}{1 + B\omega(z)} \tag{1.7}$$

which leads to

$$p(z)(1 + B\omega(z)) = e^{i\gamma} + \omega(z)(A\cos\gamma + iB\sin\gamma)$$

or we see that

$$Bp(z)\,\omega(z) - (A\cos\gamma + iB\sin\gamma)\,\omega(z) = -p(z) + e^{i\gamma}$$

which on simplifications yields

$$\omega(z) = \omega_p(z) = \frac{e^{i\gamma} - p(z)}{Bp(z) - (A\cos\gamma + iB\sin\gamma)}. \tag{1.8}$$

In view of (1.6), on differentiating (1.7), we note that

$$p'(z) = \frac{(1 + B\omega(z))(A\cos\gamma + iB\sin\gamma)\omega'(z) - \left((A\cos\gamma + iB\sin\gamma)\omega(z) + e^{i\gamma}\right)}{(1 + B\omega(z))^2}$$

which on simplifications proves that

$$p'(z) = (A - B)\left(\frac{\omega'(z)}{(1 + B\omega(z))^2}\right)\cos\gamma, \tag{1.9}$$

Then by using classical Schwarz lemma [5] and a related result as seen in [4], we note that $|\omega'(0)| \leq 1$. So we obtain

$$p'(0) = (A - B)\xi\cos\gamma, \xi = \omega_p'(0) \in \overline{\mathbb{U}}.$$

Again on differentiating (1.9), we can write

$$p''(z) = (A - B)\left(\frac{\left[1 + B\omega_p(z)\right]\omega_p''(z) - 2\left(\omega_p'(z)\right)^2}{\left[1 + B\omega_p(z)\right]^3}\right)\cos\gamma$$

where

$$p''(0) = (A - B)\left(\omega_p''(0) - 2\left[\omega_p'(0)\right]^2\right)\cos\gamma$$

and

$$\frac{p''(0)}{(A - B)\cos\gamma} = \omega_p''(0) - 2\left(\frac{\omega_p''(0)}{(A - B)\cos\gamma}\right)^2.$$

Also we see that

$$\omega_p''(0) = \frac{p''(0)}{(A - B)\cos\gamma} + 2\xi^2. \tag{1.10}$$

Now, we let

$$g(z) = \begin{cases} \frac{\omega_p(z) - \xi z}{z - \bar{\xi}\omega_p(z)}, & |\xi| < 1 \\ 0, & |\xi| = 1 \end{cases}.$$

This shows that

$$g'(0) = \begin{cases} \frac{1}{1 - |\xi|^2}\frac{\omega_p''(0)}{2}, & |\xi| < 1 \\ 0, & |\xi| = 1 \end{cases}. \tag{1.11}$$

Using Schwarz lemma for $|\xi| < 1$, we write $|g(z)| < |z|$ and $|g'(0)| < 1$. For equality, we see that $g(z) = e^{i\varepsilon}z$, $\varepsilon \in \mathbb{R}$. Now $|g'(0)| < 1$ shows that there is $l \in \overline{\mathbb{U}} : g'(0) = l$. Thus by using (1.10) and (1.11), we find that

$$l = \frac{1}{1 - |\xi|^2}\left(\frac{p''(0)}{2(A - B)\cos\gamma} + \xi^2\right)$$

which shows that

$$p''(0) = 2\left(l\left(1-|\xi|^2\right)-\xi^2\right)(A-B)\cos\gamma.$$

It follows from (1.5) that for each fixed $z_0 \in \mathbb{U}$, the region of variability $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ is a set defined as:

$$\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B] = \left\{f(z_0) : f \in \mathcal{B}_\gamma\left[\psi, \xi, \beta, A; B\right]\right\}, \tag{1.12}$$

when f ranges over the family $\mathcal{B}_\gamma\left[\psi, \xi, \beta, A; B\right]$.

The region of variability problems provide accurate information about the family $\mathcal{S}$, than theorems about bounds on functions, their derivatives and rotation theorems. It typically refers to a range or interval within which a certain variable can vary or fluctuate. It is commonly used in research to describe the extent variation of region in a particular set. It is important to note that the region of variability can vary depending on the context and the specific variable being analyzed. Different methods and techniques can be used to determine and characterize the region of variability for different types of sets. Here, we find the region of variability $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ for $f(z_0) = \left(\beta \int_0^{z_0} q(t)dt\right)^{\frac{1}{\beta}}$, where f ranges over the family $\mathcal{B}_\gamma\left[\psi, \xi, \beta, A; B\right]$ is defined above by (1.7) and

$$q(z) = \frac{1+(A\cos\gamma + iB\sin\gamma)\,\xi z e^{-i\gamma}}{z\,(1+B\xi z)}\left[\psi\,(z)\right]^\beta,$$

where q is described as a part of (1.5). As special cases, region of variability $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ can also described for different choices of parameters. As described above, Mohsan et al. [16] found $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ for a certain related subfamily of holomorphic functions. Ponnusamy et al. [8, 9] studied these regions for $f \in \mathcal{S}$. Yanagihara [22] determined such region for the family of convex functions. For work on such regions, see [10, 11, 13–16, 19, 21] and others.

For any $p \in \mathbb{N}$, we let $\mathcal{S}_p^* = \{f\,(z) = [f_0\,(z)]^p : f_0 \in \mathcal{S}^*\}$. Also $f \in C_p \Longleftrightarrow zf' \in \mathcal{S}_p^*$.

Lemma 1.1. *Let $f : f(z) = z^p + ... \in \mathcal{A}_p \subset \mathcal{H}, \mathcal{A}_1 = \mathcal{A}$. Then $f \in \mathcal{S}_p^*$ iff*

$$\operatorname{Re}\left(\frac{zf'(z)}{pf(z)}\right) > 0, \quad z \in \mathbb{U}. \tag{1.13}$$

In the case that $p = 1$, f is starlike univalent. We refer to [21, 22] for the Lemma 1.2 given subsequently.

Lemma 1.2. *For $f : f(z) = z^p + ...$ such that $\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > -1$, we have $f \in \mathcal{S}_p^*$.*

2. Main results

We initiate our investigations by studying certain properties of the family $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ such as compactness and convexity.

Theorem 2.1. *(i) $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ is compact in $\mathbb{C}$. (ii) $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ is convex in $\mathbb{C}$. (iii) If $|\xi| = 1$ or $z_0 = 0$, then*

$$\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B] = \left\{\left(\beta \int_0^{z_0} \frac{1+(A\cos\gamma + iB\sin\gamma)\,\xi t e^{-i\gamma}}{t\,(1+B\xi t)}\left[\psi\,(t)\right]^\beta dt\right)^{\frac{1}{\beta}}\right\}. \tag{2.1}$$

(*iv*) *If* $|\xi| < 1$ *and* $z_0 \neq 0$, *then*

$$\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B] = \left\{ \left(\beta \int_0^{z_0} \frac{1 + (A\cos\gamma + iB\sin\gamma)\,\xi t e^{-i\gamma}}{t\,(1 + B\xi t)} \left[\psi\,(t)\right]^\beta dt \right)^{\frac{1}{\beta}} \right\}$$

has an interior point.

Proof. (*i*) Since $\mathcal{P}_\gamma[\xi, A; B]$ is a compact set of $\mathbb{C}$. This shows that $\mathcal{B}_\gamma\,[\psi, \xi, \beta, A; B]$ is compact, which leads to the compactness of the set $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$. (*ii*) Now, we show that $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ is convex in $\mathbb{C}$. For this end in view, we take $q_1, q_2 \in \mathcal{P}_\gamma[\xi, A; B]$, $t \in [0, 1]$ and note that

$$tq_1\,(z) + (1 - t)q_2\,(z) \in \mathcal{P}_\gamma[\xi, A; B].$$

Hence $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ is convex. (*iii*) For $|\xi| = \left|\omega_f'(0)\right| = 1$, using Schwarz lemma [3], we have $\omega_f(z) = \xi z$. This leads to the function

$$\left(\beta \int_0^{z} \left[\frac{\{1 + (A\cos\gamma + iB\sin\gamma)\}\,u e^{-i\gamma}}{u\,(1 + Bu)} \right] [\psi(u)]^\beta\, du \right)^{\frac{1}{\beta}},$$

which proves (2.1). Since $z_0 = 0$, so $\mathcal{V}_\gamma[\psi, 0, \xi, \beta, A; B] = \{0\}$ is trivially satisfied. (*iv*) For $|\xi| < 1$, $\xi \in \mathbb{U}$ and $\gamma \in \overline{\mathbb{U}}$, we can write

$$q_\xi\,(z) = \frac{z + \xi}{1 + \bar{\xi} z},$$

and also we note that

$$\Psi_{l,\xi}(z) = \left(\beta \int_0^{z} \left[\frac{\left\{1 + \bar{\xi} l u + (A\cos\gamma + iB\sin\gamma)\,(lu + \xi)\right\} u e^{-i\gamma}}{u\left(1 + \bar{\xi} l u + B\,(lu + \xi)\,u\right)} \right] [\psi(u)]^\beta\, du \right)^{\frac{1}{\beta}}, \tag{2.2}$$

as seen by the Eq (1.5) is in the family $\mathcal{B}_\gamma\,[\psi, \xi, \beta, A; B]$. Next, we claim that $\Psi_{l,\xi}(z)$ is a nonconstant analytic function of l for each fixed $z_0 \in \mathbb{U}\backslash\{0\}$ and $\xi \in \mathbb{U}$. Put

$$h(z) = \frac{1}{\left(\phi_\gamma\,(A; B)\,e^{-i\gamma} - B\right)(1 - \xi^2)} \frac{\partial}{\partial l} \left(\frac{1}{\beta} \left(\Psi_{l,\xi}(z)\right)^\beta \right) = \int_0^{z} \frac{u\,[\psi(u)]^\beta}{(1 + Bu\xi)^2} du = z^{\beta+2} + ...,$$

where $\phi_\gamma\,(A; B) = A\cos\gamma + iB\sin\gamma$. It is easy to see that

$$1 + \frac{zh''(z)}{h'(z)} = 2 - \frac{2B\xi}{1 + Bz\xi} + \beta \frac{z\psi'(z)}{\psi(z)} = \frac{2}{1 + Bz\xi} + \beta \frac{z\psi'(z)}{\psi(z)},$$

and since ψ is starlike, it follows that

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > 0.$$

By Lemma 1.2, there exists a function $h_0 \in \mathcal{S}^*$ with $h = h_0^{\beta+2}$. The univalence of h_0 and $h_0(0) = 0$ implies that $h_0(z_0) \neq 0$ for all $z_0 \in \mathbb{U}\backslash\{0\}$. Consequently, the mapping $\mathbb{U} \ni l \to \Psi_{l,\xi}(z_0)$ is a nonconstant function and hence it is an open mapping. Thus, $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ contains the open set $\left\{\Psi_{l,\xi}(z) : |\xi| < 1\right\}$. For $l = 0$, we get

$$\Psi_{0,\xi}(z) = \left(\beta \int_0^z \left[\frac{1 + (A\cos\gamma + iB\sin\gamma)\,\xi u e^{-i\gamma}}{u\,(1 + Bu)}\right] [\psi(u)]^\beta \,du\right)^{\frac{1}{\beta}}.$$

$\Psi_{0,\xi}(z_0)$ is an interior point $\left\{\Psi_{l,\xi}(z) : l \in \mathbb{U}\right\} \subset \mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$. Therefore, $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ is a simple closed domain bounded by a simple closed curve $\partial\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$. □

We now prove that for $f \in \mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$, the $[f(z)]^\beta$ is contained in some closed disk with center $\chi(\xi, \gamma, z)$ and radius R_ξ^γ.

Theorem 2.2. *If $f \in \mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$, then*

$$\left|[f(z)]^\beta - \chi(\xi, \gamma, z)\right| \le R_\xi^\gamma, \tag{2.3}$$

$$\chi(\xi, \gamma, z) = \int_0^1 C(\phi(z,\xi), \phi_1(z,\xi))\, |\psi(z(t))|^\beta\, |z'(t)|\, dt,$$

and

$$R_\xi^\gamma = \int_0^1 R(\phi(z,\xi), \phi_1(z,\xi))\, |\psi(z(t))|^\beta\, |z'(t)|\, dt,$$

where

$$C(\phi(z,\xi), \phi_1(z,\xi)) = \frac{\phi(z,\xi) + |z|^2 \left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|^2 \phi_1(z,\xi)}{\left(1 - |z|^2 \left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|^2\right)},$$

$$R(\phi(z,\xi), \phi_1(z,\xi)) = \frac{\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right| |\phi(z,\xi) + \phi_1(z,\xi)|}{\left(1 - |z|^2 \left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|^2\right)},$$

and

$$\phi(z,\xi) = \frac{1 + \xi z e^{-i\gamma}(A\cos\gamma + iB\sin\gamma)}{z(1 + B\xi z)\, e^{i\gamma}}, \phi_1(z,\xi) = \frac{\overline{\xi} + z e^{-i\gamma}(A\cos\gamma + iB\sin\gamma)}{-z\left(zB + \overline{\xi}\right) e^{i\gamma}}.$$

Proof. Since $f \in \mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$. Then by using Schwarz lemma [3] for $\omega_p \in \mathcal{B}$ with $\omega_p'(0) = \xi$, we have

$$\left|\frac{\frac{\omega_p(z)}{z} - \xi}{1 - \overline{\xi}\frac{\omega_p(z)}{z}}\right| \le |z|. \tag{2.4}$$

Now, by substituting $\omega_p(z)$ from (1.8) and after calculation we get

$$\left|\frac{p(z)-\frac{\left(1+\xi z e^{-i\gamma}(A\cos\gamma+iB\sin\gamma)\right)}{1+B\xi z}}{p(z)+\frac{\overline{\xi}+ze^{-i\gamma}(A\cos\gamma+iB\sin\gamma)}{-\left(zB+\overline{\xi}\right)}}\right|\leq|z|\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|.$$

On substitution of $p(z)$ from (1.6), we write

$$\left|\frac{\left[\frac{f(z)}{\psi(z)}\right]^{\beta}\frac{zf'(z)}{f(z)}-\frac{\left(1+\xi z e^{-i\gamma}(A\cos\gamma+iB\sin\gamma)\right)e^{-i\gamma}}{(1+B\xi z)}}{\left[\frac{f(z)}{\psi(z)}\right]^{\beta}\frac{zf'(z)}{f(z)}+\frac{\left(\overline{\xi}+ze^{-i\gamma}(A\cos\gamma+iB\sin\gamma)\right)e^{-i\gamma}}{-\left(zB+\overline{\xi}\right)}}\right|\leq|z|\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|,$$

or we obtain

$$\left|\frac{\left[\frac{f(z)}{\psi(z)}\right]^{\beta}\frac{f'(z)}{f(z)}-\frac{\left(1+\xi z e^{-i\gamma}(A\cos\gamma+iB\sin\gamma)\right)e^{-i\gamma}}{(1+B\xi z)z}}{\left[\frac{f(z)}{\psi(z)}\right]^{\beta}\frac{f'(z)}{f(z)}+\frac{\left(\overline{\xi}+ze^{-i\gamma}(A\cos\gamma+iB\sin\gamma)\right)e^{-i\gamma}}{-z\left(zB+\overline{\xi}\right)}}\right|\leq|z|\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|.$$

Now, by letting

$$\phi(z,\xi)=\frac{1+\xi z e^{-i\gamma}(A\cos\gamma+iB\sin\gamma)}{z(1+B\xi z)}, \tag{2.5}$$

and

$$\phi_1(z,\xi)=\frac{\overline{\xi}+ze^{-i\gamma}(A\cos\gamma+iB\sin\gamma)}{-z\left(zB+\overline{\xi}\right)}, \tag{2.6}$$

we get

$$\left|\frac{\left[\frac{f(z)}{\psi(z)}\right]^{\beta}\frac{f'(z)}{f(z)}-\phi(z,\xi)}{\left[\frac{f(z)}{\psi(z)}\right]^{\beta}\frac{f'(z)}{f(z)}+\phi_1(z,\xi)}\right|\leq|z|\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|. \tag{2.7}$$

After simplifying (2.7), we get

$$\left|\frac{f'(z)}{[f(z)]^{1-\beta}}-\frac{\left(\phi(z,\xi)+|z|^2\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|^2\phi_1(z,\xi)\right)[\psi(z)]^{\beta}}{1-|z|^2\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|^2}\right|$$
$$\leq\frac{|z|\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|\left|\phi(z,\xi)+\phi_1(z,\xi)\right|\left|[\psi(z)]^{\beta}\right|}{1-|z|^2\left|\frac{zB+\overline{\xi}}{1+B\xi z}\right|^2}. \tag{2.8}$$

Thus, for a parametrized C^1-curve γ defined by $z=z(t),\ 0\leq t\leq 1\ :\ z(0)=0$ and $z(1)=z_0$, we may write

$$\left|[f(z)]^{\beta}-\beta\int_0^1 C\left(\phi(z,\xi),\phi_1(z,\xi)\right)|\psi(z(t))|^{\beta}\left|z'(t)\right|dt\right|$$
$$\leq\beta\int_0^1 R\left(\phi(z,\xi),\phi_1(z,\xi)\right)|\psi(z(t))|^{\beta}\left|z'(t)\right|dt.$$

On some calculations and simplification, we see that

$$
\begin{aligned}
&C\left(\phi\left(z,\xi\right),\phi_1\left(z,\xi\right)\right)\\
&=\frac{\phi\left(z,\xi\right)+|z|^2\left|\frac{zB+\bar{\xi}}{1+B\xi z}\right|^2\phi_1\left(z,\xi\right)}{1-|z|^2\left|\frac{zB+\bar{\xi}}{1+B\xi z}\right|^2}\\
&=\frac{(1+B\xi z)\left(1+\xi z e^{-i\gamma}\phi_\gamma\left(A;B\right)\right)e^{-i\gamma}-\left(|z|^2|\xi|^2(1-Bz)+|z|^2 z e^{-i\gamma}\phi_\gamma\left(A;B\right)\left(\bar{\xi}+B\xi z\right)\right)}{\left(1-|z|^2|\xi|^2-B^2|z|^2\left(|z|^2-|\xi|^2\right)+2B\left(1-|z|^2\right)\operatorname{Re}\xi z\right)z},
\end{aligned}
$$

and

$$
\begin{aligned}
&R\left(\phi\left(z,\xi\right),\phi_1\left(z,\xi\right)\right)\\
&=\frac{\left|\frac{zB+\bar{\xi}}{1+B\xi z}\right|\left|\phi\left(z,\xi\right)+\phi_1\left(z,\xi\right)\right|}{1-|z|^2\left|\frac{zB+\bar{\xi}}{1+B\xi z}\right|^2}\\
&=\frac{\left|zB+\bar{\xi}\right|^2\left|\left(1-|\xi|^2B\xi z+\left(|\xi|^2-1\right)z e^{-i\gamma}\phi_\gamma\left(A;B\right)+\xi z^2 B e^{-i\gamma}\phi_\gamma\left(A;B\right)(\xi-1)\right)e^{-i\gamma}\right|}{1-|z|^2|\xi|^2-B^2|z|^2\left(|z|^2-|\xi|^2\right)+2B\left(1-|z|^2\right)\operatorname{Re}\xi z},
\end{aligned}
$$

where $\phi_\gamma\left(A;B\right)=A\cos\gamma+iB\sin\gamma$. The relation (2.3) occurs from (2.4) and the above relations. Equality is attained in (2.3) when $f\left(z\right)=\Psi_{e^{i\theta},\xi}\left(z\right)$, for some $z\in\mathbb{U}$. Conversely, if equality occurs in (2.3) for some $z\in\mathbb{U}\backslash\{0\}$, then equality must hold in (2.4). Thus, by applying Schwarz lemma, for $\theta\in\mathbb{R}$, we write $\omega_p(z)=z\delta(e^{i\theta}z,\xi)$, $z\in\mathbb{U}$. This shows $f\left(z\right)=\Psi_{e^{i\theta},\xi}\left(z\right)$. □

The choice of $\xi=0$ in Theorem 2.2, leads to the following corollary:

Corollary 2.1. *If* $f\in\mathcal{B}_\gamma\left[\psi,0,\beta,A;B\right]$, *then*

$$\left|[f(z)]^\beta-\chi(0,\gamma,z)\right|\le R_0^\gamma,$$

$$\chi(0,\gamma,z)=\int_0^1 C(\phi\left(z,0\right),\phi_1\left(z,0\right))\left|\psi(z(t))\right|^\beta\left|z'\left(t\right)\right|dt,$$

and

$$R_\xi^\gamma=\int_0^1 R(\phi\left(z,0\right),\phi_1\left(z,0\right))\left|\psi(z(t))\right|^\beta\left|z'\left(t\right)\right|dt,$$

where

$$C(\phi\left(z,0\right),\phi_1\left(z,0\right))=\frac{\phi\left(z,0\right)+|z|^2|zB|^2\phi_1\left(z,\xi\right)}{\left(1-|z|^2|zB|^2\right)},$$

$$R(\phi\left(z,0\right),\phi_1\left(z,0\right))=\frac{|zB|\left|\phi\left(z,0\right)+\phi_1\left(z,0\right)\right|}{\left(1-|z|^2|zB|^2\right)},$$

and

$$\phi\left(z,\xi\right)=\frac{1}{z},\phi_1\left(z,\xi\right)=\frac{e^{-i\gamma}\left(A\cos\gamma+iB\sin\gamma\right)}{-zB}.$$

If we put $\beta = 1$ and $\psi = z\phi'$, where $\phi \in C$, the class of convex function in Theorem 2.2, we get the following result:

Corollary 2.2. *If $f \in \mathcal{B}_\gamma [\psi, \xi, 1, A; B]$, then*

$$|f(z) - \chi(\xi, \gamma, z)| \leq R_\xi^\gamma,$$

$$\chi(\xi, \gamma, z) = \int_0^1 C(\phi(z, \xi), \phi_1(z, \xi)) \left|(z(t)\phi'(z(t)))'\right| |z'(t)| \, dt,$$

and

$$R_\xi^\gamma = \int_0^1 R(\phi(z, \xi), \phi_1(z, \xi)) \left|(z(t)\phi'(z(t)))'\right| |z'(t)| \, dt,$$

where

$$C(\phi(z, \xi), \phi_1(z, \xi)) = \frac{\phi(z, \xi) + |z|^2 \left|\frac{zB+\bar{\xi}}{1+B\xi z}\right|^2 \phi_1(z, \xi)}{\left(1 - |z|^2 \left|\frac{zB+\bar{\xi}}{1+B\xi z}\right|^2\right)},$$

$$R(\phi(z, \xi), \phi_1(z, \xi)) = \frac{\left|\frac{zB+\bar{\xi}}{1+B\xi z}\right| |\phi(z, \xi) + \phi_1(z, \xi)|}{\left(1 - |z|^2 \left|\frac{zB+\bar{\xi}}{1+B\xi z}\right|^2\right)},$$

and

$$\phi(z, \xi) = \frac{1 + \xi z e^{-i\gamma}(A\cos\gamma + iB\sin\gamma)}{(1 + B\xi z)}, \phi_1(z, \xi) = \frac{\bar{\xi} + z e^{-i\gamma}(A\cos\gamma + iB\sin\gamma)}{-\left(zB + \bar{\xi}\right)}.$$

We need the following lemma which ensures the existence of a normalized starlike function, useful in the proof of next result.

Lemma 2.1. *Let θ be a real numbers and z belong to unit disk $\mathbb{U}$. Then*

$$H(z) = \int_0^z \frac{e^{i\theta}\varepsilon^2}{\left(1 + \left(\bar{\xi}e^{i\theta} + B\xi\right)\varepsilon + Be^{i\theta}\varepsilon^2\right)^2} d\varepsilon, |\xi| < 1,$$

where $H(0) = H'(0) = 0$ and $H(z) \neq 0$ elsewhere in $\mathbb{U}$. Moreover, there exists a starlike normalized univalent function $H_0 \in \mathcal{S}^*$ in $\mathbb{U}$: $H(z) = \frac{1}{2}e^{i\theta}H_0^2(z)$.

This lemma is proved by Ponnusamy et al. as found in [8]. In the theorem below, we show that $\Psi_{e^{i\theta},\xi}(z_0)$ lies on the boundary of $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$.

Theorem 2.3. *Let $z_0 \in \mathbb{U}\backslash\{0\}$ and $\Psi_{e^{i\theta},\xi}(z)$ is given by (2.2). Then for $\theta \in (-\pi, \pi]$, we have $\Psi_{e^{i\theta},\xi}(z_0) \in \partial\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$. Furthermore, if $\frac{f'(z)}{[f(z)]^{1-\beta}} = \frac{\Psi'_{e^{i\theta},\xi}(z)}{\left(\Psi_{e^{i\theta},\xi}(z)\right)^{1-\beta}}$ for $f \in \mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$, then $f(z) = \Psi_{e^{i\theta},\xi}(z)$.*

Proof. Now (2.2) gives that

$$\Psi_{l,\xi}(z)=\left(\beta\int_0^z\left[\frac{\left\{1+\phi_\gamma(A;B)\,q_\xi(lu)\,ue^{-i\gamma}\right\}}{\left(1+Bq_\xi(lu)\,u\right)}\right]\frac{[\psi(u)]^\beta}{u}du\right)^{\frac{1}{\beta}},$$

or

$$\frac{\left(\Psi_{l,\xi}(z)\right)^\beta}{\beta}=\int_0^z\left[\frac{\left\{1+\phi_\gamma(A;B)\,q_\xi(lu)\,ue^{-i\gamma}\right\}}{\left(1+Bq_\xi(lu)\,u\right)}\right]\frac{[\psi(u)]^\beta}{u}du. \tag{2.9}$$

On differentiating (2.9), we get

$$\frac{\Psi'_{l,\xi}(z)\left(\Psi_{l,\xi}(z)\right)^{\beta-1}}{[\psi(z)]^\beta}=\frac{\left\{1+\phi_\gamma(A;B)\,q_\xi(lz)\,ze^{-i\gamma}\right\}}{z\left(1+Bq_\xi(lz)\,z\right)}.$$

From (2.5), it follows that

$$\begin{aligned}
&\frac{\Psi'_{l,\xi}(z)\left(\Psi_{l,\xi}(z)\right)^{\beta-1}}{[\psi(z)]^\beta}-\phi(z,\xi)\\
&=\frac{\left\{1+\phi_\gamma(A;B)\,q_\xi(lz)\,ze^{-i\gamma}\right\}}{\left(1+Bq_\xi(lz)\,z\right)z}-\frac{\left(1+\xi ze^{-i\gamma}\phi_\gamma(A;B)\right)}{(1+B\xi z)\,z}\\
&=\frac{\left\{1+\phi_\gamma(A;B)\left(\frac{lz+\xi}{1+\bar{\xi}lz}\right)ze^{-i\gamma}\right\}}{\left(1+B\left(\frac{lz+\xi}{1+\bar{\xi}lz}\right)z\right)z}-\frac{\left(1+\xi ze^{-i\gamma}\phi_\gamma(A;B)\right)}{(1+B\xi z)\,z}\\
&=\frac{\left\{1+\bar{\xi}lz+\phi_\gamma(A;B)\,(lz+\xi)\,ze^{-i\gamma}\right\}}{\left(1+\bar{\xi}lz+B\,(lz+\xi)\,z\right)z}-\frac{\left(1+\xi ze^{-i\gamma}\phi_\gamma(A;B)\right)}{(1+B\xi z)\,z}\\
&=\frac{\left\{1+\phi_\gamma(A;B)\,q_\xi(lz)\,ze^{-i\gamma}\right\}}{\left(1+Bq_\xi(lz)\,z\right)z}+\frac{\bar{\xi}+ze^{-i\gamma}\phi_\gamma(A;B)}{-z\left(zB+\bar{\xi}\right)}\\
&=\frac{\left\{1+\phi_\gamma(A;B)\left(\frac{lz+\xi}{1+\bar{\xi}lz}\right)ze^{-i\gamma}\right\}}{\left(1+B\left(\frac{lz+\xi}{1+\bar{\xi}lz}\right)z\right)z}+\frac{\bar{\xi}+ze^{-i\gamma}\phi_\gamma(A;B)}{-z\left(zB+\bar{\xi}\right)}\\
&=\frac{\left\{1+\bar{\xi}lz+\phi_\gamma(A;B)\,(lz+\xi)\,ze^{-i\gamma}\right\}}{\left(1+\bar{\xi}lz+B\,(lz+\xi)\right)z)z}+\frac{\bar{\xi}+ze^{-i\gamma}\phi_\gamma(A;B)}{-z\left(zB+\bar{\xi}\right)}.
\end{aligned}$$

Moreover, we find that

$$\frac{\Psi'_{l,\xi}(z)\left(\Psi_{l,\xi}(z)\right)^{\beta-1}}{[\psi(z)]^\beta}-\phi(z,\xi)=\frac{\left(1-|\xi|^2\right)\left(\phi_\gamma(A;B)\,e^{-i\gamma}-B\right)lz^2}{z\left(1+\left(\bar{\xi}l+B\xi\right)z+Blz^2\right)(1+B\xi z)},$$

and

$$\frac{\Psi'_{l,\xi}(z)\left(\Psi_{l,\xi}(z)\right)^{\beta-1}}{[\psi(z)]^{\beta}}+\phi_1(z,\xi)=\frac{\left(1-|\xi|^2\right)\left(\phi_\gamma(A;B)e^{-i\gamma}-B\right)lz^2}{z\left(1+\left(\overline{\xi}l+B\xi\right)z+Blz^2\right)\left(zB+\overline{\xi}\right)}.$$

Therefore, we can write

$$\begin{aligned}D=&\frac{\Psi'_{l,\xi}(z)\left(\Psi_{l,\xi}(z)\right)^{\beta-1}}{[\psi(z)]^{\beta}}-C\left(\phi(z,\xi),\phi_1(z,\xi)\right)\\=&\frac{\left(\frac{\Psi'_{l,\xi}(z)\left(\Psi_{l,\xi}(z)\right)^{\beta-1}}{[\psi(z)]^{\beta}}-\phi(z,\xi)\right)-|z|^2\left|\frac{\overline{\xi}+B\xi z}{1+B\xi z}\right|^2\left(\frac{\Psi'_{l,\xi}(z)\left(\Psi_{l,\xi}(z)\right)^{\beta-1}}{[\psi(z)]^{\beta}}+\phi_1(z,\xi)\right)}{1-|z|^2\left|\frac{\overline{\xi}+B\xi z}{1+B\xi z}\right|^2}.\end{aligned}$$

Moreover, we see that

$$\begin{aligned}D=&\frac{\left(1-|\xi|^2\right)\left(\phi_\gamma(A;B)e^{-i\gamma}-B\right)\overline{\left(1+\left(\overline{\xi}l+B\xi\right)z+Blz^2\right)}}{z\left(1+\left(\overline{\xi}l+B\xi\right)z+Blz^2\right)(1+B\xi z)}\\=&\frac{lz^2\left(\phi_\gamma(A;B)e^{-i\gamma}-B\right)\overline{\left(1+\left(\overline{\xi}l+B\xi\right)z+Blz^2\right)}}{|z|^2\left(1+\left(\overline{\xi}l+B\xi\right)z+Blz^2\right)^2}.\end{aligned}$$

Putting $l=e^{i\theta}$, we have

$$\begin{aligned}&\frac{\Psi'_{e^{i\theta},\xi}(z)}{\left(\Psi_{e^{i\theta},\xi}(z)\right)^{1-\beta}}-C\left(\phi(z,\xi),\phi_1(z,\xi)\right)[\psi(z)]^{\beta}\\&=\frac{e^{i\theta}R\left(\phi(z,\xi),\phi_1(z,\xi)\right)\left(\phi_\gamma(A;B)e^{-i\gamma}-B\right)\left|\left(1+\left(\overline{\xi}e^{i\theta}+B\xi\right)z+Be^{i\theta}z^2\right)\right|^2[\psi(z)]^{\beta}}{\left(1+\left(\overline{\xi}e^{i\theta}+B\xi\right)z+Be^{i\theta}z^2\right)^2}.\end{aligned}\tag{2.10}$$

Thus

$$D=R\left(\phi(z,\xi),\phi_1(z,\xi)\right)\left(\phi_\gamma(A;B)e^{-i\gamma}-B\right)[\psi(z)]^{\beta}\frac{e^{i\theta}H'(z)}{|H'(z)|},\tag{2.11}$$

where $\psi\in\mathcal{S}^*$ and $\operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right)>0$. By Lemma 1.2, we have $H(z)=2^{-1}e^{i\theta}H_0^2(z)\in\mathcal{S}_2^*$, where H_0 is starlike in $\mathbb{U}$ with $H_0(0)=H_0'(0)-1=0$, for any $z_0\in\mathbb{U}\backslash\{0\}$, the line segment joining origin to $H_0(z_0)$ lies in $H_0(\mathbb{U})$. Assume that γ_0 is defined by

$$\gamma_0:z=z(t)=H_0^{-1}\left(tH_0(z_0)\right),\ 0\leq t\leq 1.$$

It follows that $H(z(t))=2^{-1}e^{i\theta}(H_0(z(t)))^2=2^{-1}e^{i\theta}(tH_0(z_0))^2=t^2H(z_0)$. Differentiation over t gives us $H(z(t))=tH(z_0)$ and hence,

$$H'(z(t))z'(t)=2tH(z_0),\tag{2.12}$$

so that

$$\left(\left(\Psi_{e^{i\theta},\xi}(z)\right)^{\beta-1}\Psi'_{e^{i\theta},\xi}(z)-C\left(\phi(z(t),\xi),\phi_1(z(t),\xi)\right)[\psi(z)]^{\beta}(z(t))\right)z'(t)$$

$$= R\left(\phi\left(z(t),\xi\right),\phi_1\left(z(t),\xi\right)\right)\left(\phi_\gamma\left(A;B\right)e^{-i\gamma}-B\right)\frac{|\psi(z(t))|^\beta\,|z'(t)|\,H(z_0)}{|H(z_0)|}.$$

Integrating the above equality yields the result

$$\begin{aligned}&\left(\Psi_{e^{i\theta},\xi}(z)\right)^\beta-\chi\left(\gamma_0,\xi\right)\\&=\frac{\left(\phi_\gamma\left(A;B\right)e^{-i\gamma}-B\right)H(z_0)}{|H(z_0)|}\int_0^1 R\left(\phi\left(z(t),\xi\right),\phi_1\left(z(t),\xi\right)\right)|\psi(z(t))|^\beta\,|z'(t)|\,dz\\&=\frac{\left(\phi_\gamma\left(A;B\right)e^{-i\gamma}-B\right)H(z_0)}{|H(z_0)|}R_\xi^{\gamma_0}.\end{aligned}\tag{2.13}$$

Or we see that

$$\left(\Psi_{e^{i\theta},\xi}(z)\right)^\beta-\chi\left(\gamma_0,\xi\right)=\frac{\left(\phi_\gamma\left(A;B\right)e^{-i\gamma}-B\right)H(z_0)}{|H(z_0)|}R_\xi^{\gamma_0},\tag{2.14}$$

and so we have

$$\left(\Psi_{e^{i\theta},\xi}(z)\right)^\beta\in\partial\mathbb{U}\left(\chi\left(\gamma_0,\xi\right),R_\xi^{\gamma_0}\right).$$

Since

$$\left(\Psi_{e^{i\theta},\xi}(z)\right)^\beta\in\mathcal{V}_\gamma[\psi,z_0,\xi,\beta,A;B]\subset\overline{\mathbb{U}}(\chi\left(\gamma_0,\xi\right),R_\xi^{\gamma_0}),$$

we have

$$\left(\Psi_{e^{i\theta},\xi}(z)\right)^\beta\in\partial\mathcal{V}_\gamma[\psi,z_0,\xi,\beta,A;B].$$

Now, we find that $[f(z)]^\beta=\left(\Psi_{e^{i\theta},\xi}(z)\right)^\beta$ for some $f\in\mathcal{P}_\gamma[\xi,A;B]$ and $c\in\partial\mathbb{U}$. Let

$$F_1(t)=\frac{|H(z_0)|\,f'(z\left(t\right))-|H(z_0)|\,C\left(\phi\left(z(t),\xi\right),\phi_1\left(z(t),\xi\right)\right)\left[\psi(z\left(t\right))\right]^\beta\left[f(z\left(t\right)\right]^{1-\beta}}{\left(\phi_\gamma\left(A;B\right)e^{-i\beta}-B\right)H(z_0)\left[f(z\left(t\right))\right]^{1-\beta}}z'(t),\tag{2.15}$$

and

$$k_1(t)=\frac{|H(z_0)|}{\left(\phi_\gamma\left(A;B\right)e^{-i\gamma}-B\right)H(z_0)}\left(\frac{\Psi'_{e^{i\theta},\xi}(z)}{\left(\Psi_{e^{i\theta},\xi}(z)\right)^{1-\beta}}-\left[\psi(z\left(t\right))\right]^\beta C\left[\phi\left(z(t),\xi\right),\phi_1\left(z(t),\xi\right)\right]\right)z'(t),$$

where $\gamma_0:z=z(t)$, $0\le t\le 1$. Thus $F_1(t)$ is a continuous function of t. As in [7], we see from (2.10), (2.13) and (2.15) that

$$|F_1(t)|=\frac{1}{\left|\phi_\gamma\left(A;B\right)e^{-i\gamma}-B\right|}\left|\frac{f'(z)}{f^{1-\beta}(z)}-C\left(\phi\left(z(t),\xi\right),\phi_1\left(z(t),\xi\right)\right)\left[\psi(z)\right]^\beta\right||z'(t)|\,,$$

or we can see that

$$|F_1(t)|\le\frac{1}{\left|\phi_\gamma\left(A;B\right)e^{-i\gamma}-B\right|}R\left(\phi\left(z(t),\xi\right),\phi_1\left(z(t),\xi\right)\right)|\psi(z)|^\beta\,|z'(t)|\,.$$

From (2.13), we obtain (2.11) and (2.12). This proves that $\frac{f'(z)}{[f(z)]^{1-\beta}}=\frac{\Psi'_{e^{i\theta},\xi}(z)}{\left(\Psi_{e^{i\theta},\xi}(z)\right)^{1-\beta}}$ on γ_0. On applying the identity theorem, we have $f\left(z\right)=\Psi_{e^{i\theta},\xi}\left(z\right)$. □

Theorem 2.4. *Let $z_0 \in \mathbb{U}$ and $\mathrm{Re}\beta > 0$. If $z_0 = 0$, then $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B] = \{0\}$. The boundary is the closed Jordan curve defined by $\theta \in (-\pi, \pi] \rightarrow \Psi_{e^{i\theta},\xi}(z_0)$, where*

$$\Psi_{l,\xi}(z) = \left(\beta \int_0^z \left[\frac{\left\{1 + \phi_\gamma(A; B)\, q_\xi(lu)\, ue^{-i\gamma}\right\}[\psi(u)]^\beta}{u\left(1 + Bq_\xi(lu)\, u\right)}\right] du\right)^{\frac{1}{\beta}}, \quad z \in \mathbb{U}. \tag{2.16}$$

Moreover, if $f(z_0) = \Psi_{e^{i\theta},\xi}(z_0)$ for $f \in \mathcal{P}_\gamma[\xi, A; B]$, then $f(z) = \Psi_{e^{i\theta},\xi}(z)$.

Proof. Finally, suppose that the mapping $\theta \in \partial\mathbb{U} \rightarrow \Psi_{e^{i\theta},\xi}(z_0)$ is not injective. Then there exists θ_1, $\theta_2 \in \partial\mathbb{U}$ with $\theta_1 \neq \theta_2$ such that $\Psi_{e^{i\theta_1},\xi}(z_0) = \Psi_{e^{i\theta_2},\xi}(z_0)$. Since $\Psi_{e^{i\theta_1},\xi}, \Psi_{e^{i\theta_2},\xi} \in \mathcal{P}_\gamma[\xi, A; B]$, we have $\Psi_{e^{i\theta_1},\xi} = \Psi_{e^{i\theta_2},\xi}$ from uniqueness. This contradicts $c_1 \neq c_2$. For proof of the theorem, we combine the results of Theorem 2.1 as well as Theorem 2.2 and it can be seen that a simple closed curve $\partial\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ obviously comprises of $\theta \in \partial\mathbb{U} \rightarrow \Psi_{e^{i\theta},\xi}$. As any simple closed curve cannot surrounds such a curve other than itself. Therefore, $\partial\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ is coincident with $\theta \in (-\pi, \pi] \rightarrow \Psi_{e^{i\theta},\xi}$. □

3. Concluding remarks

The region of variability problems are more useful for the family $\mathcal{S}$, than the related classical theorems about this family $\mathcal{S}$. In this study, we discussed the region of variability $\mathcal{V}_\gamma[\psi, z_0, \xi, \beta, A; B]$ for $f(z_0)$, where f ranges over the family $\mathcal{B}_\gamma[\psi, \xi, \beta, A; B]$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

This research does not involve any conflicts of interest.

References

1. B. Bhowmik, Regions of variability for a class of analytic and locally univalent functions defined by subordination, *Proc. Math. Sci.*, **125** (2015), 511–519. https://doi.org/10.1007/s12044-015-0252-5
2. S. Chen, H. Aiwu, Region of variabilty for generalized α-convex and β-stalike functions and their extreme points, *Commun. Korean Math. Soc.*, **25** (2010), 557–569. https://doi.org/10.4134/CKMS.2010.25.4.557
3. S. Dineen, *The Schwarz lemma*, Oxford: Clarendon Press, 1989.

4. P. L. Duren, *Univalent functions*, New York: Springer, 2001.

5. A. W. Goodman, *Univalent functions, Vol. I & II*, Washing, New Jersey: Polygonal Publishing House, 1983.

6. H. Grunsky, Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen, *Math. Z.*, **45** (1939), 29–61. https://doi.org/10.1007/bf01580272

7. D. J. Hallenbeck, A. E. Livingston , Applications of extreme point theory to classes of multivalent functions, *Trans. Amer. Math. Soc.*, **221** (1976), 339–359. https://doi.org/10.1090/s0002-9947-1976-0407257-2

8. S. Ponnusamy, Foundations of Complex Analysis, *Math. Gaz.*, 83 (2005), 183–183. https://doi.org/10.2307/3618748

9. S. Ponnusamy, A. Vasudevarao, Region of variability for close-to-convex functions, *Complex Var. Elliptic Equ.*, **53** (2008), 709–716. https://doi.org/10.1080/17476930801996346

10. S. Ponnusamy, A. Vasudevarao, Region of variability of two subclasses of univalent functions, *J. Math. Anal. Appl.*, **332** (2007), 1323–1334. https://doi.org/10.1016/j.jmaa.2006.11.019

11. S. Ponnusamy, A. Vasudevarao, Region of variability for functions with positive real part, *Ann. Pol. Math.*, **99** (2010), 225–245. https://doi.org/10.4064/ap99-3-2

12. S. Ponnusamy, A. Vasudevarao, M. Vuorinen, Region of variability for exponentially convex univalent functions, *Complex Anal. Oper. Theory*, **5** (2011), 955–966. https://doi.org/10.1007/s11785-010-0089-y

13. S. Ponnusamy, A. Vasudevarao, M. Vuorinen, Region of variability for certain classes of univalent functions satisfying differential inequalities, *Complex Var. Elliptic Equ.*, **54** (2009), 899–922. https://doi.org/10.1080/17476930802657616

14. S. Ponnusamy, A. Vasudevarao, H. Yanagihara, Region of variability for close-to-convex functions, *Complex Var. Elliptic Equ.*, **53** (2008), 709–716. https://doi.org/10.1080/17476930801996346

15. S. Ponnusamy, A. Vasudevarao, H. Yanagihara, Region of variability of univalent functions f for which zf' is spirallike, *Houston J. Math.*, **34** (2008), 1037–1048.

16. M. Raza, W. Ul-Haq, S. Noreen, Region of Variability for Janowski Functions, *Miskolc Math. Notes*, **16** (2015), 1117–1127. https://doi.org/10.18514/mmn.2015.1344

17. L. Spacek, Contribution a la theorie des fonctions univalentes (in Czech), *Casop. Pest. Mat.*, **62** (1932), 12–19.

18. S. S. Varma, T. Rosy, Region of variability of subclasses of univalent functions, *Asia Pac. J. Math.*, **1** (2014), 213–224.

19. W. Ul-Haq, Variability regions for janowski convex functions, *Complex Var. Elliptic Equ.*, **59** (2014), 355–361. https://doi.org/10.1080/17476933.2012.725164

20. M. Raza, W. Ul-Haq, J. L. Liu, S. Noreen, Regions of variability for a subclass of analytic functions, *AIMS Mathematics*, **5** (2020), 3365–3377. https://doi.org/10.3934/math.2020217

21. H. Yanagihara, Regions of variability for functions of bounded derivatives, *Kodai Math. J.*, **28** (2005), 452–462. https://doi.org/10.2996/kmj/1123767023

22. H. Yanagihara, Regions of variability for convex function, *Math. Nachr.*, **279** (2006), 1723–1730. https://doi.org/10.1002/mana.200310449

23. H. Yanagihara, Variability regions for families of convex functions, *Comptut. Methods Funct. Theory*, **10** (2010), 291–302. https://doi.org/10.1007/bf03321769