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Extremal solutions for fractional evolution equations of order $1 < \gamma < 2$

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Abstract: This manuscript considers a class of fractional evolution equations with order $1 < \gamma < 2$ in ordered Banach space. Based on the theory of cosine operators, this paper extends the application of monotonic iterative methods in this type of equation. This method can be applied to some physical problems and phenomena, providing new tools and ideas for academic research and practical applications. Under the assumption that the linear part is an m-accretive operator, the positivity of the operator families of fractional power solutions is obtained by using Mainardi's Wright-type function. By virtue of the positivity of the family of fractional power solution operators, we establish the monotone iterative technique of the solution of the equation and obtain the existence of extremal mild solutions under the assumption that the upper and lower solutions exist. Moreover, we investigate the positive mild solutions without assuming the existence of upper and lower solutions. In the end, we give an example to illustrate the applied value of our study.

Keywords: fractional evolution equation; monotone iterative technique; lower and upper solutions; positive solution; operator sine and cosine functions; Kuratowski measure of noncompactness
Mathematics Subject Classification: 34A08, 47H07, 47D09, 47H08

1. Introduction

Let $(X, \|\cdot\|, \leq)$ be an ordered Banach space, whose positive cone $K = \{u \in X | u \geq \theta\}$ is normal with normal constant N, θ is the zero element of X. In this paper, we discuss the existence and uniqueness of mild solutions for fractional evolution equations

$$\begin{cases} {}^cD_t^\gamma u(t) + Au(t) = f(t, u(t)), & t \in [0, a], \\ u(0) = x_0, u'(0) = x_1, \end{cases} \tag{1.1}$$

where ${}^{c}D_t^{\gamma}$ is the Caputo fractional derivative of order $\gamma \in (1, 2)$, $-A$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ of uniformly bounded linear operators in X, $f : [0, a]\times X \to X$ is a given function which will be specified later, $x_0, x_1 \in X$.

Fractional calculus is an important branch of mathematics, which was born in 1695. It is generally recognized that integral calculus and fractional calculus appear almost simultaneously. They have plenty of similarities. In the fields of turbulence models, Brownian motion, and viscoelastic materials, the researchers found that integer calculus cannot accurately describe the historical memory and spatial correlation of relevant models. However, fractional calculus can comprehensively reflect its behavior. Therefore, fractional calculus has become an indispensable method to deal with practical problems in a number of disciplines. Fractional calculus is widely used in fractional evolution equations. Because differential equations containing fractional derivatives (called fractional differential equations) are abstract formulas for many problems in engineering, flow control system, biological tissue, stochastic process, genetic mechanics, newtonian fluid mechanics, anomalous diffusion, etc, the subject is frequently discussed at home and abroad by scholars. The theory and application of fractional differential equations have yielded several remarkable results (see [14, 16, 20, 28, 31, 52] and the references therein).

In the natural world, diffusion phenomena occur frequently. For example, the transmission of odor, sound, and light, as well as the transmission of temperature, etc. It has wide applications in many fields. In the realm of physics, it can be harnessed to modulate the resistivity of semiconductors. In industry, diffusion phenomena can optimize the speed and efficiency of chemical reactions. In biology, it allows for the investigation of cellular interactions, thereby enhancing our understanding of life processes. The diffusion phenomena refer to the spontaneous migration of biomass resulting from the thermal motion of particles, including atoms, molecules, and molecular clusters. There exists a correlation between the diffusion coefficient of particles and mean squared displacement. The mean squared displacement depends on time, and the specific relationship is manifested as $\langle x^2(t)\rangle \sim const \cdot t^{\gamma}, \gamma > 0$. When $\gamma = 1$, it represents a slow diffusion phenomenon. When $\gamma = 2$, it represents a fast diffusion phenomenon, such as the wave equation. When $\gamma \in (1, 2)$ represents the super-diffusion equation, which can be applied to information signal processing. The edge detection based on fractional differential equation can improve the standard of edge detection and the community's standard of noise[35]. It can also be applied to other fields such as viscosity. Additionally, this type of equation and its various variants are applicable to mathematical models of viscoelasticity. Therefore, the study of such equations is very meaningful.

The study of fractional evolution equations has captured the interest of numerous scholars. However, the prevailing research trend involves converting equations into alternative forms, disregarding the constraints of certain equations, and failing to explore iterative sequences. In addition, the majority of the existing literature focuses on integer derivatives or fractional derivatives of $0 < \gamma < 1$. Recently, for instance, Abdollahi et al. [2] analyzed the generalized two-dimensional fractional Volterra integral equations and transformed the proposed equation into an algebraic equation using the two-dimensional Haar wavelet operation matrix method. They derived some sufficient conditions for existence and uniqueness of the equation. Alipour [5] solved for the numerical solution of the two-dimensional time fractional diffusion wave equation using the double interactive boundary element method and provided the iterative format of the time derivative. The author suggested that discretization could convert fractional differential equations into non-homogeneous Helmholtz equations. Avazzadeh et al. [6] studied fractional differential systems for order $\gamma \in (0, 1)$ under the Caputo sense, with initial value conditions. The existence and uniqueness of the system were proven using the GFP model and derivative operation matrix.

In response to the issues mentioned in the above analysis, this paper proposes an effective method, which is the monotonic iteration technique for upper and lower solutions. This method can be briefly described as: if the problem that we consider has a pair of ordered lower and upper solutions, we will construct a simple iterative sequence through these lower and upper solutions, so that it can converges uniformly to the minimum and maximum solutions between lower and upper solutions of problem under certain conditions. This method can obtain both the existence of the solution and the corresponding approximate iteration sequence. This method is comprehensively summarized in document [22]. Many scholars use this method to treat all kinds of fractional evolution equations. In 2009, McRae [36] applied the monotone iteration technique of upper and lower solutions to study the existence of extremal solutions for Riemann-Liouville fractional differential equation. Later, Cabada and Kisela [9] concerned with the existence and uniqueness of solutions for a nonlinear fractional differential equation involving Riemann-Liouville fractional derivative supplied with periodic boundary condition. Abdelouahed and Zakia [1] established the existence of maximal and minimal solutions for two types of fractional nonlinear reaction diffusion problems with periodic conditions or with initial conditions. For more applications of the upper and lower solutions, see [3, 38] and the references therein.

In practical operation, utilizing this technology poses certain challenges. If the A is infinitesimal generator of semigroup $T(t)$, we can use $T(t)$ and probability density function to express the operator families of fractional power solutions, which is a very common method. The positivity of the operator families of fractional power solutions is naturally generated by the positivity of the semigroup $T(t)$. A new semigroup $T_b(t)$ is generated after $T(t)$ perturbation and $T_b(t) = e^{-bt}T(t)$. Thus, we can still obtain the positivity of $T_b(t)$ using the positivity of $T(t)$. In this paper, we hope to adopt a similar approach. However, we encountered two very critical issues. On the one hand, we are not sure if the strong continuous cosine family and the strong continuous sine family generated after perturbation are positive; On the other hand, we don't know that the operator families of fractional power solutions described by cosine function theory and Minardi's Wright-type function are positive.

In addition, we focus on the fact that only positive solutions have practical significance in some mathematical and physical models, such as the reaction-diffusion equation, neutron transport equation and heat transfer equation. When $0 < \gamma < 1$, a host of scholars have paid attention to the positive solutions and have achieved many results. Li et al. [30] discussed a class of nonlinear fractional differential equations by constructing the upper and lower control functions of nonlinear terms. They obtained the existence of positive solutions by employing the method of upper and lower solutions and Schauder's fixed point theorem. For more applications, see [9, 12, 14, 25, 33, 42] and the references therein. When $1 < \gamma < 2$, it is rare to study positive solutions. Li et al. [24] investigated the boundary value problems described by fractional differential equation with Riemann-Liouville fractional derivative. The existence of positive solutions is obtained by using Krasnoselskii's fixed point theorem.

This article attempts to expand Eq (1.1) research from various perspectives. Firstly, the current literature predominantly employs the theory of analytic semigroup to analyze problems while attaching importance to specific fractional evolution equations with order between 0 and 1. Thus, this study classifies and compares these equations, deriving the applicability of upper and lower bound methods to fractional evolution equations with order between 1 and 2. Secondly, this article concentrates on the uniqueness of the cosine family theory of operators, using the translation of cosine function generators and combining it with dissipative operator theory to enhance the understanding of the positivity of fractional power solution operators. This further extends the existing theory of fractional power solution operators. Finally, we overcome the usual constraint that monotonic iterative methods require the existence of upper and lower solutions of the system. Our research methodology proves effective in addressing these limitations, making monotonic iterative methods widely applicable to practical problems. Overall, this paper contributes substantially to the research on Eq (1.1), deepening its understanding while providing new concepts and techniques for further research and application.

The remainder of this paper is organized as follows. In Section 2, the symbols, concepts and lemma required are introduced for this article. By introducing the concept of accretive operator and related properties, the positivity of the strong continuous cosine family and the strong continuous sine family generated after perturbation are obtained. Furthermore, combined with the definition and properties of Mainardi's Wright-type function, the positivity of the operator families of fractional power solutions is obtained. In Section 3, under the condition that the strongly continuous sine family is compact, we use the monotonic iterative technique of upper and lower solutions to demonstrate the existence of mild solutions. In the case that the strongly continuous sine family is noncompact, we explore the existence of the extreme solution of Eq (1.1). By introducing the Kuratowski measure of noncompactness, we establish the uniqueness of solution of Eq (1.1). Finally, the existence of positive solution of Eq (1.1) is obtained without assuming the existence of upper and lower solutions. In Section 4, we present an example to illustrate the applied value of our study.

2. Preliminaries

In this section, we will list some concepts and definitions used to show our main results. Assume that X is an ordered Banach space with norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $K = \{x \in X | x \geq \theta\}$ is normal with normal constant N, θ is the zero element of X. We denote that $L_b(X, Y)$ are the spaces of all bounded linear operators from X into Y equipped with the norm $\|\cdot\|_{L_b}$. If $X = Y$, $L_b(X, X)$ is denoted $L_b(X)$. We define the domain of $-A$ by $D(-A)$ and the range of $-A$ by $R(-A)$. If $-A$ is a linear operator, then the resolvent set and resolvent of $-A$ are defined by $\rho(-A)$ and $R(\lambda, -A) = (\lambda I + A)^{-1} \in L_b(X)$, respectively.

Let $C([0, a], X)$ be the Banach space of all continuous functions from $[0, a]$ into X endowed with $\|u\|_C = \sup_{t\in[0,a]} \|u(t)\|$. $C([0, a], X)$ is also an ordered Banach space, the positive cone $K_C = \{u \in C([0, a], X) | \, u(t) \in K, t \in [0, a]\}$ is also normal. K, K_C have the same normal constant N.

For $v, w \in C([0, a], X)$, $v \leq w \Leftrightarrow v(t) \leq w(t)$ for all $t \in [0, a]$. we denote the order interval $[v, w] = \{u \in C([0, a], X) | v \leq u \leq w\}$ and $[v(t), w(t)] = \{u(t) \in X | v(t) \leq u(t) \leq w(t)\}$.

A significant amount of literature refers to the definition of Caputo fractional derivation, see [13, 46] and its references.

Now, we propose definitions of the strongly continuous cosine and sine families, recalling a few of their characteristics.

Definition 2.1 ([44]) *A one parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if* *(i)* $C(0) = I$*;*

(ii) $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $s, t \in \mathbb{R}$*;*

(iii) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$*.*

The sine family associated with $\{C(t)\}_{t\in\mathbb{R}}$ is defined by $\{S(t)\}_{t\in\mathbb{R}}$, where

$$S(t)x = \int_0^t C(s)xds, \quad x \in X, \quad t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator $-A$ of cosine family $\{C(t)\}_{t\in\mathbb{R}}$ is defined by

$$-Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad \text{for } x \in D(-A),$$

where

$$D(-A) = \{x \in X | C(t)x \text{ is a twice continuously differentiable function of } t\}.$$

Clearly, $-A$ is a closed dense operator in X.

Lemma 2.2 ([44]) *Let* $\{C(t)\}_{t\geq 0}$ *be a strongly continuous cosine family in X. Then there exist constants* $M_0 > 0$ *and* $\varpi > 0$ *such that*

$$\|C(t)\|_{L_b} \leq M_0 e^{\varpi |t|}, \quad \text{for all } t \geq 0. \tag{2.2}$$

Lemma 2.3 ([44]) *Let* $\{C(t)\}_{t\geq 0}$ *be a strongly continuous cosine family in X. Then*

$$\lim_{t\to 0} \frac{S(t)}{t} = I.$$

Lemma 2.4 ([44]) *Let* $-A$ *generate the strongly continuous cosine family* $\{C(t)\}_{t\geq 0}$ *in X satisfying (2.2). Then for* $Re\lambda \geq \varpi$*,* $\lambda^2 \in \rho(-A)$*, here* $\rho(-A)$ *is the resolvent set of* $-A$*, and*

$$\lambda R(\lambda^2; -A)x = \int_0^\infty e^{-\lambda t} C(t)xdt, \quad x \in X, \tag{2.3}$$

$$R(\lambda^2; -A)x = \int_0^\infty e^{-\lambda t} S(t)xdt, \quad x \in X. \tag{2.4}$$

Moreover, according to [8, Proposition 2.11], we have the following result.

Lemma 2.5 ([8]) *Let* $-A$ *generate a strongly continuous cosine family* $\{C(t)\}_{t\geq 0}$ *satisfying (2.2). Then*

$$C(t)x = \lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{n+1} b^2_{k,n+1} \Big(I + (\frac{t}{n})^2 A\Big)^{-k} x,$$

and the convergence is uniform on bounded subsets of $t \geq 0$ *for any fixed* $x \in X$*, where* $b^2_{1,1} = 1$*,* $b^2_{k,n} = (n-1-2k)b^2_{k,n-1} + 2(k-1)b^2_{k-1,n-2} (1 \leq k \leq n, n = 2, 3, \cdots)$*, and* $b^2_{k,n} = 0 (k > n, n = 1, 2, \cdots)$*.*

According to [44, Lemma 4.1], note that $-(A + L^2 I)$ can also generate a strongly continuous cosine families $C_L(t)$ in X for arbitrary constant L. And

$$\|C_L(t)\|_{L_b} \leq M_0 e^{(\varpi + |L|)t}, \quad t \geq 0, \tag{2.5}$$

and

$$\lambda R(\lambda^2;-A-L^2I)x=\lambda R(\lambda^2+L^2;-A)x=\int_0^{\infty}e^{-\lambda t}C_L(t)xdt \tag{2.6}$$

for $Re\lambda>\varpi+|L|$ and $x\in X$. Let $S_L(t)$ denote the strongly continuous sine family associated with $C_L(t)$. In view of [43, Proposition 2.1] and (2.6), it is easy to obtain for

$$R(\lambda^2;-A-L^2I)x=R(\lambda^2+L^2;-A)x=\int_0^{\infty}e^{-\lambda t}S_L(t)xdt, \tag{2.7}$$

for $Re\lambda>\varpi+|L|$ and $x\in X$.

Next, we will introduce solution operators and their related properties. We first introduce the Mainardi's Wright-type function

$$\xi_\varrho(z)=\sum_0^{\infty}\frac{(-z)^n}{n!\Gamma(1-\varrho(n+1))},\quad \text{for } z\in\mathbb{C},\ \ \varrho\in(0,1).$$

For any $t>0$, the Mainardi's Wright-type function has the properties

$$\xi_\varrho(t)\geq 0,\ \ \int_0^{\infty}\tau^{\sigma}\xi_\varrho(\tau)d\tau=\frac{\Gamma(1+\sigma)}{\Gamma(1+\varrho\sigma)},\ \ -1<\sigma<\infty. \tag{2.8}$$

In what follows, we always suppose that $-A$ is the infinitesimal generator of a strongly continuous cosine family of uniformly bounded linear operators $\{C(t)\}_{t\geq 0}$, namely, there exist a constant $M_0\geq 1$ such that $\|C(t)\|_{L_b}\leq M_0$ for $t\geq 0$. For the sake of convenience in writing, we also set $q=\frac{\gamma}{2}$ for $\gamma\in(1,2)$. Thus, we can define three families of operators $C_q(t)(t\geq 0), N_q(t)(t\geq 0)$ and $T_q(t)(t\geq 0)$ in X as following

$$C_q(t)=\int_0^{\infty}\xi_q(\tau)C(t^q\tau)d\tau,\quad N_q(t)=\int_0^{t}C_q(s)ds, \tag{2.9}$$

$$T_q(t)=\int_0^{\infty}q\tau\xi_q(\tau)S(t^q\tau)d\tau. \tag{2.10}$$

We have the following useful properties for operator families $C_q(t)(t\geq 0), N_q(t)(t\geq 0)$ and $T_q(t)(t\geq 0)$.

Lemma 2.6 ([44]) *The operators $C_q(t)$, $N_q(t)$ and $T_q(t)$ admit the following properties:*

(i) for all $t\geq 0$, $C_q(t)$, $N_q(t)$ and $T_q(t)$ are linear operators;

(ii) for all $t\geq 0$ and for any $x\in X$,

$$\|C_q(t)x\|\leq M_0\|x\|,\ \ \|N_q(t)x\|\leq M_0\|x\|t,\ \ \|T_q(t)x\|\leq\frac{M_0}{\Gamma(2q)}\|x\|t^q;$$

(iii) $\{C_q(t),t\geq 0\},\{N_q(t),t\geq 0\},\{t^{q-1}T_q(t),t\geq 0\}$ are strongly continuous;

To prove the positivity of the proposed operator families, we need to introduce the definition of accretive operator and related lemmas.

Definition 2.7 *An operator family $\{C(t)\}_{t\geq 0}$ in X is called to be positive if $C(t)x\geq\theta$ for any $x\geq\theta$ and $t\geq 0$.*

Definition 2.8 ([8]) *$-A : D(A) \to X$ is said to be nonnegative if and only if it satisfies both of the following conditions:*

(i) There exists $P \geq 0$ such that, for every value of $\lambda > 0$ and every $x \in D(A)$,

$$\lambda\|x\| \leq P\|\lambda x - Ax\|;$$

(ii) $R(\lambda I - A) = X$ for every value of $\lambda > 0$.

Definition 2.9 ([8]) *An operator $-A$ satisfying (i) of Definition 2.8 with $P = 1$ is called accretive. If $-A$ moreover satisfies (ii) then $-A$ is called m-accretive.*

Lemma 2.10 *Let $-A$ be a linear m-accretive operator and generate a uniformly bounded strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ in X. Then strongly continuous cosine family $C_L(t)$ generated by $-A - L^2 I$ is positive.*

Proof Since $-A$ is a m-accretive operator, from Definition 2.8 and Definition 2.9, it is easy to obtain that $-A$ is nonnegative. From (2.6), it follows that $\lambda^2 + L^2 \in \rho(-A)$ and for $Re\lambda > \varpi + |L|$, one can see

$$\begin{aligned} R(\lambda^2; -A - L^2 I)x &= R(\lambda^2 + L^2; -A)x \\ &= ((\lambda^2 + L^2)I + A))^{-1}x \\ &= \sum_{n=0}^{\infty} \frac{(-A)^n}{(\lambda^2 + L^2)^{n+1}} x \geq \theta, \quad \text{for } x \geq \theta, \end{aligned}$$

which implies that $R(\lambda^2; -A - L^2 I)$ is positive. Thus, according to Lemma 2.5, we can get that

$$C_L(t)x = \lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{n+1} b^2_{k,n+1} \Big(I + (\frac{t}{n})^2 (A + L^2)\Big)^{-k} x.$$

From Definition 2.7, we can obtain that the cosine family $C_L(t)$ is positive. Similarly, by (2.7), it is easily to obtain that the sine family $S_L(t)$ is positive. This completes the proof of Lemma 2.10 . □

Define three operators $C_q(t)(t \geq 0)$, $\mathcal{N}_q(t)(t \geq 0)$ and $\mathcal{T}_q(t)(t \geq 0)$ in X as follows

$$C_q(t)x = \int_0^{\infty} \xi_q(\tau) C_L(t^q \tau) x d\tau, \quad \mathcal{N}_q(t)x = \int_0^t C_q(s) x ds, \tag{2.11}$$

$$\mathcal{T}_q(t)x = \int_0^{\infty} q\tau \xi_q(\tau) S_L(t^q \tau) x d\tau, \tag{2.12}$$

where $x \in X$, and $\xi_q(\tau)$ is the Mainardi's Wright-type function. Employing (2.5) and $t \in [0, a]$, derive the relation

$$\|C_L(t)\|_{L_b} \leq M_0 e^{(\varpi+|L|)a} := M.$$

The operators $C_q(t)$, $\mathcal{N}_q(t)$ and $\mathcal{T}_q(t)$ have properties (i) and (iii) in Lemma 2.6. At the same time, for each $t \geq 0$,

$$\|C_q(t)x\| \leq M\|x\|, \ \ \|\mathcal{N}_q(t)x\| \leq M\|x\|t, \ \ \|\mathcal{T}_q(t)x\| \leq \frac{M}{\Gamma(2q)}\|x\|t^q. \tag{2.13}$$

Taking account to the properties of the Mainardi's Wright-type function, the positivity of $\{C_L(t)\}_{t\geq 0}$, we can deduce that $C_q(t)$, $\mathcal{N}_q(t)$ and $\mathcal{T}_q(t)$ are positive operators for all $t \geq 0$.

Lemma 2.11 ([11]) *Let $-A$ be a closed operator from $D(-A)$ into X such that $R_\lambda(-A)$ exists and is compact for some λ. Then $R_\lambda(-A)$ is compact for any $\lambda \in \rho(-A)$.*

Lemma 2.12 *Assume that $-A$ is a closed operator. If the strongly continuous sine operator $S(t)$ generated by $-A$ is compact, then the strongly continuous sine operator $S_L(t)$ generated by $-A - L^2 I$ is also compact for every $t \geq 0$.*

Proof The following two properties are equivalent for every $t \geq 0$: (i) The strongly continuous sine operator $S(t)$ is compact; (ii) The resolvent operator $R(\lambda^2, -A)$ is compact. Our conclusion can be found by applying a similar proof technique in [10, Theorem 3.1]. We have therefore omitted the details here. A similar result holds that the compactness of $S_L(t)$ is equivalent to the compactness of $R(\lambda^2 + L^2, -A)$ for all $t \geq 0$. Since Lemma 2.4, $\lambda^2 \in \rho(-A)$. Thus, $\lambda^2 + L^2 \in \rho(-A)$. By Lemma 2.11 and the compactness of $R(\lambda^2, -A)$, we deduce that $R(\lambda^2 + L^2, -A)$ is compact. This completes the proof of Lemma 2.12. □

Before introducing the definitions of the lower and upper solution for Eq (1.1), we need to defined the set $C^\gamma([0, a], X)$ by

$$C^\gamma([0, a], X) = \{u \in C([0, a], X) | {}^cD_t^\gamma u \text{ exists, and } {}^cD_t^\gamma u \in C([0, a], X)\},$$

and denote by X_1 the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$.

Definition 2.13 *A function $v \in C([0, a], X)$ is said to be a lower solution of Eq (1.1) if $v \in C^\gamma([0, a], X) \cap C([0, a], X_1)$ and satisfies*

$$\begin{cases} {}^cD_t^\gamma v(t) + Av(t) \leq f(t, v(t)), & t \in [0, a], \\ v(0) \leq x_0, v'(0) \leq x_1, \end{cases} \tag{2.14}$$

If the inequality of (2.14) is inverse, we call it is an upper solution.

We define the mild solution of Eq (1.1) as follows:

Definition 2.14 *A function $u \in C([0, a], X)$ is said to be a mild solution of Eq (1.1) if satisfies*

$$u(t) = C_q(t)x_0 + N_q(t)x_1 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, u(s)) ds.$$

Moreover, if $u(t) \geq \theta$ for $t \in [0, a]$, then it is said to be a positive mild solution of Eq (1.1).

Next, we recall some facts about the measure of noncompactness. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For any $D \subset C([a, b], X)$ and $t \in [a, b]$, set $D(t) = \{u(t) | u \in D\}$. If D is bounded in $C([a, b], X)$, then $D(t)$ is bounded in X, and $\alpha(D(t)) \leq \alpha(D)$.

Lemma 2.15 ([39]) *Let S_1, S_2 be a nonempty bounded set in X and $a \in \mathbb{R}$, then the noncompactness measure $\alpha(\cdot)$ has the following properties*

(i) $\alpha(S_1) = 0 \Leftrightarrow S_1$ is a relative compact set;

(ii) $S_1 \subset S_2 \Rightarrow \alpha(S_1) \leq \alpha(S_2)$;

(iii) $\alpha(\overline{S_1}) = \alpha(S_1)$;

(iv) $\alpha(S_1 \cup S_2) \leq max\{\alpha(S_1), \alpha(S_2)\}$*;*

(v) $\alpha(aS_1) = |a|\alpha(S_1)$*, where* $aS_1 = \{x|x = ay, y \in S\}$*;*

(vi) $\alpha(S_1 + S_2) \leq \alpha(S_1) + \alpha(S_2)$*, where* $S_1 + S_2 = \{x|x = y + z, y \in S_1, z \in S_2\}$*;*

(vii) $\alpha(\overline{co}S_1) = \alpha(S_1)$*.*

Lemma 2.16 ([7]) *Let X be a Banach space and let* $D \subset C([a, b], X)$ *be bounded and equicontinuous. Then* $\alpha(D(t))$ *is continuous on* $[a, b]$*, and*

$$\alpha(D) = \max_{t \in [a,b]} \alpha(D(t)).$$

Lemma 2.17 ([19]) *Let X be a Banach space,* $D = \{u_n\} \subset C([a, b], X)$ *be a bounded and countable set. Then* $\alpha(D(t))$ *is Lebesgue integrable on* $[a, b]$*, and*

$$\alpha(\{\int_a^b u_n(s)ds\}) \leq 2\int_a^b \alpha(D(t))dt.$$

To illustrate our final results, the following lemma is also needed.

Lemma 2.18 ([51]) *Assume that* $f(t)$ *is a nonnegative function locally integrable on* $0 \leq t < T$ *(some* $T < \infty$*),* $g(t)$ *is a nonnegative, nondecreasing continuous bounded function on* $0 \leq t < T, p, q > 0$*. Suppose that* $h(t)$ *is nonnegative and locally integrable on* $0 \leq t < T$ *with*

$$h(t) \leq f(t) + g(t)\int_0^t (t - s)^{q-1}h(s)ds.$$

Then,

$$h(t) \leq f(t) + \int_0^t (\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(q))^n}{\Gamma(nq)}(t - s)^{nq-1}f(s))ds.$$

3. Main results

Throughout this part, we always assume that X is an ordered Banach space, whose positive cone K is normal. $-A$ is a linear m-accretive operator and $-A$ generates a strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ in X.

We present and demonstrate several results in this location. First, under the condition that the strongly continuous sine family is compact, we apply monotone iterative method of the lower and upper solutions to consider the existence of the extreme solutions of Eq (1.1).

Theorem 3.1. *Let* $-A$ *be a linear m-accretive operator and* $-A$ *generate a compact strongly continuous sine family* $\{S(t)\}_{t\geq 0}$ *in X. Assume that Eq (1.1) has upper and lower solutions* w_0, v_0 *with* $v_0 \leq w_0$*,* $f : [0, a] \times X \to X$ *is a continuous function. If the following assumptions are established:*

(H1) There is a constant number L such that

$$f(t, x_2) - f(t, x_1) \geq -L^2(x_2 - x_1),$$

for all $t \in [0, a]$ *and* $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$*.*

Then Eq (1.1) has minimal and maximal mild solutions $\underline{u}, \overline{u} \in [v_0, w_0]$*.*

Proof It follows easily from Eq (1.1)

$$\begin{cases} {}^cD_t^\gamma u(t) + Au(t) + L^2u(t) = f(t,u(t)) + L^2u(t), & t \in [0,a], \\ u(0) = x_0, u^{'}(0) = x_1, \end{cases} \tag{3.1}$$

where constant L is decided by condition (H1).

According to [44, Lemma 4.1], $-(A + L^2 I)$ can generate a strongly continuous cosine families $C_L(t)$ in X. And $\|C_L(t)\|_{L_b} \le M$.

We define an operator Q on $[v_0, w_0]$ by

$$\begin{aligned} Qu(t) &= C_q(t)x_0 + \mathcal{N}_q(t)x_1 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s) \\ &\quad \times [f(s,u(s)) + L^2u(s)]ds, \quad t \in [0,a]. \end{aligned} \tag{3.2}$$

From the hypothesis (H1) and the normality of the cone K, we can easily obtain that there exists a constant $M_1 > 0$ such that

$$\sup_{t\in[0,a]} \{\|f(t,u(t)) + L^2u(t)\|\} \le M_1, \tag{3.3}$$

for each $u \in [v_0, w_0]$. Thus, $Q : [v_0, w_0] \to C([0,a], X)$ is well defined. Taking into account Definition 2.14, (3.1) and (3.2), we claim that $u \in [v_0, w_0]$ is a mild solution of Eq (1.1) if and only if u is a fixed point of Q. Now, we prove that u is a fixed point of Q.

Step 1. Prove the following properties of operator Q.

(i) $v_0 \le Qv_0, Qw_0 \le w_0$.

(ii) Q is a monotonic increasing operator.

On the one hand, let

$${}^cD_t^\gamma v_0(t) + (A + L^2)v_0(t) := h(t), \quad t \ge 0. \tag{3.4}$$

In view of (3.1), (3.2), (3.4), Definition 2.13, Definition 2.14 and the positivity of $C_q(t)(t \ge 0)$, $\mathcal{N}_q(t)(t \ge 0)$ and $\mathcal{T}_q(t)(t \ge 0)$, we find that

$$\begin{aligned} v_0(t) &= C_q(t)v_0(0) + \mathcal{N}_q(t)v_0^{'}(0) + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)h(s)ds \\ &\le C_q(t)x_0 + \mathcal{N}_q(t)x_1 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s,v_0(s))ds \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)L^2v_0(t)ds \\ &= Qv_0(t) \end{aligned}$$

for each $t \in [0,a]$. Similarly, we can prove that $Qw_0(t) \le w_0(t)$ for each $t \in [0,a]$. Thus, $v_0 \le Qv_0, Qw_0 \le w_0$.

On the other hand, for every $u_1, u_2 \in [v_0, w_0]$ with $u_1 \le u_2$ and $t \in [0,a]$, then $v_0(t) \le u_1(t) \le u_2(t) \le w_0(t)$. Make use of condition (H1) and the positivity of $\mathcal{T}_q(t)(t \ge 0)$ to obtain the inequality

$$\int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)[f(s,u_1(s)) + L^2u_1(s)]ds$$

$$\leq \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)[f(s,u_2(s))+L^2u_2(s)]ds.$$

Using this fact and the positivity of $C_q(t)(t\geq 0)$ and $\mathcal{N}_q(t)(t\geq 0)$, we can deduce that $Qu_1\leq Qu_2$.

Therefore, Q is a continuous monotonic increasing operator.

Next, we define two sequences by

$$v_i = Qv_{i-1},\ \ w_i = Qw_{i-1},\ \ i=1,2,\ldots. \tag{3.5}$$

Since the monotonicity of Q, we get

$$v_0\leq v_1\leq v_2\leq\cdots\leq v_i\leq\cdots\leq w_i\leq\cdots\leq w_2\leq w_1\leq w_0. \tag{3.6}$$

Step 2. Show that $\{v_i(t)\}$ and $\{w_i(t)\}$ are relatively compact on X for every $t\in[0,a]$.

Let $\Omega=\{v_i\}(i=1,2,\ldots)$ and $\Omega_0=\{v_i\}\cup\{v_0\}$. It follows that $\Omega(t)=(Q\Omega_0)(t)$ for $t\in[0,a]$. Let $t\in[0,a]$ be fixed, the set defined by

$$Q_{\varepsilon,\delta}\Omega_0(t):=\{Q_{\varepsilon,\delta}v_i(t)|v_i\in\Omega_0,\ 0\leq t\leq a\}$$

for each $0<\varepsilon<t$ and $\delta>0$, where

$$\begin{aligned}
&(Q_{\varepsilon,\delta}v_i)(t)\\
= \ & C_q(t)x_0+\mathcal{N}_q(t)x_1+\frac{S_L(\varepsilon^q\delta)}{\varepsilon^q\delta}\int_0^{t-\varepsilon}\int_\delta^\infty (t-s)^{q-1}q\tau\xi_q(\tau)\\
&\times S_L((t-s)^q\tau-\varepsilon^q\delta)[f(s,v_{i-1}(s))+L^2v_{i-1}(s)]d\tau ds.
\end{aligned}$$

According to (2.1), (2.8), (3.3) and $\|C_L(t)\|_{L_b}\leq M$, we can prove that

$$\begin{aligned}
&\Big\|\frac{S_L(\varepsilon^q\delta)}{\varepsilon^q\delta}\int_0^{t-\varepsilon}\int_\delta^\infty (t-s)^{q-1}q\tau\xi_q(\tau)\\
&\times S_L((t-s)^q\tau-\varepsilon^q\delta)[f(s,v_{i-1}(s))+L^2v_{i-1}(s)]d\tau ds\Big\|\\
\leq\ & qM^2M_1\int_0^{t-\varepsilon}\int_\delta^\infty (t-s)^{q-1}\tau\xi_q(\tau)\Big((t-s)^q\tau-\varepsilon^q\delta\Big)d\tau ds\\
\leq\ & 2qM^2M_1\int_0^{t-\varepsilon}(t-s)^{2q-1}ds\int_\delta^\infty\tau^2\xi_q(\tau)d\tau\\
\leq\ & \frac{3M^2M_1a^{2q}}{q\Gamma(2q)}.
\end{aligned}$$

Since the compactness of $\{S(t)\}_{t>0}$ and Lemma 2.12, $Q_{\varepsilon,\delta}\Omega_0(t)$ is relatively compact in X for any $\delta>0$ and $0<\varepsilon<t$. Moreover, for every $v_i\in\Omega_0$ and $t\in[0,a]$, from (2.12), (3.2), (3.3), we can derive the following inequality

$$\|(Qv_i)(t)-(Q_{\varepsilon,\delta}v_i)(t)\|$$

$$
\begin{aligned}
&\leq \Big\| \int_0^t \int_0^\infty (t-s)^{q-1} q\tau\xi_q(\tau) S_L((t-s)^q\tau)[f(s,v_{i-1}(s)) + L^2 v_{i-1}(s)] d\tau ds \\
&\quad - \frac{S_L(\varepsilon^q\delta)}{\varepsilon^q\delta} \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{q-1} q\tau\xi_q(\tau) \\
&\quad \times S_L((t-s)^q\tau - \varepsilon^q\delta)[f(s,v_{i-1}(s)) + L^2 v_{i-1}(s)] d\tau ds \Big\| \\
&\leq \Big\| \int_0^t \int_0^\delta (t-s)^{q-1} q\tau\xi_q(\tau) S_L((t-s)^q\tau)[f(s,v_{i-1}(s)) + L^2 v_{i-1}(s)] d\tau ds \Big\| \\
&\quad + \Big\| \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{q-1} q\tau\xi_q(\tau) S_L((t-s)^q\tau)[f(s,v_{i-1}(s)) + L^2 v_{i-1}(s)] d\tau ds \Big\| \\
&\quad + \Big\| \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{q-1} q\tau\xi_q(\tau)[f(s,v_{i-1}(s)) + L^2 v_{i-1}(s)] \\
&\quad \times \Big[S_L((t-s)^q\tau) - \frac{S_L(\varepsilon^q\delta)}{\varepsilon^q\delta} S_L((t-s)^q\tau - \varepsilon^q\delta) \Big] d\tau ds \Big\| \\
&:= \chi_1 + \chi_2 + \chi_3.
\end{aligned}
$$

Results (2.1), (3.3) and $\|C_L(t)\|_{L_b} \leq M$ together imply that

$$
\begin{aligned}
\chi_1 &= \Big\| \int_0^t \int_0^\delta (t-s)^{q-1} q\tau\xi_q(\tau) S_L((t-s)^q\tau) \\
&\quad \times [f(s,v_{i-1}(s)) + L^2 v_{i-1}(s)] d\tau ds \Big\| \\
&\leq qMM_1 \int_0^t (t-s)^{2q-1} ds \int_0^\delta \tau^2 \xi_q(\tau) d\tau \\
&\to 0 \quad \text{as } \delta \to 0.
\end{aligned}
$$

Similarly, by (2.8), we obtain

$$
\begin{aligned}
\chi_2 &= \Big\| \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{q-1} q\tau\xi_q(\tau) S_L((t-s)^q\tau) \\
&\quad \times [f(s,v_{i-1}(s)) + L^2 v_{i-1}(s)] d\tau ds \Big\| \\
&\leq qMM_1 \int_{t-\varepsilon}^t (t-s)^{2q-1} ds \int_\delta^\infty \tau^2 \xi_q(\tau) d\tau \\
&\leq \frac{3MM_1\varepsilon^{2q}}{\Gamma(1+2q)} \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

Finally, in view of Lemma 2.3 and the strong continuous of $S_L(t)$ for every $t \geq 0$, we obtain

$$
\Big\| S_L((t-s)^q\tau) - \frac{S_L(\varepsilon^q\delta)}{\varepsilon^q\delta} S_L((t-s)^q\tau - \varepsilon^q\delta) \Big\|_{L_b}
$$

$$
\begin{aligned}
&\leq \|S_L((t-s)^q\tau) - S_L((t-s)^q\tau - \varepsilon^q\delta)\Big\|_{L_b} \\
&\quad + \Big\|S_L((t-s)^q\tau - \varepsilon^q\delta) - \frac{S_L(\varepsilon^q\delta)}{\varepsilon^q\delta} S_L((t-s)^q\tau - \varepsilon^q\delta)\Big\|_{L_b} \\
&\to 0 \quad \text{as } \varepsilon, \delta \to 0.
\end{aligned}
$$

Using this result, we get

$$
\begin{aligned}
\chi_3 &= \Big\| \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{q-1} q\tau\xi_q(\tau)[f(s, v_{i-1}(s)) + L^2 v_{i-1}(s)] \\
&\quad \times \Big[S_L((t-s)^q\tau) - \frac{S_L(\varepsilon^q\delta)}{\varepsilon^q\delta} S_L((t-s)^q\tau - \varepsilon^q\delta)\Big] d\tau ds \Big\| \\
&\leq qM_1 \int_0^{t-\varepsilon} (t-s)^{q-1} \int_\delta^\infty \tau\xi_q(\tau) \\
&\quad \times \Big\|S_L((t-s)^q\tau) - \frac{S_L(\varepsilon^q\delta)}{\varepsilon^q\delta} S_L((t-s)^q\tau - \varepsilon^q\delta)\Big\|_{L_b} d\tau ds \\
&\to 0 \quad \text{as } \varepsilon, \delta \to 0.
\end{aligned}
$$

Consequently, one can deduce that the set $(Q\Omega_0)(t)$ is relatively compact, which means that $\{v_i(t)\}$ is relatively compact on X for each $t \in [0, a]$. Therefore, $\{v_i(t)\}$ is relatively compact on X for every $t \in [0, a]$. Similarly, we can prove that $\{w_i(t)\}$ is relatively compact on X for every $t \in [0, a]$.

Combining the normality of the cone and monotonicity, it follows that $\{v_i\}$ themselves is convergent, that is, there exists $\underline{u} \in C([0, a], X)$ such that $v_i \to \underline{u}, i \to \infty$. Similarly, there exists $\overline{u} \in C([0, a], X)$ such that $w_i \to \overline{u}, i \to \infty$. Taking the limit of (3.5), we can assert that

$$
\underline{u}(t) = Q\underline{u}(t), \quad \overline{u}(t) = Q\overline{u}(t).
$$

Hence, two fixed points of Q, $\underline{u}$ and $\overline{u}$, are mild solutions of Eq (1.1).

Step 3. Minimal and maximal properties of $\underline{u}, \overline{u}$.

Assume that $\widetilde{u}$ is a fixed point of Q with $\widetilde{u} \in [v_0, w_0]$, then for every $t \in [0, a]$, $v_0(t) \leq \widetilde{u}(t) \leq w_0(t)$ and

$$
v_1(t) = (Qv_0)(t) \leq (Q\widetilde{u})(t) = \widetilde{u}(t) \leq (Qw_0)(t) = w_1(t),
$$

that is, $v_1 \leq \widetilde{u} \leq w_1$, In general,

$$
v_i \leq \widetilde{u} \leq w_i, \quad i = 1, 2, \ldots.
$$

It is clear that $\underline{u} \leq \widetilde{u} \leq \overline{u}$ as $i \to \infty$, which means that $\underline{u}$ and $\overline{u}$ are minimal and maximal mild solutions of Eq (1.1), and $\underline{u}$ and $\overline{u}$ can be obtained by the iterative sequences defined in (3.5) starting from v_0 and w_0. This completes the proof of Theorem 3.1. □

Furthermore, we delete the compactness of $S(t)$ and investigate the existence of the solution of Eq (1.1) under the Kuratowski measure of noncompactness.

Theorem 3.2. *Assume that Eq (1.1) has upper and lower solutions w_0, v_0 with $v_0 \leq w_0$. $f : [0, a] \times X \to X$ is a continuous function and satisfied condition (H1). If the following assumption is established:*

(H2) There is an integrable function $h_f(t) : [0, a] \to [0, \infty)$ such that

$$\alpha(\{f(t, D\}) \le h_f(t)\alpha(D)$$

for any $t \ge 0$ and bounded subset $D \subset C([0, a], X)$.
Then Eq (1.1) has maximal and minimal solutions $\overline{u}, \underline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, respectively.

Proof We define the operator Q by (3.2). In view of Theorem 3.1, it is easy to derive that $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuous monotone increasing operator, $v_0 \le Qv_0$ and $Qw_0 \le w_0$. Next, we define two sequences $\{v_i\}$ and $\{w_i\}$ by (3.5) satisfy (3.6).

We show that $\{v_i\}$ and $\{w_i\}$ are equicontinuous in $t \in [0, a]$. In fact, let $0 \le t_1 < t_2 \le a$. Taking account to (3.2), for every $u \in [v_0, w_0]$, it is evident that

$$\begin{aligned}
& \|Qu(t_2) - Qu(t_1)\| \\
= & \Big\|C_q(t_2)x_0 + \mathcal{N}_q(t_2)x_1 - C_q(t_1)x_0 - \mathcal{N}_q(t_1)x_1 \\
& + \int_0^{t_2} (t_2 - s)^{q-1}\mathcal{T}_q(t_2 - s)[f(s, u(s)) + L^2u(s)]ds \\
& - \int_0^{t_1} (t_1 - s)^{q-1}\mathcal{T}_q(t_1 - s)[f(s, u(s)) + L^2u(s)]ds\Big\| \\
\le & \|C_q(t_2)x_0 - C_q(t_1)x_0\| + \|\mathcal{N}_q(t_2)x_1 - \mathcal{N}_q(t_1)x_1\| \\
& + \int_{t_1}^{t_2} (t_2 - s)^{q-1}\Big\|\mathcal{T}_q(t_2 - s)[f(s, u(s)) + L^2u(s)]\Big\|ds \\
& + \int_0^{t_1} \Big\|\Big((t_2 - s)^{q-1}\mathcal{T}_q(t_2 - s) - (t_1 - s)^{q-1}\mathcal{T}_q(t_1 - s)\Big) \\
& \times[f(s, u(s)) + L^2u(s)]\Big\|ds \\
:= & I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We need to check $I_n \to 0 (n = 1, 2, 3, 4)$ independently of $u \in [v_0, w_0]$ as $t_2 - t_1 \to 0$. By virtue of Lemma 2.6 (iii), one can obtain that $\{C_q(t), t \ge 0\}$ and $\{\mathcal{N}_q(t), t \ge 0\}$ are strongly continuous. Thus, $I_1, I_2 \to 0$ as $t_2 - t_1 \to 0$. From (2.13) and (3.3), we can obtain that

$$\begin{aligned}
I_3 &= \int_{t_1}^{t_2} (t_2 - s)^{q-1}\Big\|\mathcal{T}_q(t_2 - s)[f(s, u(s)) + L^2u(s)]\Big\|ds \\
&\le \frac{MM_1}{\Gamma(2q)}\int_{t_1}^{t_2} (t_2 - s)^{2q-1}ds \\
&\le \frac{MM_1}{\Gamma(1 + 2q)}(t_2 - t_1)^{2q}
\end{aligned}$$

$$\to \quad 0 \text{ as } t_2 - t_1 \to 0.$$

Let $P_q(t) = t^{q-1}\mathcal{T}_q(t)$ for $t \in [0, a]$, from which we conclude that

$$\|P_q(t)x\| \leq \frac{M}{\Gamma(2q)}\|x\|t^{2q-1} \tag{3.7}$$

for all $t \in [0, a]$ and $x \in X$. In view of Lemma 2.6, we obtain that $P_q(t)$ is a strongly continuity operator. For I_4, taking $\sigma > 0$ small enough and using (2.13), (3.3) and (3.7), one can get that

$$\begin{aligned}
I_4 &= \int_0^{t_1} \Big\|((t_2 - s)^{q-1}\mathcal{T}_q(t_2 - s) \\
&\quad -(t_1 - s)^{q-1}\mathcal{T}_q(t_1 - s))[f(s, u(s)) + L^2u(s)]\Big\|ds \\
&= \int_0^{t_1-\sigma} \|P_q(t_2 - s) - P_q(t_1 - s)\|_{L_b} \cdot \|f(s, u(s)) + L^2u(s)\|ds \\
&\quad + \int_{t_1-\sigma}^{t_1} \|P_q(t_2 - s) - P_q(t_1 - s)\|_{L_b} \cdot \|f(s, u(s)) + L^2u(s)\|ds \\
&\leq t_1M_1 \sup_{s\in[0,t_1-\sigma]} \|P_q(t_2 - s) - P_q(t_1 - s)\|_{L_b} \\
&\quad +M_1 \int_{t_1-\sigma}^{t_1} \|P_q(t_2 - s)\| + \|P_q(t_1 - s)\|_{L_b}ds \\
&\leq t_1M_1 \sup_{s\in[0,t_1-\sigma]} \|P_q(t_2 - s) - P_q(t_1 - s)\|_{L_b} \\
&\quad +\frac{MM_1}{\Gamma(1 + 2q)}\Big((t_2 - t_1 + \sigma)^{2q} + (t_2 - t_1)^{2q} + \sigma^{2q}\Big) \\
&\to 0 \text{ as } t_2 - t_1 \to 0, \ \sigma \to 0.
\end{aligned}$$

In summary, we can derive that $\|Qu(t_2) - Qu(t_1)\| \to 0$ as $t_2 - t_1 \to 0$ independently of $u \in [v_0, w_0]$. Thus, Q is equicontinuous.

Next, we prove that $\{v_i\}$ and $\{w_i\}$ are convergent on X.

Set Ω and Ω_0 as Theorem 3.1. By (3.5), (3.6) and the normality of the positive cone K, we obtain that Ω and Ω_0 are bounded, which implies that

$$\alpha(\Omega(t)) = \alpha(\Omega_0(t))$$

for all $t \in [0, a]$. Due to (2.13), (3.2), Lemma 2.15, Lemma 2.16, Lemma 2.17 and (H2), one can deduce that

$$\begin{aligned}
&\alpha(\{v_i(t)\}) = \alpha(\{Qv_{i-1}(t)\}) \\
= \ &\alpha\Big(C_q(t)x_0 + \mathcal{N}_q(t)x_1 + \int_0^t (t - s)^{q-1}\mathcal{T}_q(t - s)[f(s, v_{i-1}(s)) + L^2v_{i-1}(s)]ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\Big(\int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)[f(s,v_{i-1}(s))+L^2v_{i-1}(s)]ds\Big)\\
&\leq 2\int_0^t (t-s)^{q-1}\|\mathcal{T}_q(t-s)\|_{L_b}\alpha(f(s,v_{i-1}(s))+L^2v_{i-1}(s)))ds\\
&\leq \frac{2M}{\Gamma(2q)}\int_0^t (t-s)^{2q-1}\Big(h_f(s)+L^2\Big)\alpha(v_i(s))ds.
\end{aligned}
$$

Therefore, from Lemma 2.18, we can obtain $\alpha(\{v_i(t)\}) \equiv 0$ for each $t \in [0,a]$. Consequently, based on Lemma 2.15 (i), it follows that $\{v_i(t)\}$ is relatively compact in $C([0,a],X)$.

Combining uniform boundedness and equicontinuous of $\{v_i\}$ on $[0,a]$, $\{v_i\}$ is relatively compact in $C([0,a],X)$, then the sequence $\{v_i\}$ has a convergent subsequence. According to the monotonicity and the normality of the cone, one can deduce that $\{v_i\}$ itself is convergent. Then $\{v_i\}$ themselves is convergent, that is, there exists $\underline{u} \in C([0,a],X)$ such that $v_i \to \underline{u}$ as $i \to \infty$. Similarly, there exists $\overline{u} \in C([0,a],X)$ such that $w_i \to \overline{u}$ as $i \to \infty$.

From the proof of Theorem 3.1, $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of Eq (1.1). This completes the proof of Theorem 3.2. □

Corollary 3.3. *Assume that Eq (1.1) has upper and lower solutions w_0, v_0 with $v_0 \leq w_0$. $f : [0,a]\times X \to X$ is a continuous function and satisfied condition (H1). If the following assumption is established:*

(H3) The sequence $\{u_i\} \subset [v_0(t), w_0(t)] \subset C([0,a],X)$ is increasing monotonic, and there is a constant $h_1 > 0$ such that

$$\alpha(\{f(t,u_i(t)+L^2u_i(t)\}) \leq h_1\alpha(\{u_i(t)\})$$

for each $t \in [0,a]$.
Then Eq (1.1) has maximal and minimal solutions $\overline{u}, \underline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, respectively.

Proof Using the same method as Theorem 3.2, we can obtain

$$\alpha(v_i(t)) = \alpha(Qv_{i-1}(t)) \leq \frac{2h_1M}{\Gamma(2q)}\int_0^t (t-s)^{2q-1}\alpha(v_i(s))ds.$$

According to Lemma 2.18, we can obtain that $\{v_i(t)\}$ is relatively compact in $C([0,a],X)$. This completes the proof of Corrollary 3.3. □

On the basis of Theorem 3.2, we discuss the uniqueness of the solution of Eq (1.1).

Theorem 3.4. *Suppose that Eq (1.1) has lower and upper solutions v_0, w_0 with $v_0 \leq w_0$. $f : [0,a]\times X \to X$ is a continuous function. If the condition (H1), (H2) and the following assumptions are established:*
(H4) There is a constant h_2 such that

$$f(t,x_2) - f(t,x_1) \leq h_2(x_2 - x_1)$$

for all $t \in [0,a]$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$.
Then Eq (1.1) has a unique mild solution between v_0 and w_0 provided that

$$\frac{M(h_2+L^2)a^{2q}}{\Gamma(1+2q)} \leq 1.$$

Proof We establish two iterative sequences v_i and w_i by (3.5), and its satisfy (3.6). By Theorem 3.2, we know that Eq (1.1) has maximal and minimal mild mild solution on interval $[v_0, w_0]$, that is, the operator Q has fixed point defined by (3.2). For any $t \in [0, a]$, from (3.2), (3.5) and (H4), one can obtain that

$$\begin{aligned} \theta &\leq w_i(t) - v_i(t) = Qw_{i-1}(t) - Qv_{i-1}(t) \\ &= \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)[f(s, w_{i-1}(s)) + L^2 w_{i-1}(s)]ds \\ &\quad - \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)[f(s, v_{i-1}(s)) + L^2 v_{i-1}(s)]ds \\ &\leq (h_2 + L^2)\int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)(w_{i-1}(s) - v_{i-1}(s))ds. \end{aligned}$$

From the normality of the cone K, (2.13) and (3.3), we get that

$$\|w_i(t) - v_i(t)\| \leq \frac{M(h_2 + L^2)a^{2q}}{\Gamma(1+2q)}\|w_{i-1}(s) - v_{i-1}(s)\|$$

for any $t \in [0, a]$, that is,

$$\|w_i - v_i\| \leq \frac{M(h_2 + L^2)a^{2q}}{\Gamma(1+2q)}\|w_{i-1} - v_{i-1}\|.$$

Then

$$\|w_i - v_i\| \leq (\frac{M(h_2 + L^2)a^{2q}}{\Gamma(1+2q)})^i\|w_0 - v_0\| \to 0, i \to \infty.$$

Thus, there exists a unique mild solution $\widehat{u}$ of Eq (1.1) in $[v_0, w_0]$, i.e. $\lim_{i\to\infty} w_i = \lim_{i\to\infty} v_i = \widehat{u}$. Taking limit in (3.5) as $i \to \infty$, we can get that $\widehat{u} = Q\widehat{u}$, which means that $\widehat{u}$ is unique mild solution $\widehat{u} \in [v_0, w_0]$ of Eq (1.1). This completes the proof of Theorem 3.4. □

By adding appropriate assumptions, we overcome the condition that the upper and lower solutions of the equation must exist, and obtain the existence of positive solutions of Eq (1.1).

Theorem 3.5. *Let $-A$ be a linear m-accretive operator and $-A$ generate a compact strongly continuous sine family $\{S(t)\}_{t\geq 0}$ in X. Let $x_0, x_1 \geq \theta$. $f : [0, a] \times X \to X$ is a continuous function. If the condition (H1) and the following assumptions are established:*

(H5) For $t \in [0, a]$, $x_2 \geq x_1 \geq \theta$,

$$f(t, x_2) + L^2 x_2 \geq f(t, x_1) + L^2 x_1 \geq \theta.$$

(H6) There exist nonnegative constants $c \in (0, \frac{\varpi^3}{M(1-e^a)})$ and $d \geq 0$, such that

$$\|f(t, x) + L^2 x\| \leq c\|x\| + d, \quad t \geq 0, \quad x \in X.$$

Then Eq (1.1) has minimal positive mild solutions u^*.

Proof We define the operator Q by (3.2). The positive mild solution of Eq (1.1) is equivalent to the fixed point of the operator Q. In view of the continuity of f, it is easy to derive that $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous.

For every $u, v \in K$ with $u \leq v$, from (H5), (3.2), the positivity of $C_q(t), \mathcal{N}_q(t), \mathcal{T}_q(t)$, and $x_0, x_1 \geq \theta$, it follows that for all $t \in [0, a]$,

$$\theta \leq Qu(t) \leq Qv(t). \tag{3.8}$$

Define the sequence $\{v_i\}$ by

$$v_i = Qv_{i-1}, i = 1, 2, \ldots, \tag{3.9}$$

Due to (H6), (2.12), (2.13) and (3.2), we get

$$\begin{aligned}
\|Qu(t)\| &\leq M\|x_0\| + M\|x_1\|t + (c\|u\| + d)\int_0^t \int_0^\infty q\tau\xi_q(\tau) \\
&\quad \times (t-s)^{q-1}\int_0^{(t-s)^q\tau} Me^{\varpi\rho}d\rho d\tau ds \\
&\leq M\|x_0\| + M\|x_1\|t + M(c\|u\| + d)\int_0^t \int_0^\infty q\tau\xi_q(\tau) \\
&\quad \times (t-s)^{q-1}\frac{1}{\varpi}e^{\varpi(t-s)^q\tau}d\tau ds \\
&\leq M\|x_0\| + Ma\|x_1\| + \frac{M(1-e^a)}{\varpi^3}(c\|u\| + d) \\
&= \beta_1 + \beta_2\|u\|,
\end{aligned}$$

where $\beta_1 = M\|x_0\| + aM\|x_1\| + \frac{dM(1-e^a)}{\varpi^3}, \beta_2 = \frac{cM(1-e^a)}{\varpi^3} < 1$. From (3.8) and (3.9), one can evident that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_i \leq \cdots,$$

$$\|v_i\| \leq \beta_1 + \beta_2\|v_{i-1}\|.$$

Thus,

$$\|v_i\| \leq \beta_1 + \beta_2\beta_1 + \ldots + \beta_2^{i-1}\beta_1 = \beta_1\frac{1-\beta_2^i}{1-\beta_2} \leq \frac{\beta_1}{1-\beta_2}. \tag{3.10}$$

i.e., the sequence $\{v_i\}$ is uniformly bounded.

By Theorem 3.1, we know that $\{v_i\}$ is relatively compact and equicontinuous in $[0, a]$. According to the monotonicity of sequence and the normality of cone, $\{v_i\}$ itself is uniformly converge, which means that there is $u^* \in C([0, a], X)$ such that $v_i \to u^*$ as $i \to \infty$.

$$u^* = \lim_{i\to\infty} v_i = \lim_{i\to\infty} Qv_i = Qu^*.$$

Thus, u^* is fixed point of Q, which means that u^* is a positive mild solution of Eq (1.1).

Finally, we prove that u^* is the minimal positive mild solution. In fact, let u' be a positive mild solution of Eq (1.1). Then, $u'(t) = Qu'(t)$ for $t \in [0, a]$. It is easy to find that $u'(t) \geq v_0$ and

$$u'(t) = Qu'(t) \geq Qv_0(t) = v_1(t),$$

so, $u' \geq v_1$. In general, $u' \geq v_i, i = 1, 2, \ldots$. Thus, $u' \geq u^*$ as $i \to \infty$. It follows that u^* is the minimal positive mild solution of Eq (1.1). This completes the proof of Theorem 3.5. □

4. Example

In this section, we provide an application example. We consider the following fractional partial differential system

$$\begin{cases} \frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}} u(t,z) + \triangle u(t,z) = f(t,z,u(t,z)), t \in [0,1], \ z \in \Omega, \\ u(t,0) = u(t,\pi) = 0, \\ u(0,z) = u_0(z), u^{'}(0,z) = u_1(z), z \in \Omega \end{cases} \tag{4.1}$$

where D_t^{γ} is the Caputo fractional partial derivative, Δ is a Laplace operator. Take $X = L^2(\Omega)$ and $K = \{u \in C([0,1], X) | u(t,x) \geq 0, a.e. \ (t,z) \in [0,1] \times \Omega\}$.

Obviously, X is an ordered Banach space and K is a normal cone. We define the operator $-A$ by $-A = \triangle$ and

$$D(-A) = H_0^1(\Omega) \cap H^2(\Omega).$$

This seems obvious, $-A$ generates a uniformly bounded strongly continuous cosine family $C(t)$ for $t \geq 0$. Thus, $\|C(t)\|_{L_b} \leq M$ for each $t \geq 0$.

Let $\lambda_k = k^2\pi^2$ and $\Psi_k(y) = \sqrt{\frac{2}{\pi}} \sin(k\pi y)$ is the normalized feature vector of the corresponding eigenvalue. Let $\{-\lambda_k \Psi_k\}_{k=1}^{\infty}$ be the eigensystem of $-A$, so $0 < \lambda_1 < \lambda_2 \leq \cdots \lambda_k \to \infty$ as $k \to \infty$, and $\{\Psi_k\}_{k=1}^{\infty}$ is the orthonormal set of X. The cosine function is defined by

$$C(t)x = \sum_{k=1}^{\infty} \cos(\sqrt{\lambda_k} t) \langle x, \Psi_k \rangle \Psi_k, x \in X,$$

and the associated sine family is given by

$$S(t)x = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \langle x, \Psi_k \rangle \Psi_k, x \in X.$$

From [44, Example 5.1], it is easy to check that $S(t)$ is compact.

We need the following assumptions:

(A1) There exists $v(t,z) \in C^{\gamma}([0,1], X) \cap (H^2(\Omega) \cup H_0^1(\Omega))$ satisfies

$$\begin{cases} \frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}} v(t,z) + \Delta v(t,z) \leq f(t,z,v(t,z)), \\ v(t,0) \leq v(t,\pi) = 0, \\ v(0,z) \leq v_0(z), v^{'}(0,z) \leq u_1(z), \end{cases}$$

(A2) There is a constant number L, such that

$$f(t,z,x_2) - f(t,z,x_1) \geq -L^2(x_2 - x_1)$$

for every $t \in [0,1]$ and $0 \leq x_1 \leq x_2 \leq u(t,z)$,

Theorem 4.1. *Let $f : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Conditions (A1) and (A2) guarantee that system (4.1) has minimal and maximal mild solutions between* $[0,w]$.

Proof It is obvious that Eq (4.1) equivalent to

$$\begin{cases} \frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}}u(t,z) + \triangle u(t,z) + L^2u(t,z) = f(t,z,u(t,z)) + L^2u(t,z), \\ u(t,0) = u(t,\pi) = 0, \\ u(0,z) = u_0(z), u^{'}(0,z) = u_1(z), \end{cases} \tag{4.2}$$

where L is determined by the condition (A2).

As an elliptic operator, Laplace operator has the maximum principle, so we can easily know the operator $\lambda^2 I + A$ has a positive bounded inverse. By Lemma 2.10, we can obtain that $-A - L^2 I$ generate a positive strongly continuous cosine operator $C_L(t)$ for each $t \geq 0$. As consequence, the operator families $C_q(t)(t \geq 0), \mathcal{N}_q(t)(t \geq 0)$ and $\mathcal{T}_q(t)(t \geq 0)$ are positive. By Theorem 3.1, (4.1) has minimal and maximal mild solutions between $[0, w]$. This completes the proof of Theorem 4.1. □

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5. Conclusions

In this paper, we investigate three properties of the fractional diffusion equation: existence of solutions, uniqueness of solutions, and existence of positive solutions. The existence of solutions is established through the use of upper and lower solutions, which enable us to construct monotonic iterative sequences that converge to the minimum and maximum solutions of the problem. Uniqueness of the solution is obtained using this iterative scheme. To validate our theoretical findings, we provide an example of a reaction-diffusion problem.

The combination of upper and lower solutions with monotonic iteration techniques can be applied to various types of fractional evolution equations, for modeling and analyzing phenomena in fields such as physics, engineering, and biology. Further research can explore the use of these methods in specific physical problems and phenomena, providing new tools and ideas for academic research and practical applications. The method proposed in this study has high reliability and practicality and can provide useful theoretical basis and practical applications for researchers in related fields.

Conflict of interest

All authors declare that they have no competing interests.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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