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Research article

The lifespan of classical solutions of one dimensional wave equations with semilinear terms of the spatial derivative

Takiko Sasaki[1,3], **Shu Takamatsu**[2] **and Hiroyuki Takamura**[3,*]

[1] Department of Mathematical Engineering, Faculty of Engineering, Musashino University, 3-3-3 Ariake, Koto-ku, Tokyo 135-8181, Japan
[2] Master course, Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan
[3] Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan

* **Correspondence:** Email: hiroyuki.takamura.a1@tohoku.ac.jp; Tel: +81227953891; Fax: +81227956400.

Abstract: This paper is devoted to the lifespan estimates of small classical solutions of the initial value problems for one dimensional wave equations with semilinear terms of the spatial derivative of the unknown function. It is natural that the result is same as the one for semilinear terms of the time-derivative. But there are so many differences among their proofs. Moreover, it is meaningful to study this problem in the sense that it may help us to investigate its blow-up boundary in the near future.

Keywords: semilinear wave equation; one dimension; classical solution; lifespan
Mathematics Subject Classification: primary 35L71, secondary 35B44

1. Introduction

In this paper, we consider the initial value problems

$$\begin{cases} u_{tt} - u_{xx} = |u_x|^p & \text{in} \quad \mathbb{R} \times (0,T), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $p > 1$, and $T > 0$. We assume that f and g are given smooth functions of compact support and a parameter $\varepsilon > 0$ is "small enough". We are interested in the lifespan $T(\varepsilon)$, the maximal existence time, of classical solutions of (1.1). Our result is that there exists positive constants C_1, C_2 independent of ε such that $T(\varepsilon)$ satisfies

$$C_1 \varepsilon^{-(p-1)} \le T(\varepsilon) \le C_2 \varepsilon^{-(p-1)}. \tag{1.2}$$

We note that, even if $|u_x|^p$ is replaced with $|u_t|^p$, (1.2) still holds. Such a result is due to Zhou [18] for the upper bound of $T(\varepsilon)$, and Kitamura, Morisawa and Takamura [7] for the lower bound of $T(\varepsilon)$.

As model equations to ensure the optimality of the general theory for nonlinear wave equations by Li, Yu and Zhou [8, 9], the nonlinear term $|u_t|^p$ is sufficient to be studied except for the "combined effect" case. See Morisawa, Sasaki and Takamura [10, 11] and Kido, Sasaki, Takamatsu and Takamura [6] for this direction with a possibility to improve the general theory. See also Takamatsu [17] for such an improvement. But it is quite meaningful to deal with also $|u_x|^p$ because their proofs are technically different from each others. Moreover, there is no result on its blow-up boundary due to lack of the monotonicity of the solution, while the one for $|u_t|^p$ is well-studied by Sasaki [14,15], and Ishiwata and Sasaki [2,3]. See Remark 2.1 below. It is also remarkable that it can be studied if the nonlinear term has a special form of both u_t and u_x. See Sasaki [16] for this direction. Our research may help us to study the blow-up boundary for the equation in (1.1) near future.

This paper is organized as follows. In the next section, the preliminaries are introduced. Moreover, (1.2) is divided into two theorems. Section 3 is devoted to the proof of the existence part, the lower bound of $T(\varepsilon)$, of (1.2). The main strategy is the iteration method for the system of integral equations for (u, u_x), which is essentially due to Kitamura, Morisawa and Takamura [7]. They employed it for the system of integral equations for (u, u_t) to construct a classical solution of the wave equation with nonlinear term $|u_t|^p$, which is originally introduced by John [4]. In the Section 4, following Rammaha [12, 13], we prove the blow-up part, the upper bound of $T(\varepsilon)$, of (1.2). We note that the method to be reduced to u-closed integral inequality by Zhou [18] for the nonlinear term $|u_t|^p$ cannot be applicable to (1.1) because a time delay appears in the reduced ordinary differential inequality. Rammaha [12,13] overcomes this difficulty by employing weighted functionals along with the characteristic direction in studying two or three dimensional wave equations with nonlinear terms of spatial derivatives.

2. Preliminaries and main results

Throughout this paper, we assume that the initial data $(f, g) \in C_0^2(\mathbb{R}) \times C_0^1(\mathbb{R})$ satisfies

$$\text{supp } f,\ \text{supp } g \subset \{x \in \mathbb{R} : |x| \le R\}, \quad R \ge 1. \tag{2.1}$$

Let u be a classical solution of (1.1) in the time interval $[0, T]$. Then the support condition of the initial data, (2.1), implies that

$$\text{supp } u(x,t) \subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \le t + R\}. \tag{2.2}$$

For example, see Appendix of John [5] for this fact.

It is well-known that u satisfies the integral equation

$$u(x,t) = \varepsilon u^0(x,t) + L(|u_x|^p)(x,t), \tag{2.3}$$

where u^0 is a solution of the free wave equation with the same initial data

$$u^0(x,t) := \frac{1}{2}\{f(x+t) + f(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy, \tag{2.4}$$

and a linear integral operator L for a function $v = v(x,t)$ in Duhamel's term is defined by

$$L(v)(x,t) := \frac{1}{2}\int_0^t ds \int_{x-t+s}^{x+t-s} v(y,s)dy. \tag{2.5}$$

Then, one can apply the time-derivative to (2.3) to obtain

$$u_t(x,t) = \varepsilon u_t^0(x,t) + L'(|u_x|^p)(x,t) \tag{2.6}$$

and

$$u_t^0(x,t) = \frac{1}{2}\{f'(x+t) - f'(x-t) + g(x+t) + g(x-t)\}, \tag{2.7}$$

where L' for a function $v = v(x,t)$ is defined by

$$L'(v)(x,t) := \frac{1}{2}\int_0^t \{v(x+t-s,s) + v(x-t+s,s)\}ds. \tag{2.8}$$

Therefore, u_t is expressed by u_x. On the other hand, applying the space-derivative to (2.3), we have

$$u_x(x,t) = \varepsilon u_x^0(x,t) + \overline{L'}(|u_x|^p)(x,t), \tag{2.9}$$

and

$$u_x^0(x,t) = \frac{1}{2}\{f'(x+t) + f'(x-t) + g(x+t) - g(x-t)\}, \tag{2.10}$$

where $\overline{L'}$ for a function $v = v(x,t)$ is defined by

$$\overline{L'}(v)(x,t) := \frac{1}{2}\int_0^t \{v(x+t-s,s) - v(x-t+s,s)\}ds. \tag{2.11}$$

Remark 2.1. *In view of (2.9), it is almost impossible to obtain a point-wise positivity of u_x. This fact prevents us from studying its blow-up boundary as stated in Introduction.*

Moreover, applying one more time-derivative to (2.9) yields that

$$u_{xt}(x,t) = \varepsilon u_{xt}^0(x,t) + L'(p|u_x|^{p-2}u_x u_{xt})(x,t), \tag{2.12}$$

and

$$u_{xt}^0(x,t) = \frac{1}{2}\{f''(x+t) - f''(x-t) + g'(x+t) + g'(x-t)\}. \tag{2.13}$$

Similarly, we have that

$$u_{tt}(x,t) = \varepsilon u_{tt}^0 + |u_x|^p(x,t) + \overline{L'}(p|u_x|^{p-2}u_x u_{xt})(x,t).$$

Therefore, u_{tt} is expressed by u_x, u_{xt} and so is u_{xx}.

First, we note the following fact.

Proposition 2.1. *Assume that $(f,g) \in C_0^2(\mathbb{R}) \times C_0^1(\mathbb{R})$. Let w be a C^1 solution of (2.9) in which u_x is replaced with w. Then,*

$$u(x,t) := \int_{-\infty}^x w(y,t)dy$$

is a classical solution of (1.1) in $\mathbb{R} \times [0,T]$.

Proof. This is easy along with the computations above in this section. □

Our results are divided into the following two theorems.

Theorem 2.1. *Assume (2.1). Then, there exists a positive constant* $\varepsilon_1 = \varepsilon_1(f, g, p, R) > 0$ *such that a classical solution* $u \in C^2(\mathbb{R} \times [0, T])$ *of (1.1) exists as far as* T *satisfies*

$$T \leq C_1 \varepsilon^{-(p-1)}, \tag{2.14}$$

where $0 < \varepsilon \leq \varepsilon_1$*, and* C_1 *is a positive constant independent of* ε*.*

Theorem 2.2. *Assume (2.1) and*

$$f(x), g(x) \geq 0, \text{ and } f(x) \not\equiv 0. \tag{2.15}$$

Then, there exists a positive constant $\varepsilon_2 = \varepsilon_2(f, p, R) > 0$ *such that any classical solution of (1.1) in the time interval* $[0, T]$ *cannot exist as far as* T *satisfies*

$$T > C_2 \varepsilon^{-(p-1)}, \tag{2.16}$$

where $0 < \varepsilon \leq \varepsilon_2$*, and* C_2 *is a positive constant independent of* ε*.*

The proofs of above theorems are given in following sections.

3. Proof of Theorem 2.1

According to Proposition 2.1, we shall construct a C^1 solution of (2.9) in which $u_x = w$ is the unknown function. Let $\{w_j\}_{j\in\mathbb{N}}$ be a sequence of $C^1(\mathbb{R} \times [0, T])$ defined by

$$\begin{cases} w_{j+1} = \varepsilon u_x^0 + \overline{L'}(|w_j|^p), \\ w_1 = \varepsilon u_x^0. \end{cases} \tag{3.1}$$

Then, in view of (2.12), $(w_j)_t$ has to satisfy

$$\begin{cases} (w_{j+1})_t = \varepsilon u_{xt}^0 + L'(p|w_j|^{p-2} w_j (w_j)_t), \\ (w_1)_t = \varepsilon u_{xt}^0, \end{cases} \tag{3.2}$$

so that the functional space in which $\{w_j\}$ converges is

$$X := \{w \in C^1(\mathbb{R} \times [0, T]) : \|w\|_X < \infty, \ \text{supp } w \subset \{(x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R\}\},$$

which is equipped with a norm

$$\|w\|_X := \|w\| + \|w_t\|,$$

where

$$\|w\| := \sup_{(x,t)\in\mathbb{R}\times[0,T]} |w(x, t)|.$$

We note that (2.9) implies that

$$\text{supp } w_j \subset \{(x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R\} \Longrightarrow \text{supp } w_{j+1} \subset \{(x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R\}.$$

The following lemma provides us a priori estimate.

Proposition 3.1. *Let $w \in C(\mathbb{R} \times [0,T])$ and supp $w \subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \le t + R\}$. Then, the following a priori estimate holds:*

$$\|L'(|w|^p)\| \le C\|w\|^p(T+R), \tag{3.3}$$

where C is a positive constant independent of T and ε.

Proof. The proof of Proposition 3.1 is completely same as the one of Proposition 3.1 in Morisawa, Sasaki and Takamura [10]. □

Let us continue to prove Theorem 2.1. Set

$$M := \sum_{\alpha=0}^{2} \|f^{(\alpha)}\|_{L^\infty(\mathbb{R})} + \sum_{\beta=0}^{1} \|g^{(\beta)}\|_{L^\infty(\mathbb{R})}.$$

The convergence of the sequence $\{w_j\}$

First we note that $\|w_1\| \le M\varepsilon$ by (2.10). (3.1) and (3.3) yield that

$$\|w_{j+1}\| \le M\varepsilon + C\|w_j\|^p(T+R)$$

because it is trivial that

$$|\overline{L'}(v)| \le L'(|v|).$$

Therefore, the boundedness of $\{w_j\}$, i.e.,

$$\|w_j\| \le 2M\varepsilon \quad (j \in \mathbb{N}), \tag{3.4}$$

follows from

$$C(2M\varepsilon)^p(T+R) \le M\varepsilon. \tag{3.5}$$

Assuming (3.5), one can estimate $\|w_{j+1} - w_j\|$ as follows:

$$\begin{aligned}
\|w_{j+1} - w_j\| = \|\overline{L'}(|w_j|^p - |w_{j-1}|^p)\| &\le \|L'(||w_j|^p - |w_{j-1}|^p|)\| \\
&\le 2^{p-1}p\|L'((|w_j|^{p-1} + |w_{j-1}|^{p-1})|w_j - w_{j-1}|)\| \\
&\le 2^{p-1}pC(\|w_j\|^{p-1} + \|w_{j-1}\|^{p-1})(T+R)\|w_j - w_{j-1}\| \\
&\le 2^p pC(2M\varepsilon)^{p-1}(T+R)\|w_j - w_{j-1}\|.
\end{aligned}$$

Therefore, the convergence of $\{w_j\}$ follows from

$$\|w_{j+1} - w_j\| \le \frac{1}{2}\|w_j - w_{j-1}\|$$

provided (3.5) and

$$2^p pC(2M\varepsilon)^{p-1}(T+R) \le \frac{1}{2} \tag{3.6}$$

are fulfilled.

The convergence of the sequence $\{(w_j)_t\}$

First we note that $\|(w_1)_t\| \le M\varepsilon$ by (2.13). Assume that (3.5) and (3.6) are fulfilled. Since (3.2) and (3.3) yield that

$$\begin{aligned}
\|(w_{j+1})_t\| &\le M\varepsilon + \|L'(p|w_j|^{p-2}w_j(w_j)_t)\| \\
&\le M\varepsilon + \|L'(p|w_j|^{p-1}|(w_j)_t|)\| \\
&\le M\varepsilon + pC\|w_j\|^{p-1}(T+R)\|(w_j)_t\| \\
&\le M\varepsilon + pC(2M\varepsilon)^{p-1}(T+R)\|(w_j)_t\|,
\end{aligned}$$

the boundedness of $\{(w_j)_t\}$, i.e.,

$$\|(w_j)_t\| \le 2M\varepsilon,$$

follows as long as it is fulfilled that

$$pC(2M\varepsilon)^{p-1}(T+R) \le 1. \tag{3.7}$$

Assuming (3.7), one can estimate $\{(w_{j+1})_t - (w_j)_t\}$ as follows. Noting that

$$\||w_j|^{p-2}w_j - |w_{j-1}|^{p-2}w_{j-1}| \le \begin{cases} (p-1)2^{p-2}(|w_j|^{p-2} + |w_{j-1}|^{p-2})|w_j - w_{j-1}| & \text{for } p \ge 2, \\ 2|w_j - w_{j-1}|^{p-1} & \text{for } 1 < p < 2, \end{cases}$$

we have

$$\begin{aligned}
&\|(w_{j+1})_t - (w_j)_t\| = \|L'(p|w_j|^{p-2}w_j(w_j)_t - p|w_{j-1}|^{p-2}w_{j-1}(w_{j-1})_t)\| \\
&\le p\|L'(|w_j|^{p-1}|(w_j)_t - (w_{j-1})_t|)\| + p\|L'(\||w_j|^{p-2}w_j - |w_{j-1}|^{p-2}w_{j-1}\|(w_{j-1})_t|)\| \\
&\le pC\|w_j\|^{p-1}(T+R)\|(w_j)_t - (w_{j-1})_t\| \\
&\quad + \begin{cases} L'(p(p-1)2^{p-2}(|w_j|^{p-2} + |w_{j-1}|^{p-2})|w_j - w_{j-1}\|(w_{j-1})_t| & \text{for } p \ge 2, \\ L'(2p|w_j - w_{j-1}|^{p-1}|(w_{j-1})_t|) & \text{for } 1 < p < 2, \end{cases} \\
&\le pC\|w_j\|^{p-1}(T+R)\|(w_j)_t - (w_{j-1})_t\| \\
&\quad + \begin{cases} p(p-1)2^{p-2}C(\|w_j\|^{p-2} + \|w_{j-1}\|^{p-2})\|w_j - w_{j-1}\|\|(w_{j-1})_t\| & \text{for } p \ge 2, \\ 2pC\|w_j - w_{j-1}\|^{p-1}\|(w_{j-1})_t\| & \text{for } 1 < p < 2, \end{cases} \\
&\le pC(2M\varepsilon)^{p-1}(T+R)\|(w_j)_t - (w_{j-1})_t\| + O\left(\frac{1}{2^{j\min(p-1,1)}}\right).
\end{aligned}$$

Therefore, we obtain the convergence of $\{(w_j)_t\}$ provided

$$pC(2M\varepsilon)^{p-1}(T+R) \le \frac{1}{2}. \tag{3.8}$$

Continuation of the proof

The convergence of the sequence $\{w_j\}$ to w in the closed subspace of X satisfying $\|w\|, \|w_t\| \le 2M\varepsilon$ is established by (3.5)–(3.8), which follow from

$$2^{p+1}pC(2M)^{p-1}\varepsilon^{p-1}(T+R) \le 1.$$

Therefore, the statement of Theorem 2.1 is established with

$$\varepsilon_1 := (2^{p+2}pC(2M)^{p-1}R)^{-1/(p-1)}, \quad C_1 := 2^{p+1}pC(2M)^{p-1}$$

because $R \le (2C_1)^{-1}\varepsilon^{-(p-1)}$ holds for $0 < \varepsilon \le \varepsilon_1$. □

4. Proof of Theorem 2.2

Following Rammaha [12], set

$$H(t) := \int_0^t (t-s)ds \int_{s+R_0}^{s+R} u(x,s)dx,$$

where R_0 is some fixed point with $0 < R_0 < R$. We may assume that there exists a point $x_0 \in (R_0, R)$ such that $f(x_0) > 0$ because of the assumption (2.15) and of a possible shift of x-variable.

Then it follows that

$$H''(t) = \int_{t+R_0}^{t+R} u(x,s)dx = \frac{\varepsilon}{2}\int_{t+R_0}^{t+R}\left\{f(x+t)+f(x-t)+\frac{1}{2}\int_{x-t}^{x+t} g(y)dy\right\}dx + \frac{1}{2}F(t), \tag{4.1}$$

where

$$F(t) := \int_{t+R_0}^{t+R} dx \int_0^t ds \int_{x-t+s}^{x+t-s} |u_x(y,s)|^p dy.$$

By virtue of (2.15) and (4.1), we have that

$$H''(t) \geq \frac{\varepsilon}{2}\int_{t+R_0}^{t+R} f(x-t)dx \geq 2C_f\varepsilon,$$

where

$$C_f := \frac{1}{4}\int_{R_0}^{R} f(y)dy > 0.$$

Integrating this inequality in $[0,t]$ twice and noting that $H'(0) = H(0) = 0$, we have

$$H(t) \geq C_f \varepsilon t^2. \tag{4.2}$$

On the other hand, $F(t)$ can be rewritten as

$$F(t) = \int_0^t ds \int_{t+R_0}^{t+R} dx \int_{x-t+s}^{x+t-s} |u_x(y,s)|^p dy.$$

From now on, we assume that

$$t \geq R_1 := \frac{R-R_0}{2} > 0. \tag{4.3}$$

Then, inverting the order on (y,x)-integral, for $0 \leq s \leq t - R_1$, we have that

$$\begin{aligned}
&\int_{t+R_0}^{t+R} dx \int_{x-t+s}^{x+t-s} |u_x(y,s)|^p dy \\
&= \left(\int_{s+R_0}^{s+R}\int_{t+R_0}^{y+t-s} + \int_{s+R}^{2t-s+R_0}\int_{t+R_0}^{t+R} + \int_{2t-s+R_0}^{2t-s+R}\int_{y-t+s}^{t+R}\right)|u_x(y,s)|^p dxdy \\
&\geq \int_{s+R_0}^{s+R} dy \int_{t+R_0}^{y+t-s} |u_x(y,s)|^p dx.
\end{aligned}$$

Similarly, for $t - R_1 \le s \le t$, we also have that

$$
\begin{aligned}
&\int_{t+R_0}^{t+R} dx \int_{x-t+s}^{x+t-s} |u_x(y,s)|^p dy \\
=&\left(\int_{s+R_0}^{2t-s+R_0} \int_{t+R_0}^{y+t-s} + \int_{2t-s+R_0}^{s+R} \int_{y-t+s}^{y+t-s} + \int_{s+R}^{2t-s+R} \int_{y-t+s}^{t+R}\right) |u_x(y,s)|^p dx dy \\
\ge& \int_{s+R_0}^{2t-s+R_0} dy \int_{t+R_0}^{y+t-s} |u_x(y,s)|^p dx + \int_{2t-s+R_0}^{s+R} dy \int_{y-t+s}^{y+t-s} |u_x(y,s)|^p dx.
\end{aligned}
$$

Hence, we obtain that

$$
\begin{aligned}
F(t) \ge& \int_0^{t-R_1} ds \int_{s+R_0}^{s+R} (y - s - R_0)|u_x(y,s)|^p dy \\
&+ \int_{t-R_1}^{t} ds \int_{s+R_0}^{2t-s+R_0} (y - s - R_0)|u_x(y,s)|^p dy \\
&+ \int_{t-R_1}^{t} ds \int_{2t-s+R_0}^{s+R} 2(t-s)|u_x(y,s)|^p dy.
\end{aligned}
$$

Therefore, it follows from (4.3) and

$$
1 = \frac{y-s-R_0}{y-s-R_0} \ge \frac{y-s-R_0}{R-R_0} \ge \frac{y-s-R_0}{2t}
$$

that

$$
\begin{aligned}
F(t) \ge& \int_0^{t-R_1} \frac{t-s}{t} ds \int_{s+R_0}^{s+R} (y - s - R_0)|u_x(y,s)|^p dy + \int_{t-R_1}^{t} \frac{t-s}{t} ds \int_{s+R_0}^{2t-s+R_0} (y - s - R_0)|u_x(y,s)|^p dy \\
&+ \int_{t-R_1}^{t} 2(t-s) ds \int_{2t-s+R_0}^{s+R} \frac{y-s-R_0}{2t} |u_x(y,s)|^p dy \\
=&\frac{1}{t}\int_0^t (t-s) ds \int_{s+R_0}^{s+R} (y - s - R_0)|u_x(y,s)|^p dy.
\end{aligned}
$$

In this way, (2.15), (4.1) and the estimate of $F(t)$ above yield that

$$
H''(t) \ge \frac{1}{2} F(t) \ge \frac{1}{2t}\int_0^t (t-s) ds \int_{s+R_0}^{s+R} (y - s - R_0)|u_x(y,s)|^p dy \quad \text{for } t \ge R_1.
$$

Moreover, it follows from (2.2), integration by parts and Hölder's inequality that

$$
\begin{aligned}
|H(t)| =& \left|\int_0^t (t-s) ds \int_{s+R_0}^{s+R} \partial_y (y - s - R_0) u(y,s) dy\right| \\
=& \left|\int_0^t (t-s) ds \int_{s+R_0}^{s+R} (y - s - R_0) u_x(y,s) dy\right| \\
\le& \int_0^t (t-s) ds \int_{s+R_0}^{s+R} (y - s - R_0) |u_x(y,s)| dy \\
\le& \left(\int_0^t (t-s) ds \int_{s+R_0}^{s+R} (y - s - R_0) |u_x(y,s)|^p dy\right)^{1/p} I(t)^{1-1/p},
\end{aligned}
$$

where

$$I(t) := \int_0^t (t-s)ds \int_{s+R_0}^{s+R} (y-s-R_0)dy = \frac{1}{4}t^2(R-R_0)^2 = t^2R_1^2.$$

Hence, we obtain that

$$H''(t) \geq \frac{1}{2}R_1^{-2(p-1)}t^{1-2p}|H(t)|^p \quad \text{for } t \geq R_1. \tag{4.4}$$

Therefore, the argument in Rammaha [12] can be applied to (4.2) and (4.4) to ensure that there exist positive constants $\varepsilon_2 = \varepsilon_2(f, p, R)$ and C_2 independent of ε such that a contradiction appears provided

$$T > C_2\varepsilon^{-(p-1)}$$

holds for $0 < \varepsilon \leq \varepsilon_2$. The proof is now completed. □

5. Conclusions

Our theorems could be extended to higher dimensional case basically along with our method, but we have to assume that the solution is radially symmetric at least, which is closely related to "Glassey's conjecture" for nonlinear term of $|u_t|^p$. See Hidano, Wang and Yokoyama [1] for this direction.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests in this paper.

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