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Research article

Generalized common best proximity point results in fuzzy multiplicative metric spaces

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Abstract: In this manuscript, we prove the existence and uniqueness of a common best proximity point for a pair of non-self mappings satisfying the iterative mappings in a complete fuzzy multiplicative metric space. We consider the pair of non-self mappings $X : \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ and the mappings do not necessarily have a common fixed-point. In a complete fuzzy multiplicative metric space, if φ satisfy the condition $\varphi(b, Zb, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(b, Xb, \varsigma)$, then b is a common best proximity point. Further, we obtain the common best proximity point for the real valued functions $\mathcal{L}, \mathcal{M} : (0, 1] \to \mathbb{R}$ by using a generalized fuzzy multiplicative metric space in the setting of $(\mathcal{L}, \mathcal{M})$-iterative mappings. Furthermore, we utilize fuzzy multiplicative versions of the $(\mathcal{L}, \mathcal{M})$-proximal contraction, $(\mathcal{L}, \mathcal{M})$-interpolative Reich-Rus-Ciric type proximal contractions, $(\mathcal{L}, \mathcal{M})$-Kannan type proximal contraction and $(\mathcal{L}, \mathcal{M})$-interpolative Hardy-Rogers type proximal contraction to examine the common best proximity points in fuzzy multiplicative metric space. Moreover, we provide differential non-trivial examples to support our results.

Keywords: fuzzy multiplicative metric space; fixed-point; best proximity point theorems; contraction mappings; uniqueness
Mathematics Subject Classification: 26E05, 26E25, 47H10

1. Introduction

Fixed-point theory has been studied by several researchers since 1922 with the celebrated Banach fixed-point theorem. Fixed-point theory has several applications in non linear analysis and even in mathematics in general. Its results can be applied to an extensive set of distinct type of equations (integral, differential, metrical, etc.) in order to prove the existence and uniqueness of several classes of nonlinear problems. Numerous researchers have examined fixed-point theory and specifically established and publicized many spaces the area of fixed-point theory more interesting. It has expended the possibilities to obtaining a unique solution in the field of mathematics. Let a non self mapping $X : \mathcal{P} \to \mathcal{G}$ where $\mathcal{P} \cap \mathcal{G} = \Phi$; then, a point $m \in \mathcal{P}$ is said to be the best proximity point (BPP) if $\varphi(m, Xm) = \varphi(\mathcal{P}, X\mathcal{G})$. First of all, Fan [1] established the best approximation theorem. The BPP theorem has an optimal solution, so it is more relevant than the best approximation theorem.

In 1968, Karapinar [2] introduced a new kind of contraction for discontinuous mappings and proved several fixed-points results. Altun et al. [3] gave some BPP results for p-proximal contractions. Further, Altun and Tasdemir [4] proved some BPP results for interpolative proximal contractions. Shazad et al. [5] provided some common best proximity point (CBPP) results. Basha [6] developed CBPP results for a global minimal solution. Moreover, Basha [7] examined CBPPs for multi-objective functions. Deep and Betra [8] introduced some CBPP results for the proximal F-contraction. Mondal and Dey [9] proved some CBPP results in complete metric spaces (CMSs). Shayanpour and Nematizadeh [10] presented some CBPP results in complete fuzzy metric space.

Hierro et al. [11] presented Proinov type fixed-point results in FMS. Then, Zhou et al. [12] modified the results of [11] and introduced new Proinov-type fixed-point results in FMS. Vetro and Salimi [13] established BPP theorems in non-Archimedean FMS. Paknazar [14] established some BPP theorms in FMS. In the second half of 17th century, sir Isaac Newton and Gottfried Wilhelm Leibniz established the most reliable theorems of differential and integral calculus. These two operations are the basic tools of calculus. It provided a new way for researchers. Hence, many authors have worked on it and proved several types of BPP theorems for it. Due to the development of new calculus by Grossman and Katz [15], known as multiplicative calculus, Bashirov et al. [16] established a new calculus, i.e., multiplicative calculus. Mongkolkeha and Sintunavarat [17] proved several proximity points for multiplicative proximal contraction mappings. Farheen et al. [18] introduced fuzzy multiplicative metric space (FMMS) and discussed the topological properties of FMMS. Every FMMS is Hausdorff. Uddin et al. [19] gave the concept of an controlled intuitionistic fuzzy metric-like space through the use of a continuous t-norm. Saleem et al. [20] presented a result on the graphical FMS, which is a type of FMS and proved a Banach fixed-point results in the graphical FMS. Ishtiaq et al. [21] established several fixed-point results for a generalized fuzzy rectangular metric like-space and rectangular b-metric-like spaces. There are many applications of fixed-point theory, such as establishing the well-posedness of partial differential equations and algorithms design for optimization and inverse problems. Based on the fixed-point theory, Shcheglov et al. [22] proved the uniqueness of an inverse problems for parabolic partial differential equations. Zhang and Hofmann [23] established fixed-point iterations in combination with preconditioning ideas, and they introduced new iterative regularization algorithms for inverse problems with non-negative constraints. Lin et al. [24] constructed a contraction mapping such that its fixed-point is just the gradient of a solution to the elliptic partial differential equations. By using the Schauder fixed-point theory, Baravdish et al. [25] proved the existence and uniqueness of a second order (in time) hyperbolic equation.

Inspired by the work of [10], we introduce the fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-proximal mapping, which is an extension of a mapping provided in [10]. We have divided this paper into three main parts. In the first part, we give some basic definitions and results that will help to understand our main results. In the second part, we prove theorems through the use of differential types of contractions, and examples are given to verify the results. In the third part, we give the conclusion of this paper.

2. Preliminaries

In this section, we revise some basic definition and results to introduce the main results.

Definition 2.1. *[6] The mappings $X : \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ are proximally commutative if they satisfy the condition*

$$[\varphi(\hbar, Xm) = \varphi(d, Zm) = \varphi(\mathcal{P}, \mathcal{G})] \Rightarrow Xd = Z\hbar,$$

for all $m, \hbar, d$ in $\mathcal{P}$.

Definition 2.2. *[6] A mapping $Z : \mathcal{P} \to \mathcal{G}$ proximally dominate to a mapping $X : \mathcal{P} \to \mathcal{G}$ if there exists a non-negative number $\alpha < 1$ such that*

$$\begin{aligned} \varphi(\hbar_1, Xm_1) &= \varphi(\mathcal{P}, \mathcal{G}) = \varphi(d_1, Zm_1) \\ \varphi(\hbar_2, Xm_2) &= \varphi(\mathcal{P}, \mathcal{G}) = \varphi(d_2, Zm_2) \end{aligned}$$

$$\varphi(\hbar_1, \hbar_2) \leq \alpha\varphi(d_1, d_2),$$

for all $\hbar_1, \hbar_2, d_1, d_2, m_1, m_2 \in \mathcal{P}$.

Definition 2.3. *[16] Suppose that $C \neq \Phi$. A mapping $\varphi : C \times C \to \mathbb{R}$ is said to be a multiplicative metric space if it satisfies the following conditions:*

D_1: $\varphi(\hbar_1, \hbar_2) > 1$ for all $\hbar_1, \hbar_2 \in C$ and $\varphi(\hbar_1, \hbar_2) = 1$ if and only if $\hbar_1 = \hbar_2$;
D_2: $\varphi(\hbar_1, \hbar_2) = \varphi(\hbar_2, \hbar_1)$;
D_3: $\varphi(\hbar_1, \hbar_3) \leq \varphi(\hbar_1, \hbar_2) .\varphi(\hbar_2, \hbar_3)$ for all $\hbar_1, \hbar_2, \hbar_3 \in C$.

Definition 2.4. *[18] A binary operation $* : G \times G \to G$ (where $G = [0, 1]$) is said to be a continuous t-norm (ctn) if it verifies the below axioms:*

(1) $\hbar_1 * \hbar_2 = \hbar_1 * \hbar_2$ and $\hbar_1 * (\hbar_2 * \hbar_3) = (\hbar_1 * \hbar_2) * \hbar_3$ for all $\hbar_1, \hbar_2, \hbar_3 \in G$;
(2) $*$ is continuous;
(3) $\hbar_1 * 1 = \hbar_1$ for all $\hbar_1 \in G$;
(4) $\hbar_1 * \hbar_2 \leq \hbar_3 * \hbar_4$ when $\hbar_1 \leq \hbar_3$ and $\hbar_2 \leq \hbar_4$, with $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \in G$.

Definition 2.5. *[18] A triplet $(C, \varphi, *)$ is termed an FMMS if $*$ is a ctn, C is an arbitrary set, and φ is a fuzzy set on $C \times C \times (1, \infty)$ fulfilling the below conditions for all $\hbar_1, \hbar_2, \hbar_3 \in C$ and $\varsigma, \varpi > 1$:*

(i) $\varphi(\hbar_1, \hbar_2, \varsigma) > 0$;
(ii) $\varphi(\hbar_1, \hbar_2, \varsigma) = 1$ if and only if $\hbar_1 = \hbar_2$;
(iii) $\varphi(\hbar_1, \hbar_2, \varsigma) = \varphi(\hbar_2, \hbar_1, \varsigma)$;
(iv) $\varphi(\hbar_1, \hbar_3, \varsigma.\varpi) \geq \varphi(\hbar_1, \hbar_2, \varsigma) * \varphi(\hbar_2, \hbar_3, \varpi)$;
(v) $\varphi(\hbar_1, \hbar_2, .) : (1, \infty) \to [0, 1]$.

Example 2.1. *Suppose* $C = \mathbb{R}^+$ *and* $\varphi(\hbar_1, \hbar_2, \varsigma) = \frac{\varsigma+1}{\varsigma+e^{|\hbar_1-\hbar_2|}}$, *with a ctn as* $s*t = st$. *Then,* C *is an FMMS.*

Definition 2.6. *[18] A sequence* $\{\hbar_n\}$ *in an FMMS* $(C, \varphi, *)$ *is said to be convergent to a point* $\hbar \in C$ *if and only if for each* $\varepsilon > 0$ *and* $\zeta \in (0,1)$, *there exists* $a_0(\varepsilon, \zeta) \in \mathbb{N}$ *such that* $\varphi(\hbar, \hbar_n, \varsigma) > 1 - \zeta$ *for all* $n \geq a_0(\varepsilon, \zeta)$ *or* $\lim_{n\to\infty} \varphi(\hbar, \hbar_n, \varsigma) = 1$ *for all* $\varsigma > 1$*; in this case we say that the sequence* $\{\hbar_n\}$ *is convergent.*

Definition 2.7. *[18] A sequence* $\{\hbar_n\}$ *in an FMMS* $(C, \varphi, *)$ *is said to be Cauchy if and only if for each* $\varepsilon > 0$ *and* $\zeta \in (0,1)$, *there exists* $a_0(\varepsilon, \zeta) \in \mathbb{N}$ *such that* $\varphi\left(\hbar_n, \hbar_{n+p}, \varsigma\right) > 1 - \zeta$ *for all* $n \geq a_0(\varepsilon, \zeta)$ *and every* $p \in \mathbb{N}$ *or* $lim_{n\to\infty}\varphi\left(\hbar_n, \hbar_{n+p}, \varsigma\right) = 1$, *for all* $\varsigma > 1$ *and* $p \in \mathbb{N}$.

Also, an FMMS $(C, \varphi, *)$ is said to be complete if and only if every Cauchy sequence (CS) in C is convergent.

Definition 2.8. *[18] Let* $(C, \varphi, *)$ *be a FMMS and* $\mathcal{P}, \mathcal{G} \subset C$. *We define the following sets.*

$$\begin{aligned} \mathcal{P}_0 &= \{\hbar_1 \in \mathcal{P} : \text{there exists } \hbar_2 \in \mathcal{G} \text{ such that for all } \varsigma > 1, \varphi(\hbar_1, \hbar_2, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma)\}, \\ \mathcal{G}_0 &= \{\hbar_2 \in \mathcal{G} : \text{there exists } \hbar_1 \in \mathcal{P} \text{ such that for all } \varsigma > 1, \varphi(\hbar_1, \hbar_2, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma)\}, \end{aligned}$$

where,

$$\varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \sup\{\varphi(\hbar_1, \hbar_2, \varsigma), \hbar_1 \in \mathcal{P}, \hbar_2 \in \mathcal{G}\}.$$

Definition 2.9. *[18] Let* $(C, \varphi, *)$ *be a FMMS and* $\mathcal{P}, \mathcal{G} \subset C$ *if every sequence* $\{\hbar_n\}$ *of* $\mathcal{P}$ *verifying the condition that* $\varphi(\hbar, \hbar_n, \varsigma) \to \varphi(\hbar, \mathcal{P}, \varsigma)$ *for some* $\hbar \in \mathcal{G}$ *and for all* $\varsigma > 1$, *has a convergent subsequence then,* $\mathcal{P}$ *is termed as approximately compact with respect to* $\mathcal{G}$.

Definition 2.10. *[18] Let* $(C, \varphi, *)$ *be a FMMS and* $\mathcal{P}, \mathcal{G} \subset C$. *A mapping* $Z : \mathcal{P} \to \mathcal{G}$ *is named a multiplicative contraction if there exists* $\alpha \in [0,1)$, *such that for all* $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \in \mathcal{P}$

$$\begin{aligned} \varphi(\hbar_1, Z\hbar_2, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(\hbar_3, Z\hbar_4, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma); \end{aligned}$$

hence,

$$\varphi(\hbar_1, \hbar_3, \varsigma^\alpha) \geq \varphi(\hbar_2, \hbar_4, \varsigma).$$

Definition 2.11. *[10] Let* $(C, \varphi, *)$ *be a FMS and* $\mathcal{P}, \mathcal{G} \subset C$. *Let* $Z, X : \mathcal{P} \to \mathcal{G}$ *be two mappings. We say that an element* $\hbar \in \mathcal{P}$ *is a CBPP of the mappings* Z *and* X, *if*

$$\varphi(\hbar, Z\hbar, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar, X\hbar, \varsigma).$$

Definition 2.12. *[10] Let* $(C, \varphi, *)$ *be a FMS and* $\mathcal{P}, \mathcal{G} \subset C$. *Let* $Z, X : \mathcal{P} \to \mathcal{G}$ *be two mappings. We say that* Z, X *are proximally commutative if*

$$\varphi(\hbar_1, Z\hbar, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_2, X\hbar, \varsigma), \forall \varsigma > 0;$$

then, $Z\hbar_2 = X\hbar_1$, *where* $\hbar, \hbar_1, \hbar_2 \in \mathcal{P}$.

Definition 2.13. *[10] Let $(C, \varphi, *)$ be a FMS and $\mathcal{P}, \mathcal{G} \subset C$. Let $Z, X : \mathcal{P} \to \mathcal{G}$ be two mappings. We say that the mapping Z dominates X proximally if*

$$\begin{aligned} \varphi(\hbar_1, Zh_1, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_1, Xh_2, \varsigma), \\ \varphi(\hbar_2, Zh_1, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_2, Xh_2, \varsigma), \end{aligned}$$

for all $\varsigma > 0$; then, there exists $\alpha \in (0, 1)$ such that for all $\varsigma > 0$,

$$\varphi(\hbar_1, \hbar_2, \alpha\varsigma) \geq \varphi(\hbar_1, \hbar_2, \varsigma),$$

where $\hbar_1, \hbar_2, \hbar_1, \hbar_2, h_1, h_2 \in \mathcal{P}$.

Definition 2.14. *[12] We denote by $\acute{L}$ the family of the pairs $(\mathcal{L}, \mathcal{M})$ of a functions $\mathcal{L}, \mathcal{M}$:$(0, 1] \to \mathbb{R}$ satisfying the given bellow properties:*

(s_1) $\mathcal{L}$ is non-decreasing;
(s_2) $\mathcal{M}(\hbar) > \mathcal{L}(\hbar)$ for any $\hbar \in (0, 1)$;
(s_3) $\lim_{\hbar \to T^-} \inf \mathcal{M}(\hbar) > \lim_{s \to T^-} \mathcal{L}(\hbar)$ for any $T^- \in (0, 1)$;
(s_4) If $\hbar \in (0, 1)$ is such that $\mathcal{M}(\hbar) \geq \mathcal{L}(1)$ then $\hbar = 1$.

3. Main results

In this section, we prove several CBPP results by utilizing generalized fuzzy multiplicative interpolative contractions, as well as prove non-trivial examples.

3.1. Fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-proximal contraction:

Let $(C, \varphi, *)$ be an FMMS and $\mathcal{P}, \mathcal{G} \subset C$. The mappings $Z : \mathcal{P} \to \mathcal{G}$ and $X : \mathcal{P} \to \mathcal{G}$ are called fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-proximal, if

$$\begin{aligned} \varphi(\hbar_1, Xm_1, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(d_1, Zm_1, \varsigma), \\ \varphi(\hbar_2, Xm_2, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(d_2, Zm_2, \varsigma), \\ \mathcal{L}(\varphi(\hbar_1, \hbar_2, \varsigma)) &\geq \mathcal{M}(\varphi(d_1, d_2, \varsigma)), \end{aligned} \tag{3.1}$$

holds for all $\hbar_1, \hbar_2, d_1, d_2, m_1, m_2 \in \mathcal{P}$ and $\varsigma > 1$.

Example 3.1. *Let $(C, \varphi, *)$ be an FMMS. Define $\varphi(m, n, \varsigma) = \frac{\varsigma+1}{\varsigma+e^{|m-n|}}$ with a ctn as $s * t = st$.*

Let $\mathcal{P} = \{0, 2, 4, 6, 8, 10\}$ and $\mathcal{G} = \{1, 3, 5, 7, 9, 11\}$.

Define the mappings $X : \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$, respectively as

$$Z(0) = 3, Z(2) = 5, Z(4) = 7, Z(6) = 3, Z(8) = 9, Z(10) = 11,$$

and

$$X(0) = 3, X(2) = 1, X(4) = 9, X(6) = 7, X(8) = 5, X(10) = 11.$$

Then, $\varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \frac{\varsigma+1}{\varsigma+e}$, $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{G}_0 = \mathcal{G}$. Then clearly $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $Z(\mathcal{P}_0) \subseteq \mathcal{G}_0$. Define the functions $\mathcal{L}, \mathcal{M} : (0, 1] \to \mathbb{R}$ by

$$\mathcal{L}(l) = \left\{ \begin{array}{c} \frac{1}{2^{\ln 2l}} \text{ if } 0 < l < 1 \\ 1 \text{ if } l = 1 \end{array} \right\} \text{and } \mathcal{M}(l) = \left\{ \begin{array}{c} \frac{1}{2^{\ln l}} \text{ if } 0 < l < 1 \\ 2 \text{ if } l = 1 \end{array} \right\}.$$

This shows that the mappings X and Z are fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-proximal. However, consider $\hbar_1 = 0, \hbar_2 = 8, d_1 = 4, d_2 = 6, m_1 = 2, m_2 = 4,$ and $\varsigma = 2$, which shows that X and Z are not proximal in FMMS. Hence,

$$\begin{aligned} \varphi(0, X2, 1) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(4, Z2, 1), \\ \varphi(8, X4, 1) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(6, Z4, 1). \end{aligned}$$

For $\alpha = \frac{1}{2} \in (0, 1)$, it follows that

$$\begin{aligned} \varphi(\hbar_1, \hbar_2, \varsigma^{\alpha}) &\geq \varphi(d_1, d_2, \varsigma), \\ \varphi\left(0, 8, 2^{\frac{1}{2}}\right) &\geq \varphi(4, 6, 1), \\ 0.0008 &\geq 0.3195, \end{aligned}$$

which is contradiction. Hence, the mappings X and Z are not fuzzy multiplicative proximal.

Lemma 3.1. *Let* $(C, \varphi, *)$ be an FMMS and $\{\hbar_n\} \subset C$ be a sequence verifying that $\lim_{n\to\infty} \varphi(\hbar_n, \hbar_{n+1}, \varsigma) = 1$. If the sequence $\{q_n\}$ is not a CS, then there are sub-sequences $\{\hbar_{n_k}\}$, $\{\hbar_{q_k}\}$ and $w > 0$ such that

$$\lim_{k\to\infty} \varphi(\hbar_{n_k+1}, \hbar_{q_k+1}, \varsigma) = w, \tag{3.2}$$

$$\lim_{k\to\infty} \varphi(\hbar_{n_k}, \hbar_{n_{q_k}}, \varsigma) = \varphi(\hbar_{n_k+1}, \hbar_{q_k}, \varsigma) = \varphi(\hbar_{n_k}, \hbar_{q_k+1}, \varsigma) = w. \tag{3.3}$$

Lemma 3.2. *Let* $\mathcal{L} : (0, 1] \to \mathbb{R}$. *Then the following conditions are equivalent:*

(i) $\inf_{l>\varepsilon} \mathcal{L}(l) > -\infty$ for every $\varepsilon \in (0, 1)$,
(ii) $\lim_{l\to\varepsilon-} \inf \mathcal{L}(l) > -\infty$ for any $\varepsilon \in (0, 1)$,
(iii) $\lim_{n\to\infty} \mathcal{L}(l_n) = -\infty$ implies that $\lim_{n\to\infty} l_n = 1$.

Lemma 3.3. *Assume that* $\{\hbar_n\}$ *is a sequence such that* $\lim_{n\to\infty} \varphi(\hbar_n, \hbar_{n+1}, \varsigma) = 1$ *and the mappings* $Z : \mathcal{P} \to \mathcal{G}$ *and* $X : \mathcal{P} \to \mathcal{G}$ *satisfy (3.1). If the functions* $\mathcal{L}, \mathcal{M} : (0, 1] \to \mathbb{R}$ *fulfill the following condition:*

(i) $\limsup_{l\to\epsilon+} \mathcal{M}(l) > \mathcal{L}(\epsilon+)$ *for any* $\epsilon > 0$,

then $\{\hbar_n\}$ *is a CS.*

Proof. Suppose that a sequence $\{\hbar_n\}$ is not a CS; then, by Lemma (3.1), there exist two sub-sequences $\{\hbar_{n_k}\}$, $\{\hbar_{q_k}\}$ of $\{\hbar_n\}$ and $\epsilon > 0$ such that (3.2) and (3.3) hold. By (3.2), we get that $\varphi(\hbar_{n_k+1}, \hbar_{q_k+1}) > \epsilon$. Now, for $\hbar_{n_k}, \hbar_{n_k+1}, \hbar_{q_k}, \hbar_{q_k+1}, m_{n_k}, m_{q_k}, m_{n_k+1}, m_{q_k+1} \in \mathcal{P}$, we have

$$\begin{aligned} \varphi(\hbar_{n_k+1}, X(m_{n_k+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi\left(\hbar_{q_k+1}, X\left(m_{q_k+1}\right), \varsigma\right), \\ \varphi(\hbar_{n_k}, Z(m_{n_k+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi\left(\hbar_{q_k}, Z\left(m_{q_k+1}\right), \varsigma\right). \end{aligned}$$

Thus, by (3.1), we have

$$\mathcal{L}(\varphi(\hbar_{n_k+1}, \hbar_{q_k+1}), \varsigma) \geq \mathcal{M}(\varphi(\hbar_{n_k}, \hbar_{q_k}, \varsigma)),$$

for all $k \geq 1$. Let $q_k = \varphi(\hbar_{n_k+1}, \hbar_{q_k+1}, \varsigma)$ and $q_{k-1} = \varphi(\hbar_{n_k}, \hbar_{q_k}, \varsigma)$; we have

$$\mathcal{L}(q_k) \geq \mathcal{M}(q_{k-1}) \text{ for any } k \geq 1. \tag{3.4}$$

By (3.2) and (3.3), we have that $\lim_{k\to\infty} q_k = \epsilon$ and $\lim_{k\to\infty} n_k = \epsilon$. By (3.4), we get that

$$\mathcal{L}(\epsilon+) = \lim_{k\to\infty} \mathcal{L}(q_k) \geq \lim_{k\to\infty}\inf \mathcal{M}(q_{k-1}) \geq \lim_{c\to k}\inf \mathcal{M}(c). \tag{3.5}$$

This is a contradiction to condition (i). That is, $\{\hbar_n\}$ is a CS in $\mathcal{P}$.

Theorem 3.1. *Let $(C, \varphi, *)$ be a complete FMMS (CFMMS) and $\mathcal{P}, \mathcal{G} \subset C$ such that $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$. Also, assume that $\lim_{k\to\infty} \varphi(\hbar_1, \hbar_2, \varsigma) = 1$ and $\mathcal{P}_0$ and $\mathcal{G}_0 \neq \Phi$. Let $X \colon \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ satisfy the following conditions:*

(i) Z dominates X and is fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$ −proximal,
(ii) X and Z proximally commutative,
(iii) $\mathcal{L}$ is a non-decreasing function and $\lim\inf_{l\to\epsilon+} \mathcal{M}(l) > \mathcal{L}(\epsilon+)$ for any $\epsilon > 0$,
(iv) X and Z are continuous,
(v) $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.
Then, Z and X have a unique element $m \in \mathcal{P}$ such that

$$\begin{aligned} \varphi(m, Zm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(m, Xm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned}$$

Proof. Suppose that $m_0 \in \mathcal{P}_0$. From (v), we have that $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$; then, there exists an element $m_1 \in \mathcal{P}_0$ such that, $Xm_0 = Zm_1$. Again, from (v), there exists an element $m_2 \in \mathcal{P}_0$ such that $Xm_1 = Zm_2$. This process of existence of points in $\mathcal{P}_0$ implies the existence of a sequence $\{m_n\} \subseteq \mathcal{P}_0$ such that

$$Xm_{n-1} = Zm_n$$

for all positive integral values of n, since $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

Since $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$, there exists an element $\hbar_n$ in $\mathcal{P}_0$ such that

$$\varphi(\hbar_n, Xm_n, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \text{ for all } n \in \mathbb{N}.$$

Further, it follows from the choice of m_n and $\hbar_n$ that

$$\begin{aligned} \varphi(\hbar_{n+1}, X(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_n, Z(m_{n+1}), \varsigma), \\ \varphi(\hbar_n, Xm_n, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma). \end{aligned}$$

This implies that

$$\varphi(\hbar_n, Xm_n, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma). \tag{3.6}$$

Thus, if there exists some $n \in \mathbb{N}$ such that $\hbar_n = \hbar_{n-1}$; then, by (3.6), the point $\hbar_n$ is a CBPP of the mappings X and Z. On the other hand, if $\hbar_{n-1} \neq \hbar_n$ for all $n \in \mathbb{N}$, then, by (3.6), we have

$$\begin{aligned} \varphi(\hbar_{n+1}, X(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_n, Z(m_{n+1}), \varsigma), \\ \varphi(\hbar_n, X(m_n), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma). \end{aligned}$$

Thus, by (3.1), we have

$$\mathcal{L}(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) \geq \mathcal{M}(\varphi(\hbar_n, \hbar_{n-1}, \varsigma)), \tag{3.7}$$

for all $\hbar_{n-1}, \hbar_n, \hbar_{n+1}, m_{n+1}, m_n \in \mathcal{P}$. Let $\varphi(\hbar_{n+1}, \hbar_n, \varsigma) = q_n$; we have

$$\mathcal{L}(q_n) \geq \mathcal{M}(q_{n-1}) > \mathcal{L}(q_{n-1}).$$

Since $\mathcal{L}$ is non-decreasing, by (3.7), we get that $q_n > q_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{q_n\}$ is positive and strictly non-decreasing. Hence, it converges to some element $q \geq 1$. We show that $q = 1$. Suppose on the contrary that, $q > 1$; then, by (3.7), we get the following:

$$\mathcal{L}(\varepsilon+) = \lim_{n\to\infty} \mathcal{L}(q_n) \geq \lim_{n\to\infty} \mathcal{M}(q_{n-1}) \geq \lim_{n\to q+} \inf \mathcal{M}(l).$$

This contradicts assumption (iii); hence, $q = 1$ and $\lim_{n\to\infty} \varphi(\hbar_n, \hbar_{n+1}, \varsigma) = 1$. By assumption (iii) and Lemma (3.3), we deduce that $\{\hbar_n\}$ is a CS. From the completeness of $(C, \varphi, *)$; and by using (v), there exists an element $\hbar^*$ in $\mathcal{P}$ such that $\lim_{n\to\infty} \varphi(\hbar_n, \hbar^*, \varsigma) = 1$. Moreover,

$$\varphi(\hbar^*, X(m_n), \varsigma) \geq \varphi(\hbar^*, \hbar_n, \varsigma) * \varphi(\hbar_n, X(m_n), \varsigma).$$

Also,

$$\varphi(\hbar^*, Z(m_n), \varsigma) \geq \varphi(\hbar^*, \hbar_n, \varsigma) * \varphi(\hbar_n, Z(m_n), \varsigma).$$

Therefore, $\varphi(\hbar^*, Z(m_n), \varsigma) \to \varphi(\hbar^*, \mathcal{G}, \varsigma)$ and also $\varphi(\hbar^*, X(m_n), \varsigma) \to \varphi(\hbar^*, \mathcal{G}, \varsigma)$ as $n \to \infty$. Because X and Z proximally commutative, $Z\hbar^*$ and $X\hbar^*$ are the same. Since $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$, there exists sub-sequences $\{Z(m_{n_k})\}$ of $\{Z(m_n)\}$ and $\{X(m_{n_k})\}$ of $\{X(m_n)\}$ such that $Z(m_{n_k}) \to d^* \in \mathcal{G}$ and $X(m_{n_k}) \to d^* \in \mathcal{G}$ as $k \to \infty$. Moreover, supposing that $k \to \infty$ in the below equations:

$$\begin{aligned} \varphi(d^*, X(m_{n_k}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(d^*, Z(m_{n_k}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \end{aligned} \tag{3.8}$$

we have

$$\varphi(d^*, \hbar^*, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma).$$

The fact that $\hbar^* \in \mathcal{P}_0$ implies that $X(\hbar^*) \in X(\mathcal{P}_0)$; also, by using (v) there exists an element $w \in \mathcal{P}_0$. Similarly $\hbar^* \in \mathcal{P}_0$, so $Z(\hbar^*) \in Z(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and there exists $w \in \mathcal{P}_0$ such that

$$\begin{aligned} \varphi(\hbar^*, X(\hbar^*), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, Z(\hbar^*), \varsigma), \\ \varphi(w, X(\hbar^*), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(w, Z(\hbar^*), \varsigma). \end{aligned} \tag{3.9}$$

Now, by (3.1), (3.8) and (3.9), we have

$$\mathcal{L}(\varphi(\hbar^*, w, \varsigma)) \geq \mathcal{M}(\varphi(\hbar^*, w, \varsigma)) < \mathcal{L}(\varphi(\hbar^*, w, \varsigma)).$$

Since, $\mathcal{L}$ is a non-decreasing function, we have

$$\varphi(\hbar^*, w, \varsigma^\alpha) \geq \varphi(\hbar^*, w, \varsigma) > \varphi(\hbar^*, w, \varsigma^\alpha).$$

This implies that $\hbar^*$ and w are the same. Finally, by (3.6), we have

$$\varphi(\hbar^*, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, X(\hbar^*), \varsigma).$$

This shows that the point $\hbar^*$ is a CBPP of the pair of mappings Z and X.

Theorem 3.2. *Let $(C, \varphi, *)$ be a CFMMS and $\mathcal{P}, \mathcal{G} \subset C$ such that $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$. Also, assume that $\lim_{k\to\infty} \varphi(\hbar_1, \hbar_2, \varsigma) = 1$ and $\mathcal{P}_0, \mathcal{G}_0 \neq \Phi$. Let $X\colon \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ satisfy the following conditions:*

(i) Z dominates X and is fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-proximal,
(ii) X and Z are proximally commutative,
(iii) $\mathcal{L}$ is non-decreasing and $\{\mathcal{L}(l_n)\}$ and $\{\mathcal{M}(l_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \mathcal{L}(l_n) = \lim_{n\to\infty} \mathcal{M}(l_n)$; then, $\lim_{n\to\infty} l_n = 1$,
(iv) X and Z are continuous,
(v) $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.
Then Z and X have a unique element $m \in \mathcal{P}$ such that

$$\begin{aligned} \varphi(m, Zm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(m, Xm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned}$$

Proof. Proceeding as in the proof of Theorem (3.1), we get

$$\mathcal{L}(q_n) \geq \mathcal{M}(q_{n-1}) < \mathcal{L}(q_{n-1}). \tag{3.10}$$

By (3.10), we infer that $\{\mathcal{L}(q_n)\}$ is a strictly non-decreasing sequence. We have two cases here, i.e., the sequence $\{\mathcal{L}(q_n)\}$ is either bounded above or not. If $\{\mathcal{L}(q_n)\}$ is not bounded above, then

$$\inf_{w_n > \varepsilon} \mathcal{L}(q_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows from Lemma (3.1), that $q_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\mathcal{L}(q_n)\}$ is bounded above, then it is a convergent sequence. By (3.10), the sequence $\{\mathcal{M}(q_n)\}$ is also convergent. Furthermore, both have the same limit. By condition (iii), we get that $\lim_{n\to\infty} q_n = 1$, or that $\lim_{n\to\infty} \varphi(\hbar_n, m_{n+1}, \varsigma) = 1$, for any sequence $\{\hbar_n\}$ in $\mathcal{P}$. Now, following the proof of Theorem (3.1), we have

$$\varphi(\hbar^*, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, X(\hbar^*), \varsigma).$$

This shows that the point $\hbar^*$ is a CBPP of the pair of the mappings Z and X.

3.2. Fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-interpolative Reich-Rus-Ciric type proximal contractions:

Let $(C, \varphi, *)$ be an FMMS and $\mathcal{P}, \mathcal{G} \subset C$. The mappings $Z : \mathcal{P} \to \mathcal{G}$ and $X : \mathcal{P} \to \mathcal{G}$ are called fuzzy $(\mathcal{L}, \mathcal{M})$-interpolative Reich-Rus-Ciric type proximal contractions if

$$\begin{aligned} \varphi(\hbar_1, Xm_1, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(d_1, Zm_1, \varsigma), \\ \varphi(\hbar_2, Xm_2, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(d_2, Zm_2, \varsigma), \\ \mathcal{L}(\varphi(\hbar_1, \hbar_2, \varsigma)) &\geq \mathcal{M}\Big((\varphi(d_1, d_2, \varsigma))^{\alpha} (\varphi(d_1, \hbar_1, \varsigma))^{\beta} (\varphi(d_2, \hbar_2, \varsigma))^{1-\alpha-\beta}\Big), \end{aligned} \tag{3.11}$$

holds for all $\hbar_1, \hbar_2, d_1, d_2, m_1, m_2 \in \mathcal{P}$.

Example 3.2. *Let $(C, \varphi, *)$ be an FMMS. Define $\varphi(m, n, \varsigma) = \frac{\varsigma+1}{\varsigma+e^{|m_1-m_2|+|n_1-n_2|}}$ with a ctn as $s * t = st$.*

Let $\mathcal{P} = \{(0, n) ; n \in \mathbb{R}\}$ and $\mathcal{G} = \{(1, n) ; n \in \mathbb{R}\}$.

Define the mappings $X : \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$, respectively as

$$X(0, n) = \left(1, \frac{n}{2}\right),$$

and

$$Z(0, n) = \left(1, \frac{n}{3}\right).$$

Then, $\varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(m, n, \varsigma) = \frac{\varsigma+1}{\varsigma+e}$, $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{G}_0 = \mathcal{G}$. Then, clearly $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $Z(\mathcal{P}_0) \subseteq \mathcal{G}_0$. Define the functions $\mathcal{L}, \mathcal{M} : (0, 1] \to \mathbb{R}$ by

$$\mathcal{L}(l) = \left\{ \begin{array}{c} \frac{1}{\ln l} \text{ if } 0 < l < 1 \\ 1 \text{ if } \ l = 1 \end{array} \right\} \text{and } \mathcal{M}(l) = \left\{ \begin{array}{c} \frac{1}{\ln l^2} \text{ if } 0 < l < 1 \\ 2 \text{ if } l = 1 \end{array} \right\}.$$

This shows that the mappings X and Z are fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-interpolative Reich-Rus-Ciric type proximal. However, Consider $\hbar_1 = (0, 0)$, $\hbar_2 = (0, 3)$, $d_1 = (0, 0)$, $d_2 = (0, 2)$, $m_1 = (0, 0)$, $m_2 = (0, 6)$, $\varsigma = 2$, $\alpha = \frac{1}{2}$, and $\beta = \frac{1}{3}$, which shows that X and Z are not fuzzy multiplicative interpolative Reich-Rus-Ciric type proximal. Then

$$\begin{aligned} \varphi(0, X2, 1) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(4, Z2, 1), \\ \varphi(8, X4, 1) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(6, Z4, 1). \end{aligned}$$

For $\lambda \in (0, \frac{1}{2}]$, it follows that

$$\begin{aligned} \varphi\left(\hbar_1, \hbar_2, \varsigma^{\lambda}\right) &\geq (\varphi(d_1, d_2, \varsigma))^{\alpha} (\varphi(d_1, \hbar_1, \varsigma))^{\beta} (\varphi(d_2, \hbar_2))^{1-\alpha-\beta}, \\ \left(\varphi\left((0, 0), (0, 3), 2^{0.2}\right)\right) &\geq (\varphi((0, 0), (0, 2), 2))^{\frac{1}{2}} (\varphi((0, 0), (0, 0), 2))^{\frac{1}{3}} (\varphi((0, 2), (0, 3), 2))^{1-\frac{1}{2}-\frac{1}{3}}, \\ 0.1125 &\geq 0.5627, \\ 0.1125 &\geq 0.5627, \end{aligned}$$

which is a contradiction. Hence, X and Z are not fuzzy multiplicative interpolative Reich-Rus-Ciric type proximal.

Theorem 3.3. *Let $(C, \varphi, *)$ be a CFMMS and $\mathcal{P}, \mathcal{G} \subset C$ such that $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$. Also, assume that $\lim_{k\to\infty} \varphi(\hbar_1, \hbar_2, \varsigma) = 1$ and $\mathcal{P}_0$, $\mathcal{G}_0 \neq \Phi$. Let $X\colon \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ satisfy the following conditions:*

(i) Z dominates X and is fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-interpolative Riech-Rus-Ciric type proximal,
(ii) X and Z are proximally commutative,
(iii) $\mathcal{L}$ is a non-decreasing function and $\liminf_{l\to\epsilon+} \mathcal{M}(l) > \mathcal{L}(\epsilon+)$ for any $\epsilon > 0$,
(iv) X and Z are continuous,
(v) $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

Then, Z and X have a unique element $m \in \mathcal{P}$ such that

$$\begin{aligned} \varphi(m, Zm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(m, Xm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned}$$

Proof. Suppose that $m_0 \in \mathcal{P}_0$. From (v), we have that $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$; then, there exists an element $m_1 \in \mathcal{P}_0$ such that $Xm_0 = Zm_1$. Again by using (v) there exists an element $m_2 \in \mathcal{P}_0$ such that $Xm_1 = Zm_2$. This process of establishing the existence of points in $\mathcal{P}_0$ implies that there is a sequence $\{m_n\} \subseteq \mathcal{P}_0$ such that

$$Xm_{n-1} = Zm_n$$

for all positive integral values of n, because $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

Since $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$, there exists an element $\hbar_n$ in $\mathcal{P}_0$ such that

$$\varphi(\hbar_n, Xm_n, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \text{ for all } n \in \mathbb{N}.$$

Further, it follows from the choice of m_n and $\hbar_n$ that

$$\begin{aligned} \varphi(\hbar_{n+1}, X(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_n, Z(m_{n+1}), \varsigma), \\ \varphi(\hbar_n, X(m_n), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma), \end{aligned}$$

if

$$\varphi(\hbar_n, Xm_n, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma). \tag{3.12}$$

Thus, if there exists some $n \in \mathbb{N}$ such that $\hbar_n = \hbar_{n-1}$; then, by (3.12), the point $\hbar_n$ is a CBPP of the mappings X and Z. On the other hand, if $\hbar_{n-1} \neq \hbar_n$ for all $n \in \mathbb{N}$, then by (3.12), we get

$$\begin{aligned} \varphi(\hbar_{n+1}, X(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_n, Z(m_{n+1}), \varsigma), \\ \varphi(\hbar_n, X(m_n), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma). \end{aligned}$$

Thus, by (3.11), we have

$$\begin{aligned} \mathcal{L}(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) &\geq \mathcal{M}\left((\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{\alpha} (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\beta} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\alpha-\beta}\right), \\ \mathcal{L}(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) &\geq \mathcal{M}\left((\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\beta} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\alpha}\right) \end{aligned} \tag{3.13}$$

for all $\hbar_{n-1}, \hbar_n, \hbar_{n+1}, m_n, m_{n+1} \in \mathcal{P}$. Since, $\mathcal{M}(l) > \mathcal{L}(l)$ for all $l > 0$, by (3.13), we have

$$\mathcal{L}(\varphi(\hbar_{n+1}, \hbar_n)) > \mathcal{L}\left((\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\beta} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\beta}\right).$$

Thus, $\mathcal{L}$ is a non-decreasing function; we get

$$\varphi(\hbar_{n+1}, \hbar_n, \varsigma) > (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\beta} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\beta}.$$

This implies that

$$(\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{1-\beta} > (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\beta}.$$

Let $(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) = q_n$; we have

$$\mathcal{L}(q_n) \geq \mathcal{M}\left((q_n)^{\beta} (q_{n-1})^{1-\beta}\right) > \mathcal{L}\left((q_n)^{\beta} (q_{n-1})^{1-\beta}\right).$$

This implies that $q_n > q_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{q_n\}$ is positive and strictly non-decreasing. Thus, it converges to some element $q \geq 1$. We show that $q = 1$. Suppose that $q > 1$. Then, by (3.13) we get the following

$$\mathcal{L}(\varepsilon+) = \lim_{n\to\infty} \mathcal{L}(q_n) \geq \lim_{n\to\infty} \mathcal{M}\left((q_n)^{\beta} (q_{n-1})^{1-\beta}\right) \geq \lim_{l\to q+} \inf \mathcal{M}(l).$$

This contradicts the condition (iii); hence, $q = 1$ and $\lim_{n\to\infty} \varphi(\hbar_n, \hbar_{n+1}, \varsigma) = 1$. By the condition (iii) and Lemma (3.3), we deduce that $\{\hbar_n\}$ is a CS. From the completeness of $(C, \varphi, *)$, and by using (v), there exists an element $\hbar^*$ in $\mathcal{P}$ such that $\lim_{n\to\infty} \varphi(\hbar_n, \hbar^*, \varsigma) = 1$. Furthermore, we have

$$\varphi(\hbar^*, X(m_n), \varsigma) \geq \varphi(\hbar^*, \hbar_n, \varsigma) * \varphi(\hbar_n, X(m_n), \varsigma)$$

and

$$\varphi(\hbar^*, Z(m_n), \varsigma) \geq \varphi(\hbar^*, \hbar_n, \varsigma) * \varphi(\hbar_n, Z(m_n), \varsigma).$$

Moreover, $\varphi(\hbar^*, Z(m_n), \varsigma) \to \varphi(\hbar^*, \mathcal{G}, \varsigma)$ and also $\varphi(\hbar^*, X(m_n), \varsigma) \to \varphi(\hbar^*, \mathcal{G}, \varsigma)$ as $n \to \infty$. Because X and Z proximally commutative, $Z\hbar^*$ and $X\hbar^*$ are identical. Since $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$, there exists sub-sequences $\{Z(m_{n_k})\}$ of $\{Z(m_n)\}$ and $\{X(m_{n_k})\}$ of $\{X(m_n)\}$ such that $Z(m_{n_k}) \to d^* \in \mathcal{G}$ and $X(m_{n_k}) \to d^* \in \mathcal{G}$ as $k \to \infty$. Letting $k \to \infty$ in the following equations:

$$\begin{aligned} \varphi(d^*, X(m_{n_k}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(d^*, Z(m_{n_k}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \end{aligned} \tag{3.14}$$

we get

$$\varphi(d^*, \hbar^*, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma).$$

The fact that $\hbar^* \in \mathcal{P}_0$ implies that $X(\hbar^*) \in X(\mathcal{P}_0)$, and by using (v), there exists an element $w \in \mathcal{P}_0$. Similarly, $\hbar^* \in \mathcal{P}_0$, so $Z(\hbar^*) \in Z(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and there exists $w \in \mathcal{P}_0$ such that

$$\begin{aligned} \varphi(\hbar^*, X(\hbar^*), \varsigma) &= \varphi(w, X(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(\hbar^*, Z(\hbar^*), \varsigma) &= \varphi(w, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned} \tag{3.15}$$

Now, from (3.14) and (3.15), and by (3.11), we have

$$\begin{aligned} \mathcal{L}(\varphi(\hbar^*, w, \varsigma)) &\geq \mathcal{M}\left((\varphi(\hbar^*, w, \varsigma))^{\alpha} (\varphi(\hbar^*, w, \varsigma))^{\beta} (\varphi(\hbar^*, w, \varsigma))^{1-\alpha-\beta}\right) \\ &\geq \mathcal{M}(\varphi(\hbar^*, w, \varsigma)) > \varphi(\hbar^*, w, \varsigma). \end{aligned}$$

Since $\mathcal{L}$ is a non-decreasing function, we have

$$\varphi(\hbar^*, w, \varsigma^{\alpha}) \geq \varphi(\hbar^*, w, \varsigma) > \varphi(\hbar^*, w, \varsigma^{\alpha}).$$

This implies that $\hbar^*$ and w are equal. Finally, by (3.12), we have

$$\varphi(\hbar^*, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, X(\hbar^*), \varsigma).$$

This shows that the point $\hbar^*$ is a CBPP of the pair of mappings Z and X.

Theorem 3.4. *Let $(C, \varphi, *)$ be a CFMMS and $\mathcal{P}, \mathcal{G} \subset C$ such that $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$. Also, assume that $\lim_{k\to\infty} \varphi(\hbar_1, \hbar_2, \varsigma) = 1$ and $\mathcal{P}_0$, $\mathcal{G}_0 \neq \Phi$. Let $X\colon \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ satisfy the following conditions:*

(i) Z dominates X and is fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$ −interpolative Riech-Rus-Ciric type proximal,

(ii) X and Z proximally commutative,

(iii) $\mathcal{L}$ is non-decreasing and $\{\mathcal{L}(l_n)\}$ and $\{\mathcal{M}(l_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \mathcal{L}(l_n) = \lim_{n\to\infty} \mathcal{M}(l_n)$; then, $\lim_{n\to\infty} l_n = 1$,

(iv) X and Z are continuous,

(v) $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

Then, Z and X have a unique element $m \in \mathcal{P}$ such that

$$\begin{aligned} \varphi(m, Zm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(m, Xm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned}$$

Proof. Proceeding as in the proof of Theorem (3.3), we have

$$\mathcal{L}(q_n) \geq \mathcal{M}\left((q_{n-1})^{1-\beta}(q_n)^{\beta}\right) > \mathcal{L}\left((q_{n-1})^{1-\beta}(q_n)^{\beta}\right). \tag{3.16}$$

By (3.16), we infer that $\{\mathcal{L}(q_n)\}$ is a strictly non-decreasing sequence. We have two cases here, i.e., either the sequence $\{\mathcal{L}(q_n)\}$ is bounded above or not. If $\{\mathcal{L}(q_n)\}$ is not bounded, then

$$\inf_{w_n > \varepsilon} \mathcal{L}(q_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows from Lemma (3.1), that $q_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\mathcal{L}(q_n)\}$ is bounded above, then it is a convergent sequence. By (3.16), the sequence $\{\mathcal{M}(q_n)\}$ is also convergent. Furthermore, both have the same limit. By condition (iii), we get that $\lim_{n\to\infty} q_n = 1$, or that $\lim_{n\to\infty} \varphi(\hbar_n, m_{n+1}, \varsigma) = 1$ for any sequence $\{\hbar_n\}$ in $\mathcal{P}$. Now, following the proof of Theorem (3.3), we obtain

$$\varphi(\hbar^*, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, X(\hbar^*), \varsigma).$$

This shows that the point $\hbar^*$ is a CBPP of the pair of mappings Z and X.

3.3. *Fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-Kannan type proximal contraction:*

Let $(C, \varphi, *)$ be an FMMS and $\mathcal{P}, \mathcal{G} \subset C$. The mappings $Z : \mathcal{P} \to \mathcal{G}$ and $X : \mathcal{P} \to \mathcal{G}$ are called fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-Kannan type proximal contractions if

$$\begin{aligned} \varphi(\hbar_1, Xm_1, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(d_1, Zm_1, \varsigma), \\ \varphi(\hbar_2, Xm_2, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(d_2, Zm_2, \varsigma), \\ \mathcal{L}(\varphi(\hbar_1, \hbar_2)) &\geq \mathcal{M}\left((\varphi(d_1, \hbar_1))^{\alpha}(\varphi(d_2, \hbar_2))^{1-\alpha}\right), \end{aligned} \tag{3.17}$$

holds for all $\hbar_1, \hbar_2, d_1, d_2, m_1, m_2 \in \mathcal{P}$.

Example 3.3. *Let $(C, \varphi, *)$ be an FMMS. Define $\varphi(m, n, \varsigma) = \frac{\varsigma+1}{\varsigma+e^{|m_1-m_2|+|n_1-n_2|}}$ with the ctn as $s * t = st$.*

Let $\mathcal{P} = \{(0, n); n \in \mathbb{R}\}$ and $\mathcal{G} = \{(1, n); n \in \mathbb{R}\}$.

Define the mappings $X : \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$, respectively as

$$X(0, n) = \left(1, \frac{n}{2}\right),$$

and
$$Z(0,n) = \left(1, \frac{n}{3}\right).$$
Then, $\varphi(\mathcal{P},\mathcal{G},\varsigma) = \varphi(m,n,\varsigma) = \frac{\varsigma+1}{\varsigma+e}$, $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{G}_0 = \mathcal{G}$. Then, clearly $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $Z(\mathcal{P}_0) \subseteq \mathcal{G}_0$. Define the functions $\mathcal{L}, \mathcal{M} : (0,1] \to \mathbb{R}$ by
$$\mathcal{L}(l) = \left\{ \begin{array}{c} \frac{1}{2^{\ln 2l}} \text{ if } 0 < l < 1 \\ 1 \text{ if } l = 1 \end{array} \right\} \text{and } \mathcal{M}(l) = \left\{ \begin{array}{c} \frac{1}{2^{\ln l}} \text{ if } 0 < l < 1 \\ 2 \text{ if } l = 1 \end{array} \right\}.$$
This shows that the mappings X and Z are fuzzy $(\mathcal{L},\mathcal{M})-$interpolative Kannan type proximal. However, if we consider $\hbar_1 = (0,0)$, $\hbar_2 = (0,3)$, $d_1 = (0,0)$, $d_2 = (0,2)$, $m_1 = (0,0)$, $m_2 = (0,6)$, $\varsigma = 2$, and $\alpha = \frac{1}{2}$, then X and Z are not fuzzy multiplicative interpolative Kannan type proximal. We know that
$$\begin{aligned} \varphi((0,0), X(0,0), 2) &= \varphi(\mathcal{P},\mathcal{G},\varsigma) = \varphi((0,0), Z(0,6), 2), \\ \varphi((0,3), X(0,6), 2) &= \varphi(\mathcal{P},\mathcal{G},\varsigma) = \varphi((0,2), Z(0,6), 2). \end{aligned}$$
For $\lambda = 0.2$, it follows that
$$\begin{aligned} \varphi\left(\hbar_1, \hbar_2, \varsigma^{\lambda}\right) &\geq \left((\varphi(d_1,\hbar_1,\varsigma))^{\alpha} (\varphi(d_2,\hbar_2,\varsigma))^{1-\alpha}\right), \\ \varphi\left((0,0),(0,3),2^{\lambda}\right) &\geq \left((\varphi((0,0),(0,0),2))^{\frac{1}{2}} (\varphi((0,2),(0,3),2))^{\frac{1}{2}}\right), \\ 0.1012 &\geq 0.7974, \\ 0.1012 &\geq 0.7974, \end{aligned}$$
which is contradiction. Hence, X and Z are not fuzzy multiplicative interpolative Kannan type proximal.

Theorem 3.5. *Let $(C,\varphi,*)$ be a CFMMS and $\mathcal{P},\mathcal{G} \subset C$ such that $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$. Also, assume that $\lim_{k\to\infty} \varphi(\hbar_1,\hbar_2,\varsigma) = 1$ and $\mathcal{P}_0$, $\mathcal{G}_0 \neq \Phi$. Let $X\colon \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ satisfy the following conditions:*

(i) Z dominates X and is fuzzy multiplicative $(\mathcal{L},\mathcal{M})$-interpolative Kannan type proximal contraction,

(ii) X and Z are proximally commutative,

(iii) $\mathcal{L}$ is a non-decreasing function and $\liminf_{l\to\epsilon+} \mathcal{M}(l) > \mathcal{L}(\epsilon+)$ for any $\epsilon > 0$,

(iv) X and Z are continuous,

(v) $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

Then, Z and X have a unique element $m \in \mathcal{P}$ such that
$$\begin{aligned} \varphi(m, Zm, \varsigma) &= \varphi(\mathcal{P},\mathcal{G},\varsigma), \\ \varphi(m, Xm, \varsigma) &= \varphi(\mathcal{P},\mathcal{G},\varsigma). \end{aligned}$$

Proof. Suppose that $m_0 \in \mathcal{P}_0$. From (v), we have that $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$; then, there exists an element $m_1 \in \mathcal{P}_0$ such that $Xm_0 = Zm_1$. Again, by using (v), there exists an element $m_2 \in \mathcal{P}_0$ such that $Xm_1 = Zm_2$. This process of establishing the existence of points in $\mathcal{P}_0$ gives a sequence $\{m_n\} \subseteq \mathcal{P}_0$ such that
$$Xm_{n-1} = Zm_n$$

for all positive intergral values of n, because $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

Since $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$, there exists an element $\hbar_n$ in $\mathcal{P}_0$ such that

$$\varphi(\hbar_n, Xm_n, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \text{ for all } n \in \mathbb{N}.$$

Further, it follows from the choice of m_n and $\hbar_n$ that

$$\begin{aligned} \varphi(\hbar_{n+1}, X(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_n, Z(m_{n+1}), \varsigma), \\ \varphi(\hbar_n, Xm_n, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma), \end{aligned}$$

if

$$\varphi(\hbar_n, Xm_n, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma). \tag{3.18}$$

Notice that, if there exists some $n \in \mathbb{N}$ such that $\hbar_n = \hbar_{n-1}$, then by (3.18), the point $\hbar_n$ is a CBPP of the mappings X and Z. On the other hand, if $\hbar_{n-1} \neq \hbar_n$ for all $n \in \mathbb{N}$, then, by (3.18), we get

$$\begin{aligned} \varphi(\hbar_{n+1}, X(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_n, Z(m_n), \varsigma), \\ \varphi(\hbar_n, X(m_n), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_{n-1}), \varsigma). \end{aligned}$$

Thus, by (3.17), we have

$$\mathcal{L}(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) \geq \mathcal{M}\left((\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\alpha} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\alpha}\right) \tag{3.19}$$

for all $\hbar_{n-1}, \hbar_n, \hbar_{n+1}, m_n, m_{n+1} \in \mathcal{P}$. Since, $\mathcal{M}(l) > \mathcal{L}(l)$ for all $l > 0$, by (3.19), we have

$$\mathcal{L}(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) > \mathcal{L}\left((\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\alpha} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\alpha}\right).$$

Thus, $\mathcal{L}$ is a non-decreasing function; we get

$$\varphi(\hbar_{n+1}, \hbar_n, \lambda\varsigma) > (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\alpha} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\alpha}.$$

This implies that

$$(\varphi(\hbar_{n+1}, \hbar_n, \lambda\varsigma))^{1-\alpha} > (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{1-\alpha}.$$

Let $\varphi(\hbar_{n+1}, \hbar_n, \varsigma) = q_n$; we have

$$\mathcal{L}(q_n) \geq \mathcal{M}\left((q_n)^{\alpha} (q_{n-1})^{1-\alpha}\right) > \mathcal{L}((q_n)^{\alpha} (q_{n-1})^{1-\alpha}).$$

This implies that $q_n > q_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{q_n\}$ is positive and strictly non-decreasing. Hence, it converges to some element $q \geq 1$. We show that $q = 1$. Let $q > 1$, by (3.19); we get the following:

$$\mathcal{L}(\varepsilon+) = \lim_{n\to\infty} \mathcal{L}(q_n) \geq \lim_{n\to\infty} \mathcal{M}\left((q_n)^{\alpha} (q_{n-1})^{1-\alpha}\right) \geq \lim_{l\to q+} \inf \mathcal{M}(l).$$

This contradicts assumption (iii). Hence, $q = 1$ and $\lim_{n\to\infty} \varphi(\hbar_n, \hbar_{n+1}, \varsigma) = 1$. By the condition (iii) and Lemma (3.3), we deduce that $\{\hbar_n\}$ is a CS. Therefore $(C, \varphi, *)$ is a CFMMS, $\mathcal{P} \subseteq C$ and $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$; there exists an element $\hbar^*$ in $\mathcal{P}$ such that $\lim_{n\to\infty} \varphi(\hbar_n, \hbar^*, \varsigma) = 1$. Moreover,

$$\varphi(\hbar^*, X(m_n), \varsigma^{\alpha}) \geq \varphi(\hbar^*, \hbar_n, \varsigma) * \varphi(\hbar_n, X(m_n), \varsigma)$$

and

$$\varphi(\hbar^*, Z(m_n), \varsigma^\alpha) \geq \varphi(\hbar^*, \hbar_n, \varsigma) * \varphi(\hbar_n, Z(m_n), \varsigma).$$

Therefore, $\varphi(\hbar^*, Z(m_n), \varsigma) \to \varphi(\hbar^*, \mathcal{G}, \varsigma)$ and also $\varphi(\hbar^*, X(m_n), \varsigma) \to \varphi(\hbar^*, \mathcal{G}, \varsigma)$ as $n \to \infty$. Because X and Z are proximally commutative, $Z\hbar^*$ and $X\hbar^*$ are equal. Since $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$, there exists sub-sequences $\{Z(m_{n_k})\}$ of $\{Z(m_n)\}$ and $\{X(m_{n_k})\}$ of $\{X(m_n)\}$ such that $Z(m_{n_k}) \to d^* \in \mathcal{G}$ and $X(m_{n_k}) \to d^* \in \mathcal{G}$ as $k \to \infty$. Moreover, supposing that $k \to \infty$ in the following equations:

$$\begin{aligned} \varphi(d^*, X(m_{n_k}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(d^*, Z(m_{n_k}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \end{aligned} \tag{3.20}$$

we have

$$\varphi(d^*, \hbar^*, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma).$$

The fact that $\hbar^* \in \mathcal{P}_0$ implies that $X(\hbar^*) \in X(\mathcal{P}_0)$, and by using (v), there exists an element $w \in \mathcal{P}_0$. Similarly, $\hbar^* \in \mathcal{P}_0$, so $Z(\hbar^*) \in Z(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and there exists $w \in \mathcal{P}_0$ such that

$$\begin{aligned} \varphi(\hbar^*, X(\hbar^*), \varsigma) &= \varphi(w, X(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(\hbar^*, Z(\hbar^*), \varsigma) &= \varphi(w, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned} \tag{3.21}$$

Now, from (3.20) and (3.21), and by (3.17), we have

$$\begin{aligned} \mathcal{L}(\varphi(\hbar^*, w), \varsigma) &\geq \mathcal{M}\left((\varphi(\hbar^*, w, \varsigma))^\alpha (\varphi(\hbar^*, w, \varsigma))^{1-\alpha}\right) \\ &\geq \mathcal{M}(\varphi(\hbar^*, w, \varsigma)) > \varphi(\hbar^*, w, \varsigma). \end{aligned}$$

Since $\mathcal{L}$ is a non-decreasing function, we have

$$\varphi(\hbar^*, w, \varsigma^\alpha) \geq \varphi(\hbar^*, w, \varsigma) > \varphi(\hbar^*, w, \varsigma^\alpha).$$

This implies that $\hbar^*$ and w are equal. Finally, by (3.18), we have

$$\varphi(\hbar^*, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, X(\hbar^*), \varsigma).$$

This shows that the point $\hbar^*$ is a CBPP of the pair of mappings Z and X.

Theorem 3.6. *Let $(C, \varphi, *)$ be a CFMMS and $\mathcal{P}, \mathcal{G} \subset C$ such that $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$. Suppose that $\lim_{k\to\infty} \varphi(\hbar_1, \hbar_2, \varsigma) = 1$ and $\mathcal{P}_0$, $\mathcal{G}_0 \neq \Phi$. Let $X\colon \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ satisfy the following conditions:*

(i) Z dominates X and is fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-interpolative Kannan type proximal,

(ii) X and Z are proximally commutative,

(iii) $\mathcal{L}$ is non-decreasing and $\{\mathcal{L}(l_n)\}$ and $\{\mathcal{M}(l_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \mathcal{L}(l_n) = \lim_{n\to\infty} \mathcal{M}(l_n)$; then, $\lim_{n\to\infty} l_n = 1$,

(iv) X and Z are continuous,

(v) $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

Then, Z and X have a unique element $m \in \mathcal{P}$ such that

$$\begin{aligned} \varphi(m, Zm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(m, Xm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned}$$

Proof. Proceeding as in the proof of Theorem (3.5), we get

$$\mathcal{L}(q_n) \geq \mathcal{M}\left((q_{n-1})^{1-\beta}(q_n)^{\beta}\right) > \mathcal{L}\left((q_{n-1})^{1-\beta}(q_n)^{\beta}\right). \tag{3.22}$$

By (3.16), we infer that $\{\mathcal{L}(q_n)\}$ is a strictly non-decreasing sequence. We have two cases here, i.e., either the sequence $\{\mathcal{L}(q_n)\}$ is bounded above or not. If $\{\mathcal{L}(q_n)\}$ is not bounded above, then

$$\inf_{w_n>\varepsilon} \mathcal{L}(q_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows from Lemma (3.1), that $q_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\mathcal{L}(q_n)\}$ is bounded above, then it is a convergent sequence. By (3.16), the sequence $\{\mathcal{M}(q_n)\}$ is also convergent. Furthermore, both have the same limit. By condition (iii), we get that $\lim_{n\to\infty} q_n = 1$, or that $\lim_{n\to\infty} \varphi(\hbar_n, m_{n+1}, \varsigma) = 1$, for any sequence $\{\hbar_n\}$ in $\mathcal{P}$. Now, following the proof of Theorem (3.5), we have

$$\varphi(\hbar^*, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, X(\hbar^*), \varsigma).$$

This shows that the point $\hbar^*$ is a CBPP of the pair of the mapping Z and X.

3.4. Fuzzy $(\mathcal{L}, \mathcal{M})$-interpolative Hardy-Rogers type proximal contraction

Let $(C, \varphi, *)$ be an FMMS and $\mathcal{P}, \mathcal{G} \subset C$. The mappings $Z : \mathcal{P} \to \mathcal{G}$ and $X : \mathcal{P} \to \mathcal{G}$ are called fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-interpolative Hardy-Rogers type proximal contractions if

$$\begin{aligned}
\varphi(\hbar_1, Xm_1, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(d_1, Zm_1, \varsigma),\\
\varphi(\hbar_2, Xm_2, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(d_2, Zm_2, \varsigma),\\
\mathcal{L}(\varphi(\hbar_1, \hbar_2, \varsigma)) &\leq \mathcal{M}\left((\varphi(d_1, d_2, \varsigma))^{\alpha}(\varphi(d_1, \hbar_1, \varsigma))^{\beta}(\varphi(d_2, \hbar_2, \varsigma))^{\gamma}\left(\left((\varphi(d_1, \hbar_2, \varsigma))^{\delta}\varphi(d_2, \hbar_1, \varsigma)\right)\right)^{1-\alpha-\beta-\gamma}\right),
\end{aligned} \tag{3.23}$$

holds for all $\hbar_1, \hbar_2, d_1, d_2, m_1, m_2 \in \mathcal{P}$.

Example 3.4. *Let $(C, \varphi, *)$ be an FMMS. Define $\varphi(m, n, \varsigma) = \frac{\varsigma+1}{\varsigma+e^{|m_1-m_2|+|n_1-n_2|}}$ with a ctn as $s * t = st$.*

Let $\mathcal{P} = \{(0, u); 0 \leq u < \infty\}$ and $\mathcal{G} = \{(1, u); 0 \leq u < \infty\}$.

Define the mappings $X : \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$, respectively as

$$X(0, u) = (1, u - 1),$$

and

$$Z(0, u) = (1, u + 1).$$

Then, $\varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \frac{\varsigma+1}{\varsigma+e}$, $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{G}_0 = \mathcal{G}$. Then, clearly $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $Z(\mathcal{P}_0) \subseteq \mathcal{G}_0$. Define the functions $\mathcal{L}, \mathcal{M} : (0, 1] \to \mathbb{R}$ by

$$\mathcal{L}(l) = \left\{ \begin{array}{c} \frac{1}{\ln l} \text{ if } 0 < l < 1 \\ 1 \text{ if } l = 1 \end{array} \right\} \text{and } \mathcal{M}(l) = \left\{ \begin{array}{c} \frac{1}{\ln l^2} \text{ if } 0 < l < 1 \\ 2 \text{ if } l = 1 \end{array} \right\}.$$

This shows that the mappings X and Z are fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-interpolative Hardy-Rogers type proximal. However, consider $\hbar_1 = (0, 4)$, $\hbar_2 = (0, 2)$, $d_1 = (0, 6)$, $d_2 = (0, 4)$, $m_1 = (0, 5)$, $m_2 =$

$(0,3)$, $\varsigma = 2, \alpha = 0.01, \beta = 0.02, \gamma = 0.03$, and $\delta = 0.04$, which shows that X and Z are not fuzzy multiplicative interpolative Hardy-Rogers type proximal. Hence,

$$\begin{aligned} \varphi((0,4), X(0,5), 2) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi((0,6), Z(0,5), 2), \\ \varphi((0,2), X(0,3), 2) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi((0,4), Z(0,3), 2). \end{aligned}$$

For $\lambda = 0.2$, it follows that

$$\begin{aligned} \left(\varphi\left(\hbar_1, \hbar_2, \varsigma^\lambda\right)\right) &\geq (\varphi(d_1, d_2, \varsigma))^\alpha (\varphi(d_1, \hbar_1, \varsigma))^\beta (\varphi(d_2, \hbar_2, \varsigma))^\gamma \left(\left(\varphi(d_1, \hbar_2, \varsigma)\right)^\delta \varphi(d_2, \hbar_1, \varsigma)\right)\right)^{1-\alpha-\beta-\gamma}, \\ \varphi\left((0,4), (0,2), 2^\lambda\right) &\geq \begin{array}{c} \varphi((0,4), (0,6), 2)^{0.01} (\varphi((0,4), (0,2), 2))^{0.02} (\varphi((0,6), (0,4), 2))^{0.03}, \\ (\varphi((0,6), (0,2), 2))^{0.04} (\varphi((0,4), (0,4), 2))^{0.9}, \end{array} \\ 0.0431 &\geq 0.8286, \end{aligned}$$

which is a contradiction. Hence, the mappings X and Z are not fuzzy multiplicative interpolative Hardy-Rogers type proximal.

Theorem 3.7. *Let $(C, \varphi, *)$ be a CFMMS and $\mathcal{P}, \mathcal{G} \subset C$ such that $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$. Also, assume that $\lim_{k\to\infty} \varphi(\hbar_1, \hbar_2, \varsigma) = 1$ and $\mathcal{P}_0$, $\mathcal{G}_0 \neq \Phi$. Let $X \colon \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$ satisfy the following conditions:*

(i) Z dominates X and is fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-interpolative Hardy-Rogers type proximal,
(ii) X and Z are proximally commutative,
(iii) $\mathcal{L}$ is a non-decreasing function and $\limsup_{l\to\epsilon+} \mathcal{M}(l) < \mathcal{L}(\epsilon+)$ for any $\epsilon > 0$,
(iv) X and Z are continuous,
(v) $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.
Then, Z and X have a unique element $m \in \mathcal{P}$ such that

$$\begin{aligned} \varphi(m, Zm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(m, Xm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned}$$

Proof. Suppose that $m_0 \in \mathcal{P}_0$. From (v), we have that $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$; then, there exists an element $m_1 \in \mathcal{P}_0$ such that $Xm_0 = Zm_1$. Again, by using (v), there exists an element $m_2 \in \mathcal{P}_0$ such that $Xm_1 = Zm_2$. This process of establishing the existence of points in $\mathcal{P}_0$ implies that there is a sequence $\{m_n\} \subseteq \mathcal{P}_0$ such that

$$Xm_{n-1} = Zm_n$$

for all positive integral values of n, because $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

From $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$, there exists an element $\hbar_n$ in $\mathcal{P}_0$ such that

$$\varphi(\hbar_n, Xm_n, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) \text{ for all } n \in \mathbb{N}.$$

Further, it follows from the choice of m_n and $\hbar_n$ that

$$\begin{aligned} \varphi(\hbar_{n+1}, X(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_n, Xm_n, \varsigma), \\ \varphi(\hbar_n, Z(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_{n-1}, Z(m_n), \varsigma) \end{aligned}$$

if

$$\varphi\left(\hbar_n, Xm_n, \varsigma\right) = \varphi(\mathcal{P}, \mathcal{G}) = \varphi\left(\hbar_{n-1}, Z\left(m_n\right), \varsigma\right). \tag{3.24}$$

Notice that, if there exists some $n \in \mathbb{N}$ such that $\hbar_n = \hbar_{n-1}$, then by (3.24), the point $\hbar_n$ is a CBPP of the mappings X and Z. On the other hand, if $\hbar_{n-1} \neq \hbar_n$ for all $n \in \mathbb{N}$, then, by (3.24), we obtain

$$\begin{aligned}
\varphi(\hbar_{n+1}, X(m_{n+1}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar_n, Z(m_n), \varsigma), \\
\varphi\left(\hbar_n, X\left(m_n\right), \varsigma\right) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi\left(\hbar_{n-1}, Z\left(m_{n-1}\right), \varsigma\right).
\end{aligned}$$

Thus, by (3.23), we have

$$\begin{aligned}
\mathcal{L}(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) &\geq \mathcal{M}\left(\begin{array}{c} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{\alpha} (\varphi(\hbar_n, \hbar_{n+1}, \varsigma))^{\beta} \\ (\varphi(\hbar_{n-1}, \hbar_n, \varsigma))^{\gamma} \left(\left(\varphi(\hbar_n, \hbar_n, \varsigma)\right)^{\delta} (\varphi(\hbar_{n-1}, \hbar_{n+1}, \varsigma))\right)^{1-\alpha-\beta-\gamma-\delta} \end{array}\right) \\
&\geq \mathcal{M}\left(\begin{array}{c} (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{\alpha} (\varphi(\hbar_n, \hbar_{n+1}, \varsigma))^{\beta} \\ (\varphi(\hbar_{n-1}, \hbar_n, \varsigma))^{\gamma} ((\varphi(\hbar_{n-1}, \hbar_{n+1}, \varsigma))^{1-\alpha-\beta-\gamma-\delta} \end{array}\right),
\end{aligned} \tag{3.25}$$

for all $\hbar_{n-1}, \hbar_n, \hbar_{n+1}, m_n, m_{n+1} \in \mathcal{P}$. Since, $\mathcal{M}(l) > \mathcal{L}(l)$ for all $l > 0$, by (3.25), we have

$$\mathcal{L}(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) > \mathcal{L}\left((\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{\alpha} (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\beta} (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\gamma} ((\varphi(\hbar_{n-1}, \hbar_{n+1}, \varsigma))^{1-\alpha-\beta-\gamma-\delta}\right).$$

Since $\mathcal{L}$ is a non-decreasing function, we obtain

$$\begin{aligned}
\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) &> (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{\alpha} (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\beta} (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\gamma} ((\varphi(\hbar_{n-1}, \hbar_{n+1}, \varsigma))^{1-\alpha-\beta-\gamma-\delta}, \\
\varphi(\hbar_{n+1}, \hbar_n, \varsigma) &> (\varphi(\hbar_n, \hbar_{n-1}, \varsigma))^{\alpha} (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\beta} (\varphi(\hbar_{n+1}, \hbar_n, \varsigma))^{\gamma} ((\varphi(\hbar_{n-1}, \hbar_n, \varsigma).\varphi(\hbar_n, \hbar_{n+1}, \varsigma))^{1-\alpha-\beta-\gamma-\delta}, \\
\varphi(\hbar_{n+1}, \hbar_n, \varsigma) &> (\varphi(\hbar_{n-1}, \hbar_n, \varsigma))^{1-\beta-\gamma-\delta} (\varphi(\hbar_{n-1}, \hbar_n, \varsigma))^{1-\alpha-\delta}.
\end{aligned}$$

This implies that

$$\varphi(\hbar_{n+1}, \hbar_n, \varsigma^{\alpha}) > (\varphi(\hbar_{n-1}, \hbar_n, \varsigma))^{1-\beta-\gamma-\delta} (\varphi(\hbar_{n-1}, \hbar_n, \varsigma))^{1-\alpha-\delta}.$$

Let $(\varphi(\hbar_{n+1}, \hbar_n, \varsigma)) = q_n$; we have

$$\begin{aligned}
\mathcal{L}(q_n) &\geq \mathcal{M}\left((q_{n-1})^{1-\beta-\gamma-\delta} (q_n)^{1-\alpha-\delta}\right), \\
&> \mathcal{L}\left(\left((q_{n-1})^{1-\beta-\gamma-\delta} (q_n)^{1-\alpha-\delta}\right)\right).
\end{aligned}$$

Assume that $q_n > q_{n-1}$ for some $n \geq 1$. Since $\mathcal{L}$ is non-decreasing, by (3.25), we get that $(q_n) > \left(\left((q_{n-1})^{1-\beta-\gamma-\delta} (q_n)^{1-\alpha-\delta}\right)\right)$. This is not possible. Hence, we obtain that $q_n > q_{n-1}$ for all $n \geq 1$. Thus, it converges to some element $q \geq 1$. We show that $q = 1$. Let $q > 1$, so that, by (3.25), we get the following:

$$\mathcal{L}(\varepsilon+) = \lim_{n\to\infty} \mathcal{L}(q_n) \geq \lim_{n\to\infty} \mathcal{M}\left(\left((q_{n-1})^{1-\beta-\gamma-\delta} (q_n)^{1-\alpha-\delta}\right)\right) \geq \lim_{l\to q+} \inf \mathcal{M}(l).$$

This contradicts condition (iii); hence, $q = 1$ and $\lim_{n\to\infty} \varphi(\hbar_n, \hbar_{n+1}, \varsigma) = 1$. By the condition (iii) and Lemma (3.3), we deduce that $\{\hbar_n\}$ is a CS. Therefore $(C, \varphi, *)$ is a CFMMS, $\mathcal{P} \subseteq C$ and $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$; then, there exists an element $\hbar^*$ in $\mathcal{P}$ such that $\lim_{n\to\infty} \varphi(\hbar_n, \hbar^*, \varsigma) = 1$. Moreover,

$$\varphi(\hbar^*, X(m_n), \varsigma) \geq \varphi\left(\hbar^*, \hbar_n, \varsigma\right) * \varphi\left(\hbar_n, X\left(m_n\right), \varsigma\right).$$

Also,

$$\varphi(\hbar^*, Z(m_n), \varsigma) \geq \varphi(\hbar^*, \hbar_n, \varsigma) * \varphi(\hbar_n, Z(m_n), \varsigma).$$

Therefore, $\varphi(\hbar^*, Z(m_n), \varsigma) \to \varphi(\hbar^*, \mathcal{G}, \varsigma)$ and $\varphi(\hbar^*, X(m_n), \varsigma) \to \varphi(\hbar^*, \mathcal{G}, \varsigma)$ as $n \to \infty$. Because X and Z are proximally commutative, $Z\hbar^*$ and $X\hbar^*$ are equal. Since $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$, there exists sub-sequences $\{Z(m_{n_k})\}$ of $\{Z(m_n)\}$ and $\{X(m_{n_k})\}$ of $\{X(m_n)\}$ such that $Z(m_{n_k}) \to d^* \in \mathcal{G}$ and $X(m_{n_k}) \to d^* \in \mathcal{G}$ as $k \to \infty$. Moreover, supposing that $k \to \infty$ in the following equations:

$$\begin{aligned} \varphi(d^*, X(m_{n_k}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(d^*, Z(m_{n_k}), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \end{aligned} \tag{3.26}$$

we have

$$\varphi(d^*, \hbar^*, \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma).$$

Since, $\hbar^* \in \mathcal{P}_0$, $X(\hbar^*) \in X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and there exists $w \in \mathcal{P}_0$. Similarly, $\hbar^* \in \mathcal{P}_0$, so $Z(\hbar^*) \in Z(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and there exists $w \in \mathcal{P}_0$ such that

$$\begin{aligned} \varphi(\hbar^*, X(\hbar^*), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, Z(\hbar^*), \varsigma), \\ \varphi(w, X(\hbar^*), \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(w, Z(\hbar^*), \varsigma). \end{aligned} \tag{3.27}$$

Now, from (3.26) and (3.27), and by (3.23), we have

$$\begin{aligned} \mathcal{L}(\varphi(\hbar^*, w, \varsigma)) &\geq \mathcal{M}\left((\varphi(\hbar^*, w, \varsigma))^{\alpha} (\varphi(\hbar^*, w, \varsigma))^{\beta}\right) \\ &\geq \mathcal{M}(\varphi(\hbar^*, w, \varsigma)) \\ &> \varphi(\hbar^*, w, \varsigma). \end{aligned}$$

Since $\mathcal{L}$ is a non-decreasing function, we have

$$\varphi(\hbar^*, w, \varsigma^{\alpha}) \geq \varphi(\hbar^*, w, \varsigma) > \varphi(\hbar^*, w, \varsigma^{\alpha}).$$

This implies that $\hbar^*$ and w are the same. Hence, by (3.24), we have

$$\varphi(\hbar^*, Z(\hbar^*), \varsigma^{\alpha}) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, X(\hbar^*), \varsigma).$$

This shows that the point $\hbar^*$ is a CBPP of the pair of mappings X and Z.

Theorem 3.8. *Let $(C, \varphi, *)$ be a CFMMS and $\mathcal{P}, \mathcal{G} \subset C$ such that $\mathcal{G}$ is approximately compact with respect to $\mathcal{P}$. Also, assume that $\lim_{k\to\infty} \varphi(\hbar_1, \hbar_2, \varsigma) = 1$ and $\mathcal{P}_0$, $\mathcal{G}_0 \neq \Phi$. Let $X\colon \mathcal{P} \to \mathcal{G}$ and $Z: \mathcal{P} \to \mathcal{G}$ satisfy the following conditions:*

(i) Z dominates X and is fuzzy multiplicative $(\mathcal{L}, \mathcal{M})$-interpolative Hardy-Rogers type proximal,

(ii) X and Z are proximally commutative,

(iii) $\mathcal{L}$ is non-decreasing and $\{\mathcal{L}(l_n)\}$ and $\{\mathcal{M}(l_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \mathcal{L}(l_n) = \lim_{n\to\infty} \mathcal{M}(l_n)$; then, $\lim_{n\to\infty} l_n = 1$,

(iv) X and Z are continuous,

(v) $X(\mathcal{P}_0) \subseteq \mathcal{G}_0$ and $X(\mathcal{P}_0) \subseteq Z(\mathcal{P}_0)$.

Then, Z and X have a unique element $m \in \mathcal{P}$ such that

$$\begin{aligned} \varphi(m, Zm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma), \\ \varphi(m, Xm, \varsigma) &= \varphi(\mathcal{P}, \mathcal{G}, \varsigma). \end{aligned}$$

Proof. Proceeding as in the proof of Theorem (3.7), we have

$$\begin{aligned}\mathcal{L}(q_n) &\geq \mathcal{M}\left(\left((q_{n-1})^{1-\beta-\gamma-\delta}(q_n)^{1-\alpha-\delta}\right)\right)\\ &> \mathcal{L}\left(\left((q_{n-1})^{1-\beta-\gamma-\delta}(q_n)^{1-\alpha-\delta}\right)\right).\end{aligned} \tag{3.28}$$

By (3.28), we infer that $\{\mathcal{L}(q_n)\}$ is a strictly non-decreasing sequence. We have two cases here, i.e., either the sequence $\{\mathcal{L}(q_n)\}$ is bounded above or not. If $\{\mathcal{L}(q_n)\}$ is not bounded above, then

$$\inf_{w_n>\varepsilon} \mathcal{L}(q_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows from Lemma (3.1), that $q_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\mathcal{L}(q_n)\}$ is bounded above, then it is a convergent sequence. By (3.28), the sequence $\{\mathcal{M}(q_n)\}$ is convergent. Furthermore, both have the same limit. By condition (iii), we get that $\lim_{n\to\infty} q_n = 1$, or that $\lim_{n\to\infty} \varphi(\hbar_n, m_{n+1}, \varsigma) = 1$, for any sequence $\{\hbar_n\}$ in $\mathcal{P}$. Now, following the proof of Theorem (3.7), we obtain

$$\varphi(\hbar^*, Z(\hbar^*), \varsigma) = \varphi(\mathcal{P}, \mathcal{G}, \varsigma) = \varphi(\hbar^*, X(\hbar^*), \varsigma).$$

This shows that the point $\hbar^*$ is a CBPP of the pair of mappings Z and X.

4. Conclusions

In this manuscript, we introduced generalized iterative contractive mappings for a pair of non-self-mappings $X\colon \mathcal{P} \to \mathcal{G}$ and $Z : \mathcal{P} \to \mathcal{G}$. We proved some CBPP theorems for generalized iterative mappings in a CFMMS. Further, we proved fuzzy multiplicative versions of the $(\mathcal{L}, \mathcal{M})$-proximal contraction, $(\mathcal{L}, \mathcal{M})$-interpolative Reich-Rus-Ciric type proximal contraction, $(\mathcal{L}, \mathcal{M})$-interpolative Kannan type proximal contraction, and $(\mathcal{L}, \mathcal{M})$-interpolative Hardy-Rogers type proximal contraction to examine the CBPP in the setting of FMMS. Furthermore, we provided several non-trivial examples to show the validity of our main results. The contraction conditions (3.1), (3.11), (3.17) and (3.23) can be used to demonstrate the existence of solutions to the models of linear and nonlinear dynamic systems, depending on their nature (linear or nonlinear). This paper's study expands on the worthwhile research that was previously published in [4, 5, 8–10].

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they do not have any conflict of interests. All authors have read and approved the final manuscript.

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