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Research article

Some identities on degenerate hyperbolic functions arising from p-adic integrals on $\mathbb{Z}_p$

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Abstract: The aim of this paper is to introduce several degenerate hyperbolic functions as degenerate versions of the hyperbolic functions, to evaluate Volkenborn and the fermionic p-adic integrals of the degenerate hyperbolic cosine and the degenerate hyperbolic sine functions and to derive from them some identities involving the degenerate Bernoulli numbers, the degenerate Euler numbers and the Cauchy numbers of the first kind.

Keywords: Volkenborn integral; fermionic p-adic integral; degenerate hyperbolic functions; degenerate Bernoulli numbers; degenerate Euler numbers; Cauchy numbers of the first kind
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1. Introduction

Volkenborn and the fermionic p-adic integrals of the powers of x give respectively the Bernoulli numbers and the Euler numbers, while Volkenborn and the fermionic p-adic integrals of the generalized falling factorials yield respectively the degenerate Bernoulli numbers and the degenerate Euler numbers (see (1.12)). Thus the latter may be viewed as degenerate versions of the former.

The aim of this paper is to derive some identities on degenerate hyperbolic functions arising from Volkenborn and the fermionic p-adic integrals on $\mathbb{Z}_p$. In more detail, we introduce degenerate hyperbolic functions as natural degenerate versions of the usual hyperbolic functions and derive some identities relating to them. Then we evaluate Volkenborn and the fermionic p-adic integrals of the degenerate hyperbolic cosine and the degenerate hyperbolic sine functions. From those results, we derive some identities involving the degenerate Bernoulli numbers, the degenerate Euler numbers and the Cauchy numbers of the first kind.

The outline of this paper is as follows. In Section 1, we recall Volkenborn integral of uniformly differentiable functions on $\mathbb{Z}_p$ and the fermionic p-adic integral of continuous functions on $\mathbb{Z}_p$, together with their integral equations and examples in the case of exponential functions. We remind the reader of Cauchy numbers of the first kind. We recall the degenerate exponentials as a degenerate version of the usual exponentials. Then we show that the degenerate Bernoulli and the degenerate Euler numbers arise naturally respectively from the Volkenborn and the fermionic p-adic integrals on $\mathbb{Z}_p$ of the degenerate exponentials. Section 2 is the main result of this paper. We introduce degenerate versions of the hyperbolic functions, namely the degenerate hyperbolic cosine $\cosh_\lambda(x : a)$, the degenerate hyperbolic sine $\sinh_\lambda(x : a)$, degenerate hyperbolic tangent $\tanh_\lambda(x : a)$ and the degenerate hyperbolic cotangent $\coth_\lambda(x : a)$. Then we derive several identities connecting those degenerate hyperbolic functions. We evaluate Volkenborn integrals of $\cosh_\lambda(x : a)$ and $\sinh_\lambda(x : a)$. From those results, we obtain some identities involving the degenerate Bernoulli numbers and the Cauchy numbers of the first kind. We compute the fermionic p-adic integrals of the same degenerate hyperbolic functions, from which we derive an identity involving the degenerate Euler numbers. Finally, we conclude this paper in Section 3. In the rest of this section, we recall the necessary facts that are needed throughout this paper.

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. The Volkenborn integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)\, d\mu(x) &= \lim_{N\to\infty} \sum_{x=0}^{P^N-1} f(x)\mu(x+p^N\mathbb{Z}_p)\\ &= \lim_{N\to\infty} \frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x), \quad \text{(see [1–7]).}\end{aligned} \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} f(x+1)\, d\mu(x) - \int_{\mathbb{Z}_p} f(x)\, d\mu(x) = f'(0), \quad \text{(see [3,7]),} \tag{1.2}$$

where f is a $\mathbb{C}_p$-valued uniform differentiable function on $\mathbb{Z}_p$.

In [5], the *fermionic* p-adic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\, \mu_{-1}(x+p^N\mathbb{Z}_p)\\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x,\end{aligned} \tag{1.3}$$

where f is a $\mathbb{C}_p$-valued continuous function on $\mathbb{Z}_p$.

Thus, by (1.3), we get

$$\int_{\mathbb{Z}_p} f(x+1)\, d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = 2f(0) \quad \text{(see [1,5,6]).} \tag{1.4}$$

Adopting the notation from [7], we let E denote the additive group given by

$$E = \left\{ x \in \mathbb{C}_p \,\middle|\, |x|_p < p^{-\frac{1}{p-1}} \right\}. \tag{1.5}$$

It is known that there is an isomorphism from the multiplicative group $1 + E$ to the additive group E given by $1 + x \mapsto \log(1 + x)$, where $\log = \log_p$ is the p-adic logarithm. Moreover, the inverse map is given by $x \mapsto e^x = \exp x$, where $\exp = \exp_p$ is the p-adic exponential (see [7]). We note that

$$\left| \frac{1}{x} \log(1 + x) \right|_p \leq 1, \quad \text{for } x \in E. \tag{1.6}$$

The Cauchy numbers of the first kind (also called Bernoulli numbers of the second) are defined by

$$\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}, \quad (|t|_p < 1,\ t \neq 0).$$

From (1.2) and (1.4), we note that

$$\int_{\mathbb{Z}_p} e^{xt}\, d\mu(x) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad (t \in E, t \neq 0), \quad (\text{see [1–18]}), \tag{1.7}$$

and

$$\int_{\mathbb{Z}_p} e^{xt}\, d\mu_{-1}(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (t \in E), \quad (\text{see [1,3,5]}), \tag{1.8}$$

where B_n and E_n are the n-th Bernoulli number and the n-th Euler number, respectively.

Throughout this paper, we let λ be any nonzero element in $\mathbb{C}_p$.

The degenerate exponentials are defined by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} = (1 + \lambda t)^{\frac{x}{\lambda}}, \quad (t \in \frac{1}{\lambda} E \cap \frac{1}{x} E), \quad (\text{see [8,9]}), \tag{1.9}$$

where the generalized falling factorials (also called the λ-falling factorials) are given by $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x - \lambda) \cdots (x - (n-1)\lambda),\ (n \geq 1)$.

From (1.6), we see that $\frac{xt}{\lambda t} \log(1+\lambda t)$ is in the region of convergence E of e^x, when $\lambda t \in E$ and $xt \in E$. Thus $e_\lambda^x(t) = \exp\left(\frac{xt}{\lambda t} \log(1 + \lambda t)\right)$ is convergent when $t \in \frac{1}{\lambda} E \cap \frac{1}{x} E$. Here we understand $\frac{1}{\lambda} E \cap \frac{1}{x} E = \frac{1}{\lambda} E$, for $x = 0$. Especially, $e_\lambda(t) = e_\lambda^1(t)$ is convergent for $t \in \frac{1}{\lambda} E \cap E$.

From (1.2), we observe that

$$\int_{\mathbb{Z}_p} e_\lambda^x(t)\, d\mu(x) = \frac{\frac{1}{\lambda} \log(1 + \lambda t)}{e_\lambda(t) - 1} = \sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{t^n}{n!}, \quad (t \in \frac{1}{\lambda} E \cap E), \tag{1.10}$$

where $\beta_{n,\lambda}$ are called the (fully) degenerate Bernoulli numbers (see [3, 10]).

Note that $\lim_{\lambda \to 0} \beta_{n,\lambda} = B_n,\ (n \geq 0), (\text{see [11]})$.

From (1.4), we note that

$$\int_{\mathbb{Z}_p} e_\lambda^x(t)\, d\mu_{-1}(x) = \frac{2}{e_\lambda(t)+1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}\frac{t^n}{n!}, \quad \left(t \in \frac{1}{\lambda}E \cap E\right), \tag{1.11}$$

where $\mathcal{E}_{n,\lambda}$ are called the degenerate Euler numbers (see [3, 10]).

Note that $\lim_{\lambda\to 0}\mathcal{E}_{n,\lambda} = E_n,\ (n \geq 0)$.

Thus, by (1.10) and (1.11), we get

$$\int_{\mathbb{Z}_p} (x)_{n,\lambda}\, d\mu(x) = \beta_{n,\lambda}, \quad \int_{\mathbb{Z}_p} (x)_{n,\lambda}\, d\mu_{-1}(x) = \mathcal{E}_{n,\lambda},\ (n \geq 0). \tag{1.12}$$

The hyperbolic functions of real or complex variables are given by

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \\ \tanh x &= \frac{\sinh x}{\cosh x}, \quad \text{and} \quad \coth x = \frac{\cosh x}{\sinh x}. \end{aligned} \tag{1.13}$$

2. Some identities of degenerate hyperbolic functions arising from p-adic integrals on $\mathbb{Z}_p$

Let λ be any nonzero element in $\mathbb{C}_p$, and let $a \in \frac{1}{\lambda}E \cap \frac{1}{x}E$, (see (1.5)). Let us consider the degenerate hyperbolic functions given by

$$\begin{aligned} \cosh_\lambda(x:a) &= \frac{1}{2}(e_\lambda^x(a) + e_\lambda^{-x}(a)), \\ \sinh_\lambda(x:a) &= \frac{1}{2}(e_\lambda^x(a) - e_\lambda^{-x}(a)), \\ \tanh_\lambda(x:a) &= \frac{\sinh_\lambda(x:a)}{\cosh_\lambda(x:a)}, \\ \coth_\lambda(x:a) &= \frac{\cosh_\lambda(x:a)}{\sinh_\lambda(x:a)}, \quad (x \neq 0,\ a \neq 0). \end{aligned} \tag{2.1}$$

Note that $\lim_{\lambda\to 0}\cosh_\lambda(x:a) = \cosh(xa)$, $\lim_{\lambda\to 0}\sinh_\lambda(x:a) = \sinh(xa)$, $\lim_{\lambda\to 0}\tanh_\lambda(x:a) = \tanh(xa)$, and $\lim_{\lambda\to 0}\coth_\lambda(x:a) = \coth(xa)$.

From (2.1), we note that

$$\begin{aligned} \cosh_\lambda(2x:a) &= \frac{e_\lambda^{2x}(a) + e_\lambda^{-2x}(a)}{2} = 1 + \frac{e_\lambda^{2x}(a) + e_\lambda^{-2x}(a) - 2}{2} \\ &= 2\left(\frac{e_\lambda^{x}(a) - e_\lambda^{-x}(a)}{2}\right)^2 + 1 = 1 + 2\sinh_\lambda^2(x:a). \end{aligned} \tag{2.2}$$

On the other hand, by (2.1), we get

$$\begin{aligned} 2\cosh_\lambda^2(x:a) - 1 &= 2 \times \frac{e_\lambda^{2x}(a) + e_\lambda^{-2x}(a) + 2}{4} - 1 \\ &= \frac{e_\lambda^{2x}(a) + e_\lambda^{-2x}(a)}{2} = \cosh_\lambda(2x:a). \end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following proposition.

Proposition 1. *Let $a \in \frac{1}{\lambda}E \cap \frac{1}{x}E$. Then the following identity holds true.*

$$\cosh_\lambda(2x : a) = 2\cosh_\lambda^2(x : a) - 1 = 1 + 2\sinh_\lambda^2(x : a).$$

By (2.1), we get

$$\begin{aligned}
\sinh_\lambda(2x : a) &= \frac{e_\lambda^{2x}(a) - e_\lambda^{-2x}(a)}{2} \\
&= 2\frac{e_\lambda^{x}(a) - e_\lambda^{-x}(a)}{2}\frac{e_\lambda^{x}(a) + e_\lambda^{-x}(a)}{2} \\
&= 2\sinh_\lambda(x : a)\cosh_\lambda(x : a).
\end{aligned} \tag{2.4}$$

Thus we have the following proposition.

Proposition 2. *Let $a \in \frac{1}{\lambda}E \cap \frac{1}{x}E$. Then the following identity holds true.*

$$\sinh_\lambda(2x : a) = 2\sinh_\lambda(x : a)\cosh_\lambda(x : a).$$

Now, we observe that

$$\begin{aligned}
&\sinh_\lambda(x + y : a) = \frac{1}{2}(e_\lambda^{x+y}(a) - e_\lambda^{-x-y}(a)) \\
&= \frac{e_\lambda^{x+y}(a) - e_\lambda^{-x-y}(a) + e_\lambda^{x-y}(a) - e_\lambda^{-x+y}(a)}{4} + \frac{e_\lambda^{x+y}(a) - e_\lambda^{-x-y}(a) - e_\lambda^{x-y}(a) + e_\lambda^{-x+y}(a)}{4} \\
&= \frac{e_\lambda^{x}(a) - e_\lambda^{-x}(a)}{2} \times \frac{e_\lambda^{y}(a) + e_\lambda^{-y}(a)}{2} + \frac{e_\lambda^{x}(a) + e_\lambda^{-x}(a)}{2} \times \frac{e_\lambda^{y}(a) - e_\lambda^{-y}(a)}{2} \\
&= \sinh_\lambda(x : a)\cosh_\lambda(y : a) + \cosh_\lambda(x : a)\sinh_\lambda(y : a), \quad (a \in \frac{1}{\lambda}E \cap \frac{1}{x}E \cap \frac{1}{y}E).
\end{aligned} \tag{2.5}$$

On the other hand, by (2.1), we get

$$\begin{aligned}
&\cosh_\lambda(x : a)\cosh_\lambda(y : a) + \sinh_\lambda(x : a)\sinh_\lambda(y : a) \\
&= \frac{e_\lambda^{x}(a) + e_\lambda^{-x}(a)}{2} \times \frac{e_\lambda^{y}(a) + e_\lambda^{-y}(a)}{2} + \frac{e_\lambda^{x}(a) - e_\lambda^{-x}(a)}{2} \times \frac{e_\lambda^{y}(a) - e_\lambda^{-y}(a)}{2} \\
&= \frac{e_\lambda^{x+y}(a) + e_\lambda^{-x-y}(a) + e_\lambda^{-x+y}(a) + e_\lambda^{x-y}(a)}{4} + \frac{e_\lambda^{x+y}(a) + e_\lambda^{-x-y}(a) - e_\lambda^{x-y}(a) - e_\lambda^{-x+y}(a)}{4} \\
&= \frac{2(e_\lambda^{x+y}(a) + e_\lambda^{-(x+y)}(a))}{4} = \frac{e_\lambda^{x+y}(a) + e_\lambda^{-(x+y)}(a)}{2} = \cosh_\lambda(x + y : a),
\end{aligned} \tag{2.6}$$

where $a \in \frac{1}{\lambda}E \cap \frac{1}{x}E \cap \frac{1}{y}E$.

Therefore, by (2.5) and (2.6), we obtain the following proposition.

Proposition 3. *Let $a \in \frac{1}{\lambda}E \cap \frac{1}{x}E \cap \frac{1}{y}E$. Then the following identities are valid.*

$$\begin{aligned}
&\cosh_\lambda(x + y : a) = \cosh_\lambda(x : a)\cosh_\lambda(y : a) + \sinh_\lambda(x : a)\sinh_\lambda(y : a), \\
\text{and} \quad &\sinh_\lambda(x + y : a) = \sinh_\lambda(x : a)\cosh_\lambda(y : a) + \cosh_\lambda(x : a)\sinh_\lambda(y : a).
\end{aligned}$$

Noting that $\frac{d}{dx}\cosh_\lambda(x:a)=\frac{1}{\lambda}\log(1+\lambda a)\sinh_\lambda(x:a)$ and using (1.2), we have

$$\begin{aligned}0&=\int_{\mathbb{Z}_p}\cosh_\lambda(x+1:a)\,d\mu(x)-\int_{\mathbb{Z}_p}\cosh_\lambda(x:a)\,d\mu(x)\\&=(\cosh_\lambda(1:a)-1)\int_{\mathbb{Z}_p}\cosh_\lambda(x:a)\,d\mu(x)+\sinh_\lambda(1:a)\int_{\mathbb{Z}_p}\sinh_\lambda(x:a)\,d\mu(x),\end{aligned}\tag{2.7}$$

where $a\in\frac{1}{\lambda}E\cap E$.

Observing that $\frac{d}{dx}\sinh_\lambda(x:a)=\frac{1}{\lambda}\log(1+\lambda a)\cosh_\lambda(x:a)$ and making use of (1.2), we get

$$\begin{aligned}\frac{1}{\lambda}\log(1+\lambda a)&=\int_{\mathbb{Z}_p}\sinh_\lambda(x+1:a)d\mu(x)-\int_{\mathbb{Z}_p}\sinh_\lambda(x:a)d\mu(x)\\&=(\cosh_\lambda(1:a)-1)\int_{\mathbb{Z}_p}\sinh_\lambda(x:a)\,d\mu(x)\\&\quad+\sinh_\lambda(1:a)\int_{\mathbb{Z}_p}\cosh_\lambda(x:a)\,d\mu(x),\end{aligned}\tag{2.8}$$

where $a\in\frac{1}{\lambda}E\cap E$.

By solving the system of linear equations in (2.7) and (2.8), we get

$$\begin{aligned}\int_{\mathbb{Z}_p}\cosh_\lambda(x:a)\,d\mu(x)&=\frac{1}{\lambda}\log(1+\lambda a)\frac{\sinh_\lambda(1:a)}{2(\cosh_\lambda(1:a)-1)},\\\int_{\mathbb{Z}_p}\sinh_\lambda(x:a)\,d\mu(x)&=-\frac{1}{2}\frac{1}{\lambda}\log(1+\lambda a),\quad\left(a\in\frac{1}{\lambda}E\cap E\right).\end{aligned}\tag{2.9}$$

From (2.9) and Propositions 1 and 2, we have

$$\begin{aligned}\int_{\mathbb{Z}_p}\cosh_\lambda(x:a)d\mu(x)&=\frac{1}{\lambda}\log(1+\lambda a)\frac{2\sinh_\lambda(\frac{1}{2}:a)\cosh_\lambda(\frac{1}{2}:a)}{4\sinh^2_\lambda(\frac{1}{2}:a)}\\&=\frac{a}{2}\coth_\lambda\left(\frac{1}{2}:a\right)\frac{\log(1+\lambda a)}{\lambda a},\quad\left(a\in\frac{1}{\lambda}E\cap E\right).\end{aligned}\tag{2.10}$$

Therefore, from (2.9) and (2.10) we obtain the following theorem.

Theorem 4. *Let $a\in\frac{1}{\lambda}E\cap E$. Then the following relations hold true.*

$$\begin{aligned}\frac{\lambda a}{\log(1+\lambda a)}\int_{\mathbb{Z}_p}\cosh_\lambda(x:a)\,d\mu(x)&=\frac{a}{2}\coth_\lambda\left(\frac{1}{2}:a\right),\\\frac{\lambda a}{\log(1+\lambda a)}\int_{\mathbb{Z}_p}\sinh_\lambda(x:a)\,d\mu(x)&=-\frac{a}{2}.\end{aligned}$$

By Taylor expansion, we get

$$\begin{aligned}\cosh_\lambda(x:a)&=\frac{1}{2}(e^x_\lambda(a)+e^{-x}_\lambda(a))=\frac{1}{2}(e^x_\lambda(a)+e^x_{-\lambda}(-a))\\&=\frac{1}{2}\sum_{m=0}^{\infty}((x)_{m,\lambda}+(-1)^m(x)_{m,-\lambda})\frac{a^m}{m!},\quad\left(a\in\frac{1}{\lambda}E\cap\frac{1}{x}E\right).\end{aligned}\tag{2.11}$$

From Theorem 4, (2.11) and (1.12), we have

$$\begin{aligned}\frac{a}{2}\coth_{\lambda}\Big(\frac{1}{2}:a\Big)&=\frac{\lambda a}{\log(1+\lambda a)}\int_{\mathbb{Z}_p}\cosh_{\lambda}(x:a)\,d\mu(x)\\&=\Big(\sum_{l=0}^{\infty}\lambda^{l}C_l\frac{a^l}{l!}\Big)\frac{1}{2}\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}((x)_{m,\lambda}+(-1)^m(x)_{m,-\lambda})d\mu(x)\frac{a^m}{m!}\\&=\sum_{l=0}^{\infty}\lambda^{l}C_l\frac{a^l}{l!}\sum_{m=0}^{\infty}\Big(\frac{\beta_{m,\lambda}+(-1)^m\beta_{m,-\lambda}}{2}\Big)\frac{a^m}{m!}\\&=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}\Big(\frac{\beta_{m,\lambda}+(-1)^m\beta_{m,-\lambda}}{2}\Big)C_{n-m}\lambda^{n-m}\Big)\frac{a^n}{n!},\end{aligned}\tag{2.12}$$

where $a\in\frac{1}{\lambda}E\cap E$. By Taylor expansion, we get

$$\sinh_{\lambda}(x:a)=\frac{1}{2}\sum_{m=0}^{\infty}((x)_{m,\lambda}-(-1)^m(x)_{m,-\lambda})\frac{a^m}{m!},\quad (a\in\frac{1}{\lambda}E\cap\frac{1}{x}E).\tag{2.13}$$

Thus, by (2.9) and (2.13), we get

$$\begin{aligned}-\frac{1}{2\lambda}\log(1+\lambda a)&=\int_{\mathbb{Z}_p}\sinh_{\lambda}(x:a)\,d\mu(x)\\&=\sum_{n=0}^{\infty}\Big(\frac{\beta_{n,\lambda}-(-1)^n\beta_{n,-\lambda}}{2}\Big)\frac{a^n}{n!}=\sum_{n=1}^{\infty}\Big(\frac{\beta_{n,\lambda}-(-1)^n\beta_{n,-\lambda}}{2}\Big)\frac{a^n}{n!},\end{aligned}\tag{2.14}$$

$(a\in\frac{1}{\lambda}E\cap E)$,

and

$$-\frac{1}{2\lambda}\log(1+\lambda a)=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\lambda^{n-1}}{n}a^n=-\frac{1}{2}\sum_{n=1}^{\infty}(n-1)!(-1)^{n-1}\lambda^{n-1}\frac{a^n}{n!},\quad(|\lambda a|_p\leq 1).\tag{2.15}$$

By (2.12), (2.14) and (2.15), we obtain the following theorem.

Theorem 5. *For $n\in\mathbb{N}$, we have the identity:*

$$-(n-1)!(-\lambda)^{n-1}=\beta_{n,\lambda}-(-1)^n\beta_{n,-\lambda}.$$

In addition, we have the following relation:

$$\frac{a}{2}\coth_{\lambda}\Big(\frac{1}{2}:a\Big)=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}\Big(\frac{\beta_{m,\lambda}+(-1)^m\beta_{m,-\lambda}}{2}\Big)C_{n-m}\lambda^{n-m}\Big)\frac{a^n}{n!},$$

where $a\in\frac{1}{\lambda}E\cap E$.

From (1.4), we note that

$$\begin{aligned}2 &= \int_{\mathbb{Z}_p} \cosh_\lambda(x+1:a)\,d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \cosh_\lambda(x:a)\,d\mu_{-1}(x)\\ &= (\cosh_\lambda(1:a)+1)\int_{\mathbb{Z}_p} \cosh_\lambda(x:a)\,d\mu_{-1}(x) + \sinh_\lambda(1:a)\int_{\mathbb{Z}_p} \sinh_\lambda(x:a)\,d\mu_{-1}(x),\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}0 &= \int_{\mathbb{Z}_p} \sinh_\lambda(x+1:a)\,d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \sinh_\lambda(x:a)\,d\mu_{-1}(x)\\ &= (\cosh_\lambda(1:a)+1)\int_{\mathbb{Z}_p} \sinh_\lambda(x:a)\,d\mu_{-1}(x) + \sinh_\lambda(1:a)\int_{\mathbb{Z}_p} \cosh_\lambda(x:a)\,d\mu_{-1}(x),\end{aligned} \tag{2.17}$$

where $a \in \frac{1}{\lambda}E \cap E$.

By solving the system of linear equations in (2.16) and (2.17), we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p} \cosh_\lambda(x:a)\,d\mu_{-1}(x) &= 1,\\ \int_{\mathbb{Z}_p} \sinh_\lambda(x:a)\,d\mu_{-1}(x) &= -\frac{\sinh_\lambda(1:a)}{\cosh_\lambda(1:a)+1},\end{aligned} \tag{2.18}$$

where $a \in \frac{1}{\lambda}E \cap E$.

Thus, by (2.18) and Propositions 1 and 2, we get

$$\int_{\mathbb{Z}_p} \sinh_\lambda(x:a)\,d\mu_{-1}(x) = -\frac{2\sinh_\lambda(\frac{1}{2}:a)\cosh_\lambda(\frac{1}{2}:a)}{2\cosh_\lambda^2(\frac{1}{2}:a)} = -\tanh_\lambda\Big(\frac{1}{2}:a\Big), \tag{2.19}$$

where $a \in \frac{1}{\lambda}E \cap E$.

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 6. *Let $a \in \frac{1}{\lambda}E \cap E$. Then we have the following relations.*

$$\int_{\mathbb{Z}_p} \sinh_\lambda(x:a)\,d\mu_{-1}(x) = -\tanh_\lambda\Big(\frac{1}{2}:a\Big),$$
$$\int_{\mathbb{Z}_p} \cosh_\lambda(x:a)\,d\mu_{-1}(x) = 1.$$

Thus, by Theorem 6, (2.13) and (1.12), we get

$$\begin{aligned}-\tanh_\lambda(\frac{1}{2}:a) &= \int_{\mathbb{Z}_p} \sinh_\lambda(x:a)\,d\mu_{-1}(x)\\ &= \sum_{n=0}^{\infty}\Big(\frac{\mathcal{E}_{n,\lambda}-(-1)^n\mathcal{E}_{n,-\lambda}}{2}\Big)\frac{a^n}{n!}, \quad (a \in \frac{1}{\lambda}E \cap E).\end{aligned} \tag{2.20}$$

In addition, by Theorem 6, (2.11) and (1.12), we get

$$1=\int_{\mathbb{Z}_p}\cosh_\lambda(x;a)\,d\mu_{-1}(x)=\sum_{n=0}^{\infty}\left(\frac{\mathcal{E}_{n,\lambda}+(-1)^n\mathcal{E}_{n,-\lambda}}{2}\right)\frac{a^n}{n!},\quad \left(a\in\frac{1}{\lambda}E\cap E\right). \tag{2.21}$$

Thus, by(2.21), we get

$$\frac{1}{2}(\mathcal{E}_{n,\lambda}+(-1)^n\mathcal{E}_{n,-\lambda})=\begin{cases}1, & \text{if} \quad n=0,\\ 0, & \text{if} \quad n>0.\end{cases} \tag{2.22}$$

For $n\geq 1$, we have

$$\mathcal{E}_{n,\lambda}=(-1)^{n-1}\mathcal{E}_{n,-\lambda}. \tag{2.23}$$

From (2.20) and (2.23), we obtain the following theorem.

Theorem 7. *Let $a\in\frac{1}{\lambda}E\cap E$. Then we have the following relation.*

$$-\tanh_\lambda\left(\frac{1}{2}:a\right)=\sum_{n=1}^{\infty}\mathcal{E}_{n,\lambda}\frac{a^n}{n!}.$$

3. Conclusions

We introduced several degenerate hyperbolic functions which are degenerate versions of the usual hyperbolic functions. We computed Volkenborn and the fermionic p-adic integrals of the degenerate hyperbolic cosine and the degenerate hyperbolic sine functions. From those results, we were able to derive some identities regarding the degenerate hyperbolic tangent and the degenerate hyperbolic cotangent functions.

In recent years, various kinds of tools, like generating functions, combinatorial methods, p-adic analysis, umbral calculus, differential equations, probability theory, special functions, analytic number theory and operator theory, have been used in studying special numbers and polynomials, and degenerate versions of them.

It is one of our future research projects to continue to explore many special numbers and polynomials and their applications to physics, science and engineering as well as to mathematics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Taekyun Kim, Hye Kyung Kim and Dae Sa Kim are the Guest Editors of special issue "Number theory, combinatorics and their applications: theory and computation" for AIMS Mathematics. Taekyun Kim, Hye Kyung Kim and Dae Sa Kim were not involved in the editorial review and the decision to publish this article.

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