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Research article

Stability of a mixed type additive-quadratic functional equation with a parameter in matrix intuitionistic fuzzy normed spaces

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Abstract: The purpose of this paper is first to introduce the notation of matrix intuitionistic fuzzy normed spaces, and then by virtue of this notation to study the Hyers-Ulam stability results concerning the mixed type additive-quadratic functional equation

$$\begin{aligned}2k[f(x+ky)+f(kx+y)] &= k(1-s+k+ks+2k^2)f(x+y)+k(1-s-3k+ks+2k^2)f(x-y)\\ &\quad +2kf(kx)+2k(s+k-ks-2k^2)f(x)+2(1-k-s)f(ky)+2ksf(y)\end{aligned}$$

in the setting of matrix intuitionistic fuzzy normed spaces by applying two different methods, where s is a parameter, $k>1$ and $s\neq 1-2k$. Moreover, the interdisciplinary relation between the theory of matrix intuitionistic fuzzy normed spaces and the theory of functional equations are also presented in this paper.

Keywords: fixed point method; Hyers-Ulam stability; matrix intuitionistic fuzzy normed spaces; mixed type additive-quadratic functional equation
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1. Introduction

In 1940, Ulam [24] posed the stability problem concerning group homomorphisms. For Banach spaces, the problem was solved by Hyers [7] in the case of approximate additive mappings. And then Hyers' result was extended by Aoki [1] and Rassias [18] for additive mappings and linear mappings, respectively. In 1994, another further generalization, the so-called generalized Hyer-Ulam stability, was obtained by Gavruta [6]. Later, the stability of several functional equations has been extensively discussed by many mathematicians and there are many interesting results concerning this problem (see [2,8–10,19,20] and references therein); also, some stability results of different functional equations and inequalities were studied and generalized [5,11,12,15–17,26] in various matrix normed spaces like matrix fuzzy normed spaces, matrix paranormed spaces and matrix non-Archimedean random normed spaces.

In 2017, Wang and Xu [25] introduced the following functional equation

$$\begin{aligned} 2k[f(x+ky)+f(kx+y)] = k(1-s+k+ks+2k^2)f(x+y)+k(1-s-3k+ks+2k^2)f(x-y) \\ +2kf(kx)+2k(s+k-ks-2k^2)f(x)+2(1-k-s)f(ky)+2ksf(y) \end{aligned} \tag{1.1}$$

where s is a parameter, $k > 1$ and $s \neq 1-2k$. It is easy to verify that $f(x) = ax + bx^2 (x \in \mathbb{R})$ satisfies the functional Eq (1.1), where a, b are arbitrary constants. They considered the general solution of the functional Eq (1.1), and then determined the generalized Hyers-Ulam stability of the functional Eq (1.1) in quasi-Banach spaces by applying the direct method.

The main purpose of this paper is to employ the direct and fixed point methods to establish the Hyers-Ulam stability of the functional Eq (1.1) in matrix intuitionistic fuzzy normed spaces. The paper is organized as follows: In Sections 1 and 2, we present a brief introduction and introduce related basic definitions and preliminary results, respectively. In Section 3, we prove the Hyers-Ulam stability of the functional Eq (1.1) in matrix intuitionistic fuzzy normed spaces by applying the direct method. In Section 4, we prove the Hyers-Ulam stability of the functional Eq (1.1) in matrix intuitionistic fuzzy normed spaces by applying the fixed point method. Our results may be viewed as a continuation of the previous contribution of the authors in the setting of fuzzy stability (see [14, 17]).

2. Preliminaries

For the sake of completeness, in this section, we present some basic definitions and preliminary results, which will be useful to investigate the Hyers-Ulam stability results in matrix intuitionistic fuzzy normed spaces. The notions of continuous t-norm and continuous t-conorm can be found in [14, 22]. Using these, an intuitionistic fuzzy normed space (for short, IFNS) is defined as follows:

Definition 2.1. ([14, 21]) *The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an IFNS if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfy the following conditions. For every $x, y \in X$ and $s, t > 0$,*

(i) $\mu(x, t) + \nu(x, t) \leq 1$;

(ii) $\mu(x, t) > 0$, (iii) $\mu(x, t) = 1$ if and only if $x = 0$;

(iii) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (v) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$;

(iv) $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous;

(v) $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$;

(vi) $\nu(x, t) < 1$, (ix) $\nu(x, t) = 0$ if and only if $x = 0$;

(vii) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (xi) $\nu(x, t) \diamond \ \nu(y, s) \geq \nu(x + y, t + s)$;

(xiii) $\nu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous;

(ix) $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to 0} \nu(x, t) = 1$.

In this case, (μ, ν) is called an intuitionistic fuzzy norm.

The following concepts of convergence and Cauchy sequences are considered in [14, 21]:

Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $\{x_k\}$ is said to be intuitionistic fuzzy convergent to $x \in X$ if for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - x, t) > 1 - \varepsilon$$

and

$$\nu(x_k - x, t) < \varepsilon$$

for all $k \geq k_0$. In this case we write

$$(\mu, \nu) - \lim x_k = x.$$

The sequence $\{x_k\}$ is said to be an intuitionistic fuzzy Cauchy sequence if for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - x_\ell, t) > 1 - \varepsilon$$

and

$$\nu(x_k - x_\ell, t) < \varepsilon$$

for all $k, \ell \geq k_0$. $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$.

Following [11, 12], we will also use the following notations: The set of all $m \times n$-matrices in X will be denoted by $M_{m,n}(X)$. When $m = n$, the matrix $M_{m,n}(X)$ will be written as $M_n(X)$. The symbols $e_j \in M_{1,n}(\mathbb{C})$ will denote the row vector whose jth component is 1 and the other components are 0. Similarly, $E_{ij} \in M_n(\mathbb{C})$ will denote the $n \times n$ matrix whose (i, j)-component is 1 and the other components are 0. The $n \times n$ matrix whose (i, j)-component is x and the other components are 0 will be denoted by $E_{ij} \otimes x \in M_n(X)$.

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and

$$\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$$

holds for $A \in M_{k,n}$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

Following [23], we introduce the concept of a matrix intuitionistic fuzzy normed space as follows:

Definition 2.2. ([23]) *Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space, and the symbol θ for a rectangular matrix of zero elements over X. Then:*

(1) *$(X, \{\mu_n\}, \{\nu_n\}, *, \diamond)$ is called a matrix intuitionistic fuzzy normed space (briefly, MIFNS) if for each positive integer n, $(M_n(X), \mu_n, \nu_n, *, \diamond)$ is an intuitionistic fuzzy normed space, μ_n and ν_n satisfy the following conditions:*

(i) *$\mu_{n+m}(\theta + x, t) = \mu_n(x, t), \nu_{n+m}(\theta + x, t) = \nu_n(x, t)$ for all $t > 0$, $x = [x_{ij}] \in M_n(X)$, $\theta \in M_n(X)$;*

(ii) *$\mu_k(AxB, t) \geq \mu_n(x, \frac{t}{\|A\|\cdot\|B\|})$, $\nu_k(AxB, t) \leq \nu_n(x, \frac{t}{\|A\|\cdot\|B\|})$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$.*

(2) *$(X, \{\mu_n\}, \{\nu_n\}, *, \diamond)$ is called a matrix intuitionistic fuzzy Banach space if $(X, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy Banach space and $(X, \{\mu_n\}, \{\nu_n\}, *, \diamond)$ is a matrix intuitionistic fuzzy normed space.*

The following Lemma 2.3 was found in [23].

Lemma 2.3. ([23]) *Let* $(X, \{\mu_n\}, \{\nu_n\}, *, \diamond)$ *be a matrix intuitionistic fuzzy normed space. Then,*
(1) $\mu_n(E_{kl} \otimes x, t) = \mu(x, t)$*,* $\nu_n(E_{kl} \otimes x, t) = \nu(x, t)$ *for all* $t > 0$ *and* $x \in X$*.*
(2) *For all* $[x_{ij}] \in M_n(X)$ *and* $t = \sum_{i,j=1}^{n} t_{ij} > 0$*,*

$$\mu(x_{kl}, t) \geq \mu_n([x_{ij}], t) \geq \min\{\mu(x_{ij}, t_{ij}) : i, j = 1, 2, \ldots, n\},$$
$$\mu(x_{kl}, t) \geq \mu_n([x_{ij}], t) \geq \min\{\mu(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \ldots, n\},$$

and

$$\nu(x_{kl}, t) \leq \nu_n([x_{ij}], t) \leq \max\{\nu(x_{ij}, t_{ij}) : i, j = 1, 2, \ldots, n\},$$
$$\nu(x_{kl}, t) \leq \nu_n([x_{ij}], t) \leq \max\{\nu(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \ldots, n\}.$$

(3) $\lim_{m\to\infty} x_m = x$ *if and only if* $\lim_{m\to\infty} x_{ijm} = x_{ij}$ *for* $x_m = [x_{ijm}], x = [x_{ij}] \in M_n(X)$*.*

For explicit later use, we also recall the following Lemma 2.4 is due to Diaz and Margolis [4], which will play an important role in proving our stability results in this paper.

Lemma 2.4. (The fixed point alternative theorem [4]) *Let* (E, d) *be a complete generalized metric space and* $J\colon E \to E$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$*. Then for each fixed element* $x \in E$*, either*

$$d(J^n x, J^{n+1} x) = \infty, \quad \forall n \geq 0,$$

or

$$d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0,$$

for some natural number n_0*. Moreover, if the second alternative holds then:*

(*i*) *The sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J*.*

(*ii*) y^* *is the unique fixed point of* J *in the set* $E^* := \{y \in E \mid d(J^{n_0} x, y) < +\infty\}$ *and* $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy), \ \forall x, y \in E^*$*.*

3. Stability of the functional Eq (1.1): direct method

From now on, let $(X, \{\mu_n\}, \{\nu_n\}, *, \diamond)$ be a matrix intuitionistic fuzzy normed space and $(Y, \{\mu_n\}, \{\nu_n\}, *, \diamond)$ be a matrix intuitionistic fuzzy Banach space. In this section, we will prove the Hyers-Ulam stability of the functional Eq (1.1) in matrix intuitionistic fuzzy normed spaces by using the direct method. For the sake of convenience, given mapping $f\colon X \to Y$, we define the difference operators $Df\colon X^2 \to Y$ and $Df_n\colon M_n(X^2) \to M_n(Y)$ of the functional Eq (1.1) by

$$\begin{aligned}
Df(a, b) :&= 2k[f(a + kb) + f(ka + b)] \\
&\quad - k(1 - s + k + ks + 2k^2)f(a + b) - k(1 - s - 3k + ks + 2k^2)f(a - b) \\
&\quad - 2kf(ka) - 2k(s + k - ks - 2k^2)f(a) - 2(1 - k - s)f(kb) - 2ksf(b), \\
Df_n([x_{ij}], [y_{ij}]) :&= 2k[f_n([x_{ij}] + k[y_{ij}]) + f_n(k[x_{ij}] + [y_{ij}])] \\
&\quad - k(1 - s + k + ks + 2k^2)f_n([x_{ij}] + [y_{ij}]) - k(1 - s - 3k + ks + 2k^2)f_n([x_{ij}] - [y_{ij}])
\end{aligned}$$

$$- 2kf_n(k[x_{ij}]) - 2k(s + k - ks - 2k^2)f_n([x_{ij}]) - 2(1 - k - s)f_n(k[y_{ij}]) - 2ksf_n([y_{ij}])$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

We start with the following lemmas which will be used in this paper.

Lemma 3.1. ([25]) *Let V and W be real vector spaces. If an odd mapping $f\colon V \to W$ satisfies the functional Eq (1.1), then f is additive.*

Lemma 3.2. ([25]) *Let V and W be real vector spaces. If an even mapping $f\colon V \to W$ satisfies the functional Eq (1.1), then f is quadratic.*

Theorem 3.3. *Let $\varphi_o\colon X^2 \to [0, \infty)$ be a function such that for some real number α with $0 < \alpha < k$,*

$$\varphi_o(ka, kb) = \alpha\varphi_o(a, b) \tag{3.1}$$

for all $a, b \in X$. Suppose that an odd mapping $f\colon X \to Y$ satisfies the inequality

$$\begin{cases} \mu_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t+\sum_{i,j=1}^{n} \varphi_o(x_{ij}, y_{ij})}, \\ \nu_n(Df_n([x_{ij}], [y_{ij}]), t) \leq \frac{\sum_{i,j=1}^{n} \varphi_o(x_{ij}, y_{ij})}{t+\sum_{i,j=1}^{n} \varphi_o(x_{ij}, y_{ij})} \end{cases} \tag{3.2}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exists a unique additive mapping $A\colon X \to Y$ such that

$$\begin{cases} \mu_n(f_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t+n^2 \sum_{i,j=1}^{n} \varphi_o(0, x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - A_n([x_{ij}]), t) \leq \frac{n^2 \sum_{i,j=1}^{n} \varphi_o(0, x_{ij})}{(k-\alpha)(2k+s-1)t+n^2 \sum_{i,j=1}^{n} \varphi_o(0, x_{ij})} \end{cases} \tag{3.3}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. When $n = 1$, (3.2) is equivalent to

$$\mu(Df(a, b), t) \geq \frac{t}{t+\varphi_o(a,b)} \quad \text{and} \quad \nu(Df(a, b), t) \leq \frac{\varphi_o(a,b)}{t+\varphi_o(a,b)} \tag{3.4}$$

for all $a, b \in X$ and all $t > 0$. Putting $a = 0$ in (3.4), we have

$$\begin{cases} \mu(2(2k + s - 1)f(kb) - 2(2k + s - 1)kf(b), t) \geq \frac{t}{t+\varphi_o(0,b)}, \\ \nu(2(2k + s - 1)f(kb) - 2(2k + s - 1)kf(b), t) \leq \frac{\varphi_o(0,b)}{t+\varphi_o(0,b)} \end{cases} \tag{3.5}$$

for all $b \in X$ and all $t > 0$. Replacing a by $k^p a$ in (3.5) and using (3.1), we get

$$\begin{cases} \mu(\frac{f(k^{p+1}a)}{k^{p+1}} - \frac{f(k^p a)}{k^p}, \frac{t}{2k(2k+s-1)k^p}) \geq \frac{t}{t+\alpha^p\varphi_o(0,a)}, \\ \nu(\frac{f(k^{p+1}a)}{k^{p+1}} - \frac{f(k^p a)}{k^p}, \frac{t}{2k(2k+s-1)k^p}) \leq \frac{\alpha^p\varphi_o(0,a)}{t+\alpha^p\varphi_o(0,a)} \end{cases} \tag{3.6}$$

for all $a \in X$ and all $t > 0$. It follows from (3.6) that

$$\begin{cases} \mu(\frac{f(k^{p+1}a)}{k^{p+1}} - \frac{f(k^p a)}{k^p}, \frac{\alpha^p t}{2k(2k+s-1)k^p}) \geq \frac{t}{t+\varphi_o(0,a)}, \\ \nu(\frac{f(k^{p+1}a)}{k^{p+1}} - \frac{f(k^p a)}{k^p}, \frac{\alpha^p t}{2k(2k+s-1)k^p}) \leq \frac{\varphi_o(0,a)}{t+\varphi_o(0,a)} \end{cases} \tag{3.7}$$

for all $a \in X$ and all $t > 0$. It follows from

$$\frac{f(k^p a)}{k^p} - f(a) = \sum_{\ell=0}^{p-1} (\frac{f(k^{\ell+1}a)}{k^{\ell+1}} - \frac{f(k^\ell a)}{k^\ell})$$

and (3.7) that

$$\begin{cases} \mu(\frac{f(k^p a)}{k^p} - f(a), \sum_{\ell=0}^{p-1} \frac{\alpha^\ell t}{2k(2k+s-1)k^\ell}) \geq \prod_{\ell=0}^{p-1} \mu(\frac{f(k^{\ell+1}a)}{k^{\ell+1}} - \frac{f(k^\ell a)}{k^\ell}, \frac{\alpha^\ell t}{2k(2k+s-1)k^\ell}) \geq \frac{t}{t+\varphi_o(0,a)}, \\ \nu(\frac{f(k^p a)}{k^p} - f(a), \sum_{\ell=0}^{p-1} \frac{\alpha^\ell t}{2k(2k+s-1)k^\ell}) \leq \coprod_{\ell=0}^{p-1} \nu(\frac{f(k^{\ell+1}a)}{k^{\ell+1}} - \frac{f(k^\ell a)}{k^\ell}, \frac{\alpha^\ell t}{2k(2k+s-1)k^\ell}) \leq \frac{\varphi_o(0,a)}{t+\varphi_o(0,a)} \end{cases} \tag{3.8}$$

for all $a \in X$ and all $t > 0$, where

$$\prod_{j=0}^{p} a_j = a_1 * a_2 * \cdots * a_p, \quad \coprod_{j=0}^{p} a_j = a_1 \diamond a_2 \diamond \cdots \diamond a_p.$$

By replacing a with $k^q a$ in (3.8), we have

$$\begin{cases} \mu(\frac{f(k^{p+q}a)}{k^{p+q}} - \frac{f(k^q a)}{k^q}, \sum_{\ell=0}^{p-1} \frac{\alpha^\ell t}{2k(2k+s-1)k^{\ell+q}}) \geq \frac{t}{t+\alpha^q \varphi_o(0,a)}, \\ \nu(\frac{f(k^{p+q}a)}{k^{p+q}} - \frac{f(k^q a)}{k^q}, \sum_{\ell=0}^{p-1} \frac{\alpha^\ell t}{2k(2k+s-1)k^{\ell+q}}) \leq \frac{\alpha^q \varphi_o(0,a)}{t+\alpha^q \varphi_o(0,a)} \end{cases} \tag{3.9}$$

for all $a \in X$, $t > 0$, $p > 0$ and $q > 0$. Thus

$$\begin{cases} \mu(\frac{f(k^{p+q}a)}{k^{p+q}} - \frac{f(k^q a)}{k^q}, \sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell t}{2k(2k+s-1)k^\ell}) \geq \frac{t}{t+\varphi_o(0,a)}, \\ \nu(\frac{f(k^{p+q}a)}{k^{p+q}} - \frac{f(k^q a)}{k^q}, \sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell t}{2k(2k+s-1)k^\ell}) \leq \frac{\varphi_o(0,a)}{t+\varphi_o(0,a)} \end{cases} \tag{3.10}$$

for all $a \in X$, $t > 0$, $p > 0$ and $q > 0$. Hence

$$\begin{cases} \mu(\frac{f(k^{p+q}a)}{k^{p+q}} - \frac{f(k^q a)}{k^q}, t) \geq \frac{t}{t+\sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell}{2k(2k+s-1)k^\ell} \varphi_o(0,a)}, \\ \nu(\frac{f(k^{p+q}a)}{k^{p+q}} - \frac{f(k^q a)}{k^q}, t) \leq \frac{\sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell}{2k(2k+s-1)k^\ell} \varphi_o(0,a)}{t+\sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell}{2k(2k+s-1)k^\ell} \varphi_o(0,a)} \end{cases} \tag{3.11}$$

for all $a \in X$, $t > 0$, $p > 0$ and $q > 0$. Since $0 < \alpha < k$ and

$$\sum_{\ell=0}^{\infty} \frac{\alpha^\ell}{2k(2k+s-1)k^\ell} < \infty,$$

the Cauchy criterion for convergence in IFNS shows that $\{\frac{f(k^p a)}{k^p}\}$ is a Cauchy sequence in $(Y, \mu, \nu, *, \diamond)$. Since $(Y, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy Banach space, this sequence converges to some point $A(a) \in Y$. So one can define the mapping A: $X \to Y$ such that

$$A(a) := (\mu, \nu) - \lim_{p\to\infty} \frac{f(k^p a)}{k^p}.$$

Moreover, if we put $q = 0$ in (3.11), we get

$$\begin{cases} \mu(\frac{f(k^p a)}{k^p} - f(a), t) \geq \frac{t}{t + \sum_{\ell=0}^{p-1} \frac{\alpha^\ell}{2k(2k+s-1)k^\ell} \varphi_o(0,a)}, \\ \\ \nu(\frac{f(k^p a)}{k^p} - f(a), t) \leq \frac{\sum_{\ell=0}^{p-1} \frac{\alpha^\ell}{2k(2k+s-1)k^\ell} \varphi_o(0,a)}{t + \sum_{\ell=0}^{p-1} \frac{\alpha^\ell}{2k(2k+s-1)k^\ell} \varphi_o(0,a)} \end{cases} \tag{3.12}$$

for all $a \in X$, $t > 0$ and $p > 0$. Thus, we obtain

$$\begin{cases} \mu(f(a) - A(a), t) \geq \mu(f(a) - \frac{f(k^p a)}{k^p}, \frac{t}{2}) * \mu(\frac{f(k^p a)}{k^p} - A(a), \frac{t}{2}) \geq \frac{t}{t + \sum_{\ell=0}^{p-1} \frac{\alpha^\ell}{k(2k+s-1)k^\ell} \varphi_o(0,a)}, \\ \\ \nu(f(a) - A(a), t) \leq \nu(f(a) - \frac{f(k^p a)}{k^p}, \frac{t}{2}) * \nu(\frac{f(k^p a)}{k^p} - A(a), \frac{t}{2}) \leq \frac{\sum_{\ell=0}^{p-1} \frac{\alpha^\ell}{k(2k+s-1)k^\ell} \varphi_o(0,a)}{t + \sum_{\ell=0}^{p-1} \frac{\alpha^\ell}{k(2k+s-1)k^\ell} \varphi_o(0,a)} \end{cases} \tag{3.13}$$

for every $a \in X$, $t > 0$ and large p. Taking the limit as $p \to \infty$ and using the definition of IFNS, we get

$$\begin{cases} \mu(f(a) - A(a), t) \geq \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t + \varphi_o(0,a)}, \\ \\ \nu(f(a) - A(a), t) \leq \frac{\varphi_o(0,a)}{(k-\alpha)(2k+s-1)t + \varphi_o(0,a)}. \end{cases} \tag{3.14}$$

Replacing a and b by $k^p a$ and $k^p b$ in (3.4), respectively, and using (3.1), we obtain

$$\mu(\tfrac{1}{k^p} Df(k^p a, k^p b), t) \geq \frac{t}{t + (\frac{\alpha}{k})^p \varphi_o(a,b)} \quad \text{and} \quad \nu(\tfrac{1}{k^p} Df(k^p a, k^p b), t) \leq \frac{(\frac{\alpha}{k})^p \varphi_o(a,b)}{t + (\frac{\alpha}{k})^p \varphi_o(a,b)} \tag{3.15}$$

for all $a, b \in X$ and all $t > 0$. Letting $p \to \infty$ in (3.15), we obtain

$$\mu(DA(a,b), t) = 1 \quad \text{and} \quad \nu(DA(a,b), t) = 0 \tag{3.16}$$

for all $a, b \in X$ and all $t > 0$. This means that A satisfies the functional Eq (1.1). Since f: $X \to Y$ is an odd mapping, and using the definition A, we have $A(-a) = -A(a)$ for all $a \in X$. Thus by Lemma 3.1, the mapping A: $X \to Y$ is additive. To prove the uniqueness of A, let A': $X \to Y$ be another additive mapping satisfying (3.14). Let $n = 1$. Then we have

$$\begin{cases} \mu(A(a) - A'(a), t) = \mu(\frac{A(k^p a)}{k^p} - \frac{A'(k^p a)}{k^p}, t) \\ \geq \mu(\frac{A(k^p a)}{k^p} - \frac{f(k^p a)}{k^p}, \frac{t}{2}) * \mu(\frac{f(k^p a)}{k^p} - \frac{A'(k^p a)}{k^p}, \frac{t}{2}) \geq \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t + 2(\frac{\alpha}{k})^p \varphi_o(0,a)}, \\ \\ \nu(A(a) - A'(a), t) = \nu(\frac{A(k^p a)}{k^p} - \frac{A'(k^p a)}{k^p}, t) \\ \leq \nu(\frac{A(k^p a)}{k^p} - \frac{f(k^p a)}{k^p}, \frac{t}{2}) \diamond \nu(\frac{f(k^p a)}{k^p} - \frac{A'(k^p a)}{k^p}, \frac{t}{2}) \leq \frac{2(\frac{\alpha}{k})^p \varphi_o(0,a)}{(k-\alpha)(2k+s-1)t + 2(\frac{\alpha}{k})^p \varphi_o(0,a)} \end{cases} \tag{3.17}$$

for all $a \in X$, $t > 0$ and $p > 0$. Letting $p \to \infty$ in (3.17), we get

$$\mu(A(a) - A'(a), t) = 1 \quad \text{and} \quad \nu(A(a) - A'(a), t) = 0$$

for all $a \in X$ and $t > 0$. Hence we get $A(a) = A'(a)$ for all $a \in X$. Thus the mapping A: $X \to Y$ is a unique additive mapping.

By Lemma 2.3 and (3.14), we get

$$\begin{cases} \mu_n(f_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \min\{\mu(f(x_{ij}) - A(x_{ij}), \frac{t}{n^2}) : i, j = 1, \ldots, n\} \\ \qquad \geq \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t + n^2 \sum_{i,j=1}^{n} \varphi_o(0, x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - A_n([x_{ij}]), t) \leq \max\{\nu(f(x_{ij}) - A(x_{ij}), \frac{t}{n^2}) : i, j = 1, \ldots, n\} \\ \qquad \leq \frac{n^2 \sum_{i,j=1}^{n} \varphi_o(0, x_{ij})}{(k-\alpha)(2k+s-1)t + n^2 \sum_{i,j=1}^{n} \varphi_o(0, x_{ij})} \end{cases}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$. Thus A: $X \to Y$ is a unique additive mapping satisfying (3.3), as desired. This completes the proof of the theorem. □

Theorem 3.4. *Let φ_e: $X^2 \to [0, \infty)$ be a function such that for some real number α with $0 < \alpha < k^2$,*

$$\varphi_e(ka, kb) = \alpha \varphi_e(a, b) \tag{3.18}$$

for all $a, b \in X$. Suppose that an even mapping f: $X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\begin{cases} \mu_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t + \sum_{i,j=1}^{n} \varphi_e(x_{ij}, y_{ij})}, \\ \nu_n(Df_n([x_{ij}], [y_{ij}]), t) \leq \frac{\sum_{i,j=1}^{n} \varphi_e(x_{ij}, y_{ij})}{t + \sum_{i,j=1}^{n} \varphi_e(x_{ij}, y_{ij})} \end{cases} \tag{3.19}$$

for all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exists a unique quadratic mapping Q: $X \to Y$ such that

$$\begin{cases} \mu_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{(k^2-\alpha)(2k+s-1)t}{(k^2-\alpha)(2k+s-1)t + n^2 \sum_{i,j=1}^{n} \varphi_e(0, x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \leq \frac{n^2 \sum_{i,j=1}^{n} \varphi_e(0, x_{ij})}{(k^2-\alpha)(2k+s-1)t + n^2 \sum_{i,j=1}^{n} \varphi_e(0, x_{ij})} \end{cases} \tag{3.20}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. When $n = 1$, (3.19) is equivalent to

$$\mu(Df(a, b), t) \geq \frac{t}{t + \varphi_e(a, b)} \quad \text{and} \quad \nu(Df(a, b), t) \leq \frac{\varphi_e(a, b)}{t + \varphi_e(a, b)} \tag{3.21}$$

for all $a, b \in X$ and all $t > 0$. Letting $a = 0$ in (3.21), we obtain

$$\begin{cases} \mu(2(2k + s - 1)f(kb) - 2(2k + s - 1)k^2 f(b), t) \geq \frac{t}{t + \varphi_e(0, b)}, \\ \nu(2(2k + s - 1)f(kb) - 2(2k + s - 1)k^2 f(b), t) \leq \frac{\varphi_e(0, b)}{t + \varphi_e(0, b)} \end{cases} \tag{3.22}$$

for all $b \in X$ and all $t > 0$. Replacing a by $k^p a$ in (3.22) and using (3.18), we get

$$\begin{cases} \mu(\frac{f(k^{p+1}a)}{k^{2(p+1)}} - \frac{f(k^p a)}{k^{2p}}, \frac{t}{2k^2(2k+s-1)k^{2p}}) \geq \frac{t}{t+\alpha^p \varphi_e(0,a)}, \\ \nu(\frac{f(k^{p+1}a)}{k^{2(p+1)}} - \frac{f(k^p a)}{k^{2p}}, \frac{t}{2k^2(2k+s-1)k^{2p}}) \leq \frac{\alpha^p \varphi_e(0,a)}{t+\alpha^p \varphi_e(0,a)} \end{cases} \tag{3.23}$$

for all $a \in X$ and all $t > 0$. It follows from (3.23) that

$$\begin{cases} \mu(\frac{f(k^{p+1}a)}{k^{2(p+1)}} - \frac{f(k^p a)}{k^{2p}}, \frac{\alpha^p t}{2k^2(2k+s-1)k^{2p}}) \geq \frac{t}{t+\varphi_e(0,a)}, \\ \nu(\frac{f(k^{p+1}a)}{k^{2(p+1)}} - \frac{f(k^p a)}{k^{2p}}, \frac{\alpha^p t}{2k^2(2k+s-1)k^{2p}}) \leq \frac{\varphi_e(0,a)}{t+\varphi_e(0,a)} \end{cases} \tag{3.24}$$

for all $a \in X$ and all $t > 0$. It follows from

$$\frac{f(k^p a)}{k^{2p}} - f(a) = \sum_{\ell=0}^{p-1} (\frac{f(k^{\ell+1}a)}{k^{2(\ell+1)}} - \frac{f(k^\ell a)}{k^{2\ell}})$$

and (3.24) that

$$\begin{cases} \mu(\frac{f(k^p a)}{k^{2p}} - f(a), \sum_{\ell=0}^{p-1} \frac{\alpha^\ell t}{2k^2(2k+s-1)k^{2\ell}}) \geq \prod_{\ell=0}^{p-1} \mu(\frac{f(k^{\ell+1}a)}{k^{2(\ell+1)}} - \frac{f(k^\ell a)}{k^{2\ell}}, \frac{\alpha^\ell t}{2k^2(2k+s-1)k^{2\ell}}) \geq \frac{t}{t+\varphi_e(0,a)}, \\ \nu(\frac{f(k^p a)}{k^{2p}} - f(a), \sum_{\ell=0}^{p-1} \frac{\alpha^\ell t}{2k^2(2k+s-1)k^{2\ell}}) \leq \coprod_{\ell=0}^{p-1} \nu(\frac{f(k^{\ell+1}a)}{k^{2(\ell+1)}} - \frac{f(k^\ell a)}{k^{2\ell}}, \frac{\alpha^\ell t}{2k^2(2k+s-1)k^{2\ell}}) \leq \frac{\varphi_e(0,a)}{t+\varphi_e(0,a)} \end{cases} \tag{3.25}$$

for all $a \in X$ and all $t > 0$, where

$$\prod_{j=0}^{p} a_j = a_1 * a_2 * \cdots * a_p, \quad \coprod_{j=0}^{p} a_j = a_1 \diamond a_2 \diamond \cdots \diamond a_p.$$

By replacing a with $k^q a$ in (3.25), we have

$$\begin{cases} \mu(\frac{f(k^{p+q}a)}{k^{2(p+q)}} - \frac{f(k^q a)}{k^{2q}}, \sum_{\ell=0}^{p-1} \frac{\alpha^\ell t}{2k^2(2k+s-1)k^{2(\ell+q)}}) \geq \frac{t}{t+\alpha^q \varphi_e(0,a)}, \\ \nu(\frac{f(k^{p+q}a)}{k^{2(p+q)}} - \frac{f(k^q a)}{k^{2q}}, \sum_{\ell=0}^{p-1} \frac{\alpha^\ell t}{2k^2(2k+s-1)k^{2(\ell+q)}}) \leq \frac{\alpha^q \varphi_e(0,a)}{t+\alpha^q \varphi_e(0,a)} \end{cases} \tag{3.26}$$

for all $a \in X$, $t > 0$, $p > 0$ and $q > 0$. Thus

$$\begin{cases} \mu(\frac{f(k^{p+q}a)}{k^{2(p+q)}} - \frac{f(k^q a)}{k^{2q}}, \sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell t}{2k^2(2k+s-1)k^{2\ell}}) \geq \frac{t}{t+\varphi_e(0,a)}, \\ \nu(\frac{f(k^{p+q}a)}{k^{2(p+q)}} - \frac{f(k^q a)}{k^{2q}}, \sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell t}{2k^2(2k+s-1)k^{2\ell}}) \leq \frac{\varphi_e(0,a)}{t+\varphi_e(0,a)} \end{cases} \tag{3.27}$$

for all $a \in X$, $t > 0$, $p > 0$ and $q > 0$. Hence

$$\begin{cases} \mu(\frac{f(k^{p+q}a)}{k^{2(p+q)}} - \frac{f(k^q a)}{k^{2q}}, t) \geq \frac{t}{t+\sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell}{2k^2(2k+s-1)k^{2\ell}} \varphi_e(0,a)}, \\ \nu(\frac{f(k^{p+q}a)}{k^{2(p+q)}} - \frac{f(k^q a)}{k^{2q}}, t) \leq \frac{\sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell}{2k^2(2k+s-1)k^{2\ell}} \varphi_e(0,a)}{t+\sum_{\ell=q}^{p+q-1} \frac{\alpha^\ell}{2k^2(2k+s-1)k^{2\ell}} \varphi_e(0,a)} \end{cases} \tag{3.28}$$

for all $a \in X$, $t > 0$, $p > 0$ and $q > 0$. Since $0 < \alpha < k^2$ and

$$\sum_{\ell=0}^{\infty} \frac{\alpha^{\ell}}{2k^2(2k+s-1)k^{2\ell}} < \infty,$$

the Cauchy criterion for convergence in IFNS shows that $\{\frac{f(k^p a)}{k^{2p}}\}$ is a Cauchy sequence in $(Y, \mu, \nu, *, \diamond)$. Since $(Y, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy Banach space, this sequence converges to some point $Q(a) \in Y$. So one can define the mapping Q: $X \to Y$ such that

$$Q(a) := (\mu, \nu) - \lim_{p\to\infty} \frac{f(k^p a)}{k^{2p}}.$$

Moreover, if we put $q = 0$ in (3.28), we get

$$\begin{cases} \mu(\frac{f(k^p a)}{k^{2p}} - f(a), t) \geq \frac{t}{t+\sum_{\ell=0}^{p-1} \frac{\alpha^{\ell}}{2k^2(2k+s-1)k^{2\ell}}\varphi_e(0,a)}, \\ \nu(\frac{f(k^p a)}{k^{2p}} - f(a), t) \leq \frac{\sum_{\ell=0}^{p-1} \frac{\alpha^{\ell}}{2k^2(2k+s-1)k^{2\ell}}\varphi_e(0,a)}{t+\sum_{\ell=0}^{p-1} \frac{\alpha^{\ell}}{2k^2(2k+s-1)k^{2\ell}}\varphi_e(0,a)} \end{cases} \tag{3.29}$$

for all $a \in X$, $t > 0$ and $p > 0$. Thus, we obtain

$$\begin{cases} \mu(f(a) - Q(a), t) \geq \mu(f(a) - \frac{f(k^p a)}{k^{2p}}, \frac{t}{2}) * \mu(\frac{f(k^p a)}{k^{2p}} - Q(a), \frac{t}{2}) \\ \qquad \geq \frac{t}{t+\sum_{\ell=0}^{p-1} \frac{\alpha^{\ell}}{k^2(2k+s-1)k^{2\ell}}\varphi_e(0,a)}, \\ \nu(f(a) - Q(a), t) \leq \nu(f(a) - \frac{f(k^p a)}{k^{2p}}, \frac{t}{2}) * \nu(\frac{f(k^p a)}{k^{2p}} - Q(a), \frac{t}{2}) \\ \qquad \leq \frac{\sum_{\ell=0}^{p-1} \frac{\alpha^{\ell}}{k^2(2k+s-1)k^{2\ell}}\varphi_e(0,a)}{t+\sum_{\ell=0}^{p-1} \frac{\alpha^{\ell}}{k^2(2k+s-1)k^{2\ell}}\varphi_e(0,a)} \end{cases} \tag{3.30}$$

for every $a \in X$, $t > 0$ and large p. Taking the limit as $p \to \infty$ and using the definition of IFNS, we get

$$\begin{cases} \mu(f(a) - Q(a), t) \geq \frac{(k^2-\alpha)(2k+s-1)t}{(k^2-\alpha)(2k+s-1)t+\varphi_e(0,a)}, \\ \nu(f(a) - Q(a), t) \leq \frac{\varphi_e(0,a)}{(k^2-\alpha)(2k+s-1)t+\varphi_e(0,a)}. \end{cases} \tag{3.31}$$

Replacing a and b by $k^p a$ and $k^p b$ in (3.21), respectively, and using (3.18), we obtain

$$\mu(\tfrac{1}{k^{2p}} Df(k^p a, k^p b), t) \geq \frac{t}{t+(\frac{\alpha}{k^2})^p \varphi_e(a,b)}, \qquad \nu(\tfrac{1}{k^{2p}} Df(k^p a, k^p b), t) \leq \frac{(\frac{\alpha}{k^2})^p \varphi_e(a,b)}{t+(\frac{\alpha}{k^2})^p \varphi_e(a,b)} \tag{3.32}$$

for all $a, b \in X$ and all $t > 0$. Letting $p \to \infty$ in (3.32), we obtain

$$\mu(DQ(a,b), t) = 1 \quad \text{and} \quad \nu(DQ(a,b), t) = 0 \tag{3.33}$$

for all $a, b \in X$ and all $t > 0$. This means that Q satisfies the functional Eq (1.1). Since f: $X \to Y$ is an even mapping, and using the definition Q, we have $Q(-a) = -Q(a)$ for all $a \in X$. Thus by Lemma 3.2, the mapping Q: $X \to Y$ is quadratic. To prove the uniqueness of Q, let Q': $X \to Y$ be another quadratic mapping satisfying (3.31). Let $n = 1$. Then we have

$$\begin{cases} \mu(Q(a) - Q'(a), t) = \mu(\frac{Q(k^p a)}{k^{2p}} - \frac{Q'(k^p a)}{k^{2p}}, t) \\ \qquad \geq \mu(\frac{Q(k^p a)}{k^{2p}} - \frac{f(k^p a)}{k^{2p}}, \frac{t}{2}) * \mu(\frac{f(k^p a)}{k^{2p}} - \frac{Q'(k^p a)}{k^{2p}}, \frac{t}{2}) \\ \qquad \geq \frac{(k^2-\alpha)(2k+s-1)t}{(k^2-\alpha)(2k+s-1)t+2(\frac{\alpha}{k^2})^p \varphi_e(0,a)}, \\ \nu(Q(a) - Q'(a), t) = \nu(\frac{Q(k^p a)}{k^{2p}} - \frac{Q'(k^p a)}{k^{2p}}, t) \\ \qquad \leq \nu(\frac{Q(k^p a)}{k^{2p}} - \frac{f(k^p a)}{k^{2p}}, \frac{t}{2}) \diamond \nu(\frac{f(k^p a)}{k^{p}} - \frac{Q'(k^p a)}{k^{2p}}, \frac{t}{2}) \\ \qquad \leq \frac{2(\frac{\alpha}{k^2})^p \varphi_e(0,a)}{(k^2-\alpha)(2k+s-1)t+2(\frac{\alpha}{k^2})^p \varphi_e(0,a)} \end{cases} \tag{3.34}$$

for all $a \in X$, $t > 0$ and $p > 0$. Letting $p \to \infty$ in (3.34), we get

$$\mu(Q(a) - Q'(a), t) = 1 \quad \text{and} \quad \nu(Q(a) - Q'(a), t) = 0$$

for all $a \in X$ and $t > 0$. Hence we get $Q(a) = Q'(a)$ for all $a \in X$. Thus the mapping Q: $X \to Y$ is a unique quadratic mapping.

By Lemma 2.3 and (3.31), we get

$$\begin{cases} \mu_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \min\{\mu(f(x_{ij}) - Q(x_{ij}), \frac{t}{n^2}) : i, j = 1, \ldots, n\} \geq \frac{(k^2-\alpha)(2k+s-1)t}{(k^2-\alpha)(2k+s-1)t+n^2 \sum_{i,j=1}^n \varphi_e(0,x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \leq \max\{\nu(f(x_{ij}) - Q(x_{ij}), \frac{t}{n^2}) : i, j = 1, \ldots, n\} \leq \frac{n^2 \sum_{i,j=1}^n \varphi_e(0,x_{ij})}{(k^2-\alpha)(2k+s-1)t+n^2 \sum_{i,j=1}^n \varphi_e(0,x_{ij})} \end{cases}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$. Thus Q: $X \to Y$ is a unique quadratic mapping satisfying (3.20), as desired. This completes the proof of the theorem. □

Theorem 3.5. *Let φ: $X^2 \to [0, \infty)$ be a function such that for some real number α with $0 < \alpha < k$,*

$$\varphi(ka, kb) = \alpha\varphi(a, b) \tag{3.35}$$

for all $a, b \in X$. Suppose that a mapping f: $X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\begin{cases} \mu_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t+\sum_{i,j=1}^n \varphi(x_{ij},y_{ij})}, \\ \nu_n(Df_n([x_{ij}], [y_{ij}]), t) \leq \frac{\sum_{i,j=1}^n \varphi(x_{ij},y_{ij})}{t+\sum_{i,j=1}^n \varphi(x_{ij},y_{ij})} \end{cases} \tag{3.36}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exist a unique quadratic mapping Q: $X \to Y$ and a unique additive mapping A: $X \to Y$ such that

$$\begin{cases} \mu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t+2n^2 \sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \leq \frac{2n^2 \sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})}{(k-\alpha)(2k+s-1)t+2n^2 \sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})} \end{cases} \tag{3.37}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$, $\widetilde{\varphi}(a, b) = \varphi(a, b) + \varphi(-a, -b)$ for all $a, b \in X$.

Proof. When $n = 1$, (3.36) is equivalent to

$$\mu(Df(a,b),t) \geq \frac{t}{t+\varphi(a,b)} \quad \text{and} \quad \nu(Df(a,b),t) \leq \frac{\varphi(a,b)}{t+\varphi(a,b)} \tag{3.38}$$

for all $a, b \in X$ and all $t > 0$. Let

$$f_e(a) = \frac{f(a) + f(-a)}{2}$$

for all all $a \in X$. Then $f_e(0) = 0, f_e(-a) = f_e(a)$. And we have

$$\begin{cases} \mu(Df_e(a,b),t) = \mu(\frac{1}{2}Df(a,b) + \frac{1}{2}Df(-a,-b),t) \\ = \mu(Df(a,b) + Df(-a,-b),2t) \geq \mu(Df(a,b),t) * \mu(Df(-a,-b),t) \\ \geq \min\{\mu(Df(a,b),t), \mu(Df(-a,-b),t)\} \geq \frac{t}{t+\widetilde{\varphi}(a,b)}, \\ \nu(Df_e(a,b),t) = \nu(\frac{1}{2}Df(a,b) + \frac{1}{2}Df(-a,-b),t) \\ = \nu(Df(a,b) + Df(-a,-b),2t) \leq \nu(Df(a,b),t) \diamond \nu(Df(-a,-b),t) \\ \leq \max\{\nu(Df(a,b),t), \nu(Df(-a,-b),t)\} \leq \frac{\widetilde{\varphi}(a,b)}{t+\widetilde{\varphi}(a,b)} \end{cases} \tag{3.39}$$

for all $a \in X$ and all $t > 0$. Let

$$f_o(a) = \frac{f(a) - f(-a)}{2}$$

for all all $a \in X$. Then $f_0(0) = 0, f_o(-a) = -f_o(a)$. And we obtain

$$\begin{cases} \mu(Df_o(a,b),t) = \mu(\frac{1}{2}Df(a,b) - \frac{1}{2}Df(-a,-b),t) \\ = \mu(Df(a,b) - Df(-a,-b),2t) \geq \mu(Df(a,b),t) * \mu(Df(-a,-b),t) \\ = \min\{\mu(Df(a,b),t), \mu(Df(-a,-b),t)\} \geq \frac{t}{t+\widetilde{\varphi}(a,b)}, \\ \nu(Df_o(a,b),t) = \nu(\frac{1}{2}Df(a,b) - \frac{1}{2}Df(-a,-b),t) \\ = \nu(Df(a,b) - Df(-a,-b),2t) \leq \nu(Df(a,b),t) \diamond \nu(Df(-a,-b),t) \\ = \max\{\nu(Df(a,b),t), \nu(Df(-a,-b),t)\} \leq \frac{\widetilde{\varphi}(a,b)}{t+\widetilde{\varphi}(a,b)} \end{cases} \tag{3.40}$$

for all $a \in X$ and all $t > 0$. It follows that the definition of $\widetilde{\varphi}$ that $\widetilde{\varphi}(ka, kb) = \alpha\widetilde{\varphi}(a, b)$ for all $a, b \in X$. It is easy to check that the condition of Theorems 3.3 and 3.4 are satisfying. Then applying the proofs of Theorems 3.3 and 3.4, we know that there exists a unique quadratic mapping Q: $X \to Y$ and a unique additive mapping A: $X \to Y$ satisfying

$$\begin{cases} \mu(f_e(a) - Q(a),t) \geq \frac{(k^2-\alpha)(2k+s-1)t}{(k^2-\alpha)(2k+s-1)t+\widetilde{\varphi}(0,a)}, \\ \nu(f_e(a) - Q(a),t) \leq \frac{\widetilde{\varphi}(0,a)}{(k^2-\alpha)(2k+s-1)t+\widetilde{\varphi}(0,a)} \end{cases} \tag{3.41}$$

and

$$\begin{cases} \mu(f_o(a) - A(a),t) \geq \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t+\widetilde{\varphi}(0,a)}, \\ \nu(f_o(a) - A(a),t) \leq \frac{\widetilde{\varphi}(0,a)}{(k-\alpha)(2k+s-1)t+\widetilde{\varphi}(0,a)} \end{cases} \tag{3.42}$$

for all $a \in X$ and all $t > 0$. Therefore

$$\begin{cases} \mu(f(a) - Q(a) - A(a), t) = \mu(f_e(a) - Q(a) + f_o(a) - A(a), t) \\ \geq \mu(f_e(a) - Q(a), \frac{t}{2}) * \mu(f_o(a) - A(a), \frac{t}{2}) \\ = \min\{\mu(f_e(a) - Q(a), \frac{t}{2}), \mu(f_o(a) - A(a), \frac{t}{2})\} \\ \geq \min\{\frac{(k^2-\alpha)(2k+s-1)t}{(k^2-\alpha)(2k+s-1)t+2\widetilde{\varphi}(0,a)}, \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t+2\widetilde{\varphi}(0,a)}\} \\ = \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t+2\widetilde{\varphi}(0,a)}, \\ \nu(f(a) - Q(a) - A(a), t) = \nu(f_e(a) - Q(a) + f_o(a) - A(a), t) \\ \leq \nu(f_e(a) - Q(a), \frac{t}{2}) \diamond \nu(f_o(a) - A(a), \frac{t}{2}) \\ = \max\{\nu(f_e(a) - Q(a), \frac{t}{2}), \nu(f_o(a) - A(a), \frac{t}{2})\} \\ \leq \max\{\frac{2\widetilde{\varphi}(0,a)}{(k^2-\alpha)(2k+s-1)t+2\widetilde{\varphi}(0,a)}, \frac{2\widetilde{\varphi}(0,a)}{(k-\alpha)(2k+s-1)t+2\widetilde{\varphi}(0,a)}\} \\ = \frac{2\widetilde{\varphi}(0,a)}{(k-\alpha)(2k+s-1)t+2\widetilde{\varphi}(0,a)}. \end{cases} \tag{3.43}$$

By Lemma 2.3 and (3.43), we have

$$\begin{cases} \mu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \\ \qquad \geq \min\{\mu(f(x_{ij}) - Q(x_{ij}) - A(x_{ij}), \frac{t}{n^2}) : i, j = 1, \dots, n\} \\ \qquad \geq \frac{(k-\alpha)(2k+s-1)t}{(k-\alpha)(2k+s-1)t+2n^2\sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \\ \qquad \leq \max\{\nu(f(x_{ij}) - Q(x_{ij}) - A(x_{ij}), \frac{t}{n^2}) : i, j = 1, \dots, n\} \\ \qquad \leq \frac{2n^2\sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})}{(k-\alpha)(2k+s-1)t+2n^2\sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})} \end{cases}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$. Thus Q: $X \to Y$ is a unique quadratic mapping and a unique additive mapping A: $X \to Y$ satisfying (3.37), as desired. This completes the proof of the theorem. □

Corollary 3.6. *Let r, θ be positive real numbers with $r < 1$. Suppose that a mapping f: $X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{cases} \mu_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t+\sum_{i,j=1}^n \theta(\|x_{ij}\|^r+\|y_{ij}\|^r)}, \\ \nu_n(Df_n([x_{ij}], [y_{ij}]), t) \leq \frac{\sum_{i,j=1}^n \theta(\|x_{ij}\|^r+\|y_{ij}\|^r)}{t+\sum_{i,j=1}^n \theta(\|x_{ij}\|^r+\|y_{ij}\|^r)} \end{cases} \tag{3.44}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exist a unique quadratic mapping Q: $X \to Y$ and a unique additive mapping A: $X \to Y$ such that

$$\begin{cases} \mu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{(k-k^r)(2k+s-1)t}{(k-k^r)(2k+s-1)t+4n^2\sum_{i,j=1}^n \theta\|x_{ij}\|^r}, \\ \nu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \leq \frac{4n^2\sum_{i,j=1}^n \theta\|x_{ij}\|^r}{(k-k^r)(2k+s-1)t+4n^2\sum_{i,j=1}^n \theta\|x_{ij}\|^r} \end{cases} \tag{3.45}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. The proof follows from Theorem 3.5 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$, we obtain the desired result. □

4. Stability of the functional Eq (1.1): fixed point method

In this section, we will prove the Hyers-Ulam stability of the functional Eq (1.1) in matrix intuitionistic fuzzy normed spaces by applying the fixed point method.

Theorem 4.1. *Let φ_o: $X^2 \to [0,\infty)$ be a function such that for some real number ρ with $0 < \rho < 1$ and*

$$\varphi_o(a,b) = \frac{\rho}{k}\varphi_o(ka,kb) \tag{4.1}$$

for all $a,b \in X$. Suppose that an odd mapping f: $X \to Y$ satisfies the inequality

$$\begin{cases} \mu_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^n \varphi_o(x_{ij},y_{ij})}, \\ \\ \nu_n(Df_n([x_{ij}],[y_{ij}]),t) \leq \frac{\sum_{i,j=1}^n \varphi_o(x_{ij},y_{ij})}{t+\sum_{i,j=1}^n \varphi_o(x_{ij},y_{ij})} \end{cases} \tag{4.2}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exists a unique additive mapping A: $X \to Y$ such that

$$\begin{cases} \mu_n(f_n([x_{ij}]) - A_n([x_{ij}]),t) \geq \frac{2k(2k+s-1)(1-\rho)t}{2k(2k+s-1)(1-\rho)t+\rho n^2\sum_{i,j=1}^n \varphi_o(0,x_{ij})}, \\ \\ \nu_n(f_n([x_{ij}]) - A_n([x_{ij}]),t) \leq \frac{\rho n^2\sum_{i,j=1}^n \varphi_o(0,x_{ij})}{2k(2k+s-1)(1-\rho)t+\rho n^2\sum_{i,j=1}^n \varphi_o(0,x_{ij})} \end{cases} \tag{4.3}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. When $n = 1$, similar to the proof of Theorem 3.3, we have

$$\begin{cases} \mu(2(2k+s-1)f(ka) - 2(2k+s-1)kf(a),t) \geq \frac{t}{t+\varphi_o(0,a)}, \\ \\ \nu(2(2k+s-1)f(ka) - 2(2k+s-1)kf(a),t) \leq \frac{\varphi_o(0,a)}{t+\varphi_o(0,a)} \end{cases} \tag{4.4}$$

for all $a \in X$ and all $t > 0$.

Let $S_1 = \{g_1 : X \to Y\}$, and introduce a generalized metric d_1 on S_1 as follows:

$$d_1(g_1,h_1) := \inf\left\{\lambda \in \mathbb{R}_+ \middle| \begin{cases} \mu(g_1(a) - h_1(a),\lambda t) \geq \frac{t}{t+\varphi_o(0,a)}, \\ \\ \nu(g_1(a) - h_1(a),\lambda t) \leq \frac{\varphi_o(0,a)}{t+\varphi_o(0,a)}, \end{cases} \forall a \in X, \forall t > 0\right\}.$$

It is easy to prove that (S_1,d_1) is a complete generalized metric space ([3, 13]). Now, we define the mapping $\mathcal{J}_1$: $S_1 \to S_1$ by

$$\mathcal{J}_1 g_1(a) := kg_1(\frac{a}{k}), \quad \text{for all } g_1 \in S_1 \text{ and } a \in X. \tag{4.5}$$

Let $g_1,h_1 \in S_1$ and let $\lambda \in \mathbb{R}_+$ be an arbitrary constant with $d_1(g_1,h_1) \leq \lambda$. From the definition of d_1, we get

$$\begin{cases} \mu(g_1(a) - h_1(a),\lambda t) \geq \frac{t}{t+\varphi_o(0,a)}, \\ \\ \nu(g_1(a) - h_1(a),\lambda t) \leq \frac{\varphi_o(0,a)}{t+\varphi_o(0,a)} \end{cases}$$

for all $a \in X$ and $t > 0$. Therefore, using (4.1), we get

$$\begin{cases} \mu(\mathcal{J}_1 g_1(a) - \mathcal{J}_1 h_1(a), \lambda\rho t) = \mu(kg_1(\frac{a}{k}) - kh_1(\frac{a}{k}), \lambda\rho t) \\ = \mu(g_1(\frac{a}{k}) - h_1(\frac{a}{k}), \frac{\lambda\rho t}{k}) \geq \frac{\frac{\rho}{k}t}{\frac{\rho}{k}t + \frac{\rho}{k}\varphi_o(0,a)} = \frac{t}{t+\varphi_o(0,a)}, \\ \nu(\mathcal{J}_1 g_1(a) - \mathcal{J}_1 h_1(a), \lambda\rho t) = \nu(kg_1(\frac{a}{k}) - kh_1(\frac{a}{k}), \lambda\rho t) \\ = \nu(g_1(\frac{a}{k}) - h_1(\frac{a}{k}), \frac{\lambda\rho t}{k}) \leq \frac{\frac{\rho}{k}\varphi_o(0,a)}{\frac{\rho}{k}t + \frac{\rho}{k}\varphi_o(0,a)} = \frac{\varphi_o(0,a)}{t+\varphi_o(0,a)} \end{cases} \tag{4.6}$$

for some $\rho < 1$, all $a \in X$ and all $t > 0$. Hence, it holds that $d_1(\mathcal{J}_1 g_1, \mathcal{J}_1 h_1) \leq \lambda\rho$, that is, $d_1(\mathcal{J}_1 g_1, \mathcal{J}_1 h_1) \leq \rho d_1(g_1, h_1)$ for all $g_1, h_1 \in S_1$.

Furthermore, by (4.1) and (4.4), we obtain the inequality

$$d(f, \mathcal{J}_1 f) \leq \frac{\rho}{2k(2k+s-1)}.$$

It follows from Lemma 2.4 that the sequence $\mathcal{J}_1^p f$ converges to a fixed point A of $\mathcal{J}_1$, that is, for all $a \in X$ and all $t > 0$,

$$A : X \to Y, \quad A(a) := (\mu, \nu) - \lim_{p\to\infty} k^p f(\frac{a}{k^p}) \tag{4.7}$$

and

$$A(ka) = kA(a). \tag{4.8}$$

Meanwhile, A is the unique fixed point of $\mathcal{J}_1$ in the set

$$S_1^* = \{g_1 \in S_1 : d_1(f, g_1) < \infty\}.$$

Thus, there exists a $\lambda \in \mathbb{R}_+$ such that

$$\begin{cases} \mu(f(a) - A(a), \lambda t) \geq \frac{t}{t+\varphi_o(0,a)}, \\ \nu(f(a) - A(a), \lambda t) \leq \frac{\varphi_o(0,a)}{t+\varphi_o(0,a)} \end{cases}$$

for all $a \in X$ and all $t > 0$. Also,

$$d_1(f, A) \leq \frac{1}{1-\rho} d(f, \mathcal{J}_1 f) \leq \frac{\rho}{2k(1-\rho)(2k+s-1)}.$$

This means that the following inequality

$$\begin{cases} \mu(f(a) - A(a), t) \geq \frac{2k(2k+s-1)(1-\rho)t}{2k(2k+s-1)(1-\rho)t + \rho\varphi_o(0,a)}, \\ \nu(f(a) - A(a), t) \leq \frac{\rho\varphi_o(0,a)}{2k(2k+s-1)(1-\rho)t + \rho\varphi_o(0,a)} \end{cases} \tag{4.9}$$

holds for all $a \in X$ and all $t > 0$. It follows from (3.4) and (4.1) that

$$\mu(k^p Df(\frac{a}{k^p}, \frac{b}{k^p}), t) \geq \frac{t}{t+\rho^p\varphi_o(a,b)}, \qquad \nu(k^p Df(\frac{a}{k^p}, \frac{b}{k^p}), t) \leq \frac{\rho^p\varphi_o(a,b)}{t+\rho^p\varphi_o(a,b)} \tag{4.10}$$

for all $a, b \in X$ and all $t > 0$. Letting $p \to \infty$ in (4.10), we obtain

$$\mu(DA(a,b),t) = 1 \quad \text{and} \quad \nu(DA(a,b),t) = 0 \tag{4.11}$$

for all $a, b \in X$ and all $t > 0$. This means that A satisfies the functional Eq (1.1). Since $f\colon X \to Y$ is an odd mapping, and using the definition A, we have $A(-a) = -A(a)$ for all $a \in X$. Thus by Lemma 3.1, the mapping $A\colon X \to Y$ is additive.

By Lemma 2.3 and (4.9), we get

$$\begin{cases} \mu_n(f_n([x_{ij}]) - A_n([x_{ij}]),t) \geq \min\{\mu(f(x_{ij}) - A(x_{ij}), \frac{t}{n^2}) : i,j = 1,\cdots,n\} \\ \qquad \geq \frac{2k(2k+s-1)(1-\rho)t}{2k(2k+s-1)(1-\rho)t+\rho n^2\sum_{i,j=1}^n \varphi_o(0,x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - A_n([x_{ij}]),t) \leq \max\{\nu(f(x_{ij}) - A(x_{ij}), \frac{t}{n^2}) : i,j = 1,\ldots,n\} \\ \qquad \leq \frac{\rho n^2\sum_{i,j=1}^n \varphi_o(0,x_{ij})}{2k(2k+s-1)(1-\rho)t+\rho n^2\sum_{i,j=1}^n \varphi_o(0,x_{ij})} \end{cases}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$. Thus $A\colon X \to Y$ is a unique additive mapping satisfying (4.3), as desired. This completes the proof of the theorem. □

Theorem 4.2. *Let $\varphi_e\colon X^2 \to [0,\infty)$ be a function such that for some real number ρ with $0 < \rho < 1$ and*

$$\varphi_e(a,b) = \frac{\rho}{k^2}\varphi_e(ka,kb) \tag{4.12}$$

for all $a, b \in X$. Suppose that an even mapping $f\colon X \to Y$ satisfies the inequality

$$\begin{cases} \mu_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^n \varphi_e(x_{ij},y_{ij})}, \\ \nu_n(Df_n([x_{ij}],[y_{ij}]),t) \leq \frac{\sum_{i,j=1}^n \varphi_e(x_{ij},y_{ij})}{t+\sum_{i,j=1}^n \varphi_e(x_{ij},y_{ij})} \end{cases} \tag{4.13}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exists a unique quadratic mapping $Q\colon X \to Y$ such that

$$\begin{cases} \mu_n(f_n([x_{ij}]) - Q_n([x_{ij}]),t) \geq \frac{2k^2(2k+s-1)(1-\rho)t}{2k^2(2k+s-1)(1-\rho)t+\rho n^2\sum_{i,j=1}^n \varphi_e(0,x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - Q_n([x_{ij}]),t) \leq \frac{\rho n^2\sum_{i,j=1}^n \varphi_e(0,x_{ij})}{2k^2(2k+s-1)(1-\rho)t+\rho n^2\sum_{i,j=1}^n \varphi_e(0,x_{ij})} \end{cases} \tag{4.14}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. When $n = 1$, similar to the proof of Theorem 3.4, we obtain

$$\begin{cases} \mu(2(2k+s-1)f(ka) - 2(2k+s-1)k^2f(a),t) \geq \frac{t}{t+\varphi_e(0,a)}, \\ \nu(2(2k+s-1)f(ka) - 2(2k+s-1)k^2f(a),t) \leq \frac{\varphi_e(0,a)}{t+\varphi_e(0,a)} \end{cases} \tag{4.15}$$

for all $a \in X$ and all $t > 0$.

Let $S_2 := \{g_2 : X \to Y\}$, and introduce a generalized metric d_2 on S_2 as follows:

$$d_2(g_2, h_2) := \inf\left\{\lambda \in \mathbb{R}_+ \middle| \begin{cases} \mu(g_2(a) - h_2(a), \lambda t) \geq \frac{t}{t+\varphi_e(0,a)}, \\ \\ \nu(g_2(a) - h_2(a), \lambda t) \leq \frac{\varphi_e(0,a)}{t+\varphi_e(0,a)}, \end{cases} \quad \forall a \in X, \forall t > 0 \right\}.$$

It is easy to prove that (S_2, d_2) is a complete generalized metric space ([3, 13]). Now, we define the mapping $\mathcal{J}_2$: $S_2 \to S_2$ by

$$\mathcal{J}_2 g_2(a) := k^2 g_2(\frac{a}{k}), \quad \text{for all } g_2 \in S_2 \text{ and } a \in X. \tag{4.16}$$

Let $g_2, h_2 \in S_2$ and let $\lambda \in \mathbb{R}_+$ be an arbitrary constant with $d_2(g_2, h_2) \leq \lambda$. From the definition of d_2, we get

$$\begin{cases} \mu(g_2(a) - h_2(a), \lambda t) \geq \frac{t}{t+\varphi_e(0,a)}, \\ \\ \nu(g_2(a) - h_2(a), \lambda t) \leq \frac{\varphi_e(0,a)}{t+\varphi_e(0,a)} \end{cases}$$

for all $a \in X$ and $t > 0$. Therefore, using (4.12), we get

$$\begin{cases} \mu(\mathcal{J}_2 g_2(a) - \mathcal{J}_2 h_2(a), \lambda\rho t) = \mu(k^2 g_2(\frac{a}{k}) - k^2 h_2(\frac{a}{k}), \lambda\rho t) \\ = \mu(g_2(\frac{a}{k}) - h_2(\frac{a}{k}), \frac{\lambda\rho t}{k^2}) \geq \frac{\frac{\rho}{k^2}t}{\frac{\rho}{k^2}t + \frac{\rho}{k^2}\varphi_e(0,a)} = \frac{t}{t+\varphi_e(0,a)}, \\ \\ \nu(\mathcal{J}_2 g_2(a) - \mathcal{J}_2 h_2(a), \lambda\rho t) = \nu(k^2 g_2(\frac{a}{k}) - k^2 h_2(\frac{a}{k}), \lambda\rho t) \\ = \nu(g_2(\frac{a}{k}) - h_2(\frac{a}{k}), \frac{\lambda\rho t}{k^2}) \leq \frac{\frac{\rho}{k^2}\varphi_e(0,a)}{\frac{\rho}{k^2}t + \frac{\rho}{k^2}\varphi_e(0,a)} = \frac{\varphi_e(0,a)}{t+\varphi_e(0,a)} \end{cases} \tag{4.17}$$

for some $\rho < 1$, all $a \in X$ and all $t > 0$. Hence, it holds that $d_2(\mathcal{J}_2 g_2, \mathcal{J}_2 h_2) \leq \lambda\rho$, that is, $d_2(\mathcal{J}_2 g_2, \mathcal{J}_2 h_2) \leq \rho d_2(g_2, h_2)$ for all $g_2, h_2 \in S_2$.

Furthermore, by (4.12) and (4.15), we obtain the inequality

$$d(f, \mathcal{J}_2 f) \leq \frac{\rho}{2k^2(2k + s - 1)}.$$

It follows from Lemma 2.4 that the sequence $\mathcal{J}_2^p f$ converges to a fixed point Q of $\mathcal{J}_2$, that is, for all $a \in X$ and all $t > 0$,

$$Q : X \to Y, \quad Q(a) := (\mu, \nu) - \lim_{p\to\infty} k^{2p} f(\frac{a}{k^p}) \tag{4.18}$$

and

$$Q(ka) = k^2 Q(a). \tag{4.19}$$

Meanwhile, Q is the unique fixed point of $\mathcal{J}_2$ in the set

$$S_2^* = \{g_2 \in S_2 : d_2(f, g_2) < \infty\}.$$

Thus there exists a $\lambda \in \mathbb{R}_+$ such that

$$\begin{cases} \mu(f(a) - Q(a), \lambda t) \geq \frac{t}{t+\varphi_e(0,a)}, \\ \nu(f(a) - Q(a), \lambda t) \leq \frac{\varphi_e(0,a)}{t+\varphi_e(0,a)} \end{cases}$$

for all $a \in X$ and all $t > 0$. Also,

$$d_2(f, Q) \leq \frac{1}{1-\rho} d(f, \mathcal{J}_2 f) \leq \frac{\rho}{2k^2(1-\rho)(2k+s-1)}.$$

This means that the following inequality

$$\begin{cases} \mu(f(a) - Q(a), t) \geq \frac{2k^2(2k+s-1)(1-\rho)t}{2k^2(2k+s-1)(1-\rho)t+\rho\varphi_e(0,a)}, \\ \nu(f(a) - Q(a), t) \leq \frac{\rho\varphi_e(0,a)}{2k^2(2k+s-1)(1-\rho)t+\rho\varphi_e(0,a)} \end{cases} \tag{4.20}$$

holds for all $a \in X$ and all $t > 0$. The rest of the proof is similar to the proof of Theorem 4.1. This completes the proof of the theorem. □

Theorem 4.3. *Let φ: $X^2 \to [0, \infty)$ be a function such that for some real number ρ with $0 < \rho < k$,*

$$\varphi(a, b) = \frac{\rho}{k^2} \varphi(ka, kb) \tag{4.21}$$

for all $a, b \in X$. Suppose that a mapping f: $X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\begin{cases} \mu_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t+\sum_{i,j=1}^n \varphi(x_{ij}, y_{ij})}, \\ \nu_n(Df_n([x_{ij}], [y_{ij}]), t) \leq \frac{\sum_{i,j=1}^n \varphi(x_{ij}, y_{ij})}{t+\sum_{i,j=1}^n \varphi(x_{ij}, y_{ij})} \end{cases} \tag{4.22}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exist a unique quadratic mapping Q: $X \to Y$ and a unique additive mapping A: $X \to Y$ such that

$$\begin{cases} \mu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{k(2k+s-1)(1-\rho)t}{k(2k+s-1)(1-\rho)t+\rho n^2 \sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})}, \\ \nu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \leq \frac{\rho n^2 \sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})}{k(2k+s-1)(1-\rho)t+\rho n^2 \sum_{i,j=1}^n \widetilde{\varphi}(0,x_{ij})} \end{cases} \tag{4.23}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$, $\widetilde{\varphi}(a, b) = \varphi(a, b) + \varphi(-a, -b)$ for all $a, b \in X$.

Proof. The proof follows from Theorems 4.1 and 4.2, and a method similar to Theorem 3.5. This completes the proof of the theorem. □

Corollary 4.4. *Let r, θ be positive real numbers with $r > 2$. Suppose that a mapping f: $X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{cases} \mu_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t+\sum_{i,j=1}^n \theta(\|x_{ij}\|^r+\|y_{ij}\|^r)}, \\ \nu_n(Df_n([x_{ij}], [y_{ij}]), t) \leq \frac{\sum_{i,j=1}^n \theta(\|x_{ij}\|^r+\|y_{ij}\|^r)}{t+\sum_{i,j=1}^n \theta(\|x_{ij}\|^r+\|y_{ij}\|^r)} \end{cases} \tag{4.24}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ *and all* $t > 0$. *Then there exist a unique quadratic mapping* Q*:* $X \to Y$ *and a unique additive mapping* A*:* $X \to Y$ *such that*

$$\begin{cases} \mu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{(2k+s-1)(k^r-k^2)t}{(2k+s-1)(k^r-k^2)t+2kn^2\sum_{i,j=1}^{n}\theta\|x_{ij}\|^r}, \\ \nu_n(f_n([x_{ij}]) - Q_n([x_{ij}]) - A_n([x_{ij}]), t) \leq \frac{2kn^2\sum_{i,j=1}^{n}\theta\|x_{ij}\|^r}{(2k+s-1)(k^r-k^2)t+2kn^2\sum_{i,j=1}^{n}\theta\|x_{ij}\|^r} \end{cases} \tag{4.25}$$

for all $x = [x_{ij}] \in M_n(X)$ *and all* $t > 0$.

Proof. Taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and $\rho = k^{2-r}$ in Theorem 4.3, we get the desired result. □

5. Conclusions

We use the direct and fixed point methods to investigate the Hyers-Ulam stability of the functional Eq (1.1) in the framework of matrix intuitionistic fuzzy normed spaces. We therefore provide a link two various discipline: matrix intuitionistic fuzzy normed spaces and functional equations. We generalized the Hyers-Ulam stability results of the functional Eq (1.1) from quasi-Banach spaces to matrix intuitionistic fuzzy normed spaces. These circumstances can be applied to other significant functional equations.

Use of AI tools declaration

The author declare he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest in this paper.

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