AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 25409–25421.
DOI:10.3934/math.20231296
Received: 04 June 2023
Revised: 31 July 2023
Accepted: 22 August 2023
Published: 31 August 2023

Research article

Blow-up to a shallow water wave model including the Degasperis-Procesi equation

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Abstract: A nonlinear equation, depicting motions of shallow water waves and including the famous Degasperis-Procesi model, is considered. The key element is that we derive L^2 conservation law of solutions for the nonlinear equation, which leads to the bound of the solution itself. Using several estimates derived from the model, we obtain that when its solution blows up in the Sobolev space if and only if the space derivative of the solution tends to minus infinite.

Keywords: blow up; shallow water wave equation; generalized Degasperis-Procesi model; classical energy methods
Mathematics Subject Classification: 35G25, 35L05

1. Preliminary

This work is to probe blow-up feature of the equation

$$v_t - v_{txx} + kv_x + mvv_x = 3v_xv_{xx} + vv_{xxx} + \alpha v_{xxx}, \tag{1.1}$$

where constants $k \in \mathbb{R}, \alpha \in \mathbb{R}$ and $m > 0$. Equation (1.1) depicts the motion of shallow water waves (see Constantin and Lannes [2]). Actually, the shallow water wave model deduced in [2] includes Eq (1.1).

Setting $k = 0$, $\alpha = 0$ and $m = 4$, Eq (1.1) is turned into the famous Degasperis-Procesi (DP) model [6]

$$v_t - v_{txx} + 4vv_x = 3v_xv_{xx} + vv_{xxx}. \tag{1.2}$$

The global weak solutions and assumptions to cause the wave breaking of solution for Eq (1.2) are explored in Escher et al. [7]. For certain partial differential equations, their solution remains bounded, but its derivative about space variable tends to infinite at the blow-up time. This phenomena is called wave breaking of solutions (see Constantin [2, 3]). The dressing method is employed in Constantin and Ivanov [4] to investigate dynamical features of the DP model (1.2). The global strong solutions and wave breaking phenomena for the DP model are explored with certain functional spaces in [14, 24]. The large time asymptotic features of the periodic entropy (discontinuous) solutions for DP equation is considered in [5]. Lundmark and Szmigielski [15] study the multi-peakon solutions of the Eq (1.2) (also see Matsono [12, 19]). Lenells [18] finds out many traveling wave solutions of the DP model. Periodic and solitary wave solutions to the DP model are classified in Vakhnenko and Parkes [22]. Two conservation laws to Eq (1.2) are utilized to investigate the stability of peakons in Lin and Liu [16]. Lai and Wu [17] study L^1 local stability for a shallow water wave equation including DP equation endowed with certain conditions. The infinite propagation speed of DP model is discussed in Henry [10]. Akinyemi et al. [1] apply an efficient numerical simulation method to study the coupled nonlinear Schr*ö*dinger-Korteweg-de Vries and Maccari systems. Utilizing the properties of fractional operators and the numerical computational techniques, Veeresha et al. [23] investigate the shallow water forced Korteweg-De Vries model associated with critical flow over a hole (see [13]). For nonlinear models relating to Eq (1.2), we refer the reader to [8, 9, 11, 20, 21] and the references therein.

For Eq (1.1) endowed with $v(0, x) = v_0(x) \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, we derive that

$$\int_{\mathbb{R}} \frac{1+\xi^2}{m+\xi^2}|\hat{v}(t,\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{1+\xi^2}{m+\xi^2}|\hat{v_0}(\xi)|^2 d\xi \sim \parallel v_0 \parallel^2_{L^2(\mathbb{R})}, \tag{1.3}$$

which leads to

$$\parallel v(t,\cdot) \parallel_{L^2(\mathbb{R})} \leq \max\left(\sqrt{m}, \sqrt{\frac{1}{m}}\right) \parallel v_0 \parallel_{L^2(\mathbb{R})} .$$

The objective of this work is to study the shallow water wave Eq (1.1), which generalizes the famous Degasperis-Processi model. We find that the wave breaking of the solutions for Eq (1.1) behaves the same structure as that of the DP model (see [8, 20]). The novelty in our work is that we derive $L^2(\mathbb{R})$ conservation law (1.3), which takes a key role to derive several bounded estimates of solutions for Eq (1.1). For any constant α, we find arguments to support the bounded property of $\parallel v(t,\cdot) \parallel_{L^\infty(\mathbb{R})}$. For blow-up time T, we deduce that when the solution of Eq (1.1) blows up, namely, $\lim_{t\to T} \parallel v(t,\cdot) \parallel_{H^2} = \infty$ if and only if the space derivative of the solution tends to minus infinite. Here we state that the main technique used in this work is the classical energy estimate methods.

In Section 2, we give the local well-posedness of solution for Eq (1.1) and derive the conservation law (1.3). Several Lemmas about the bound property of the solutions are established. Section 3 provides conditions imposing on the initial value to discuss the wave breaking for Eq (1.1). Conclusions are summarized in Section 4.

2. Basic L^∞ estimate and several lemmas

We write the Cauchy problem for Eq (1.1) in the form

$$\begin{cases} v_t - v_{txx} + kv_x + mvv_x = 3v_xv_{xx} + vv_{xxx} + \alpha v_{xxx}, \\ v(0,x) = v_0(x), \end{cases} \tag{2.1}$$

where k, α are constants, constant $m > 0$ and $v_0(x) \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$.

Utilizing $\Lambda^{-2} = (1 - \frac{\partial^2}{\partial x^2})^{-1}$ and $\Lambda^{-2}h(x) = \frac{1}{2}\int_R e^{-|x-z|}h(z)dz$ for every $h \in L^p(\mathbb{R})(1 \leq p \leq \infty)$, we acquire

$$\begin{cases} v_t + vv_x = -\frac{m-1}{2}\Lambda^{-2}(v^2)_x + (\alpha - k)\Lambda^{-2}v_x - \alpha v_x, \\ v(0, x) = v_0(x). \end{cases} \tag{2.2}$$

In order to discuss the blow-up of solution, we introduce the local existence result for problem (2.1).

Lemma 2.1. *([2]) Assume $s > \frac{3}{2}$ and $v_0(x) \in H^s(\mathbb{R})$. Then, problem (2.1) exists a unique solution v satisfying*

$$v \in C([0, T);\ H^s(\mathbb{R})) \cap C^1([0, T);\ H^{s-1}(\mathbb{R})),$$

where $T = T(v_0)$ stands for the maximal existence time of $v(t, x)$.

Lemma 2.2. *Assume $m > 0$, $y = v - \frac{\partial^2 v}{\partial x^2}$ and $W = (m - \frac{\partial^2}{\partial x^2})^{-1}v$. Let $s > \frac{3}{2}$, $v_0 \in H^s(\mathbb{R})$. If v satisfies problem (2.1), then,*

$$\int_{\mathbb{R}} yvdx = \int_{\mathbb{R}} \frac{1+\xi^2}{m+\xi^2}|\hat{W}(\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{1+\xi^2}{m+\xi^2}|\hat{v_0}(\xi)|^2 d\xi \sim \| v_0 \|^2_{L^2(\mathbb{R})} . \tag{2.3}$$

Moreover,

$$\begin{cases} \| v \|_{L^2} \leq \sqrt{\frac{1}{m}} \| v_0 \|_{L^2}, & if \quad m \leq 1, \\ \| v \|_{L^2} \leq \sqrt{m} \| v_0 \|_{L^2}, & if \quad m \geq 1, \end{cases} \tag{2.4}$$

which is equivalent to

$$\| v(t, \cdot) \|_{L^2(\mathbb{R})} \leq \max\Big(\sqrt{m}, \sqrt{\frac{1}{m}}\Big) \| v_0 \|_{L^2(\mathbb{R})} .$$

Proof. We have $v = mW - W_{xx}$ and $W_{xx} = mW - v$. Using (1.1) and integration by parts yields

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} yWdx &= \int_{\mathbb{R}} y_t Wdx + \int_{\mathbb{R}} yW_t dx = 2\int_{\mathbb{R}} Wy_t dx \\ &= 2\int_{\mathbb{R}} \Big[(-\frac{m}{2}v^2)_x - kv_x + \frac{1}{2}\partial^3_{xxx}(v^2) + \alpha v_{xxx}\Big]Wdx \\ &= 2\int_{\mathbb{R}} \Big[(-\frac{m}{2}v^2)_x W - kv_x W + \frac{1}{2}(v^2)_x W_{xx} + \alpha v_x W_{xx}\Big]dx \\ &= \int_{\mathbb{R}} \Big[(-mv^2)_x W - 2kv_x W + (v^2)_x(mW - v) + 2\alpha v_x(mW - v)\Big]dx \\ &= \int_{\mathbb{R}} \Big(- 2kv_x W - (v^2)_x v + 2\alpha m v_x W\Big)dx \\ &= 2(\alpha m - k)\int_{\mathbb{R}} v_x Wdx \end{aligned}$$

$$
\begin{aligned}
&= -2(\alpha m - k) \int_{\mathbb{R}} v W_x dx \\
&= -2(\alpha m - k) \int_{\mathbb{R}} (mW - W_{xx}) W_x dx \\
&= 0,
\end{aligned}
$$

which together with the Parserval identity, we obtain (2.3). Using (2.3) derives (2.4). □

The conservation law in Lemma 2.2 takes a key role to establish the bounds of solutions for problem (2.1).

Lemma 2.3. *Provided that* $v_0(x) \in H^s(\mathbb{R})$ $(s > \frac{3}{2})$*, then,*

$$
\begin{cases}
\int_{\mathbb{R}} v_x \Lambda^{-2} v dx = \int_{\mathbb{R}} v \Lambda^{-2} v_x dx = 0, \\
\int_{\mathbb{R}} v_{xx} \Lambda^{-2} v_x dx = \int_{\mathbb{R}} v_x \Lambda^{-2} v_{xx} dx = 0, \\
\int_{\mathbb{R}} v_{xx} \Lambda^{-2} (v^2)_x dx = \int_{\mathbb{R}} v \Lambda^{-2} (v^2)_x dx.
\end{cases}
$$

Proof. Setting $\Lambda^{-2} v = V$, we have

$$
v = V - V_{xx},
$$

which together with integration by parts, yields

$$
\int_{\mathbb{R}} v \Lambda^{-2} v_x dx = \int_{\mathbb{R}} (V - V_{xx}) V_x dx = \int_{\mathbb{R}} V dV - \int_{\mathbb{R}} V_x dV_x = 0. \tag{2.5}
$$

$$
\begin{aligned}
\int_{\mathbb{R}} v_{xx} \Lambda^{-2} v_x dx &= \int_{\mathbb{R}} v \Lambda^{-2} v_{xxx} dx \\
&= \int_{\mathbb{R}} v \Lambda^{-2} (1 - \Lambda^2) v_x dx \\
&= \int_{\mathbb{R}} v \Lambda^{-2} v_x dx - \int_{\mathbb{R}} v v_x dx = 0
\end{aligned} \tag{2.6}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}} v_{xx} \Lambda^{-2} (v^2)_x dx &= \int_{\mathbb{R}} v \Lambda^{-2} \partial_x^3 (v^2) dx \\
&= \int_{\mathbb{R}} v \Lambda^{-2} (1 - \Lambda^2) \partial_x (v^2) dx \\
&= \int_{\mathbb{R}} v \Lambda^{-2} \partial_x (v^2) dx - 2 \int_{\mathbb{R}} v^2 v_x dx \\
&= \int_{\mathbb{R}} v \Lambda^{-2} (v^2)_x dx.
\end{aligned} \tag{2.7}
$$

Using Eqs (2.5)–(2.7) ends the proof. □

Utilizing Lemma 2.2, we can establish the estimates about the operator Λ^{-2}.

Lemma 2.4. *Provided that $v(t,x)$ satisfies (2.2) and $v_0(x) \in H^s(\mathbb{R})$ $(s > \frac{3}{2})$, then,*

$$\begin{cases} |\Lambda^{-2}v^2| < \frac{c_m^2}{2} \parallel v_0 \parallel_{L^2}^2, \qquad |\Lambda^{-2}\partial_x(v^2)| < \frac{c_m^2}{2} \parallel v_0 \parallel_{L^2}^2, \\ 0 \leq \int_{\mathbb{R}} \Lambda^{-2}v^2 dx < c_m^2 \parallel v_0 \parallel_{L^2}^2, \quad |\int_{\mathbb{R}} \Lambda^{-2}\partial_x(v^2)dx| < c_m^2 \parallel v_0 \parallel_{L^2}^2, \\ |\int_{\mathbb{R}} v\Lambda^{-2}(v^2)_x dx| < c_m^3 \parallel v_0 \parallel_{L^2}^3, \quad |\int_{\mathbb{R}} v_x\Lambda^{-2}(v^2)dx| < c_m^3 \parallel v_0 \parallel_{L^2}^3, \end{cases}$$

where $c_m = \max\left(\sqrt{m}, \sqrt{\frac{1}{m}}\right)$.

Proof. Utilizing $\int_{\mathbb{R}} e^{-|x-y|}dx = 2$, we obtain

$$\Lambda^{-2}v^2 = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}v^2(t,y)dy < \frac{c_m^2}{2} \parallel v_0 \parallel_{L^2}^2 .$$

The Tonelli theorem and Lemma 2.2 ensure that

$$\begin{aligned} 0 \leq \int_{\mathbb{R}} & \Lambda^{-2}v^2 dx \\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-|x-y|}v^2(t,y)dydx \\ &= \frac{1}{2}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-|x-y|}dx\Big)v^2(t,y)dy \\ &\leq \int_{\mathbb{R}} v^2(t,y)dy < c_m^2 \parallel v_0 \parallel_{L^2}^2 . \end{aligned} \tag{2.8}$$

Applying

$$\Lambda^{-2}\partial_x(v^2) = \frac{e^x}{2}\int_x^{\infty} e^{-y}v^2(t,y)dy - \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y}v^2(t,y)dy,$$

we acquire

$$\begin{aligned} |\Lambda^{-2}\partial_x(v^2)| &= |\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}sgn(y-x)v^2(t,y)dy| \\ &\leq \frac{1}{2}\int_{\mathbb{R}} v^2(t,y)dy < \frac{c_m^2}{2} \parallel v_0 \parallel_{L^2}^2 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \int_{\mathbb{R}} |\Lambda^{-2}\partial_x(v^2)|dx &= \frac{1}{2}\int_{\mathbb{R}} |\int_{\mathbb{R}} e^{-|x-y|}sgn(y-x)v^2(t,y)dy|dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}} v^2(t,y)dy\int_{-\infty}^{\infty} |e^{-|x-y|}sgn(y-x)|dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-|x-y|}dx\Big)v^2(t,y)dy \\ &\leq c_m^2 \parallel v_0 \parallel_{L^2}^2 . \end{aligned} \tag{2.10}$$

Using Eqs (2.8)–(2.10) arises

$$
\begin{aligned}
&|\int_{\mathbb{R}} v_x\Lambda^{-2}(v^2)dx| \\
&\quad = |\int_{\mathbb{R}} v\Lambda^{-2}\partial_x(v^2)dx| \\
&\quad \le \Big(\int_{\mathbb{R}} v^2dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}}[\Lambda^{-2}\partial_x(v^2)]^2dx\Big)^{\frac{1}{2}} \\
&\quad \le c_m \parallel v_0 \parallel_{L^2} \frac{c_m}{\sqrt{2}} \parallel v_0 \parallel_{L^2} \Big(\int_{\mathbb{R}}|\Lambda^{-2}\partial_x(v^2)|dx\Big)^{\frac{1}{2}} \\
&\quad \le c_m^3 \parallel v_0 \parallel_{L^2}^3 .
\end{aligned} \tag{2.11}
$$

Applying (2.8)–(2.11) ends the proof. □

Lemma 2.5. *Assume $v \in H^s(\mathbb{R})$ with $s \ge 3$. Then,*

$$
\begin{cases}
\int_{\mathbb{R}} vv_xv_{xx}dx = -\frac{1}{2}\int_{\mathbb{R}} v_x^3dx, \\
\int_{\mathbb{R}} vv_{xx}v_{xxx}dx = -\frac{1}{2}\int_{\mathbb{R}} v_xv_{xx}^2dx.
\end{cases} \tag{2.12}
$$

Proof. Utilizing integration by parts arises

$$
\int_{\mathbb{R}} vv_xv_{xx}dx = \int_{\mathbb{R}} vv_xdv_x = -\int_{\mathbb{R}} v_x(v_x^2 + vv_{xx})dx,
$$

which leads to the first identity in (2.12). Since

$$
\int_{\mathbb{R}} vv_{xx}v_{xxx}dx = \int_{\mathbb{R}} vv_{xx}dv_{xx} = -\int_{\mathbb{R}} v_{xx}(v_xv_{xx} + vv_{xxx})dx,
$$

which deduces that the second identity in (2.12) holds. □

For $t \in [0, T)$, we consider the problem

$$
\begin{cases}
q_t = v(t, q) + \alpha, \\
q(0, x) = x.
\end{cases} \tag{2.13}
$$

Lemma 2.6. *Assume that T and $v_0 \in H^s(\mathbb{R})$, $s \ge 3$ are defined as in Lemma 2.1. Then, problem (2.13) exists a unique solution q such that $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$ and $q_x(t, x) > 0$ for $(t, x) \in [0, T) \times \mathbb{R}$.*

Proof. If $(t, x) \in [0, T) \times \mathbb{R}$, using the conclusion in Lemma 2.1, we have $v_x \in C^2(\mathbb{R})$ and $v_t \in C^1[0, T)$. Thus, we know that $v(t, x)$ and $v_x(t, x)$ are Lipschitz continuous with respect to x and t. Using the well-posedness of ordinary differential equation, we obtain that (2.13) exists a unique $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$.

From (2.13), we have

$$
\begin{cases}
\frac{d}{dt}q_x = v_x(t, q)q_x, \\
q_x(0, x) = 1,
\end{cases}
$$

which results in

$$
q_x = \exp\Big(\int_0^t v_x(\tau, q(\tau, x))d\tau\Big).
$$

For every $T' < T$, we obtain $\sup_{(t,x)\in[0,T')\times\mathbb{R}} |v_x(t,x)| < \infty$. Thus, there must exist a constant $C_0 > 0$ to ensure $q_x(t,x) \geq e^{-C_0 t} > 0$. □

The isomorphic property about $q(t,x)$ is very important to prove the following Lemma.

Lemma 2.7. *Provided that $t \in [0,T)$, $v_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, then,*

$$\| v(t,\cdot) \|_{L^\infty} \leq \| v_0 \|_{L^\infty} + \Big(\frac{|\alpha - k| c_m}{2} \| v_0 \|_{L^2} + \frac{|1-m| c_m^2}{4} \| v_0 \|_{L^2}^2 \Big) t,$$

in which $c_m = \max\Big(\sqrt{m}, \sqrt{\frac{1}{m}}\Big)$.

Proof. Setting $\eta(x) = \frac{1}{2} e^{-|x|}$, we obtain $\Lambda^{-2} h = \eta \star h$ with $h \in L^p(\mathbb{R})$ $(1 \leq p \leq \infty)$. The density arguments in [14] arrow us to only consider Lemma 2.7 for $s = 3$. For $v_0 \in H^3(\mathbb{R})$, Lemma 2.1 ensures $v \in C([0,T), H^3(\mathbb{R})) \cap C^1([0,T), H^2(\mathbb{R}))$. Making use of (2.2) yields

$$v_t + (v+\alpha) v_x = (1-m)\eta \star (v v_x) + (\alpha - k)\eta \star v_x. \tag{2.14}$$

Using the Hörlder inequality yields

$$|\eta \star v_x| \leq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} |v(t,y)| dy \leq \frac{1}{2} \Big(\int_{\mathbb{R}} e^{-2|x-y|} dy \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} v^2(t,y) dy \Big)^{\frac{1}{2}} \leq \frac{1}{2} \| v \|_{L^2(\mathbb{R})} . \tag{2.15}$$

We have

$$\begin{aligned} |\eta \star (v v_x)| &= |\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} v v_y dy| \\ &= \frac{1}{2} | \int_{-\infty}^{x} e^{-x+y} v v_y dy + \frac{1}{2} \int_{x}^{+\infty} e^{x-y} v v_y dy| \\ &= | - \frac{1}{4} \int_{\infty}^{x} e^{-|x-y|} v^2 dy + \frac{1}{4} \int_{x}^{\infty} e^{-|x-y|} v^2 dy| \\ &\leq \frac{1}{4} \int_{-\infty}^{\infty} e^{-|x-y|} v^2 dy \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \frac{dv(t,q(t,x))}{dt} &= v_t(t,q(t,x)) + v_x(t,q(t,x)) \frac{dq(t,x)}{dt} \\ &= (v_t + (v+\alpha) v_x)(t,q(t,x)). \end{aligned} \tag{2.17}$$

Applying (2.14) and (2.17) yields

$$\begin{aligned} \frac{dv(t,q(t,x))}{dt} = \frac{m-1}{4} \int_{-\infty}^{q(t,x)} e^{-|q(t,x)-y|} v^2 dy \\ - \frac{m-1}{4} \int_{q(t,x)}^{\infty} e^{-|q(t,x)-y|} v^2 dy + (\alpha - k)\eta \star v_x, \end{aligned}$$

which together with (2.15) and (2.16), we get

$$| \frac{dv(t,q(t,x))}{dt} | \leq \frac{|m-1|}{4} \int_{-\infty}^{\infty} e^{-|q(t,x)-y|} v^2 dy + | (\alpha - k)\eta \star v_x |$$

$$
\begin{aligned}
&\leq \frac{|m-1|}{4}\int_{-\infty}^{\infty} v^2 dy + |\alpha-k| \,|\int_{-\infty}^{\infty} e^{-|q(t,x)-y|} v_y dy\,| \\
&\leq \frac{|m-1|}{4} \parallel v \parallel_{L^2}^2 + \frac{|\alpha-k|}{2} \parallel v \parallel_{L^2} \\
&\leq \frac{|1-m|}{4} c_m^2 \parallel v_0 \parallel_{L^2(\mathbb{R})}^2 + \frac{|\alpha-k| c_m}{2} \parallel v_0 \parallel_{L^2(\mathbb{R})}. \qquad (2.18)
\end{aligned}
$$

From (2.18), we obtain

$$
\begin{aligned}
-t\Big(\frac{|\alpha-k|c_m}{2} \parallel v_0 \parallel_{L^2(\mathbb{R})} + \frac{|1-m|c_m^2}{4} \parallel v_0 \parallel_{L^2(\mathbb{R})}^2\Big) &\leq v(t,q(t,x)) - v_0 \\
&\leq t\Big(\frac{|\alpha-k|c_m}{2} \parallel v_0 \parallel_{L^2(\mathbb{R})} + \frac{|1-m|c_m^2}{4} \parallel v_0 \parallel_{L^2(\mathbb{R})}^2\Big).
\end{aligned}
$$

Thus,

$$
\parallel v(t,q(t,x)) \parallel_{L^\infty} \leq \parallel v_0 \parallel_{L^\infty} + t\Big(\frac{|\alpha-k|c_m}{2} \parallel v_0 \parallel_{L^2(\mathbb{R})} + \frac{|1-m|c_m^2}{4} \parallel v_0 \parallel_{L^2(\mathbb{R})}^2\Big),
$$

which together with Lemma 2.6 ends the proof. □

3. Blow-up scenario

Provided that the maximal time of existence $T > 0$ for problem (2.2) is finite and $v_0(x) \in H^3(\mathbb{R})$, Lemma 2.1 guarantees existence $v(t,x) \in C([0,T);\ H^3(\mathbb{R})) \cap C^1([0,T);\ H^2(\mathbb{R}))$. When $\alpha = 0, m = 4$, it is derived in [2,14] that $\parallel v \parallel_{L^\infty(\mathbb{R})}$ is bounded as t tends to T. For an arbitrary α in Eq (1.1), Lemma 2.7 ensures the bounded feature of $\parallel v \parallel_{L^\infty(\mathbb{R})}$. We shall verify that the blow-up occurrence of Eq (1.1) is analogous to the wave breaking phenomena of the DP model.

Theorem 3.1. *Let $v_0(x) \in H^3(\mathbb{R}),\ m > 0$ and $T > 0$ be defined as in Lemma 2.1. Then, $\lim_{t\to T} \parallel v(t,\cdot) \parallel_{H^2} = \infty$ is equivalent to*

$$
\lim_{t\nearrow T} \inf_{x\in\mathbb{R}} [v_x(t,x)] = -\infty. \qquad (3.1)
$$

Proof. Lemma 2.1 ensures $v(t,x) \in C([0,T),\ H^3(\mathbb{R})) \cap C^1([0,T),\ H^2(\mathbb{R}))$.

From system (2.2), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} v^2 dx &= -\int_{\mathbb{R}} v^2 v_x dx + (\alpha-k)\int_{\mathbb{R}} v\Lambda^{-2} v_x dx \\
&\quad - \frac{m-1}{2}\int_{\mathbb{R}} v\Lambda^{-2}(v^2)_x dx + \alpha\int_{\mathbb{R}} v_x v dx \\
&= -\frac{m-1}{2}\int_{\mathbb{R}} v\Lambda^{-2}(v^2)_x dx, \qquad (3.2)
\end{aligned}
$$

in which Lemma 2.3 is used.

From (2.2), we obtain

$$
v_{tx} + (vv_x)_x = -\frac{m-1}{2}\Lambda^{-2}(v^2)_{xx} + (\alpha-k)\Lambda^{-2} v_{xx} + \alpha v_{xx}
$$

$$= -\frac{m-1}{2}\Lambda^{-2}(1-\Lambda^2)v^2 + (\alpha-k)\Lambda^{-2}v_{xx} + \alpha v_{xx}$$
$$= -\frac{m-1}{2}\Lambda^{-2}v^2 + \frac{m-1}{2}v^2 + (\alpha-k)\Lambda^{-2}v_{xx} + \alpha v_{xx}. \tag{3.3}$$

Multiplying Eq (3.3) by v_x and using Lemmas 2.3 and 2.5 yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} v_x^2dx &= \int_{\mathbb{R}} v_x\Big(-(vv_x)_x - \frac{m-1}{2}\Lambda^{-2}v^2 + \frac{m-1}{2}v^2 + (\alpha-k)\Lambda^{-2}v_{xx} + \alpha v_{xx}\Big)dx\\ &= -\int_{\mathbb{R}} v_x(vv_x)_xdx - \frac{m-1}{2}\int_{\mathbb{R}} v_x\Lambda^{-2}v^2dx\\ &= -\int_{\mathbb{R}} v_x(v_x^2 + vv_{xx})dx - \frac{m-1}{2}\int_{\mathbb{R}} v_x\Lambda^{-2}v^2dx\\ &= -\frac{1}{2}\int_{\mathbb{R}} v_x^3dx + \frac{m-1}{2}\int_{\mathbb{R}} v\Lambda^{-2}(v^2)_xdx.\end{aligned} \tag{3.4}$$

Differentiating (3.3) about x gives rise to

$$\begin{aligned}v_{txx} = -(vv_x)_{xx} + (\alpha-k)\Lambda^{-2}v_x - (\alpha-k)v_x - \frac{m-1}{2}\Lambda^{-2}(v^2)_x\\ +\frac{m-1}{2}(v^2)_x + \alpha v_{xxx}.\end{aligned} \tag{3.5}$$

Multiplying (3.5) by v_{xx}, using integration by parts and Lemmas 2.3 and 2.5, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} v_{xx}^2dx\\ &= \int_{\mathbb{R}} v_{xx}\bigg(-(vv_x)_{xx} + (\alpha-k)\Lambda^{-2}v_x - (\alpha-k)v_x - \frac{m-1}{2}\Lambda^{-2}(v^2)_x\\ &\qquad +\frac{m-1}{2}(v^2)_x + \alpha v_{xxx}\bigg)dx\\ &= -\int_{\mathbb{R}} v_{xx}(vv_x)_{xx}dx + (\alpha-k)\int_{\mathbb{R}} v_{xx}\Lambda^{-2}v_xdx - (\alpha-k)\int_{\mathbb{R}} v_{xx}v_xdx\\ &\qquad -\frac{m-1}{2}\int_{\mathbb{R}} v_{xx}\Lambda^{-2}(v^2)_xdx + \frac{m-1}{2}\int_{\mathbb{R}} v_{xx}(v^2)_xdx + \alpha\int_{\mathbb{R}} v_{xx}v_{xxx}dx\\ &= -\frac{5}{2}\int_{\mathbb{R}} v_xv_{xx}^2dx - \frac{m-1}{2}\int_{\mathbb{R}} v_x^3dx + \frac{m-1}{2}\int_{\mathbb{R}} v_x\Lambda^{-2}v^2dx\\ &= -\frac{5}{2}\int_{\mathbb{R}} v_xv_{xx}^2dx - \frac{m-1}{2}\int_{\mathbb{R}} v_x^3dx - \frac{m-1}{2}\int_{\mathbb{R}} v\Lambda^{-2}(v^2)_xdx.\end{aligned} \tag{3.6}$$

Applying (3.2), (3.4) and (3.6) yields

$$\begin{aligned}&\frac{1}{2}\Big[\frac{d}{dt}\int_{\mathbb{R}} v^2dx + \frac{d}{dt}\int_{\mathbb{R}} v_x^2dx + \frac{d}{dt}\int_{\mathbb{R}} v_{xx}^2dx\Big]\\ &\qquad = -\frac{5}{2}\int_{\mathbb{R}} v_xv_{xx}^2dx - \frac{m}{2}\int_{\mathbb{R}} v_x^3dx - \frac{m-1}{2}\int_{\mathbb{R}} v\Lambda^{-2}(v^2)_xdx.\end{aligned} \tag{3.7}$$

Provided that for any $(t,x) \in [0,T)\times\mathbb{R}$ and $\lim_{t\to T} \| v(t,\cdot) \|_{H^2(\mathbb{R})}= \infty$, we assume that there is a constant $C > 0$ satisfying

$$v_x(t,x) \geq -C. \tag{3.8}$$

Employing (3.7)–(3.8) and Lemma 2.4 gives rise to

$$\begin{aligned}&\frac{1}{2}\Big[\frac{d}{dt}\int_{\mathbb{R}} v^2dx + \frac{d}{dt}\int_{\mathbb{R}} v_x^2dx + \frac{d}{dt}\int_{\mathbb{R}} v_{xx}^2dx\Big]\\ &\quad\leq \max(\frac{5C}{2},\frac{mC}{2})\Big(\int_{\mathbb{R}} v^2dx + \int_{\mathbb{R}} v_x^2dx + \int_{\mathbb{R}} v_{xx}^2dx\Big) + \frac{|m-1|c_m^3}{2} \| v_0 \|^3_{L^2(\mathbb{R})}.\end{aligned} \tag{3.9}$$

Letting

$$H(t) = \int_{\mathbb{R}}\Big(v^2 + v_x^2 + v_{xx}^2\Big)dx$$

and using (3.9), we obtain

$$H(t) \leq \max(5C,mC)\int_0^t H(\tau)d\tau + |m-1|c_m^3 \| v_0 \|^3_{L^2(\mathbb{R})} T + H(0),$$

Utilizing the Gronwall inequality yields

$$H(t) \leq \Big(|m-1|c_m^3 \| v_0 \|^3_{L^2(\mathbb{R})} T + E(0)T + H(0)\Big)e^{\max(5C,mC)t},$$

which leads to $v(t,x) \in H^2(\mathbb{R})$, meaning that (3.8) is wrong. On the other hand, if (3.1) holds, using the inequality $\| v_x \|_{L^\infty} \leq \| v(t,\cdot) \|_{H^2(\mathbb{R})}$, we obtain $\lim_{t\to T} \| v(t,\cdot) \|_{H^2(\mathbb{R})}= \infty$. □

Theorem 3.2. *Let $s \geq 3$, $v_x(0,0) < 0$, $\alpha = k$ and $m \geq 1$. Provided that $v_0(x)$ is an odd function and $v_0(x) \in H^s(\mathbb{R})$, then, solution $v(t,x)$ of system (2.2) blows up at time T and T is bounded above $-\frac{1}{v_x(0,0)}$.*

Proof. Employing Lemma 2.1 ensures the existence $v \in C([0,T);\ H^3(\mathbb{R})) \cap C^1([0,T);\ H^2(\mathbb{R}))$.

The symmetry $(v,x) \to (-v,-x)$ holds for system (2.2) if $v_0(x)$ is an odd function. Using system (2.2) and the assumption in Theorem 3.2 yields

$$v(t,0) = v_{xx}(t,0) = 0. \tag{3.10}$$

Using (3.3) gives rise to

$$v_{tx} = -v_x^2 - vv_{xx} + \frac{m-1}{2}v^2 - \frac{m-1}{2}\Lambda^{-2}v^2 + \alpha v_{xx}. \tag{3.11}$$

Using $\Lambda^{-2}v^2 \geq 0$, $m \geq 1$, setting $Y(t) = v_x(t,0)$, from (3.10)–(3.11), we deduce that

$$\frac{dY(t)}{dt} \leq -Y^2(t),$$

which yields

$$\frac{1}{Y(0)} + t \leq \frac{1}{Y(t)} < 0.$$

Thus,

$$t \leq \frac{-1}{Y(0)} = -\frac{1}{v_x(0,0)}.$$

The proof is finished. □

4. Conclusions

For shallow water wave model (1.1) with an arbitrary constant α and Degasperis-Procesi equation (1.2), if the initial value $v_0 \in H^3(\mathbb{R})$, both of them possess the wave breaking feature. Namely, their solutions remain bounded and their slopes become infinite when their solutions blow up at finite time T. It is concluded that the shallow water wave Eq (1.1) and DP model behave the same blow-up structure in certain sense. The further question is to find other simple conditions imposing on the initial data to ensure that the wave breaking happens for Eq (1.1). Using the numerical simulation methods to discover the dynamical characteristics of the solutions for certain inhomogeneous boundary conditions for Eq (1.1) also needs to be investigated.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

This work is supported by the Younth Scientific Research Project of Yili Normal University (Grand No. 2023YSQN002) and the '14th Five-Years' Key Discipline of Xinjiang Uygur Autonomous Region.

Conflict of interest

The authors declare no conflict of interest.

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