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*Research article*

# The reproducing kernel method for nonlinear fourth-order BVPs

**Shiyv Wang**[1], **Xueqin Lv**[2,*] **and Songyan He**[3,*]

[1] School of Mathematics and Sciences, Harbin Normal University, Harbin 150025, China
[2] College of Basic Science, Tianjin Sino-German University of Applied Sciences, Tianjin 300350, China
[3] School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

* **Correspondence:** Email: xueqinhsd@163.com, hsymath@163.com.

**Abstract:** Based on the reproducing kernel theory, we solve the nonlinear fourth order boundary value problem in the reproducing kernel space $W_2^5[0,1]$. Its approximate solution is obtained by truncating the n-term of the exact solution and using the $\varepsilon$-best approximate method. Meanwhile, the approximate solution $u_n^{(i)}(x)$ converges uniformly to the exact solution $u^{(i)}(x), (i, 0, 1, 2, 3, 4)$. The validity and accuracy of this method are verified by some examples.

**Keywords:** reproducing kernel space; nonlinear fourth-order BVPs; $\varepsilon$-best approximate solution
**Mathematics Subject Classification:** 35G30, 65L10

## 1. Introduction

The initial- and boundary-value problems for ordinary differential equations have been investigated by numerous authors and by different methods [1–3]. Nonlinear fourth-order boundary value problems are applied in many scientific fields, such as nuclear physics, gas dynamics, fluid mechanics, boundary layer theory and nonlinear optics. It is an important branch of differential equation theory. It has a profound physical background and extensive theoretical application. In recent years, many authors have devoted themselves to the study of nonlinear fourth-order boundary value problems. Hence, different numerical techniques have been proposed, such as the lower and upper solution method [4], fixed point theory [5] and so on. In [6], Liu studied the existence of one or multiple positive solution of the fourth-order two point boundary value problem by using the Krasnoselskii fixed point theorem. Mustafa et al. present an iterative collocation numerical approach based on interpolating subdivision schemes for the solution of non-linear fourth order boundary value problems involving ordinary differential equations in [7]. Abd-Elhameed study two algorithms based on applying Galerkin and collocation spectral methods to obtain new approximate solutions of linear and nonlinear fourth-order two point boundary value problems [8]. In [9], a fixed-point iterative method to find the solution of the problem was also proposed by Chang to solve nonlinear fourth-order two-point boundary value problem.

Recently, reproducing kernel space theory has applied to solving system of second-order BVPs [11], heat conduction equation [12, 13], impulsive delay differential equations [14], the fractional integro-differential equation [15], nonlinear fractional Fokker-Planck differential equations [16] and other equation models [17–20]. In [12, 15, 16], the theory of reproducing kernel and $\varepsilon$-best approximate method are used to solve equations. In this paper, We construct a reproducing kernel space $W_2^5[0,1]$ with boundary value conditions, in which the boundary value conditions are in the form of linear combination and the $\varepsilon$ approximate solution method is used to solve the equation. By truncating the n-term of the exact solution, we can construct the numerical solution for fourth-order BVPs. The advantage of the approach is that the $u_n^{(i)}(x)$ converges uniformly to $u^{(i)}(x), (i, 0, 1, 2, 3, 4)$.

This paper is organized in six sections including the Introduction, the reproducing kernel spaces are constructed and the reproducing kernels are given in section 2. The representation of approximation solution of Eq (1.1) is introduced in section 3. And the implementation method for obtaining the approximation solution is described in detail. Then some numerical experiments are presented in section 4. Finally, a conclusion is generalized in the final section.

In this paper, we will consider how to solve the following nonlinear fourth order differential equation:

$$u^{(4)}(x) - \lambda q(x) f(x, u(x)) = 0, 0 \leq x \leq 1, \tag{1.1}$$

with the boundary conditions

$$\begin{cases} \alpha_1 u(0) - \beta_1 u'(0) = 0, \\ \gamma_1 u(1) + \sigma_1 u'(1) = 0, \\ \alpha_2 u''(0) - \beta_2 u^{(3)}(0) = 0, \\ \gamma_2 u''(1) + \sigma_2 u^{(3)}(1) = 0, \end{cases} \tag{1.2}$$

where $\lambda$ is non-negative real numbers, $\alpha_i, \beta_i, \sigma_i, \gamma_i (i = 1, 2)$ are constants. $f(x, u(x))$ and $q(x)$ are two continuous functions on [0,1]. In this paper, $\varepsilon$-best approximate solution is used, and the numerical solution are obtained in the reproducing kernel space $W_2^5[0,1]$.

## 2. Reproducing kernel spaces $W_2^5[0,1]$ and $W_2^1[0,1]$

### 2.1. The reproducing kernel space $W_2^5[0,1]$

**Definition 2.1.** $W_2^5[0,1] = \{u(x) \mid u^{(4)}$ is absolutely continuous, $\alpha_1 u(0) - \beta_1 u'(0) = 0, \gamma_1 u(1) + \sigma_1 u'(1) = 0, \alpha_2 u''(0) - \beta_2 u^{(3)}(0) = 0, \gamma_2 u''(1) + \sigma_2 u^{(3)}(1) = 0, u^{(5)} \in L^2[0,1]\}$, and

$$\langle u(x), v(x) \rangle_{W_2^5} = \sum_{i=0}^{4} u^{(i)}(0) v^{(i)}(0) + \int_0^1 u^{(5)}(x) v^{(5)}(x) dx, \tag{2.1}$$

$$\|u\|_{W_2^5} = \sqrt{\langle u, u \rangle_{W_2^5}}, u(x), v(x) \in W_2^5[0,1].$$

**Theorem 2.1.** $W_2^5[0,1]$ is a complete reproducing kernel space, i.e., there exists $R_x(y) \in W_2^5[0,1]$, for every $x \in [0,1]$ and $u(y) \in W_2^5[0,1]$ satisfying

$$\langle u(y), R_x(y)\rangle_{W_2^5} = u(x). \tag{2.2}$$

*2.2. The reproducing kernel space $W_2^1[0,1]$*

**Definition 2.2.** $W_2^1[0,1] = \{u(x) \mid u \text{ is absolutely continuous}, u' \in L^2[0,1]\}$, and

$$\langle u, v\rangle_{W_2^1} = u(0)v(0) + \int_0^1 u'v'dx, \tag{2.3}$$

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}, u(x), v(x) \in W_2^1[0,1]. \tag{2.4}$$

It can be easily proved that $W_2^1[0,1]$ is a reproducing kernel space, its kernel function is

$$Q_x(y) = \begin{cases} 1 + x, y \le x, \\ 1 + y, y > x. \end{cases} \tag{2.5}$$

## 3. $\varepsilon$-best approximate solution for Eq (1.1)

In this section, the solution of Eq (1.1) is given in the reproducing kernel space $W_2^5[0,1]$.

*3.1. The linear boundedness of operator L*

Here introduce the linear operation $L : W_2^5[0,1] \to W_2^1[0,1]$

$$L(u(x)) \triangleq u^{(4)}(x), \tag{3.1}$$

then Eq (1.1) is equivalent to

$$Lu(x) = g(x, u(x)), \tag{3.2}$$

where $g(x, u(x)) = \lambda q(x) f(x, u(x))$.

**Lemma 3.1.** $W_2^5[0,1]$ is a reproducing kernel space, $R_x(y)$ is the reproducing kernel function of $W_2^5[0,1]$, then $R_x^{(i)}(y), (i = 0, 1)$ is bounded on [0,1].

**Theorem 3.1.** $L$ is a bounded linear operator.

*Proof.* We only prove that $\|Lu\|^2_{W_2^1} \le M\|u\|^2_{W_2^5}$, where $M > 0$ is a fixed constant. Due to (2.3) we have

$$\begin{aligned} \|Lu\|^2_{W_2^1} &= \langle Lu(x), Lu(x)\rangle_{W_2^1} \\ &= (Lu(0))^2 + \int_0^1 ((Lu(x))')^2 dx. \end{aligned}$$

By the property of reproducing kernel space and (3.1), it is easy to know that

$$\langle u(\cdot), (LR_x)(\cdot)\rangle_{W_2^5} = \langle u(\cdot), R_x^{(4)}(\cdot)\rangle_{W_2^5} = \langle u^{(4)}(\cdot), R_x(\cdot)\rangle_{W_2^5} = u^{(4)}(x) = Lu(x),$$

$$\langle u(\cdot), ((LR_x)(\cdot))'\rangle_{W_2^5} = \langle u(\cdot), R_x^{(5)}(\cdot)\rangle_{W_2^5} = \langle u^{(5)}(\cdot), R_x(\cdot)\rangle_{W_2^5} = u^{(5)}(x) = (Lu(x))'.$$

For Lemma 3.1, we have

$$|Lu(x)| = |\langle u(\cdot), R_x^{(4)}(\cdot)\rangle_{W_2^5}| \leq M_1 \cdot \|u\|_{W_2^5},$$

$$|(Lu(x))'| = |\langle u(\cdot), R_x^{(5)}(\cdot)\rangle_{W_2^5}| \leq M_2 \cdot \|u\|_{W_2^5},$$

where $M_1, M_2$ are constants. Then

$$\|Lu\|_{W_2^1}^2 = (Lu(0))^2 + \int_0^1 ((Lu(x))')^2 dx \leq (M_1^2 + M_2^2) \cdot \|u\|_{W_2^5}^2 \leq M \cdot \|u\|_{W_2^5}^2,$$

where $M = M_1^2 + M_2^2$.

### 3.2. *ε-best approximate solution for Eq (3.2)*

Noting that $\{x_i\}_{i=1}^{\infty}$ is dense subset in $[0, 1]$, and let $\psi_x(y) = L^* Q_x(y)$, where $L^*$ is the conjugate operator of $L$ and $Q_x(y)$ is given by (2.5). Furthermore, $\psi_i(x) = \psi_{x_i}(x) = L^* Q_{x_i}(x)$.

**Definition 3.1.** Let $\{\psi_n\}_{i=1}^{\infty}$ be a standard orthogonal system of the inner product space $H$, if every $u \in H$, $\langle u, \psi_n\rangle = 0, (n = 1, 2, \ldots)$, we know $u = 0$, then $\{\psi_n\}_{i=1}^{\infty}$ is complete.

**Lemma 3.2.** $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system in $W_2^5[0, 1]$.

*Proof.* For $u(x) \in W_2^5[0, 1]$, let $\langle u(x), \psi_i(x)\rangle_{W_2^5} = 0, (i = 1, 2, \ldots)$, that is

$$\langle u(x), \psi_i(x)\rangle_{W_2^5} = \langle u(x), L^* Q_{x_i}(x)\rangle_{W_2^5} = \langle Lu(x), Q_{x_i}(x)\rangle_{W_2^1} = (Lu)(x_i) = 0.$$

Noting that $\{x_i\}_{i=1}^{\infty}$ is dense subset in $[0, 1]$, then $(Lu)(x) = 0$, due to the existence of $L^{-1}$, we get $u(x) = 0$.

$\{\overline{\psi_i}(x)\}_{i=1}^{\infty}$ can be derived from Gram−Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ of $W_2^5[0, 1]$

$$\overline{\psi_i}(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), (\beta_{ii} > 0, i = 1, 2, \ldots), \tag{3.3}$$

where $\beta_{ik}$ are orthogonal coefficients.

**Theorem 3.2.** Suppose $u(x)$ is the exact solution of Eq (3.2), then $u(x)$ can be expressed as

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} g(x_k, u(x_k)) \overline{\psi}_i(x). \tag{3.4}$$

*Proof.* From (3.3), we know

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x)\rangle_{W_2^5} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x)\rangle_{W_2^5} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), L^* Q_{x_k}(x)\rangle_{W_2^5} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Lu(x), Q_{x_k}(x)\rangle_{W_2^1} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle g(x, u(x)), Q_{x_k}(x)\rangle_{W_2^1} \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} g(x_k, u(x_k)) \overline{\psi}_i(x). \end{aligned}$$

Now, the approximate solution $u_n(x)$ can be obtained by truncating the $n$-term of the exact solution $u(x)$,

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k\overline{\psi}_i(x), \tag{3.5}$$

where $\alpha_k = g(x_k, u(x_k))$, so $u_n(x) \to u(x)$ in $W_2^5[0,1]$ as $n \to \infty$. Next we will get the concrete $\varepsilon$-approximate solution.

**Definition 3.2.** $\forall\varepsilon > 0$, if $u(x) \in W_2^5$, satisfies

$$\|Lu - g\|_{W_2^1} < \varepsilon,$$

the $u(x)$ is recorded as the $\varepsilon$-best approximate solution of $Lu = g$.

**Theorem 3.3.** For any $\varepsilon > 0$, there exists a positive integer $N$, such that for every $n > N$,

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k^*\overline{\psi}_i(x)$$

is an $\varepsilon$-best approximate solution of (3.2), and $\{\alpha_k^*\}_{k=1}^n$ satisfies

$$\|L\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k^*\overline{\psi}_i(x) - g(x,u(x))\|_{W_2^1} = \min_{\{\alpha_k\}_{k=1}^n}\|L\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k\overline{\psi}_i(x) - g(x,u(x))\|_{W_2^1}.$$

*Proof.* $u(x)$ is the exact solution, then $u(x)$ can be expressed as (3.4). So, $\forall\varepsilon > 0$, there exists a positive integer $N$, such that for every $n > N$, we have the following inequality

$$\|\sum_{i=n}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k\overline{\psi}_i(x) - u(x)\|_{W_2^5} \le \frac{\varepsilon}{\|L\|}.$$

Thus,

$$\begin{aligned}
\|Lu_n - g\|_{W_2^1} &= \|L\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k^*\overline{\psi}_i(x) - g(x,u(x))\|_{W_2^1} \\
&= \min_{\{\alpha_k\}_{k=1}^n}\|L\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k\overline{\psi}_i(x) - Lu(x)\|_{W_2^1} \\
&\le \|L\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k\overline{\psi}_i(x) - Lu(x)\|_{W_2^1} \\
&\le \|L\|\cdot\|\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k\overline{\psi}_i(x) - u(x)\|_{W_2^5} \le \varepsilon.
\end{aligned}$$

Thus, $u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k^*\overline{\psi}_i(x)$ is the $\varepsilon$-best approximate solution of Eq (3.2).

According to (3.5), if we can determine the approximate value of $\{\alpha_k\}_{k=1}^n$, we can get the approximate solution $u_n(x)$.

In order to find the minimum of $\|L\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\alpha_k\overline{\psi}_i(x) - g(x,u(x))\|_{W_2^1}$ with respect to $\{\alpha_k\}_{k=1}^n$, we bring (3.5) into $g(x_k, u_n(x_k))$ and solve

$$\min_{\{\alpha_k\}_{k=1}^n}\sum_{k=1}^{n}[g(x_k, u_n(x_k)) - \alpha_k]^2.$$

For convenience, we denote

$$J(\alpha_1, \alpha_2 \cdots \alpha_n) = \sum_{k=1}^{n} [g(x_k, u_n(x_k)) - \alpha_k]^2,$$

then

$$J(\alpha_1^*, \alpha_2^* \cdots \alpha_n^*) = \min_{\{\alpha_k\}_{k=1}^n} J(\alpha_1, \alpha_2 \cdots \alpha_n),$$

we can get the unique solution $(\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$ of Eq (3.5), so we can get an approximate solution $u_n(x)$.

We will give the concrete calculation process of applying Mathematica 11.0 to realize the above algorithm.

**Step 1.** Firstly we pick any initial set of values $\{\alpha_k^0\}_{k=1}^n$, usually we set $\{\alpha_k^0\}_{k=1}^n$ to the initial value of zero.

**Step 2.** When we pick the initial value, using the command **FindMinimum**, the lowest value point $\{\alpha_k^1\}_{k=1}^n$ of $J(\alpha_1^0, \alpha_2^0 \cdots \alpha_n^0)$ is obtained. If $J(\alpha_1^0, \alpha_2^0 \cdots \alpha_n^0) < 10^{-20}$, the program ends.

**Step 3.** Otherwise, inserting $\{\alpha_k^1\}_{k=1}^n$ into Eq (3.5) to get $\alpha_n^1(x)$. Due to $\alpha_k = g(x_k, u(x_k))$, we can get $\{\alpha_k^2\}_{k=1}^n$. Subsequently insert $\{\alpha_k^2\}_{k=1}^n$ into Eq (3.5) to get $\alpha_n^2(x)$. We can calculate $J(\alpha_1^2, \alpha_2^2 \cdots \alpha_n^2)$ with $\{\alpha_k^2\}_{k=1}^n$ and $\alpha_n^2(x)$.

**Step 4.** If $J(\alpha_1^2, \alpha_2^2 \cdots \alpha_n^2) < J(\alpha_1^1, \alpha_2^1 \cdots \alpha_n^1)$, replace $\{\alpha_k^0\}_{k=1}^n$ with $\{\alpha_k^2\}_{k=1}^n$, and proceed to the second step; Otherwise, return to the first step, select another set of $\{\alpha_k^0\}_{k=1}^n$ as the initial value, and recalculate.

**Theorem 3.4.** The approximate solution $u_n(x)$ and its derivatives uniformly converge to exact solution $u(x)$ and its derivatives.

*Proof.* By Lemma 3.3, we know that there exist positive real numbers $C_1$, such that

$$\|R_x^{(i)}(\cdot)\|_{W_2^5} \leq C_1.$$

Therefore, as $n \to \infty$ we have

$$\begin{aligned} |u_n^{(i)}(x) - u^{(i)}(x)| &= |\langle (u_n(\cdot) - u(\cdot))^{(i)}, R_x(\cdot)\rangle_{W_2^5}| \\ &= |\langle u_n(\cdot) - u(\cdot), R_x^{(i)}(\cdot)\rangle_{W_2^5}| \\ &\leq \|R_x^{(i)}(\cdot)\|_{W_2^5} \|u_n - u\|_{W_2^5} \\ &\leq C_1 \|u_n - u\|_{W_2^5}, \end{aligned}$$

where $i = 0, 1, 2, 3, 4$. So, $|u_n(x) - u(x)| \to 0$.

**Lemma 3.3.** $(Lu_n)(x_k) = g(x_k, u(x_k))$, where $\{x_k\}_{k=1}^\infty$ is the dense subset in the [0,1].

*Proof.* By (3.5), we know

$$\langle u_n, \overline{\psi}_i\rangle_{W_2^5} = \langle \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} g(x_k, u(x_k)) \overline{\psi}_i(x_k), \overline{\psi}_i(x_k)\rangle = \sum_{k=1}^{i} \beta_{ik} g(x_k, u(x_k)).$$

On the other hand,

$$\begin{aligned}\langle u_n, \overline{\psi}_i\rangle_{W_2^5} &= \sum_{k=1}^{i} \beta_{ik}\langle u_n, \psi_i\rangle_{W_2^5}\\ &= \sum_{k=1}^{i} \beta_{ik}\langle u_n, L^* Q_{x_k}(x)\rangle_{W_2^5}\\ &= \sum_{k=1}^{i} \beta_{ik}\langle Lu_n, Q_{x_k}(x)\rangle_{W_2^1}\\ &= \sum_{k=1}^{i} \beta_{ik} Lu_n(x_k).\end{aligned}$$

So,

$$\sum_{k=1}^{i} \beta_{ik} g(x_k, u(x_k)) = \sum_{k=1}^{i} \beta_{ik} Lu_n(x_k).$$

Then $Lu_n(x_k) = g(x_k, u(x_k))$.

**Theorem 3.5.** Suppose $u(x)$ is the exact solution of Eq (3.2), $e_n(x)$ is the error between the approximate solution $u_n(x)$ and the exact solution $u(x)$, $X = \{x_k\}_{k=1}^{\infty}$ is the dense subset in the [0,1], then $e_n(x) = |u(x) - u_n(x)| = o(\frac{1}{n})$.

*Proof.* For every $x \in [0, 1]$, $\exists\, x_j \in X, (j = 1, 2, \ldots)$ satisfying $|x_j - x| < \frac{1}{n}$. By Lemma 3.3, we have

$$\begin{aligned}Lu(x) - Lu_n(x) &= Lu(x) - Lu(x_j) - [Lu_n(x) - Lu_n(x_j)]\\ &= u^{(4)}(x) - u^{(4)}(x_j) - u_n^{(4)}(x) + u_n^{(4)}(x_j)\\ &= \langle u(\cdot), R_x^{(4)}(\cdot)\rangle_{W_2^5} - \langle u(\cdot), R_{x_j}^{(4)}(\cdot)\rangle_{W_2^5} - \langle u_n(\cdot), R_x^{(4)}(\cdot)\rangle_{W_2^5} + \langle u_n(\cdot), R_{x_j}^{(4)}(\cdot)\rangle_{W_2^5}\\ &= \langle u(\cdot) - u_n(\cdot), R_x^{(4)}(\cdot)\rangle_{W_2^5} - \langle u(\cdot) - u_n(\cdot), R_{x_j}^{(4)}(\cdot)\rangle_{W_2^5}\\ &= \langle u(\cdot) - u_n(\cdot), R_x^{(4)}(\cdot) - R_{x_j}^{(4)}(\cdot)\rangle_{W_2^5}\\ &= \langle u(\cdot) - u_n(\cdot), LR_x(\cdot) - LR_{x_j}(\cdot)\rangle_{W_2^5}.\end{aligned}$$

Furthermore, due to the bounded properties of $\|R_\xi^{'}(\cdot)\|_{W_2^5}$ and Lagrange mean value theorem,

$$\begin{aligned}|u(x) - u_n(x)| &= |L^{-1}L(u(x) - u_n(x))|\\ &= |\langle u(\cdot) - u_n(\cdot), L^{-1}LR_x(\cdot) - L^{-1}LR_{x_j}(\cdot)\rangle_{W_2^5}|\\ &\le \|u(\cdot) - u_n(\cdot)\|_{W_2^5}\|R_x(\cdot) - R_{x_j}(\cdot)\|_{W_2^5}\\ &\le \|u(\cdot) - u_n(\cdot)\|_{W_2^5}\|R_\xi^{'}(\cdot)(x - x_j)\|_{W_2^5}\\ &\le \|R_\xi^{'}(\cdot)\|_{W_2^5}\|u(\cdot) - u_n(\cdot)\|_{W_2^5}\|x - x_j\|_{W_2^5} = o(\tfrac{1}{n}),\end{aligned}$$

where $\xi$ is between $x$ and $x_j$

## 4. Numerical examples

Two numerical experiments are conducted to demonstrate the effectiveness of the proposed algorithm, where they satisfy (1.2), $\alpha_1 = \gamma_1 = \alpha_2 = \gamma_2 = 1, \beta_1 = \sigma_1 = \beta_2 = \sigma_2 = 0, u(0) = u(1) = u'(0) = u'(1) = u''(0) = u''(1) = u^{(3)}(0) = u^{(3)}(1) = 0$. Symbolic and numerical computations performed by using Mathematica 11.0.

**Example 1.** Consider a nonlinear equation

$$u^{(4)}(x) - \mathrm{e}^{u(x)} - \sin(\mathrm{u}^2(\mathrm{x})) = \mathrm{f(x)}, 0 \le \mathrm{x} \le 1.$$

The true solution is $u(x) = \sin\pi x$, where $f(x) = -e^{\sin\pi x} + \pi^2\sin(\pi x) - 2\sin(\sin(\pi x)^2)$. C.R. is calculated according to $C.R. = \log_{\frac{n2}{n1}} \frac{max|e_{n1}(x)|}{max|e_{n2}(x)|}$ and The errors in Table 1 show that the proposed algorithm is effective. The absolute error $e_n(x) = |u(x) - u_n(x)|, (n = 20, 40, 60, 80)$ are shown in Figure 1.

**Table 1.** $max|u(x) - u_n(x)|, (n = 20, 40, 60, 80)$ of Example 1.

| n | max $\|u(x) - u_n(x)\|$ | C.R. | Times |
|---|---|---|---|
| 20 | 1.27924E-03 | - | 6.126s |
| 40 | 4.23309E-04 | 1.59551 | 38.609s |
| 60 | 2.05152E-04 | 1.78647 | 128.937s |
| 80 | 1.20373E-04 | 1.85327 | 296.078s |

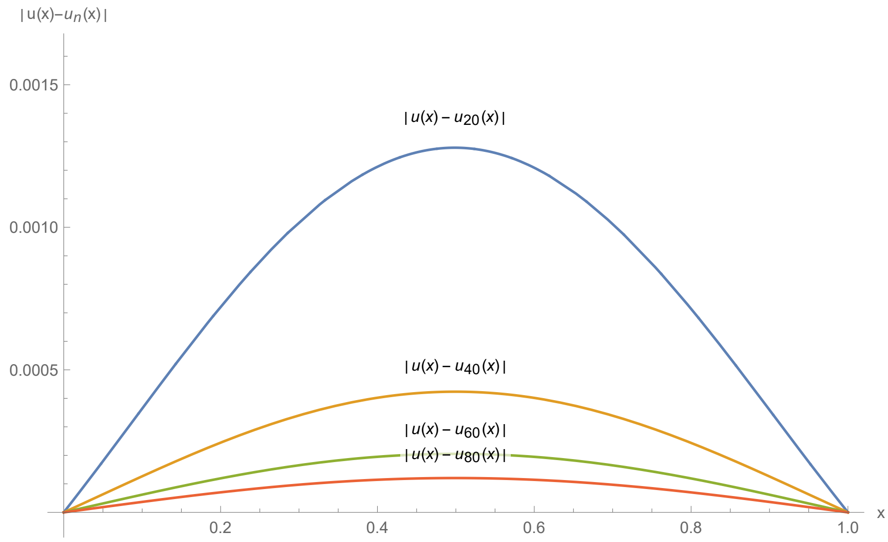


**Figure 1.** $e_n(x)$ of Example 1.

**Example 2.** Consider the following equation

$$u^{(4)}(x) - u^2(x) - \cos(u(x)) = f(x), 0 \leq x \leq 1$$

where $f(x) = -\cos(\cos(\pi x)\sin(\pi x)) + 16\pi^4\cos(\pi x)\sin(\pi x) - \cos(\pi x)^2\sin(\pi x)^2$. The exact solution of Example 2 is $u(x) = \cos(\pi x)\sin(\pi x)$. When $n = 100$, we also calculate the absolute errors $e_n(x) = |u^{(i)}(x) - u_n^{(i)}(x)|, (i = 0, 1, 2, 3, 4)$, the results are shown in Table 2.

**Table 2.** Absolute errors $e_{100}(x) = |u^{(i)}(x) - u^{(i)}_{100}(x)|, (i = 0, 1, 2, 3, 4)$.

| $x$ | $e^{(0)}_{100}(x)$ | $e^{(1)}_{100}(x)$ | $e^{(2)}_{100}(x)$ | $e^{(3)}_{100}(x)$ | $e^{(4)}_{100}(x)$ |
|---|---|---|---|---|---|
| 1/100 | 9.02632E-06 | 9.1378E-04 | 3.68894E-04 | 4.18474E-02 | 2.80605E-07 |
| 11/100 | 9.47289E-05 | 7.54092E-04 | 3.44446E-03 | 3.28231E-02 | 3.04321E-05 |
| 21/100 | 1.47926E-04 | 2.82366E-04 | 5.74113E-03 | 1.21911E-02 | 7.41203E-05 |
| 31/100 | 1.46764E-04 | 3.05592E-04 | 5.72732E-03 | 1.23501E-02 | 7.12108E-05 |
| 41/100 | 9.13207E-05 | 7.76891E-04 | 3.47212E-03 | 3.16309E-02 | 2.57648E-05 |
| 51/100 | 2.94973E-06 | 9.48553E-04 | 1.18033E-04 | 3.8452E-02 | 4.58561E-08 |
| 61/100 | 8.43218E-05 | 7.55831E-04 | 3.63901E-03 | 3.02826E-02 | 2.60113E-05 |
| 71/100 | 1.37122E-04 | 2.76229E-04 | 5.71491E-03 | 1.02046E-02 | 6.80514E-05 |
| 81/100 | 1.35792E-04 | 2.99746E-04 | 5.51299E-03 | 1.42456E-02 | 6.66736E-05 |
| 91/100 | 8.22946E-05 | 7.4009E-04 | 3.05495E-03 | 3.38999E-02 | 2.43126E-05 |

**Example 3.** Consider the nonlinear differential equation [7]

$$u^{(4)}(x) = u^2(x) - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48$$

subject to the boundary conditions

$$u(0) = u'(0) = 0, u(1) = u'(1) = 1.$$

The exact solution of Example 3 is $u(x) = x^5 - 2x^4 + 2x^2$. The obtained numerical results for this problem are presented in Table 3. The maximum absolute error obtained by the proposed method is $4.39328 \times 10^{-5}$. This is far more encouraging than the maximum error of $1.73 \times 10^{-2}$ by Mustafa et al. [7].

**Table 3.** Comparison of exact and computed solution of Example 3.

| $x$ | Exact solution | Computed solution | Error in [7] | Error ($\varepsilon$-best approximate) |
|---|---|---|---|---|
| 0.0 | 0.0000000 | 0.0000000 | 0.000000e+00 | 0.000000e+00 |
| 0.1 | 0.1981000 | 0.0198358 | 0.0004095 | 2.58282E-05 |
| 0.2 | 0.0771200 | 0.0771616 | 0.0025752 | 4.16382E-05 |
| 0.3 | 0.1662300 | 0.1662730 | 0.0066432 | 4.29637E-05 |
| 0.4 | 0.2790400 | 0.2790740 | 0.0115595 | 3.36519E-05 |
| 0.5 | 0.4062500 | 0.4062680 | 0.0156708 | 1.78605E-05 |
| 0.6 | 0.5385600 | 0.5385600 | 0.0173246 | 4.07436E-08 |
| 0.7 | 0.6678700 | 0.6678550 | 0.0154706 | 1.50789E-05 |
| 0.8 | 0.7884800 | 0.7884570 | 0.0102612 | 2.25306E-05 |
| 0.9 | 0.8982900 | 0.8982730 | 0.0036517 | 1.71761E-05 |
| 1.0 | 1.0000000 | 1.0000000 | 0.000000e+00 | 0.000000e+00 |

## 5. Conclusions

In this work, the algorithm combined the $\varepsilon$-best approximate solution and the theory of reproducing kernel space. According to the linear equations given by boundary conditions, we construct reproducing kernel space and solve reproducing kernel function and give the exact solution, denoted by series, of the nonlinear fourth-order BVPs in reproducing kernel spaces. Truncating the series, the approximate solution is obtained. The $u_n^{(i)}(x)$ converges uniformly to $u^{(i)}(x), (i = 0, 1, 2, 3, 4)$. The numerical examples illustrate the advantages of the algorithm, whose proposed algorithm can be used to deal with more complex models.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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