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Research article

Bivariate q-extended Weibull morgenstern family and correlation coefficient formulas for some of its sub-models

Rasha Abd-Elwahaab Attwa[1], **Taha Radwan**[2,3,*] **and Esraa Osama Abo Zaid**[4]

[1] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt; Email: Rasha_atwa75@hotmail.com

[2] Department of Management Information Systems, College of Business Administration, Qassim University, Buraydah, Saudi Arabia; Email: t.radwan@qu.edu.sa

[3] Department of Mathematics and Statistics, Faculty of Management Technology and Information Systems, Port Said University, Port Said, Egypt; Email: taha_ali_2003@hotmail.com

[4] Department of Mathematics and Computer Science, Faculty of Science, Suez University, Suez, Egypt; Email: esraa.abozaid@sci.suezuni.edu.eg

* **Correspondence:** Email: t.radwan@qu.edu.sa.

Abstract: A bivariate extension of a flexible univariate family is proposed. The new family is called bivariate q-extended Weibull Morgenstern family of distributions which can be constructed based on the Farlie-Gumbel-Morgenstern (FGM) copula technique. After introducing the new family, four sub-models are discussed in detail from the theoretical and numerical coefficient of correlation point of view with pointing to the effect of the q parameter.

Keywords: mathematical model; statistical model; FGM; extended Weibull family of disributions; q-extended Weibull family of disributions
Mathematics Subject Classification: 60G70, 62G30

1. Introduction

Let (X, Y) be a pair of absolutely continuous random variables (r.v.'s), formally defined by the cumulative distribution function (DF)

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)\{1 + \lambda A(F_X(x))B(F_Y(y))\},$$

where $F_X(x)$ and $F_Y(y)$ are the marginals DF's of X and Y, respectively. Moreover, the two kernels $A(x) \to 0$ and $B(x) \to 0$, as $x \to 1$, satisfy certain regularity conditions ensuring that $F_{X,Y}(x, y)$ is a DF with absolutely continuous marginals $F_X(x)$ and $F_Y(y)$. Usually, any bivariate distribution $F_{X,Y}(x,y)$ that has the above representation is referred to as FGM distribution. This model was originally introduced by Morgenstern (1956) [11] for $A(x) = 1 - x$ and $B(y) = 1 - y$ and investigated by Gumbel (1960) [7] for exponential with its multivariate case. The admissible range of association parameter λ for the distribution, with $A(x) = 1 - x$ and $B(y) = 1 - y$, is $-1 \leq \lambda \leq 1$ and the Pearson correlation coefficient between X and Y can never exceed 1/3. The classical FGM distribution is a flexible family useful in applications provided that the correlation between the variables is not too large. It can be utilized for arbitrary continuous marginals $F_X(x)$ and $F_Y(y)$.

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \lambda(1 - F_X(x))(1 - F_Y(y))], -1 \leq \lambda \leq 1, \tag{1.1}$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1 + \lambda(2F_X(x) - 1)(2F_Y(y) - 1)], -1 \leq \lambda \leq 1. \tag{1.2}$$

The maximum correlation coefficient is 1/4 when the marginal distributions are exponential (Gumbel, 1960 [7]). The maximum correlation coefficients equal 0.2921 and 0.314 for generalized exponential (Tahmasebi and Jafari, 2015 [17]) and Rayleigh distributions (Tahmasebi et al., 2018 [18]), restrictively.

In mathematical physics and probability, the q-distribution is more general than classical distribution. It was introduced by Diaz and Parguan (2009) [4] and Diaz et al. (2010) [3] in the continuous case and by Charalambides (2010) [1] in the discrete case. The construction of a q-distribution is the construction of a q-analogue of ordinary distribution. Mathai (2005) [10] introduced the q-analogue of the gamma distribution with respect to Lebesgue measure. Recently, several q-type super statistical distributions, such as the q-exponential, q-Weibull and q-logistic, were developed in the context of statistical mechanics, information theory and reliability modelling as discussed for instance by Sol and Kac (2005) [15] and Srivastave and Choi (2012) [16]. Provost et al. (2018) [14] introduced the q-generalized extreme value under linear normalization (q-GEVL) and q-Gumbel distributions.

Gurvich et al. (1997) [8] introduced a class of extended Weibull family (EWF) of distributions with survival function (SF),

$$\bar{F}(x;\theta,\eta) = e^{-\theta\Psi(x;\eta)}; \qquad x > 0, \quad \theta > 0, \tag{1.3}$$

where η is a vector of parameters and $\Psi(x;\eta)$ is a non-negative, continuous, monotone increasing and differentiable function of x, which depends on the parameter vector η. Also, $\Psi(x;\eta) \to 0^+$ as $x \to 0^+$ and $\Psi(x;\eta) \to \infty$ as $x \to \infty$. The probability density function (PDF) corresponding to (1.3) can be listed as

$$f(x;\theta,\eta) = \theta\Psi^{(1)}(x;\eta)e^{-\theta\Psi(x;\eta)}, \qquad x > 0; \quad \theta > 0, \tag{1.4}$$

where $\Psi^{(1)}(x,\eta)$ is the derivative of $\Psi(x;\eta)$ with respect to x.

Let X be a continuous random variable from the new family which is introduced by Nigm and Sadk (2021a). This family is called q-extended Weibull family and denoted by q-EWF. The SF is:

$$\bar{F}(x;q,\theta,\eta) = [1 + q\theta\Psi(x;\eta)]^{-\frac{1}{q}}; \quad x > 0, \quad \theta > 0, \quad q \neq 0, \tag{1.5}$$

where η could be a vector of parameters and $\Psi(x;\eta)$ is non-negative, monotonically increasing function of x, not depending on θ and differentiable with respect to x. The PDF corresponding to (1.5) is:

$$f(x;q,\theta,\eta) = \theta\Psi^{(1)}(x;\eta)[1 + q\theta\Psi(x;\eta)]^{-\frac{(1+q)}{q}}; \quad x > 0, \quad \theta > 0, \quad q \neq 0. \tag{1.6}$$

Note that if $q \to 0$ in (1.5) and (1.6) we have (1.3) and (1.4).
Some special cases of the q-EWF with the corresponding $\Psi(x;\eta)$ are listed in Table 1.

Table 1. Special distributions of the q-EWF.

Distribution	$\Psi(x;\eta)$	Support	θ	η
q- Exponential	x	$x \geq 0$	θ	$\emptyset$
q- Pareto	$\log(\frac{x}{b})$	$x \geq b$	θ	b
q- Weibull	x^b	$x \geq 0$	θ	b
q- Frechet	$-\log(1-e^{x^{-b}})$	$x \geq 0$	1	b
q- Lomax	$\log(1+\frac{x}{b})$	$x \geq 0$	θ	b
q- Linear failure rate	$ax + \frac{bx^2}{2}$	$x \geq 0$	1	(a,b)
q- Burr XII	$\log(1+(\frac{x}{c})^b)$	$x \geq 0$	θ	(b,c)

The inference for q-EWF were considered under Type-II progressive censoring with random removal by Nigm and Sadk (2021b) [12]. Also, a class of models called Marshall-Olkin q-extended Weibull family was proposed by Nigm and Sadk (2021c) [13]. There are a lot of papers on the same topic (Cousineau (2009) [2], Gauchman (2004) [5], Gradshteyn and Ryzhik (2007) [6] and Krugman (1996) [9]).

In this paper, we propose a bivariate extension of the q-EWF using the FGM approach. The main topic is finding the maximum correlation coefficient for some special cases of the FGM q-EWF. For some distributions, the correlation coefficient can be obtained mathematically when the r_{th} moment of the marginal distribution exists. Otherwise, the correlation coefficient can be obtained numerically. The paper is organized as follows: In Section 2, some properties of the FGM q-EWF distribution are investigated. In Section 3, some special cases of this family are studied and the maximum correlation is obtained. Finally, some conclusion remarks are reported in Section 4.

2. FGM q-EWF (θ, η, q) model

The joint CDF and PDF of (X,Y) are defined by (1.1) and (1.2), respectively, where $X \sim$ q-EWF(θ_1,η_1,q_1) and $Y \sim$ q-EWF(θ_2,η_2,q_2). Thus, it is easy to show that the (n,m)th joint moment of the FGM q-EWF $(\lambda,\theta_1,\theta_2,q_1,q_2)$ family is given by

$$\mathrm{E}(X^nY^m) = \mathrm{E}(X^n)\mathrm{E}(Y^m) + \lambda(\mathrm{E}(X^n) - \mathrm{E}(U_1^n))(\mathrm{E}(Y^m) - \mathrm{E}(U_2^m));\ n,m = 1,2,..., \tag{2.1}$$

where $U_1 \sim$ q-EWF$(2\theta_1;\eta_1,q_1/2)$ and $U_2 \sim$ q-EWF$(2\theta_2;\eta_2,q_2/2)$.
Also, the covariance of (X,Y) is

$$Cov(X,Y) = \lambda(\mathrm{E}(X) - \mathrm{E}(U_1))(\mathrm{E}(Y) - \mathrm{E}(U_2)). \tag{2.2}$$

The correlation coefficient of (X,Y) is

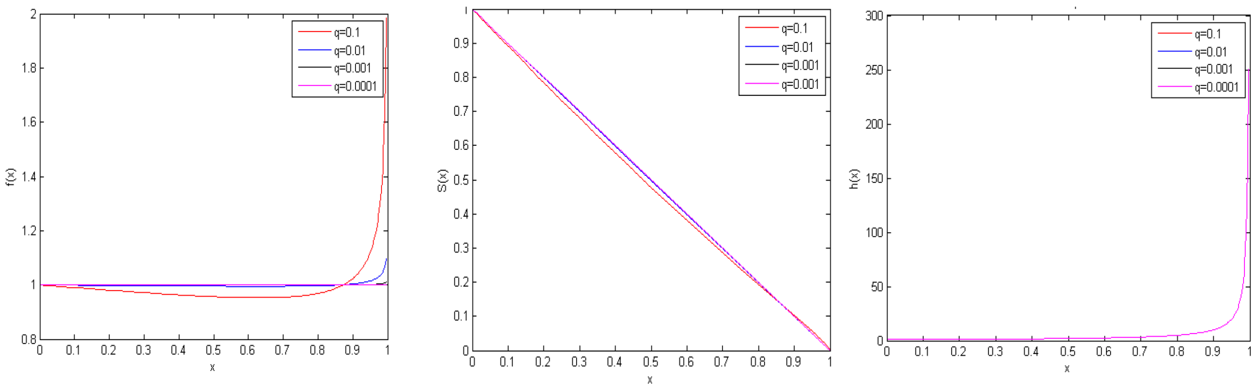
$$\rho(X,Y) = \lambda\frac{[E(X) - E(U_1)]}{\sqrt{Var(X)}}\frac{[E(Y) - E(U_2)]}{\sqrt{Var(Y)}}. \tag{2.3}$$

3. Some special cases of the FGM q-EWF (θ, η, q_i)

In this section, we obtain the maximal correlation coefficient for special cases of the FGM q-EW(θ, η, q) family distribution through the following sub-sections:

3.1. q-Uniform distribution (q-U(θ, η, q))

The q-U(θ, η, q) model can be expressed by putting $\Psi(x, \eta_1) = -\log(1-x); 0 < x < 1,\ \theta = 1$ in (1.5) and (1.6) . Figure 1 shows the PDF, SF and Hazard function (HF) of q-U(θ, η, q) for various values of the parameter q. According to Figure 1, it is noted that q-U (θ, η, q_i) cannot be used to model left and right skewed data.



(a) (b) (c)

Figure 1. (a) The PDF, (b) SF and (c) HF of q-U(θ, η, q) distribution for different values of parameter q .

The following theorem lists the new created model based on the coefficient of correlation.

Theorem 3.1. *Let* $(X, Y) \sim$ *bivariate FGM q-U* (θ_i, η_i, q_i), *for* $\Psi(x, \eta_1) = -\log(1-x)$, $\Psi(y, \eta_2) = -\log(1-y)$, $0 < x, y < 1$, $\theta_i = 1$, *and* $U_i \sim q - U(2\theta_i, \eta_i, q_i/2)$, $i = 1, 2$. *Then*

$$E(X^n) = \sum_{k=0}^{n} (-1)^k C_k^n \varphi_k(q_1, t) \ \text{and} \ E(Y^m) = \sum_{j=0}^{m} (-1)^j C_j^m \varphi_j(q_2, t), \tag{3.1}$$

$$E(U_1^n) = 2\sum_{k=0}^{n} (-1)^k C_k^n \xi_k(q_1, t) \ \text{and} \ E(U_2^m) = 2\sum_{j=0}^{m} (-1)^j C_j^m \xi_j(q_2, t), \tag{3.2}$$

where

$$\varphi_k(q_1, t) = \int_0^\infty e^{-kt}[1 + q_1 t]^{-\frac{1}{q_1}-1} dt \ , \ \varphi_j(q_2, t) = \int_0^\infty e^{-jt}[1 + q_2 t]^{-\frac{1}{q_2}-1} dt, \tag{3.3}$$

$$\xi_k(q_1, t) = \int_0^\infty e^{-kt}[1 + q_1 t]^{-\frac{2}{q_1}-1} dt \ \text{and} \ \xi_j(q_2, t) = \int_0^\infty e^{-jt}[1 + q_2 t]^{-\frac{2}{q_2}-1} dt \tag{3.4}$$

and

$$\rho(X, Y) = \lambda \frac{(2\xi_1(q_1, t) - \varphi_1(q_1, t))(2\xi_1(q_2, t) - \varphi_1(q_2, t))}{\sqrt{(\varphi_2(q_1, t) - \varphi_1^2(q_1, t))(\varphi_2(q_2, t) - \varphi_1^2(q_2, t))}}. \tag{3.5}$$

Proof. With $\Psi(x, \eta_1) = -\log(1-x)$, $\theta_1 = 1$ in (1.5) and (1.6), we have

$$\bar{F} = [1 + (-q_1 \log(1-x))]^{-\frac{1}{q_1}}, \ \frac{1}{q_1} > 0; \ 0 \le x \le 1.$$

Then

$$E(X^n) = \int_0^1 \frac{x^n}{1-x}[1 + (-q_1 \log(1-x))]^{-\frac{1}{q_1}-1} dx.$$

Put $t = -\log(1-x)$, we get

$$E(X^n) = \int_0^\infty (1-e^{-t})^n [1 + q_1 t]^{-\frac{1}{q_1}-1} dt$$

$$= \int_0^\infty [\sum_{k=0}^{n} (-1)^k C_k^n e^{-kt}][1 + q_1 t]^{-\frac{1}{q_1}-1} dt$$

$$= \sum_{k=0}^{n} (-1)^k C_k^n \int_0^\infty e^{-kt}(1 + q_1 t)^{-\frac{1}{q_1}-1} dt$$

$$= \sum_{k=0}^{n} (-1)^k C_k^n \varphi_k(q_1, t), \tag{3.6}$$

where $\varphi_k(q_1, t)$ defined in (3.3). Similarly, we have

$$E(Y^m) = \sum_{j=0}^{m} (-1)^j C_j^m \varphi_j(q_2, t). \tag{3.7}$$

where $\varphi_j(q_2, t)$ defined in (3.3). Also, $U_i \sim q - U(2\theta_i, \eta_i, \frac{q_i}{2})$, $i = 1, 2$ then:

$$E(U_1^n) = 2 \sum_{k=0}^{n} (-1)^k C_k^n \xi_k(q_1, t), \tag{3.8}$$

where $\xi_k(q_1, t)$ defined in (3.4) and

$$E(U_2^m) = 2 \sum_{j=0}^{m} (-1)^j C_j^m \xi_j(q_2, t), \tag{3.9}$$

where $\xi_j(q_2, t)$ defined in (3.4).
From (3.6) and (3.7) for $n = m = 1, 2$, we get

$$E(X) = 1 - \varphi_1(q_1, t), \qquad E(Y) = 1 - \varphi_1(q_2, t), \tag{3.10}$$

$$E(X^2) = 1 - 2\varphi_1(q_1, t) + \varphi_2(q_1, t), \qquad E(Y^2) = 1 - 2\varphi_1(q_2, t) + \varphi_2(q_2, t) \tag{3.11}$$

$$Var(X) = \varphi_2(q_1, t) - \varphi_1^2(q_1, t), \ \ Var(Y) = \varphi_2(q_2, t) - \varphi_1^2(q_2, t). \tag{3.12}$$

From (3.8) and (3.9) for $n = m = 1$ we get

$$E(U_1) = 1 - 2\xi_1(q_1, t), \qquad E(U_2) = 1 - 2\xi_1(q_2, t). \tag{3.13}$$

From (3.10), (3.12) and (3.13) in (2.3) we get

$$\rho(X, Y) = \lambda \frac{(2\xi_1(q_1, t) - \varphi_1(q_1, t))(2\xi_1(q_2, t) - \varphi_1(q_2, t))}{\sqrt{(\varphi_2(q_1, t) - \varphi_1^2(q_1, t))(\varphi_2(q_2, t) - \varphi_1^2(q_2, t))}}. \tag{3.14}$$

□

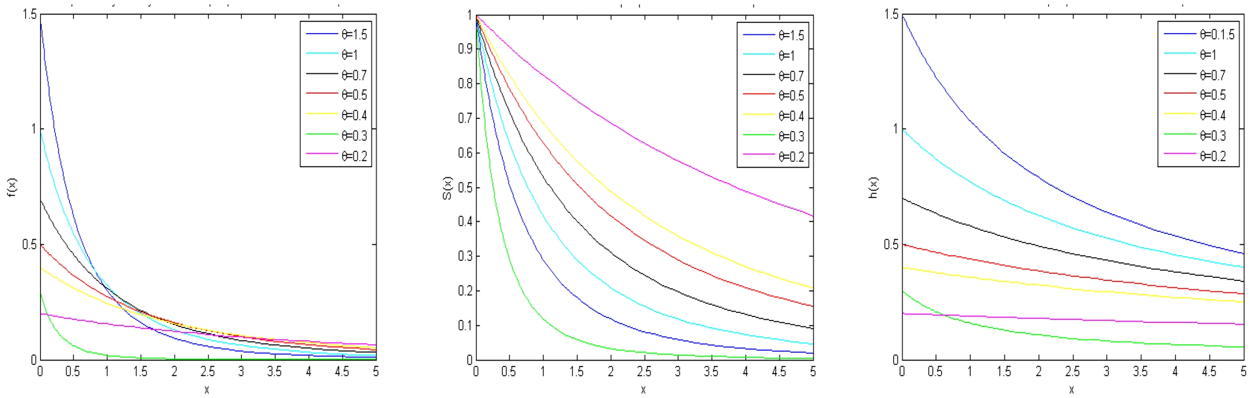
Remark 3.1. If $q_i \to 0, i = 1, 2$ we see that $\varphi_1(q_i, t) = \frac{1}{2}$, $\varphi_2(q_i, t) = \frac{1}{3}$, $\xi_1(q_i, t) = \frac{1}{3}$ and $\rho(X, Y) = \frac{\lambda}{3}$. The following table (see Table 2) gives the values of the correlation coefficient in case FGM q-Uniform distribution.

Table 2. The values of the correlation coefficient in case FGM q-Uniform distribution.

q_1	q_2	ρ
16.9	19.9	0.141
16.8	19.8	0.1415
16.6	19.6	0.1424
2.2	5.2	0.273
0.5	0.01	0.1212
-0.01	-0.01	0.333
-0.02	-0.02	0.333
0.1	0.1	0.333
0.5	0.5	0.3331
-0.1	-0.1	0.333
-0.5	-0.01	0.3323
-0.05	-0.01	0.333
-0.01	-0.001	0.333

3.2. q-exponential distribution (q-Exp(θ, η, q))

By putting $\Psi(x, \eta) = x$, $x > 0$ in (1.5), we will have $q - Exp(\theta, \eta, q)$. Figure 2 illustrates the PDF, the SF and HF for q-exponential distribution for $q = 0.3$ and different values of θ as follow:



(a) (b) (c)

Figure 2. (a) The PDF, (b) SF and (c) HF of q-exponential distribution for $q = 0.3$ and different values of θ.

We obtain the maximum correlation coefficient from special cases of FGM q-EWF (θ, η, q) distribution $[q - Exp(\theta, \eta, q)]$ through the following theorem.

Theorem 3.2. *Let* $(X,\ Y) \sim$ *bivariant FGM* q-$Exp\ (\theta_i, \eta_i, q_i)$, *for* $\Psi(x, \eta_1) = x, \Psi(y, \eta_2) = y$, $0 < x, y < \infty,\ \theta_i > 0\ ,\ U_i \sim\ \ q\text{-}Exp(2\theta_i, \eta_i, \frac{q_i}{2}),\ i = 1, 2.$ *Then*

$$E(X^n) = \frac{n!}{\theta_1^n \prod_{j=1}^{n}(1 - jq_1)} \quad and \quad E(Y^m) = \frac{m!}{\theta_2^m \prod_{k=1}^{m}(1 - kq_2)}, \qquad jq_1, kq_2 \neq 1 \tag{3.15}$$

$$E(U_1^n) = \frac{2n!}{\theta_1^n \prod_{j=1}^{n+1}(2 - jq_1)} \quad and \quad E(U_2^m) = \frac{2m!}{\theta_2^m \prod_{k=1}^{m+1}(2 - kq_2)}, \qquad jq_1, kq_2 \neq 2 \tag{3.16}$$

and

$$\rho(X, Y) = \lambda \frac{(1 - q_1)(1 - q_2)\sqrt{(1 - 2q_1)(1 - 2q_2)}}{(2 - q_1)(2 - q_2)}, \qquad q_1 \neq 2, q_2 \neq 2. \tag{3.17}$$

Proof. With $\Psi(x, \eta_1) = x$ in (1.5) and (1.6) we have,

$$\bar{F}\ = [1 + q\theta_1 x]^{-\frac{1}{q_1}}, q_1 \neq 0,\ \ 0 \leq X_i \leq \infty.$$

Then

$$E(X^n) = \int_0^\infty \theta_1 x^n [1 + q_1\theta_1 x)]^{-\frac{1}{q_1} - 1} dx.$$

Put $\quad t = q_1\theta_1 x$
we get

$$E(X^n) = \int_0^\infty q_1^{-(n+1)} \theta_1{}^{-n} t^n [1 + t]^{-\frac{1}{q_1} - 1} dt.$$

Let $u = t/(t+1)$
then,

$$E(X^n) = \int_0^1 q_1^{-(n+1)} \theta_1^{-n} u^n [1-u]^{-n-1+\frac{1}{q_1}} du$$

$$= q_1^{-(n+1)} \theta_1^{-n} \beta(n+1, \tfrac{1}{q_1} - n)$$

$$= \frac{n!}{\theta_1^n \prod_{j=1}^{n} (1 - jq_1)} \quad , \quad jq_1 \neq 1. \tag{3.18}$$

Similarly, we have

$$E(Y^m) = \frac{m!}{\theta_2^m \prod_{k=1}^{m} (1 - kq_2)} \quad , \quad kq_2 \neq 1. \tag{3.19}$$

Also, $U_i \sim q\text{-}Exp(2\theta_i, \eta_i, q_i/2),\ i = 1, 2$
then:

$$E(U_1^n) = \frac{2n!}{\theta_1^n \prod_{j=1}^{n+1} (2 - jq_1)} \quad , \quad jq_1 \neq 2 \tag{3.20}$$

and

$$E(U_2^m) = \frac{2m!}{\theta_2^m \prod_{k=1}^{m+1} (2 - kq_2)} \quad , \ kq_2 \neq 2. \tag{3.21}$$

From (3.18) and (3.19) for $n = m = 1, 2$ we get

$$E(X) = \frac{1}{\theta_1(1-q_1)} \quad , \quad E(Y) = \frac{1}{\theta_2(1-q_2)} \quad , \tag{3.22}$$

$$E(X^2) = \frac{2}{\theta_1^2(1-q_1)(1-2q_1)} \quad , \qquad E(Y^2) = \frac{2}{\theta_2^2(1-q_2)(1-2q_2)}$$

$$Var(X) = \frac{1}{\theta_1^2(1-q_1)^2(1-2q_1)} \quad , \qquad Var(Y) = \frac{1}{\theta_2^2(1-q_2)^2(1-2q_2)}. \tag{3.23}$$

From (3.20) and (3.21) for $n = m = 1$ we get

$$E(U_1) = \frac{1}{\theta_1(1-q_1)(2-q_1)} \quad , \quad E(U_2) = \frac{1}{\theta_2(1-q_2)(2-q_2)} \quad , \ q_i \neq 1,\ q_i \neq 2, i = 1, 2. \tag{3.24}$$

From (3.22)–(3.24) in (2.3) we get

$$\rho(X, Y) = \lambda \frac{(1-q_1)(1-q_2)\sqrt{(1-2q_1)(1-2q_2)}}{(2-q_1)(2-q_2)} \quad q_i \neq 1,\ q_i \neq 2, i = 1, 2. \tag{3.25}$$

□

Remark 3.2. If $q_i \to 0, i = 1, 2$, we see that $E(X) = 1/\theta_1$, $E(Y) = 1/\theta_2, Var(X) = 1/\theta_1^2$, $Var(Y) = 1/\theta_2^2, E(U_1) = 1/2\theta_1$, $E(U_2) = 1/2\theta_2$ *and* $\rho(X, Y) = \lambda/4$

The following tables (see Table 3) gives the values of the correlation coefficient in case FGM q-exponential distribution.

Table 3. (a) and (b) show the values of the correlation coefficient in case FGM q-exponential distribution.

q_1	q_2	ρ
-20	-20	0.0847
-19	-19	0.0884
-18	-18	0.0925
-17	-17	0.09695
-16	-16	0.1019
-15	-15	0.1073
-14	-14	0.1133
-13	-13	0.12
-12	-12	0.1276
-11	-11	0.1361
-10	-10	0.1458
-9	-9	0.157
-8	-8	0.17
-7	-7	0.1852
-6	-6	0.203125
-5	-5	0.2244898
-4	-4	0.25
-3	-3	0.28
-2	-2	0.3125
-1	-1	0.3333
0	0	0.25
6	6	0.6875
7	7	0.52
8	8	0.416667
9	9	0.3469388
10	10	0.296875
-18	4	0
-15	-4	0.1637578
-5	-2	0.264842
-5	-1	0.2735506
-4	-4	0.25
-3	-16	0.1688743
-3	-4	0.2020726
-3	-7	0.22771
-3	-4	0.264575
-3	-3	0.28
-3	-1	0.3055
-2	-20	0.1627025
-2	-13	0.1936492
-2	-12	0.1996489
-2	-11	0.206227
-2	-10	0.213478
-2	-7	0.2405626

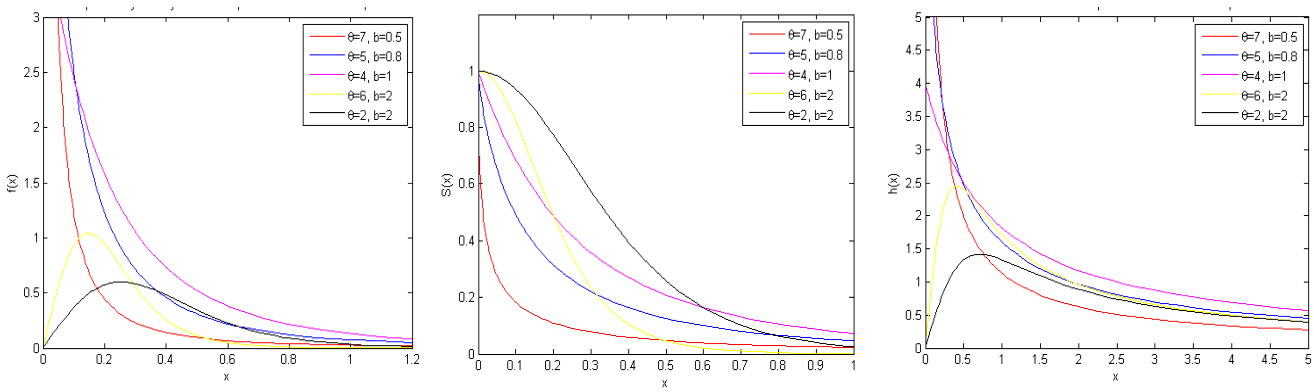
(a)

q_1	q_2	ρ
-2	-6	0.2519456
-2	-5	0.264864
-2	-4	0.2795085
-2	-3	0.295804
-2	-2	0.3125
-2	-1	0.3227
-2	-14	0.1881499
-2	0	0.2795085
-2	10	0
-1	-20	0.168038
-1	-19	0.1716929
0	-4	0.25
0	-3	0.264575
0	-2	0.2795085
0	-1	0.288675
1	6	-0.829156
1	7	-0.7211
1	8	-0.645497
1	9	-0.589015
1	10	-0.54486
3	0	0
8	1	-0.645497
8	4	0.8539126
8	5	0.645497
8	6	0.535218
8	7	0.4654747
4	7	0.953939
4	8	0.8539126
4	9	0.7791937
4	10	0.720785
5	-20	0
9	1	-0.589015
9	4	0.7791937
9	5	0.589015
9	6	0.4883855
9	7	0.4247448
9	8	0.3802076
9	10	0.320932

(b)

3.3. q-Weibull distribution (q-W(θ, η, q)))

By putting $\Psi(x, \eta) = x^b$, $x > 0$ in (1.5), we have *q-Weibull*(θ, η, q). Figure 3 illustrates the probability density function, survival function, hazard function of q-Weibull distribution for $q = 0.3$ and some different values of θ and b as follows:



(a) (b) (c)

Figure 3. (a) The PDF, (b) SF and (c) HF of q-Weibull distribution for$q = 0.3$ and some different values of θ and b.

From Figures 2 and 3, we see that the q-exponential and q-Weibull distributions can be used for modeling various data in different fields. The previous line of the PDF of the two distributions can be unimodal and bimodal. The hazard function for these distributions have various shapes, including increasing, decreasing, unimodal or bathtub.

The maximum correlation coefficient from special cases of FGM q-EWF (λ, θ, q) family distributions $[q - W(\theta, \eta, q)]$ will be obtained through the following theorem.

Theorem 3.3. *Let* $(X, Y) \sim$ *bivariant FGM q-W* (θ_i, η_i, q_i), *for,* $\Psi(x, \eta_1) = x^{b_1}, Psi(y, \eta_2) = y^{b_2}$, $0 < x, y < \infty,\ b_i > 0,\ \theta_i > 0,\ \ U_i \sim$ *q-W* $(2\theta_i, \eta_i, q_i/2), i = 1, 2$. *Then*

$$E(X^n) = \frac{\beta(\frac{n}{b_1} + 1, \frac{1}{q_1} - \frac{n}{b_1})}{\theta_1^{\frac{n}{b_1}} q_1^{\frac{n}{b_1}+1}} \quad and \quad E(Y^m) = \frac{\beta(\frac{m}{b_2} + 1, \frac{1}{q_2} - \frac{m}{b_2})}{\theta_2^{\frac{m}{b_2}} q_2^{\frac{m}{b_2}+1}} \quad , \qquad q_i \neq 1, i = 1, 2 \tag{3.26}$$

$$E(U_1^n) = \frac{2\beta(\frac{n}{b_1} + 1, \frac{2}{q_1} - \frac{n}{b_1})}{\theta_1^{\frac{n}{b_1}} q_1^{\frac{n}{b_1}+1}} \quad and \quad E(U_2^m) = \frac{2\beta(\frac{m}{b_2} + 1, \frac{2}{q_2} - \frac{m}{b_2})}{\theta_2^{\frac{m}{b_2}} q_2^{\frac{m}{b_2}+1}} \quad , \qquad q_i \neq 1, i = 1, 2 \tag{3.27}$$

and

$$\rho(X, Y) = \lambda \frac{[\beta(\frac{1}{b_1} + 1, \frac{1}{q_1} - \frac{1}{b_1}) - 2\beta(\frac{1}{b_1} + 1, \frac{2}{q_1} - \frac{1}{b_1})]}{\sqrt{q_1\beta(\frac{2}{b_1} + 1, \frac{1}{q_1} - \frac{2}{b_1}) - (\beta(\frac{1}{b_1} + 1, \frac{1}{q_1} - \frac{1}{b_1}))^2}} *$$

$$\frac{[\beta(\frac{1}{b_2} + 1, \frac{1}{q_2} - \frac{1}{b_2}) - 2\beta(\frac{1}{b_2} + 1, \frac{2}{q_2} - \frac{1}{b_2})]}{\sqrt{q_2\beta(\frac{2}{b_2} + 1, \frac{1}{q_2} - \frac{2}{b_2}) - (\beta(\frac{1}{b_2} + 1, \frac{1}{q_2} - \frac{1}{b_2}))^2}}. \tag{3.28}$$

Proof. With $\Psi(x,\eta_1) = x^{b_1}$ *in* (1.5) and (1.6) we have

$$\bar{F} = [1 + q_1\theta_1 x^{b_1}]^{-\frac{1}{q_1}}, \;\; q_1 \neq 0, \;\; 0 \leq x \leq \infty.$$

Then

$$E(X^n) = \int_0^\infty \theta_1 b_1 x^{n+b_1-1}[1 + q_1\theta_1 x^{b_1}]^{-\frac{1}{q_1}-1} dx.$$

Put $t = q_1\theta_1 x^{b_1}$

$$E(X^n) = \int_0^\infty q_1^{-(\frac{n}{b_1}+1)} \theta_1^{-\frac{n}{b_1}} t^{\frac{n}{b_1}} [1 + t]^{-\frac{1}{q_1}-1} dt$$

let $u = t/t + 1$
then

$$\begin{aligned} E(X^n) &= \int_0^1 q^{-(\frac{n}{b_1}+1)} \theta_1^{-\frac{n}{b_1}} u^{\frac{n}{b_1}} [1 - u]^{-\frac{n}{b_1}-1+\frac{1}{q_1}} du \\ &= \frac{\beta(\frac{n}{b_1} + 1, \frac{1}{q_1} - \frac{n}{b_1})}{\theta_1^{\frac{n}{b_1}} q_1^{\frac{n}{b_1}+1}} \;\;, \qquad q_1 \neq 0. \end{aligned} \tag{3.29}$$

Similarly, we have

$$E(Y^n) = \frac{\beta(\frac{m}{b_2} + 1, \frac{1}{q_2} - \frac{m}{b_2})}{\theta_2^{\frac{m}{b_2}} q_2^{\frac{m}{b_2}+1}} \;\;, \quad q_2 \neq 0. \tag{3.30}$$

Also, $U_i \sim q - W(2\theta_i, \eta_i, q_i/2), \; i = 1, 2$
then:

$$E(U_1^n) = \frac{2\beta(\frac{n}{b_1} + 1, \frac{2}{q_1} - \frac{n}{b_1})}{\theta_1^{\frac{n}{b_1}} q_1^{\frac{n}{b_1}+1}} \;\;, \tag{3.31}$$

and

$$E(U_2^m) = \frac{2\beta(\frac{n}{b_2} + 1, \frac{2}{q_2} - \frac{m}{b_2})}{\theta_2^{\frac{m}{b_2}} q_2^{\frac{m}{b_2}+1}}. \tag{3.32}$$

From (3.29) and (3.30) for $n = m = 1, 2.$ we get

$$\begin{aligned} E(X) &= \frac{\beta(\frac{1}{b_1} + 1, \frac{1}{q_1} - \frac{1}{b_1})}{\theta_1^{\frac{1}{b_1}} q_1^{\frac{1}{b_1}+1}}, \qquad E(Y) = \frac{\beta(\frac{1}{b_2} + 1, \frac{1}{q_2} - \frac{1}{b_2})}{\theta_2^{\frac{1}{b_2}} q_2^{\frac{1}{b_2}+1}}, \\ E(X^2) &= \frac{\beta(\frac{2}{b_1} + 1, \frac{1}{q_1} - \frac{2}{b_1})}{\theta_1^{\frac{2}{b_1}} q_1^{\frac{2}{b_1}+1}} \;\;, \qquad E(Y^2) = \frac{\beta(\frac{2}{b_2} + 1, \frac{1}{q_2} - \frac{2}{b_2})}{\theta_2^{\frac{2}{b_2}} q_2^{\frac{2}{b_2}+1}} \end{aligned} \tag{3.33}$$

$$\begin{aligned} Var(X) &= \theta_1^{\frac{-2}{b_1}} q_1^{\frac{-2}{b_1}-2} [q_1\beta(\tfrac{2}{b_1} + 1, \tfrac{1}{q_1} - \tfrac{2}{b_1}) - [\beta(\tfrac{1}{b_1} + 1, \tfrac{1}{q_1} - \tfrac{1}{b_1})]^2], \\ Var(Y) &= \theta_2^{\frac{-2}{b_2}} q_2^{\frac{-2}{b_2}-2} [q_2\beta(\frac{2}{b_2} + 1, \frac{1}{q_2} - \frac{2}{b_2}) - [\beta(\frac{1}{b_2} + 1, \frac{1}{q_2} - \frac{1}{b_2})]^2]. \end{aligned} \tag{3.34}$$

From (3.31) and (3.32) for $n = m = 1$ we get

$$E(U_1) = \frac{2\beta(\frac{1}{b_1}+1, \frac{2}{q_1}-\frac{1}{b_1})}{\theta_1^{\frac{1}{b_1}} q_1^{\frac{1}{b_1}+1}}, \quad E(U_2) = \frac{2\beta(\frac{1}{b_2}+1, \frac{2}{q_2}-\frac{1}{b_2})}{\theta_2^{\frac{1}{b_2}} q_2^{\frac{1}{b_2}+1}}. \tag{3.35}$$

From (3.33)–(3.35) in (2.3) we get

$$\rho(X,Y) = \lambda \frac{[\beta(\frac{1}{b_1}+1, \frac{1}{q_1}-\frac{1}{b_1}) - 2\beta(\frac{1}{b_1}+1, \frac{2}{q_1}-\frac{1}{b_1})]}{\sqrt{q_1\beta(\frac{2}{b_1}+1, \frac{1}{q_1}-\frac{2}{b_1}) - (\beta(\frac{1}{b_1}+1, \frac{1}{q_1}-\frac{1}{b_1}))^2}} *$$

$$\frac{[\beta(\frac{1}{b_2}+1, \frac{1}{q_2}-\frac{1}{b_2}) - 2\beta(\frac{1}{b_2}+1, \frac{2}{q_2}-\frac{1}{b_2})]}{\sqrt{q_2\beta(\frac{2}{b_2}+1, \frac{1}{q_2}-\frac{2}{b_2}) - (\beta(\frac{1}{b_2}+1, \frac{1}{q_2}-\frac{1}{b_2}))^2}}. \tag{3.36}$$

□

Remark 3.3. From Theorem 3.2:
(1) If $q_i \to 0, i = 1, 2$ we see that,

$$E(X) = \theta_1^{\frac{-1}{b_1}}\Gamma(1+\frac{1}{b_1}), \qquad E(Y) = \theta_2^{\frac{-1}{b_2}}\Gamma(1+\frac{1}{b_2}),$$

$$Var(X) = \theta_1^{\frac{-2}{b_1}}[\Gamma(1+\frac{2}{b_1}) - (\Gamma(1+\frac{1}{b_1}))^2], \quad Var(Y) = \theta_2^{\frac{-2}{b_2}}[\Gamma(1+\frac{2}{b_2}) - (\Gamma(1+\frac{1}{b_2}))^2],$$

$$E(U_1) = (2\theta_1)^{\frac{-1}{b_1}}\Gamma(1+\frac{1}{b_1}), \qquad E(U_2) = (2\theta_2)^{\frac{-1}{b_2}}\Gamma(1+\frac{1}{b_2})$$

then,

$$\rho(X,Y) = \lambda \frac{(1-2^{\frac{-1}{b1}})(1-2^{\frac{-1}{b2}})[\Gamma(1+\frac{1}{b_1})\Gamma(1+\frac{1}{b_2})]}{\sqrt{\Gamma(1+\frac{2}{b_1})-(\Gamma(1+\frac{1}{b_1}))^2}\sqrt{\Gamma(1+\frac{2}{b_2})-(\Gamma(1+\frac{1}{b_2}))^2}}.$$

(2) If $b_1 = b_2 = 1$ then q-Weibull distribution becomes q-exponential distribution (as in Theorem 3.2).

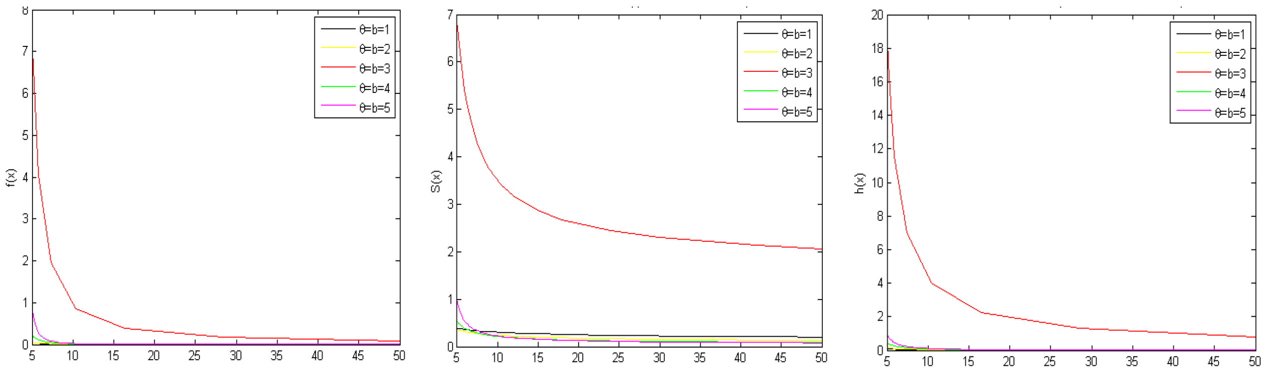
The following table (see Table 4) gives the values of the correlation coefficient in case FGM q-Weibull distribution.

Table 4. The values of the correlation coefficient in case FGM q-Weibull distribution for $b_1 = b_2 = 2$ in (a) and for $b_1 = b_2 = 3$ in (b).

q_1	q_2	ρ	q_1	q_2	ρ
0.01	0.0001	0.31356	0.01	0.002	0.3219998
0.001	0.0001	0.31389	0.01	0.0021	0.321998
0.2	1	0	0.01	0.0022	0.3219966
0.3	0.1	0.2890665	0.01	0.0023	0.321995
0.3	0.7	0.2064799	0.01	0.0024	0.3219935
0.3	0.8	0.173983	0.01	0.0082	0.3219
0.3	0.9	0.126611	0.01	0.0083	0.3218983
0.3	1	0	0.01	0.0084	0.3218967
0.4	0.9	0.1215606	0.01	0.0085	0.321895
0.4	1	0	0.01	0.0086	0.3218934
0.5	0.1	0.262496	0.01	0.0087	0.3218918
0.5	0.2	0.2586	0.01	0.0088	0.32189
0.5	0.3	0.249315	0.4	0.1	0.3053376
0.9	0.8	0.080194	0.4	0.2	0.3024359
0.9	0.9	0.058387	0.4	0.3	0.2983908
0.9	1	0	0.4	0.4	0.2930296
1	0.1	0	0.4	0.5	0.2861815
0.9	0.2	0.13044	0.01	0.0089	0.3218885
0.9	0.3	0.12661	0.01	0.009	0.3218868
0.9	0.4	0.1215606	0.01	0.0091	0.3218852
0.9	0.5	0.114973	0.01	0.0092	0.3218835
0.9	0.6	0.10641	0.01	0.0093	0.3218819
0.9	0.7	0.09521949	0.01	0.0094	0.3218803
1	0.2	0	0.01	0.0095	0.3218786
1	0.3	0	0.01	0.0096	0.321877
			0.01	0.0097	0.3218753
			0.01	0.0098	0.3218737
			0.01	0.0099	0.321872
			0.01	0.01	0.3218704
			0.9	0.1	0.2506464
			0.9	0.2	0.2480838
			0.9	0.3	0.2447657
			0.4	0.8	0.254991
			0.5	0.1	0.2982018
			0.5	0.2	0.2953679
			0.5	0.3	0.291417
	(a)			(b)	

3.4. *q-pareto distribution (q-pareto(θ, η, q))*

By putting $\Psi(x, \eta) = \log(x/b)$ in (1.5) , $\eta = b,\ x \geq b$ we will have $q - pareto(\theta, b, q)$. Figure 4 illustrates the PDF, the SF and HF for q-pareto distribution for $q = 1$ and different values of θ and b as follow:



(a) (b) (c)

Figure 4. (a) The PDF, (b) SF and (c) HF of q-pareto distribution for $q = 1$ and different values of θ and b.

The maximum correlation coefficient from special case of FGM q-pareto (θ, η, q) distribution $[q-pareto(\theta, b, q)]$ will be obtained through the following theorem.

Theorem 3.4. *Let* $(X,\ Y) \sim$ *bivariant FGM q-pareto* (θ_i, b_i, q_i), *for* $\Psi(x, \eta_1) = \log(x/b_1)$, $\Psi(y, \eta_2) = \log(y/b_2)$, $x \geq b_1$, $y \geq b_2$, $\theta_i > 0$, $U_i \sim$ *q-pareto* $(2\theta_i, b_i, q_i/2)$, $i = 1, 2$. *Then*

$$E(X^n) = b_1^n \sum_{k=0}^{\infty} \frac{n^k}{k!} \varphi_k(t; q_1, \theta_1) \ , \quad E(Y^m) = b_2^m \sum_{j=0}^{\infty} \frac{m^j}{j!} \varphi_j(t; q_2, \theta_2) \tag{3.37}$$

$$E(U_1^n) = 2b_1^n \sum_{k=0}^{\infty} \frac{n^k}{k!} \varphi_k^*(t; q_1, \theta_1), \quad and \quad E(U_2^m) = 2b_2^m \sum_{j=0}^{\infty} \frac{m^j}{j!} \varphi_j^*(t; q_2, \theta_2) \tag{3.38}$$

where,
$\varphi_k(t; q_1, \theta_1) = \theta_1 \int_0^\infty t^k (1 + q_1\theta_1 t)^{-\frac{1}{q_1}-1} dt$, $\quad \varphi_j(t; q_2, \theta_2) = \theta_2 \int_0^\infty t^j (1 + q_2\theta_2 t)^{-\frac{1}{q_2}-1} dt$,
$\varphi_k^*(t; q_1, \theta_1) = \theta_1 \int_0^\infty t^k (1 + q_1\theta_1 t)^{-\frac{2}{q_1}-1} dt$, $\quad \varphi_j^*(t; q_2, \theta_2) = \theta_2 \int_0^\infty t^j (1 + q_2\theta_2 t)^{-\frac{2}{q_2}-1} dt$,

and

$$\rho(X, Y) = \lambda \frac{[(1 + \varphi_1(t; \theta_1, q_1)) - 2(1 + \varphi_1^*(t; \theta_1, q_1))][(1 + \varphi_1(t; \theta_2, q_2)) - 2(1 + \varphi_1^*(t; \theta_2, q_2))]}{\sqrt{(2\varphi_2(t; \theta_1, q_1) - \varphi_1^2(t; \theta_1, q_1))(2\varphi_2(t; \theta_2, q_2) - \varphi_1^2(t; \theta_2, q_2))}} \ . \tag{3.39}$$

Proof. With $\Psi(x, \eta_1) = \log(\frac{x}{b_1})$ in (1.5) and (1.6) we have,

$$\bar{F} = [1 + q_1\theta_1 \log(\tfrac{x}{b_1})]^{-\frac{1}{q_1}}, \quad , x \geq b_1, \quad q_1 \neq 0.$$

Then

$$E(X^n) = \int_{b_1}^{\infty} \theta_1 x^{n-1} [1 + q_1\theta_1 \log \frac{x}{b_1}]^{-\frac{1}{q_1}-1} dx.$$

Put $t = \log x/b_1$ we get

$$E(X^n) = \theta_1 b_1^n \int_0^\infty e^{nt} [1 + q_1\theta_1 t]^{-\frac{1}{q_1}-1} dt.$$

Then,

$$E(X^n) = b_1^n \theta_1 \int_0^\infty \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} [1 + q_1\theta_1 t]^{-\frac{1}{q_1}-1} dt$$
$$= b_1^n \sum_{k=0}^{\infty} \frac{n^k}{k!} \varphi_k(t; q_1, \theta_1) \tag{3.40}$$

where, $\varphi_k(t; q_1, \theta_1) = \theta_1 \int_0^\infty t^k (1 + q_1\theta_1 t)^{-\frac{1}{q_1}-1} dt.$
Similarly, we have

$$E(Y^m) = b_2^m \sum_{j=0}^{\infty} \frac{m^j}{j!} \varphi_j(t; q_2, \theta_2) \tag{3.41}$$

where, $\varphi_j(t; q_2, \theta_2) = \theta_2 \int_0^\infty t^j (1 + q_2\theta_2 t)^{-\frac{1}{q_2}-1} dt.$
Also, $U_i \sim q - pareto(2\theta_i, \eta_i, \frac{q_i}{2})$, then,

$$E(U_1^n) = 2b_1^n \sum_{k=0}^{\infty} \frac{n^k}{k!} \varphi_k^*(t; q_1, \theta_1) \tag{3.42}$$

where, $\varphi_k^*(t; q_1, \theta_1) = \theta_1 \int_0^\infty t^k (1 + q_1\theta_1 t)^{-\frac{2}{q_1}-1} dt$
and

$$E(U_2^m) = 2b_2^m \sum_{j=0}^{\infty} \frac{m^j}{j!} \varphi_j^*(t; q_2, \theta_2) \tag{3.43}$$

where, $\varphi_j^*(t; q_2, \theta_2) = \theta_2 \int_0^\infty t^j (1 + q_2\theta_2 t)^{-\frac{2}{q_2}-1} dt.$
From (3.40) and (3.41) for $n = m = 1, 2$ we get

$$E(X) = b_1[1 + \varphi_1(t; q_1, \theta_1)], \quad E(Y) = b_2[1 + \varphi_1(t; q_2, \theta_2)], \tag{3.44}$$
$$E(X^2) = b_1^2[1 + 2\varphi_1(t; q_1, \theta_1) + 2\varphi_2(t; q_1, \theta_1)],$$

$$E(Y^2) = b_2^2[1 + 2\varphi_1(t; q_2, \theta_2) + 2\varphi_2(t; q_2, \theta_2)]$$

then,

$$Var(X) = b_1^2[2\varphi_2(t; q_1, \theta_1) - \varphi_1^2(t; q_1, \theta_1)],$$
$$Var(Y) = b_2^2[2\varphi_2(t; q_2, \theta_2) - \varphi_1^2(t; q_2, \theta_2)]. \tag{3.45}$$

From (3.42) and (3.43) for $n = m = 1$ we get

$$E(U_1) = 2b_1[1 + \varphi_1^*(t; q_1, \theta_1)], \quad E(U_2) = 2b_2[1 + \varphi_1^*(t; q_2, \theta_2)]. \tag{3.46}$$

From (3.44)–(3.46) in (2.3) we get

$$\rho(X, Y) = \lambda \frac{[(1 + \varphi_1(t; \theta_1, q_1)) - 2(1 + \varphi_1^*(t; \theta_1, q_1))][(1 + \varphi_1(t; \theta_2, q_2)) - 2(1 + \varphi_1^*(t; \theta_2, q_2))]}{\sqrt{(2\varphi_2(t; \theta_1, q_1) - \varphi_1^2(t; \theta_1, q_1))(2\varphi_2(t; \theta_2, q_2) - \varphi_1^2(t; \theta_2, q_2))}}. \tag{3.47}$$

□

Remark 3.4. If $q_i \to 0, i = 1, 2$, we see that

$$\rho(X, Y) = \frac{\lambda}{4} \frac{(1-2\theta_1)(1-2\theta_2)}{\sqrt{(4\theta_1 - 1)(4\theta_2 - 1)}}, \ \theta_i > 1/4, \ i = 1, 2$$

for $\theta_i \to \infty$ then $|\rho(X, \ Y)| < \frac{1}{4}$.

Table 5 gives the values of the correlation coefficient in case FGM q-pareto distribution.

Table 5. The values of the correlation coefficient in case FGM q-pareto distribution.

q_1	q_2	θ_1	θ_2	ρ
0.1	0.1	0.4	0.4	0.0079
0.01	0.01	0.1	0.1	0.0535
0.01	0.03	2	2	0.6862
0.1	0.2	1.7	1.7	0.2073
0.2	0.2	0.1	0.1	0.0522
0.1	0.3	2	2	0.2256
0.16	0.16	3	2	0.57
0.16	0.01	3	2	0.8436
0.6	0.05	1	2	0
0.1	0.1	2.5	2.5	0.8489
0.1	0.1	2.4	2.4	0.7625
0.1	0.1	0.1	0.1	0.054
0.05	0.05	1	1	0.0593
0.05	0.03	1	1	0
0.1	0.1	2.6	2.6	0.9398
0.1	0.1	1.5	1.5	0.1938
0.1	0.1	1.6	1.6	0.2385
0.1	0.1	2.1	2.1	0.5313
0.1	0.1	2.2	2.3	0.6808

From Tables 2–5, we see that the maximum values of the correlation coefficient for the four models, FGM q-Uniform distribution, FGM q-exponential distribution, FGM q-Weibull distribution and FGM q-pareto distributionare, are different according the values of the parameters q_1 and q_2.

Table 2 shows that the maximum value of the correlation coefficient $\rho = 0.333$ when q_1 and q_2 with negative and positive values approach zero.

From Table 3, we have the maximum correlation coefficient value $0.645 \leq \rho \leq 0.953$ when $4 \leq q_1, q_2 \leq 10$.

Table 4 shows that the maximum correlation coefficient $\rho = 0.3218$ when q_1 and q_2 take some positive values, which approaches zero. From this table, we also see that the values of ρ increases when b_1 and b_2 increase. In Table 5, we have the maximum correlation coefficient value $0.53 \leq \rho \leq 0.9389$ for different values of q_1 and q_2. The values of θ_1 is considered a pivotal role in increasing the values of correlation coefficient. Finally, the values of q_1 and q_2 play important roles to increase the maximum value of the correlation coefficient. Also, the values of q_1 and q_2 are not unique for the four models to increase the maximum correlation coefficient.

4. Conclusions

In this paper, we propose the bivariate extension of the q-EWF using the FGM approach. The main topic is finding the maximum correlation coefficient for some special cases of the FGM q-EWF. For some distributions, the correlation coefficient was obtained mathematically when the r^{th} moment of marginal distribution exists. Otherwise, the correlation coefficient is obtained numerically.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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