Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 25311–25324.
DOI: 10.3934/math.20231291
Received: 09 July 2023
Revised: 09 August 2023
Accepted: 22 August 2023
Published: 30 August 2023

Research article

Limit theorems for negatively superadditive-dependent random variables with infinite or finite means

Yongfeng Wu[1,2,*]

[1] School of Mathematics and Computer Science, Tongling University, Tongling 244000, China
[2] School of Mathematics and Finance, Chuzhou University, Chuzhou 239000, China

* **Correspondence:** Email: wyfwyf@126.com; Tel: +8618956203176.

Abstract: The author studies the laws of large numbers for weighted sums of negatively superadditive-dependent random variables. The obtained results in this paper extend and improve the corresponding theorems of Yang et al. [Commun. Stat. Theor. M., 48 (2019), 3044–3054]. Moreover, the author obtains a new theorem of mean convergence for weighted sums of negatively superadditive-dependent random variables, which was not considered in Yang et al. (2019).

Keywords: law of large numbers; mean convergence; negatively superadditive-dependent
Mathematics Subject Classification: 60F05, 60F15

1. Introduction

Notation: For two sequences of positive constants $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$, symbols $a_n \sim b_n$, $a_n = O(b_n)$ and $a_n = o(b_n)$ stand for $\lim a_n/b_n = 1$, $\lim a_n/b_n \in (0, \infty)$ and $\lim a_n/b_n = 0$, respectively. For simplicity, we shall write $\xrightarrow{\mathbb{P}}$, $\xrightarrow{a.s.}$ and $\xrightarrow{L^p}$ to express the convergence in probability, the almost certain convergence and p-mean convergence, respectively.

The following concept of superadditive function was introduced in [1].

Definition 1.1. *A function $\phi : R^n \to R$ is called superadditive if $\phi(\boldsymbol{x} \vee \boldsymbol{y}) + \phi(\boldsymbol{x} \wedge \boldsymbol{y}) \geq \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in R^n$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.*

Hu [2] introduced the concept of negatively superadditive-dependent (NSD) based on the above concept of superadditive function.

Definition 1.2. *A random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ is said to be NSD if*

$$\mathbb{E}\phi(X_1, X_2, \cdots, X_n) \leq \mathbb{E}\phi(X_1^*, X_2^*, \cdots, X_n^*),$$

where $X_1^*, X_2^*, \cdots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in the above equation exists. A sequence $\{X_n,\ n \geq 1\}$ of random variables is said to be NSD if for each $n \geq 1$, $(X_1, X_2, \cdots, X_n)$ is NSD.

Hu [2] established some basic properties and three structural theorems of NSD random variables. An interesting example was also presented in [2], which illustrated that NSD is not necessarily negatively associated (NA, [3]). Christofides and Vaggelatou [4] showed that NA is NSD. Eghbal et al. [5] derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables. Shen et al. [6] studied almost sure convergence and strong stability for weighted sums of NSD random variables. Wang et al. [7] studied complete convergence for arrays of rowwise NSD random variables, with applications to nonparametric regression. For more research of the limit theory for NSD random variables, the author can refer the reader to [8–24].

NA random variable has been studied many times and attracted extensive attention, so it is very significant to investigate the limit theorems of this wider NSD class, which is highly desirable and of considerable significance in theory and application.

A random variable X is called to be a two-tailed Pareto distribution whose density is

$$f(x) = \begin{cases} \dfrac{q}{x^2} & \text{if } x \leq -1, \\ 0 & \text{if } -1 < x < 1, \\ \dfrac{p}{x^2} & \text{if } x \geq 1, \end{cases} \tag{1.1}$$

where $p + q = 1$.

Let $\{X_n,\ n \geq 1\}$ be independent Pareto-Zipf random variables satisfying $\mathbb{P}(X_n = 0) = 1 - 1/n$,

$$\mathbb{P}(X_n \leq x) = 1 - \frac{1}{x+n} \quad \text{for all } x > 0, \tag{1.2}$$

and $f_{X_n}(x) = \frac{1}{(x+n)^2}\mathbb{I}(x > 0)$.

Obviously, if the random variable X_n satisfies Eq (1.1) or (1.2), then $\mathbb{E}|X_n| = \infty$, $n \geq 1$. Alder [25] considered independent and identically distributed (i.i.d.) random variables satisfying Eq (1.1) and studied the strong law of large numbers. Alder [26] obtained the weak law of large numbers for Pareto-Zipf random variables. For more research on laws of large numbers for i.i.d. random variables with infinite mean, the author can refer to works of Adler [27, 28] and Matsumoto and Nakata [29–31].

Yang et al. [24] investigated the law of large numbers for NSD random variables satisfying Pareto-type distributions with infinite means, and obtained the following theorems which extend and improve the corresponding ones in [25, 26]:

Theorem 1.1. *Let $\{X_n,\ n \geq 1\}$ be a nonnegative sequence of NSD random variables whose distributions are defined by $\mathbb{P}(X_n = 0) = 1 - 1/c_n$ for $n \geq 1$ and the tail probability*

$$\mathbb{P}(X_n > x) = \frac{1}{x + c_n} \quad \textit{for all } x > 0 \textit{ and } n \geq 1, \tag{1.3}$$

where $\{c_n\}$ is a nondecreasing constant sequence with $c_n \geq 1$ and

$$C_n = \sum_{j=1}^{n} \frac{1}{c_j} \to \infty. \tag{1.4}$$

Then we have

$$\frac{\sum_{j=1}^{n} c_j^{-1} X_j}{C_n \log C_n} \xrightarrow{\mathbb{P}} 1. \tag{1.5}$$

Theorem 1.2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of NSD random variables with the same distributions from a two-tailed Pareto distribution defined by Eq (1.1). Then for all $\beta > 0$ we have*

$$\frac{1}{\log^{\beta} n} \sum_{j=1}^{n} \frac{\log^{\beta-2} j}{j} X_j \xrightarrow{a.s.} \frac{p-q}{\beta}. \tag{1.6}$$

In the current work, the author studies the weak and strong laws of large numbers for NSD random variables. The obtained results in this article extend and improve Theorems 1.1 and 1.2. Meanwhile, the author investigates p-mean convergence for NSD random variables under some appropriate conditions, which was not considered in [24].

Throughout this paper, the symbol C denotes a positive constant which may differ from one place to another. The symbol $\mathbb{I}(A)$ denotes the indicator function of the event A.

2. Some lemmas and main results

To prove our main results, we first present some technical lemmas.

Lemma 2.1. *([2]) If $(X_1, X_2, \cdots, X_n)$ is NSD and $f_1, f_2, \cdots, f_n$ are all non decreasing, then $(f_1(X_1), f_2(X_2), \cdots, f_n(X_n))$ is also NSD.*

As we know, moment inequalities are very important tools in establishing the limit theorems for sequences of random variables. Shen et al. [6] presented the following Marcinkiewicz-Zygmund inequality with exponent 2.

Lemma 2.2. *([6]) Let $\{X_n,\ n \geq 1\}$ be a sequence of NSD random variables with $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n^2 < \infty$ for $n \geq 1$. Then*

$$\mathbb{E}\left(\max_{1\leq k\leq n}\left(\sum_{i=1}^{k} X_i\right)^2\right) \leq 2\sum_{i=1}^{n} \mathbb{E}X_i^2, \quad n \geq 1.$$

By means of similar methods in Shao [32], Wang et al. [7] established the following Rosenthal-type maximal inequality, which is very useful in establishing the convergence properties for NSD random variables:

Lemma 2.3. *([7]) Let $p > 1$. Let $\{X_n,\ n \geq 1\}$ be a sequence of NSD random variables with $\mathbb{E}|X_i|^p < \infty$ for each $i \geq 1$. Then for all $n \geq 1$,*

$$\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq 2^{3-p} \sum_{i=1}^{n} \mathbb{E}|X_i|^p \quad for\ 1 < p \leq 2$$

and

$$\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq 2\left(\frac{15p}{\ln p}\right)^p \left[\sum_{i=1}^{n} \mathbb{E}|X_i|^p + \left(\mathbb{E}X_i^2\right)^{p/2}\right] \quad for\ p > 2.$$

Lemma 2.4. *([6]) Let $\{X_n,\ n \geq 1\}$ be a sequence of NSD random variables. If*

$$\sum_{n=1}^{\infty} Var(X_n) < \infty,$$

then $\sum_{n=1}^{\infty}(X_n - EX_n)$ almost certainly converges.

Now we state our main results and the proofs will be presented in next section.

Theorem 2.1. *Let $\{X_n,\ n \geq 1\}$ be a nonnegative sequence of NSD random variables whose distributions are defined by $\mathbb{P}(X_n = 0) = 1 - 1/c_n$ for $n \geq 1$ and the tail probability*

$$\mathbb{P}(X_n > x) = \frac{1}{x + c_n} \quad \textit{for all } x > 0 \textit{ and } n \geq 1, \tag{2.1}$$

where $\{c_n, n \geq 1\}$ is a nondecreasing constant sequence with $c_n \geq 1$ and

$$C_n = \sum_{j=1}^{n} \frac{1}{c_j} \to \infty. \tag{2.2}$$

Let $\{D_n, n \geq 1\}$ be a sequence of constants satisfying $D_n \to \infty$ and $C_n = o(D_n)$. Then we have

$$\frac{1}{D_n} \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} c_j^{-1}(X_j - \mathbb{E}X_{nj}) \right| \xrightarrow{\mathbb{P}} 0, \tag{2.3}$$

where $X_{nj} = X_j\mathbb{I}(X_j \leq D_n c_j) + D_n c_j \mathbb{I}(X_j > D_n c_j)$, $1 \leq j \leq n$.

Take $D_n = C_n \log C_n$, then we can obtain the following corollary which extends Theorem 1.1.

Corollary 2.1. *Let $\{X_n,\ n \geq 1\}$ be a nonnegative sequence of NSD random variables whose distributions are defined by $\mathbb{P}(X_n = 0) = 1 - 1/c_n$ for $n \geq 1$ and the tail probability Eq (2.1), where $\{c_n, n \geq 1\}$ is a nondecreasing constant sequence satisfying $c_n \geq 1$ and Eq (2.2). Then*

$$\frac{1}{C_n \log C_n} \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} c_j^{-1}(X_j - \mathbb{E}X_{nj}) \right| \xrightarrow{\mathbb{P}} 0. \tag{2.4}$$

Remark 2.1. *Yang et al. [24] proved that*

$$\frac{1}{C_n \log C_n} \sum_{j=1}^{n} c_j^{-1}\mathbb{E}X_j\mathbb{I}(X_j \leq c_j C_n \log C_n) \to 1$$

and

$$\frac{1}{C_n \log C_n} \sum_{j=1}^{n} c_j^{-1}\mathbb{E}(c_j C_n \log C_n \mathbb{I}(X_j > c_j C_n \log C_n)) = \sum_{j=1}^{n} \mathbb{P}(X_j > c_j C_n \log C_n) \to 0,$$

which yields

$$\frac{1}{C_n \log C_n} \sum_{j=1}^{n} c_j^{-1} E X_{nj} \to 1.$$

Then we can find that Theorem 1.1 is a special case of Corollary 2.1 for $k = n$. Therefore, Theorem 2.1 and Corollary 2.1 extend and improve Theorem 1.1.

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed NSD random variables. Let $\{d_n, n \geq 1\}$ be a sequence of positive constants satisfying $d_n \uparrow \infty$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants such that $\varphi(n) \equiv c_n d_n$ satisfies $\varphi(n) \to \infty$ as $n \to \infty$,*

$$\sum_{m=n}^{\infty} \frac{1}{\varphi^2(m)} = O(n\varphi^{-2}(n)) \tag{2.5}$$

and

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > \varphi(n)) < \infty. \tag{2.6}$$

Then

$$\frac{1}{d_n} \sum_{j=1}^{n} c_j^{-1}(X_j - \mathbb{E}\widetilde{X}_j) \longrightarrow 0 \ \ a.s., \tag{2.7}$$

where $\widetilde{X}_j = -\varphi(j)\mathbb{I}(X_j < -\varphi(j)) + X_j\mathbb{I}(|X_j| \leq \varphi(j)) + \varphi(j)\mathbb{I}(X_j > \varphi(j))$, $1 \leq j \leq n$.

Remark 2.2. *We will show that Theorem 1.2 is a special case of Theorem 2.2. In fact, if we assume that $\{X_n, n \geq 1\}$ is a sequence of NSD random variables with the same distributions from a two-tailed Pareto distribution defined by Eq (1.1), and take $c_n = n\log^{2-\beta} n$ and $d_n = \log^{\beta} n$ $(\beta > 0)$, then $\varphi(n) = c_n d_n = n\log^2 n$. We can verify that $\varphi(n) = n\log^2 n$ satisfies the conditions stated in Theorem 2.2.*

First, it is clear that $\varphi(n) = n\log^2 n$ satisfies $\varphi(n) \to \infty$ as $n \to \infty$.

Second, we have by standard calculations that

$$\sum_{m=n}^{\infty} \frac{1}{\varphi^2(m)} \sim \int_n^{\infty} \frac{1}{x^2 \log^4 x} \mathrm{d}x = O(n^{-1}\log^{-4}(n)) = O(n\varphi^{-2}(n)),$$

which shows that Eq (2.5) is verified.

Next, we have by Eq (1.1) and $\varphi(n) = n\log^2 n$ that

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > \varphi(n)) &= \sum_{n=1}^{\infty} \mathbb{P}(|X_1| > n\log^2 n) \\
&= \sum_{n=1}^{\infty} \left(\int_{-\infty}^{-n\log^2 n} qx^{-2}\mathrm{d}x + \int_{n\log^2 n}^{\infty} px^{-2}\mathrm{d}x \right) \\
&= \sum_{n=1}^{\infty} \frac{p+q}{n\log^2 n} = \sum_{n=1}^{\infty} \frac{1}{n\log^2 n} < \infty
\end{aligned}$$

and then Eq (2.6) is verified.

Finally, we also obtain by Eq (1.1) and $\varphi(j) = j\log^2 j$ that

$$\begin{aligned}
\frac{1}{d_n}\sum_{j=1}^{n} c_j^{-1}\mathbb{E}\widetilde{X}_j &= \frac{1}{d_n}\sum_{j=1}^{n}(-d_j\mathbb{P}(X_j < -\varphi(j)) + c_j^{-1}\mathbb{E}X_j\mathbb{I}(|X_j| < \varphi(j)) + d_j\mathbb{P}(X_j > \varphi(j))) \\
&= \frac{p-q}{\log^{\beta} n}\sum_{j=1}^{n}\frac{\log^{\beta-2} j}{j} + \frac{p-q}{\log^{\beta} n}\sum_{j=1}^{n}\frac{\log^{\beta-1} j}{j} \\
&=: J_1 + J_2.
\end{aligned}$$

By similar argument as in the proof of $H \to 0$ in [24], we can obtain $J_1 \to 0$. By similar argument as in the proof of Eq (3.5) in [24], we can prove $J_2 \to \frac{p-q}{\beta}$. Then we obtain by Eq (2.7) that

$$\frac{1}{\log^{\beta} n}\sum_{j=1}^{n}\frac{\log^{\beta-2} j}{j}X_j \xrightarrow{a.s.} \frac{p-q}{\beta}.$$

To sum up, Theorem 1.2 is a special case of Theorem 2.2 and then Theorem 2.2 extends Theorem 1.2.

Next, we present a new theorem of p-mean convergence for NSD random variables under some appropriate conditions, which was not considered in [24–26].

Theorem 2.3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of NSD random variables satisfying*

$$\limsup_{x\to\infty}\sup_{j\geq 1} x^{\alpha}\mathbb{P}(|X_j| > x) < \infty, \quad \alpha \in (1,2). \tag{2.8}$$

Let $\{d_n, n \geq 1\}$ be a sequence of positive constants satisfying $d_n \uparrow \infty$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants such that $c_j \geq 1$ and

$$\sum_{j=1}^{n} c_j^{-\alpha} = o(d_n^{\alpha}). \tag{2.9}$$

Then for $p \in (1,\alpha)$,

$$\frac{1}{d_n}\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} c_j^{-1}(X_j - \mathbb{E}\widehat{X_{nj}})\right| \xrightarrow{L^p} 0, \tag{2.10}$$

where $\widehat{X_{nj}} = -d_n c_j \mathbb{I}(X_j < -d_n c_j) + X_j\mathbb{I}(|X_j| \leq d_n c_j) + d_n c_j \mathbb{I}(X_j > d_n c_j)$, $1 \leq j \leq n$.

3. The proofs

Proof of Theorem 2.1. We first observe that for every $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{P}\left(\frac{1}{D_n}\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} c_j^{-1}(X_j - \mathbb{E}X_{nj})\right| > 2\varepsilon\right) \\ \leq\ & \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} c_j^{-1}(X_j - X_{nj})\right| > D_n\varepsilon\right) + \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} c_j^{-1}(X_{nj} - \mathbb{E}X_{nj})\right| > D_n\varepsilon\right) \\ =:\ & H_1 + H_2. \end{aligned}$$

To prove Eq (2.3), we need only to show that $H_i \to 0$ as $n \to \infty$, $i = 1,2$. For H_1, we have by the definition of X_{nj}, $C_n = o(D_n)$, Eqs (2.1) and (2.2) that

$$\begin{aligned} H_1 &\leq \mathbb{P}\left(\bigcup_{j=1}^{n}(X_j \neq X_{nj})\right) \leq \sum_{j=1}^{n}\mathbb{P}(X_j > D_n c_j) \\ &= \sum_{j=1}^{n}\frac{1}{D_n c_j + c_j} = \frac{1}{D_n+1}\sum_{j=1}^{n} c_j^{-1} = \frac{C_n}{D_n+1} \to 0. \end{aligned}$$

For fixed $n \geq 1$, X_{nj} is the nondecreasing function of X_j. Hence, it follows by Lemma 2.1 that $\{X_{nj},\ 1 \leq j \leq n\}$ is a sequence of NSD random variables. Hence we have by Markov's inequality and Lemma 2.3 with $1 < p \leq 2$,

$$\begin{aligned}
H_2 &\leq \frac{C}{D_n^p}\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} c_j^{-1}(X_{nj} - \mathbb{E}X_{nj})\right|\right)^p \leq \frac{C}{D_n^p}\sum_{j=1}^{n} c_j^{-p}\mathbb{E}|X_{nj}|^p \\
&= \frac{C}{D_n^p}\sum_{j=1}^{n} c_j^{-p}\mathbb{E}|X_j|^p\mathbb{I}(X_j \leq D_n c_j) + C\sum_{j=1}^{n}\mathbb{P}(X_j > D_n c_j) \\
&= \frac{C}{D_n^p}\sum_{j=1}^{n} c_j^{-p}\int_0^{(D_n c_j)^p}\mathbb{P}(|X_j|^p\mathbb{I}(X_j \leq D_n c_j) \geq t)\mathrm{d}t + C\sum_{j=1}^{n}\frac{1}{D_n c_j + c_j} \\
&= \frac{C}{D_n^p}\sum_{j=1}^{n} c_j^{-p}\int_0^{(D_n c_j)^p}\mathbb{P}(|X_j|^p \geq t)\mathrm{d}t + C\frac{C_n}{D_n + 1} \\
&= \frac{C}{D_n^p}\sum_{j=1}^{n} c_j^{-p}\int_0^{(D_n c_j)^p}\frac{1}{t^{1/p} + c_j}\mathrm{d}t + C\frac{C_n}{D_n + 1} \qquad \text{(by (2.1))} \\
&\leq \frac{C}{D_n^p}\sum_{j=1}^{n} c_j^{-p}\int_0^{(D_n c_j)^p}\frac{1}{t^{1/p}}\mathrm{d}t + C\frac{C_n}{D_n + 1} \\
&= C\frac{C_n}{D_n} + C\frac{C_n}{D_n + 1} \to 0.
\end{aligned}$$

The proof is completed.

Proof of Theorem 2.2. Obviously, to prove Eq (2.7), we need only to show

$$\frac{1}{d_n}\sum_{j=1}^{n} c_j^{-1}(X_j - \widetilde{X}_j) \longrightarrow 0 \ \text{ a.s.} \tag{3.1}$$

and

$$\frac{1}{d_n}\sum_{j=1}^{n} c_j^{-1}(\widetilde{X}_j - \mathbb{E}\widetilde{X}_j) \longrightarrow 0 \ \text{ a.s..} \tag{3.2}$$

By Eq (2.6), $d_n \uparrow \infty$ and the Borel-Cantelli lemma, we obtain

$$\frac{1}{d_n}\sum_{j=1}^{n} c_j^{-1}|X_j|\mathbb{I}(|X_j| > \varphi(j)) \longrightarrow 0 \ \text{ a.s..}$$

Noting that

$$|X_j + \varphi(j)|\mathbb{I}(X_j < -\varphi(j)) + |X_j - \varphi(j)|\mathbb{I}(X_j > \varphi(j)) \leq |X_j|\mathbb{I}(|X_j| > \varphi(j)).$$

Then

$$\left|\frac{1}{d_n}\sum_{j=1}^{n} c_j^{-1}(X_j - \widetilde{X}_j)\right|$$

$$= \left| \frac{1}{d_n} \sum_{j=1}^{n} c_j^{-1} X_j \mathbb{I}(|X_j| > \varphi(j)) + (X_j + \varphi(j))\mathbb{I}(X_j < -\varphi(j)) + (X_j - \varphi(j))\mathbb{I}(X_j > \varphi(j)) \right|$$

$$\leq \frac{2}{d_n} \sum_{j=1}^{n} c_j^{-1} |X_j| \mathbb{I}(|X_j| > \varphi(j)) \longrightarrow 0 \text{ a.s.},$$

which yields Eq (3.1).

It follows by the definition of $\widetilde{X}_j$ that

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{1}{\varphi^2(j)} \mathbb{E}(\widetilde{X}_j - \mathbb{E}\widetilde{X}_j)^2 &\leq C \sum_{j=1}^{\infty} \frac{1}{\varphi^2(j)} \mathbb{E}X_j^2 \mathbb{I}(|X_j| \leq \varphi(j)) + C \sum_{j=1}^{\infty} \mathbb{P}(|X_j| > \varphi(j)) \\ &=: I_1 + I_2. \end{aligned}$$

We obtain directly by Eq (2.6) that $I_2 < \infty$. Let $F(x)$ be the distribution of X_1, then

$$\begin{aligned} I_1 &= C \sum_{j=1}^{\infty} \frac{1}{\varphi^2(j)} \mathbb{E}X_1^2 \mathbb{I}(|X_1| \leq \varphi(j)) \\ &= C \sum_{j=1}^{\infty} \frac{1}{\varphi^2(j)} \int_{-\infty}^{\infty} x^2 \mathbb{I}(|X_1| \leq \varphi(j)) \mathrm{d}F(x) \\ &= C \int_{-\infty}^{\infty} x^2 \sum_{j:\, \varphi(j) \geq |x|} \frac{1}{\varphi^2(j)} \, \mathrm{d}F(x). \end{aligned} \tag{3.3}$$

Define $N(|x|) = \sharp\{j : \varphi(j) < |x|\}$ and $j_* = \inf\{j : \varphi(j) \geq |x|\}$. Hence we can obtain $N(|x|) \geq j_* - 1$ and

$$\begin{aligned} \sum_{j:\, \varphi(j) \geq |x|} \frac{1}{\varphi^2(j)} &\leq \sum_{j=j_*}^{\infty} \frac{1}{\varphi^2(j)} \\ &\leq C \frac{j_*}{\varphi^2(j_*)} \qquad \text{(by Eq (2.5))} \\ &\leq C \frac{j_*}{x^2} \\ &\leq C \frac{N(|x|) + 1}{x^2}. \end{aligned} \tag{3.4}$$

It follows by Eqs (2.6), (3.3) and (3.4) that

$$\begin{aligned} I_1 &\leq C \int_{-\infty}^{\infty} (N(|x|) + 1) \, \mathrm{d}F(x) = C\mathbb{E}N(|X_1|) + C \\ &= CE\left[\sum_{j=1}^{\infty} \mathbb{I}(|X_1| > \varphi(j)) \right] + C \\ &= C \sum_{j=1}^{\infty} \mathbb{P}(|X_1| > \varphi(j)) + C < \infty. \end{aligned}$$

Now we obtain by $I_1 < \infty$ and $I_2 < \infty$ that

$$\sum_{j=1}^{\infty} \frac{1}{\varphi^2(j)} \mathbb{E}(\widetilde{X}_j - \mathbb{E}\widetilde{X}_j)^2 < \infty. \tag{3.5}$$

Consequently, by Lemma 2.4 and Eq (3.5), we get

$$\sum_{j=1}^{\infty} \frac{1}{\varphi(j)} (\widetilde{X}_j - \mathbb{E}\widetilde{X}_j) \quad \text{converges a.s.,}$$

which implies Eq (3.2) by Kronecker's lemma, together with the condition $d_n \uparrow \infty$.

The proof is completed.

Proof of Theorem 2.3. Noting that

$$\begin{aligned}
& \mathbb{E}\left\{\frac{1}{d_n} \max_{1\le k\le n}\left|\sum_{j=1}^{k} c_j^{-1}(X_j - \mathbb{E}\widehat{X_{nj}})\right|\right\}^p \\
\le\ & \frac{1}{d_n^p}\mathbb{E}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} c_j^{-1}(\widehat{X_{nj}} - \mathbb{E}\widehat{X_{nj}})\right|\right\}^p + \frac{1}{d_n^p}\mathbb{E}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} c_j^{-1}(X_j - \widehat{X_{nj}})\right|\right\}^p \\
\le\ & \frac{1}{d_n^p}\left\{\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k} c_j^{-1}(\widehat{X_{nj}} - \mathbb{E}\widehat{X_{nj}})\right|\right)^2\right\}^{p/2} + \frac{1}{d_n^p}\mathbb{E}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} c_j^{-1}(X_j - \widehat{X_{nj}})\right|\right\}^p \\
=:\ & J_1 + J_2.
\end{aligned}$$

To prove Eq (2.10), it is sufficient to prove $J_1 \to 0$ and $J_2 \to 0$. By Lemma 2.1 and the fact that $\widehat{X_{nj}}$ is the nondecreasing function of X_j, $\{\widehat{X_{nj}}, 1 \le j \le n\}$ is also a sequence of NSD random variables.

We have by Lemma 2.2 that

$$\begin{aligned}
J_1^{2/p} &= \frac{1}{d_n^2}\mathbb{E}\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} c_j^{-1}(\widehat{X_{nj}} - \mathbb{E}\widehat{X_{nj}})\right|\right\}^2 \\
&\le \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2}\mathbb{E}(\widehat{X_{nj}} - \mathbb{E}\widehat{X_{nj}})^2 \\
&\le \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2}\mathbb{E}X_j^2\mathbb{I}(|X_j| \le d_n c_j) + C\sum_{j=1}^{n}\mathbb{P}(|X_j| > d_n c_j) \\
&=: J_3 + J_4.
\end{aligned}$$

By $d_n \uparrow \infty$, Eqs (2.8) and (2.9), we have

$$J_4 \le C\frac{1}{d_n^\alpha}\sum_{j=1}^{n} c_j^{-\alpha} \to 0 \qquad \text{as} \quad n \to \infty. \tag{3.6}$$

Now we will show $J_3 \to 0$. Observing

$$J_3 = \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2}\int_0^{(d_n c_j)^2} \mathbb{P}(X_j^2\mathbb{I}(|X_j| \le d_n c_j) \ge t)\mathrm{d}t$$

$$\leq \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2}\int_0^{(d_nc_j)^2}\mathbb{P}(X_j^2 \geq t)\mathrm{d}t.$$

Let $t = u^2$, then

$$J_3 \leq \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2}\int_0^{d_nc_j} u\mathbb{P}(|X_j| \geq u)\mathrm{d}u.$$

From Eq (2.8), we know that, there exists $M > 0$ and $N_0 \in \mathbf{N}$ such that

$$\mathbb{P}(|X_j| \geq u) \leq Mu^{-\alpha} \quad \text{for} \;\; u > N_0. \tag{3.7}$$

Since $d_n \uparrow \infty$ and $c_j \geq 1$, while n is sufficiently large, we can obtain $d_nc_j > N_0$. Hence

$$\begin{aligned} J_3 &\leq \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2}\int_0^{N_0} u\mathbb{P}(|X_j| \geq u)\mathrm{d}u + \frac{CM}{d_n^2}\sum_{j=1}^{n} c_j^{-2}\int_{N_0}^{d_nc_j} u^{1-\alpha}\mathrm{d}u \\ &=: J_3' + J_3''. \end{aligned}$$

By $\alpha < 2$, $c_j \geq 1$ and Eq (2.9), we have

$$\begin{aligned} J_3' &\leq \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2}\int_0^{N_0} u\mathrm{d}u \;\leq\; \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2} \\ &\leq \frac{C}{d_n^{2-\alpha}}\frac{1}{d_n^{\alpha}}\sum_{j=1}^{n} c_j^{-\alpha} \to 0 \qquad \text{as} \quad n \to \infty \end{aligned}$$

and

$$\begin{aligned} J_3'' &\leq \frac{C}{d_n^2}\sum_{j=1}^{n} c_j^{-2}[(d_nc_j)^{2-\alpha} - N_0^{2-\alpha}] \\ &\leq \frac{C}{d_n^{\alpha}}\sum_{j=1}^{n} c_j^{-\alpha} \to 0 \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Finally, we need only to show $J_2 \to 0$ as $n \to \infty$. Let

$$Z_{nj} = X_j - \widehat{X_{nj}} = (X_j + d_nc_j)\mathbb{I}(X_j < -d_nc_j) + (X_j - d_nc_j)\mathbb{I}(X_j > d_nc_j).$$

We first prove that

$$\mathbb{E}Z_{nj} \to 0 \qquad \text{as} \quad n \to \infty. \tag{3.8}$$

Observing

$$\begin{aligned} |\mathbb{E}Z_{nj}| &\leq \mathbb{E}|Z_{nj}| \;\leq\; \mathbb{E}|X_j|\mathbb{I}(|X_j| > d_nc_j) \\ &= \left(\int_0^{d_nc_j} + \int_{d_nc_j}^{\infty}\right)\mathbb{P}(|X_j|\mathbb{I}(|X_j| > d_nc_j) \geq t)\mathrm{d}t \\ &= \int_0^{d_nc_j}\mathbb{P}(|X_j| > d_nc_j)\mathrm{d}t + \int_{d_nc_j}^{\infty}\mathbb{P}(|X_j| \geq t)\mathrm{d}t \end{aligned}$$

$$\begin{aligned} &= \quad d_n c_j \mathbb{P}(|X_j| > d_n c_j) + \int_{d_n c_j}^{\infty} \mathbb{P}(|X_j| \geq t) \mathrm{d}t \\ &=: \quad J_5 + J_6. \end{aligned}$$

By Eq (3.7) and $\alpha > 1$, we have

$$J_5 \leq \frac{M}{(d_n c_j)^{\alpha-1}} \to 0 \qquad \text{as} \quad n \to \infty$$

and

$$J_6 \leq M \int_{d_n c_j}^{\infty} t^{-\alpha} \mathrm{d}t \leq \frac{CM}{(d_n c_j)^{\alpha-1}} \to 0 \qquad \text{as} \quad n \to \infty,$$

which yields Eq (3.8). Therefore, we obtain by Lemma 2.3 that

$$\begin{aligned} J_2 \quad &\leq \quad \frac{1}{d_n^p} \mathbb{E}\left\{ \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} c_j^{-1} (Z_{nj} - \mathbb{E} Z_{nj}) \right| \right\}^p \\ &\leq \quad \frac{C}{d_n^p} \sum_{j=1}^{n} c_j^{-p} \mathbb{E}|Z_{nj}|^p \\ &\leq \quad \frac{C}{d_n^p} \sum_{j=1}^{n} c_j^{-p} \mathbb{E}|X_j|^p \mathbb{I}(|X_j| > d_n c_j). \qquad \text{(by the definition of } Z_{nj}) \end{aligned}$$

By similar arguments as in the proof of Eq (3.8), we can obtain

$$\mathbb{E}|X_j|^p \mathbb{I}(|X_j| > d_n c_j) = (d_n c_j)^p \mathbb{P}(|X_j| > d_n c_j) + \int_{(d_n c_j)^p}^{\infty} \mathbb{P}(|X_j|^p \geq t) \mathrm{d}t.$$

Then

$$\begin{aligned} J_2 \quad &\leq \quad C \sum_{j=1}^{n} \mathbb{P}(|X_j| > d_n c_j) + \frac{C}{d_n^p} \sum_{j=1}^{n} c_j^{-p} \int_{(d_n c_j)^p}^{\infty} \mathbb{P}(|X_j|^p \geq t) \mathrm{d}t \\ &=: \quad J_2' + J_2''. \end{aligned}$$

By similar arguments as the proof of $J_4 \to 0$, we obtain $J_2' \to 0$. We also have by Eq (3.7), $p < \alpha$ and Eq (2.9) that

$$\begin{aligned} J_2'' \quad &\leq \quad \frac{C}{d_n^p} \sum_{j=1}^{n} c_j^{-p} \int_{(d_n c_j)^p}^{\infty} t^{-\alpha/p} \mathrm{d}t \\ &\leq \quad \frac{C}{d_n^\alpha} \sum_{j=1}^{n} c_j^{-\alpha} \to 0 \qquad \text{as} \quad n \to \infty. \end{aligned}$$

The proof is completed.

4. Conclusions

In this work the author investigated the limit theorems for negatively superadditive-dependent random variables, and obtained some new results on the law of large numbers and mean convergence under some appropriate conditions. As a future work, we propose to consider some other strong convergence for sequence of negatively superadditive-dependent random variables.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the Social Sciences Planning Project of Anhui Province (AHSKY2018D98).

Conflict of interest

The author declares that he has no conflict of interest.

References

1. J. H. B. Kemperman, On the FKG-inequalities for measures on a partially ordered space, *Indagat. Math.*, **80** (1977), 313–331. https://doi.org/10.1016/1385-7258(77)90027-0
2. T. Z. Hu, Negatively superadditive dependence of random variables with applications, *Chinese J. Appl. Probab. Statist.*, **16** (2000), 133–144.
3. K. J. Dev, F. Proschan, Negative association of random variables with applications, *Ann. Statist.*, **11** (1983), 286–295. https://doi.org/10.1214/aos/1176346079
4. T. C. Christofides, E. Vaggelatou, A connection between supermodular ordering and positive/negative association, *J. Multivariate Anal.*, **88** (2004), 138–151. https://doi.org/10.1016/s0047-259x(03)00064-2
5. N. Eghbal, M. Amini, A. Bozorgnia, Some maximal inequalities for quadratic forms of negative superadditive dependence random variables, *Statist. Probabil. Lett.*, **80** (2010), 587–591. https://doi.org/10.1016/j.spl.2009.12.014
6. Y. Shen, X. J. Wang, W. Z. Yang, S. H. Hu, Almost sure convergence theorem and strong stability for weighted sums of NSD random variables, *Acta Math. Sin.*, **29** (2013), 743–756. https://doi.org/10.1007/s10114-012-1723-6
7. X. J. Wang, X. Deng, L. L. Zheng, S. H. Hu, Complete convergence for arrays of rowwise negatively superadditive-dependent random variables and its applications, *Statistics*, **48** (2014), 834–850. https://doi.org/10.1080/02331888.2013.800066

8. N. Eghbal, M. Amini, A. Bozorgnia, On the Kolmogorov inequalities for quadratic forms of dependent uniformly bounded random variables, *Scienceasia*, **81** (2011), 1112–1120. https://doi.org/10.1016/j.spl.2011.03.005

9. X. J. Wang, A. T. Shen, Z. Y. Chen, S. H. Hu, Complete convergence for weighted sums of NSD random variables and its application in the EV regression model, *Test*, **24** (2015), 166–184. https://doi.org/10.1007/s11749-014-0402-6

10. Y. C. Yu, H. C. Hu, L. Liu, S. Y. Huang, M-test in linear models with negatively superadditive dependent errors, *J. Inequal. Appl.*, **235** (2017), 235. https://doi.org/10.1186/s13660-017-1509-6

11. A. T. Shen, X. H. Wang, Kaplan-Meier estimator and hazard estimator for censored negatively superadditive dependent data, *Statistics*, **50** (2016), 377–388. https://doi.org/10.1080/02331888.2015.1038269

12. H. Naderi, F. Boukhari, P. Matula, A note on the weak law of large numbers for weighted negatively superadditive dependent random variables, *Commun. Stat. Theor. M.*, **51** (2022), 7465–7475. https://doi.org/10.1080/03610926.2021.1873377

13. S. C. Ta, C. M. Tran, D. V. Le, On the almost sure convergence for sums of negatively superadditive dependent random vectors in Hilbert spaces and its application, *Commun. Stat. Theor. M.*, **49** (2020), 2770–2786. https://doi.org/10.1080/03610926.2019.1584304

14. X. H. Wang, S. H. Hu, On the strong consistency of M-estimates in linear models for negatively superadditive dependent errors, *Aust. Nz. J. Stat.*, **57** (2015), 259–274. https://doi.org/10.1111/anzs.12117

15. Y. Wu, X. J. Wang, S. H. Hu, Complete convergence for arrays of rowwise negatively superadditive-dependent random variables and its applications, *Appl. Math. Ser. B*, **31** (2016), 439–457. https://doi.org/10.1007/s11766-016-3406-z

16. X. H. Wang, X. Q. Li, S. H. Hu, On the complete convergence of weighted sums for an array of rowwise negatively superadditive dependent random variables, *Scienceasia*, **42** (2016), 66–74. https://doi.org/10.2306/scienceasia1513-1874.2016.42.066

17. X. Deng, X. J. Wang, Y. Wu, Y. Ding, Complete moment convergence and complete convergence for weighted sums of NSD random variables, *Racsam. Rev. R. Acad. A.*, **110** (2016), 97–120. https://doi.org/10.1007/s13398-015-0225-7

18. A. T. Shen, X. H. Wang, H. Y. Zhu, Convergence properties for weighted sums of NSD random variables, *Commun. Stat. Theor. M.*, **45** (2016), 2402–2412. https://doi.org/10.1080/03610926.2014.881492

19. A. Kheyri, M. Amini, H. Jabbari, A. Bozorgnia, Kernel density estimation under negative superadditive dependence and its application for real data, *J. Stat. Comput. Sim.*, **89** (2019), 2373–2392. https://doi.org/10.1080/00949655.2019.1619738

20. B. Meng, Q. Y. Wu, D. C. Wang, On the strong convergence for weighted sums of negatively superadditive dependent random variables, *J. Inequal. Appl.*, **269** (2017), 269. https://doi.org/10.1080/03610918.2022.2093371

21. K. Bertin, S. Torres, L. Viitasaari, Least-square estimators in linear regression models under negatively superadditive dependent random observations, *Statistics*, **55** (2021), 1018–1034. https://doi.org/10.1080/02331888.2021.1993854

22. M. Q. Chen, K. Chen, Z. J. Wang, Z. L. Lu, X. J. Wang, Complete moment convergence for partial sums of arrays of rowwise negatively superadditive dependent random variables, *Commun. Stat. Theor. M.*, **49** (2020), 1158–1173. https://doi.org/10.1080/03610926.2018.1554136

23. Y. C. Yu, X. S. Liu, L. Liu, W. S. Liu, On adaptivity of wavelet thresholding estimators with negatively super-additive dependent noise, *Math. Slovaca*, **69** (2019), 1485–1500. https://doi.org/10.1515/ms-2017-0324

24. W. Z. Yang, L. Yang, D. Wei, S. H. Hu, The laws of large numbers for Pareto-type random variables with infinite means, *Commun. Stat. Theor. M.*, **48** (2019), 3044–3054. https://doi.org/10.1080/03610926.2018.1473602

25. A. Adler, Laws of large numbers for two tailed Pareto random variables, *Probab. Math. Stat.*, **28** (2008), 121–128.

26. A. Adler, An exact weak law of large numbers, *Bull. Inst. Math. Acad.*, **7** (2012), 417–422.

27. A. Adler, Exact weak laws and one side strong laws, *Bull. Inst. Math. Acad.*, **12** (2017), 103–124.

28. A. Adler, One sided strong laws for random variables with infnite mean, *Open Math.*, **15** (2017), 828–832. https://doi.org/10.1515/math-2017-0070

29. K. Matsumoto, T. Nakata, Limit theorems for a generalized Feller game, *J. Appl. Probab.*, **50** (2013), 54–63. https://doi.org/10.1239/jap/1363784424

30. T. Nakata, Limit theorems for nonnegative independent random variables with truncation, *Acta Math. Hung.*, **145** (2015), 1–16. https://doi.org/10.1007/s10474-014-0474-5

31. T. Nakata, Weak laws of large numbers for weighted independent random variables with infnite mean, *Statist. Probabil. Lett.*, **109** (2016), 124–129. https://doi.org/10.1016/j.spl.2015.11.017

32. Q. M. Shao, A comparison theorem on moment inequalities between negatively associated and independent random variables, *J. Theor. Probab.*, **13** (2000), 343–355. https://doi.org/10.1023/A:1007849609234