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Research article

Approximate controllability of Sobolev-type Atangana-Baleanu fractional differential inclusions with noise effect and Poisson jumps

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Abstract: In this paper, we explore the approximative controllability of fractional stochastic differential inclusions (SDIs) of Sobolev-type with fractional derivatives in Atangana-Baleanu (AB) sense and Poisson jumps. Our findings are supported by the fixed point theorem, multi-valued map theory, compact semigroup theory and stochastic analysis principles. In the later part, an illustration is provided to clarify the established outcomes.

Keywords: fractional calculus; stochastic differential inclusions; control theory; Poisson jumps
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1. Introduction

In recent years, numerous fields have recognised the effect of including random effects in modelling and analysing physical processes. The optimum control systems modelled by stochastic and partial differential equations have attracted a lot of attention (see [1–11]). Consequently, stochastic and partial differential inclusion result from these optimal control problems. Fractional-order differential equations can be used to solve some physical problems instead of integer-order differential equations. As a result, a large number of researchers have recently made significant progress in a variety of fields, including physics, fluid mechanics, control theory, image analysis, biology, engineering, porous media and others. Many authors have investigated the theoretical results based on existence and uniqueness of solutions to fractional differential equations in various forms (see [12–17]). Recently, a novel fractional derivative known as the AB fractional derivative was introduced by Atangana and Baleanu [18]). Many studies and discussion related to AB fractional derivative have appeared in several areas of applications, for example, Khan et al. [19] discussed the existence and data dependence theorems for solutions of an ABC-fractional order impulsive system. Mallika et al. [20] studied a new class of Atangana-Baleanu fractional Volterra-Fredholm integrodifferential inclusions with non-instantaneous impulses. Omaba and Enyi [21] studied the Atangana–Baleanu time-fractional stochastic integro-differential equation by using Banach fixed point theory. Panda et al. [22] discussed the results on system of Atangana-Baleanu fractional order Willis aneurysm and nonlinear singularly perturbed boundary value problems.

The notion of controllability of dynamical systems is one of the fundamental concepts in mathematical control theory which plays pivotal role in many areas of science and engineering (see [23–28]). The dynamical systems must be treated by the weaker concept of controllability, namely approximate controllability. There are many studies on the approximate controllability of stochastic and deterministic systems, for example, Liu and Li [29] studied the approximate controllability of fractional evolution systems with Riemann-Liouville fractional derivatives. Mahmudov and Mckibben [30] investigated the approximate controllability of fractional evolution equations with generalized Riemann-Liouville fractional derivative. Ahmed [31] discussed the approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space. Subramaniam [32] studied the approximate controllability of Sobolev-type nonlocal Hilfer fractional stochastic differential system. Ma et al. [33] investigated the approximate controllability of Atangana-Baleanu fractional neutral delay integrodifferential stochastic systems with nonlocal conditions.

To the best of our knowledge, no work has been reported in the literature regarding the approximate controllability of Sobolev-type Atangana-Baleanu fractional stochastic differential inclusions with fractional Brownian motion and Poisson jumps. Motivated by this, the aim of this paper is to study the approximate controllability of Sobolev-type stochastic differential inclusions with fractional Brownian motion and Poisson jumps, where the time fractional derivative is the Atangana-Baleanu fractional derivative in the Caputo sense, of the form:

$$\begin{aligned} {}^{ABC}D^{\alpha}_{0+}\Phi x(t) &\in \mathfrak{A}x(t) + \mathfrak{B}u(t) + \varrho(t, x(t)) + \sigma(t, x(t))\frac{dB^H(t)}{dt} + \int_Z \hbar(t, x(t), \xi)\tilde{N}(dt, d\xi), \quad t \in J := (0, \mathfrak{C}], \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where ${}^{ABC}D^{\alpha}_{0+}$ is AB-Caputo fractional derivative of order $\frac{1}{2} < \alpha < 1$. $x(\cdot)$ is the state variable in separable Hilbert space $\mathfrak{X}$ with $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$. Let B^H be a fBm on separable Hilbert space Y with $H \in (1/2, 1)$. Φ and $\mathfrak{A}$ are linear operators in $\mathfrak{X}$. ϱ and σ are multi-valued functions satisfying some assumptions. $\hbar : J \times \mathfrak{X} \times Z \to \mathfrak{X}$ is a nonlinear function. The control function $u(\cdot)$ is given in $\mathcal{L}_2(J, U)$, the Hilbert space of admissible control functions with U as a separable Hilbert space. The symbol $\mathfrak{B}$ stands for a bounded linear operator from U into $\mathfrak{X}$.

The contributions of the present work:

- Sobolev-type Atangana-Baleanu fractional stochastic differential inclusions with Poisson jumps are presented.
- Approximate controllability for (1.1) is investigated for the first time.
- An example is offered to define the primary results.

2. Preliminaries

The following lemmas and definitions are used in the paper.

Let $(\Omega, \mathfrak{I}, \mathbb{P})$ be a complete probability space containing the entire family of right continuous increasing sub-σ-algebras $\{\mathfrak{I}_t\}_{t\in J}$ satisfying $\mathfrak{I}_t \subset \mathfrak{I}$ and $\mathfrak{C} > 0$ be arbitrary fixed horizons. Let $(Z, \mathcal{V}, \lambda(d\xi))$ be a σ-finite measurable space. We are given a stationary Poisson point process $(\mathfrak{q}_t)_{t\geq 0}$, which is defined on $(\Omega, \mathfrak{I}, \mathbb{P})$ with values in Z and with characteristic measure λ. Let $\tilde{M}(dt, d\xi)$ be the counting measure of $\mathfrak{q}_t$ such that (s.t.) $\tilde{N}(t, \ell) = \mathbb{E}(\tilde{M}(t, \ell)) = t\lambda(\ell)$ for $\ell \in \mathcal{V}$. Define $\tilde{N}(t, d\xi) := \tilde{M}(t, d\xi) - t\lambda(d\xi)$, the Poisson martingale measure generated by $\mathfrak{q}_t$. An one-dimensional fBm with the Hurst index $H \in (1/2, 1)$ is a centred Gaussian process $\beta^H = \{\beta^H(t), 0 \leq t \leq \mathfrak{C}\}$ with covariance function

$$\mathbf{C}_H(t, s) = \mathbb{E}(\beta^H(t)\beta^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - \mid t - s \mid^{2H}).$$

Suppose $\mathcal{L}(Y, \mathfrak{X})$ be the space of bounded linear operators from Y to $\mathfrak{X}$. Then, define the infinite dimensional fBm on Y with covariance Θ as

$$B^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) e_n \sqrt{\mathfrak{Z}_n},$$

where β_n^H are real, independent fBm's. This process is a Y-valued Gaussian, which starts from zero, has zero mean and covariance

$$\mathbb{E}[\langle B^H(t), \mathfrak{d}\rangle\langle B^H(s), \mathfrak{g}\rangle] = \mathbf{C}_H(t, s)\langle\Theta(\mathfrak{d}), \mathfrak{g}\rangle, \quad \mathfrak{d}, \mathfrak{g} \in Y, \quad t, s \in [0, \mathfrak{C}].$$

We propose the separable Hilbert space $\mathfrak{L}^2_\Theta(Y, \mathfrak{X})$ of all Θ-Hilbert-Schmidt operators $\hat{\Psi} : Y \to \mathfrak{X}$ [34].

Lemma 2.1. *If* $\hat{\Psi} : [0, \mathfrak{C}] \to \mathfrak{L}^2_\Theta(Y, \mathfrak{X})$ *satisfies* $\int_0^{\mathfrak{C}} \| \hat{\Psi}(s) \|^2_{\mathfrak{L}^2_\Theta} ds < \infty$, *then*

$$\mathbb{E}\left\|\int_0^t \hat{\Psi}(s) dB^H(s)\right\|^2 \leq 2H\mathfrak{C}^{2H-1} \int_0^t \left\|\hat{\Psi}(s)\right\|^2_{\mathfrak{L}^2_\Theta} ds.$$

Here, $C(J, \mathcal{L}^2(\Omega, \mathfrak{X}))$ is the Banach space of all continuous maps from J into $\mathcal{L}^2(\Omega, \mathfrak{X})$ equipped with the supremum norm $\|x\|_C = \sup_{t\in J}\left(\mathbb{E}\|x(t)\|^2\right)^{1/2}$.

Assume, $\mathfrak{A} : \mathbf{D}(\mathfrak{A}) \subset \mathfrak{X} \to \mathfrak{X}$ and $\Phi : \mathbf{D}(\Phi) \subset \mathfrak{X} \to \mathfrak{X}$ satisfy the following hypotheses:

(1) $\mathfrak{A}$ and Φ are closed linear operators.
(2) $\mathbf{D}(\Phi) \subset \mathbf{D}(\mathfrak{A})$ and Φ is bijective.
(3) $\Phi^{-1} : \mathfrak{X} \to \mathbf{D}(\Phi)$ is continuous. Here, (1) and (2) together with the closed graph theorem imply the boundedness of $\mathfrak{A}\Phi^{-1} : \mathfrak{X} \to \mathfrak{X}$, in addition, $\mathfrak{A}\Phi^{-1}$ is the infinitesimal generator of an α-resolvent family $(S_\alpha(\tau))_{\tau\geq 0}$, $(Q_\alpha(\tau))_{\tau\geq 0}$ stands for the solution operator defined on a separable Hilbert space $\mathfrak{X}$.

Let $P(\mathfrak{X}) = \left\{\mathfrak{A} \subseteq \mathfrak{X} : \mathfrak{A} \neq \phi\right\}$ be the family of all nonempty subsets of $\mathfrak{X}$,

$$P_{cp}(\mathfrak{X}) = \left\{\mathfrak{A} \in P(\mathfrak{X}) : \mathfrak{A} \quad is \quad compact\right\}, \quad P_b(\mathfrak{X}) = \left\{\mathfrak{A} \in P(\mathfrak{X}) : \mathfrak{A} \quad is \quad bounded\right\},$$

$$P_{cl}(\mathfrak{X}) = \left\{\mathfrak{A} \in P(\mathfrak{X}) : \mathfrak{A} \quad is \quad closed\right\}, \quad P_{cv}(\mathfrak{X}) = \left\{\mathfrak{A} \in P(\mathfrak{X}) : \mathfrak{A} \quad is \quad convex\right\},$$

$P_{cp,cv}(\mathfrak{X}) = P_{cp}(\mathfrak{X}) \cap P_{cv}(\mathfrak{X})$ denotes the collection of all non-empty compact and convex subsets of $\mathfrak{X}$.

Proposition 2.1. *([35])*

(i) A multivalued map $\mathfrak{W} : \mathfrak{X} \to 2^{\mathfrak{X}}\backslash\emptyset$ is convex (closed) valued if $\mathfrak{W}(\mathfrak{g})$ is convex (closed) $\forall\ \mathfrak{g} \in \mathfrak{X}$. $\mathfrak{W}$ is bounded on bounded sets, if $\mathfrak{W}(B) = \bigcup_{\mathfrak{g}\in B} \mathfrak{W}(\mathfrak{g})$ is bounded in $\mathfrak{X}$, for any bounded set B on $\mathfrak{X}$.

(ii) A map $\mathfrak{W}$ is said to be upper semi-continuous (u.s.c.) on $\mathfrak{X}$, if for each $\mathfrak{g}_0 \in \mathfrak{X}$, the set $\mathfrak{W}(\mathfrak{g}_0)$ is a nonempty closed subset of $\mathfrak{X}$ and if for each open subset Ω of $\mathfrak{X}$ containing $\mathfrak{W}(\mathfrak{g}_0)$, there exists an open neighborhood $\hat{\Theta}$ of $\mathfrak{g}_0$ such that $\mathfrak{W}(\hat{\Theta}) \subseteq \Omega$.

(iii) A map $\mathfrak{W}$ is said to be completely continuous, if $\mathfrak{W}(B)$ is relatively compact for every $B \in P_b(\mathfrak{X})$. If the multi-valued map $\mathfrak{W}$ is completely continuous with nonempty compact values, then $\mathfrak{W}$ is u.s.c. if and only if $\mathfrak{W}$ has a closed graph, i.e., $\mathfrak{g}_n \to \mathfrak{g}, u_n \to u, u_n \in \mathfrak{W}(\mathfrak{g}_0)$ imply $u \in \mathfrak{W}(\mathfrak{g})$. We say that $\mathfrak{W}$ has a fixed point if there is $\mathfrak{g} \in \mathfrak{X}$ such that $\mathfrak{g} \in \mathfrak{W}(\mathfrak{g})$.

Lemma 2.2. *([36]) $\mathfrak{W} : I \times \mathfrak{X} \to P_{b,cl,cv}(\mathfrak{X})$ is measurable to t $\forall$ fixed $x \in \mathfrak{X}$, u.s.c. to x for each $t \in \mathfrak{J}$ and for each $x \in C(\mathfrak{J}, \mathfrak{X})$, the set $\mathbb{S}_{\mathfrak{W},x} := \{\mathfrak{f} \in L^1(\mathfrak{J}, \mathfrak{X}) : \mathfrak{f}(\mathfrak{N}) \in \mathfrak{W}(t, x(t))\}$, for a.e. $t \in \mathfrak{J}$ is nonempty. Let $\mathcal{N}$ be a linear continuous mapping from $L^1(\mathfrak{J}, \mathfrak{X})$ into $C(\mathfrak{J}, \mathfrak{X})$. Then, the operator*

$$\mathcal{N} \circ \mathbb{S}_{\mathfrak{W}} : C(\mathfrak{J}, \mathfrak{X}) \to P_{b,cl,cv}(C(\mathfrak{J}, \mathfrak{X})),$$
$$x \mapsto \mathcal{N} \circ \mathbb{S}_{\mathfrak{W}}(x) = \mathcal{N}(\mathbb{S}_{\mathfrak{W}}),$$

is a closed graph operator in $C(\mathfrak{J}, \mathfrak{X}) \times C(\mathfrak{J}, \mathfrak{X})$.

Lemma 2.3. *([35]) Let $\mathfrak{D}$ be a nonempty subset of $\mathfrak{X}$ which is bounded, closed and convex. Suppose $\mathfrak{Q} : \mathfrak{D} \to 2^{\mathfrak{X}}$ is u.s.c. with closed, convex values s.t. $\mathfrak{Q}(\mathfrak{D}) \subset \mathfrak{D}$ and $\mathfrak{Q}(\mathfrak{D})$ is compact. Then, $\mathfrak{Q}$ has a fixed point.*

Definition 2.1. *([18]) The AB fractional derivative is defied by the following in the Caputo sense: for $\mathfrak{f} \in H^1(a, b); a < b$ and at $t \in (a, b)$ of order $\alpha \in (0, 1)$, we have*

$$^{ABC}D^{\alpha}_{a+}\mathfrak{f}(t) = \frac{V(\alpha)}{1-\alpha}\int_a^t \mathfrak{f}'(s)\mathfrak{E}_{\alpha}(-\nu(t-s)^{\alpha})ds, \tag{2.1}$$

where the function $\nu = \frac{\alpha}{1-\alpha}$, $\mathfrak{E}_{\alpha}(\cdot)$ is the one parameter Mittag-Leffler function defined by

$$\mathfrak{E}_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)},$$

and the normalization function $V(\alpha) = (1-\alpha) + \frac{\alpha}{\Gamma(\alpha)}$ is any function with $V(0) = V(1) = 1$.

The fractional integral of AB is provided by

$$^{AB}I^{\alpha}_{a+}\mathfrak{f}(t) = \frac{1-\alpha}{V(\alpha)}\mathfrak{f}(t) + \frac{\alpha}{V(\alpha)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathfrak{f}(s)ds. \tag{2.2}$$

Definition 2.2. *([37]) $x \in C(J, \mathcal{L}^2(\Omega, \mathfrak{X}))$ is a mild solution of (1.1), if it satisfies the following conditions:*

(1) $x(0)=x_0\in\mathcal{L}^2(\Omega,\mathfrak{X}))$ *and* $u(\cdot)\in\mathcal{L}_2(J,U)$,

(2) $\exists\,\mathfrak{f}\in\mathbb{S}_{\varrho,x}$ *s.t.* $\mathfrak{f}(t)\in\varrho(t,x(t))$, $\overline{h}\in\mathbb{S}_{\sigma,x}$ *s.t.* $\overline{h}(t)\in\sigma(t,x(t))$, $\overline{y}\in\mathbb{S}_{\hbar,x}$ *s.t.* $\overline{y}(t,\xi)\in\hbar(t,x(t),\xi)$ *and* $x(t)$ *verifies the following equation:*

$$\begin{aligned}x(t)&=\Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\Big\{\mathfrak{B}u(s)+\mathfrak{f}(s)\Big\}ds\\&+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\overline{h}(s)dB^H(s)+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\\&+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t\Phi^{-1}Q_\alpha(t-s)\Big\{\mathfrak{B}u(s)+\mathfrak{f}(s)\Big\}ds+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t\Phi^{-1}Q_\alpha(t-s)\overline{h}(s)dB^H(s)\\&+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t\Phi^{-1}Q_\alpha(t-s)\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi),\end{aligned}$$

where $\mathfrak{R}=\vartheta^*(\vartheta^*I-\mathfrak{A})^{-1}$ *and* $\wp=-\delta^*\mathfrak{A}(\vartheta^*I-\mathfrak{A})^{-1}$, *with* $\vartheta^*=\frac{V(\alpha)}{1-\alpha}$, $\quad\delta^*=\frac{\alpha}{1-\alpha}$,

$$\begin{aligned}S_\alpha(t)&=\mathfrak{E}_\alpha(-\wp t^\alpha)=\frac{1}{2\pi i}\int_\Upsilon e^{st}s^{\alpha-1}(s^\alpha I-\wp)^{-1}ds,\\Q_\alpha(t)&=t^{\alpha-1}\mathfrak{E}_{\alpha,\alpha}(-\wp t^\alpha)=\frac{1}{2\pi i}\int_\Upsilon e^{st}(s^\alpha I-\wp)^{-1}ds,\end{aligned}$$

and the path Υ *is lying on* $\Xi_{(\chi,\mathfrak{M})}$.

We need the following assumption:

$(A0)$ $\mathfrak{A}\in\aleph^\alpha(\alpha_0,l_0)$ then $\|S_\alpha(t)\|\le\mathfrak{J}e^{lt}$ and $\|Q_\alpha(t)\|\le\mathfrak{R}e^{lt}(1+t^{\alpha-1})$, for every $t>0, l>l_0$. Thus, $\mathfrak{J}^*=\sup_{t\ge0}\|S_\alpha(t)\|$, $\mathfrak{J}_1^*=\sup_{t\ge0}\mathfrak{R}e^{lt}(1+t^{\alpha-1})$. So, we get $\|S_\alpha(t)\|\le\mathfrak{J}^*$ and $\|Q_\alpha(t)\|\le\mathfrak{J}_1^*t^{\alpha-1}$.

To study the approximate controllability for (1.1), we first consider the fractional stochastic linear system:

$$\begin{aligned}{}^{ABC}D_{0+}^{\alpha}\Phi x(t)&\in\mathfrak{A}x(t)+\mathfrak{B}u(t),\quad t\in J:=(0,\mathfrak{C}]\\x(0)&=x_0.\end{aligned}\tag{2.3}$$

Let $x(\mathfrak{C};\gamma,u)$ be the state value of (1.1) at the terminal state b, corresponding to the control u and the initial value γ. Denote by $\mathrm{R}(\mathfrak{C},\gamma)=\Big\{x(\mathfrak{C};\gamma,u):u\in\mathcal{L}_2(J,U)\Big\}$ the reachable set of (1.1) at terminal time $\mathfrak{C}$, its closure in $\mathfrak{X}$ is denoted by $\overline{\mathrm{R}(\mathfrak{C},\gamma)}$.

Definition 2.3. *([37]) (1.1) be approximately controllable on the interval* $[0,\mathfrak{C}]$ *if* $\overline{\mathrm{R}(\mathfrak{C},\gamma)}=\mathcal{L}^2(\Omega,\mathfrak{X})$.

Remark 2.1. *(2.3) is approximately controllable on J, if and only if* $\kappa\aleph(\kappa,\Delta_0^{\mathfrak{C}})\to0$ *strongly as* $\kappa\to0^+$.

It is appropriate to introduce two pertinent operators now,

$$\Delta_0^{\mathfrak{C}}=\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{B}\mathfrak{B}^*\Phi^{-1}Q_\alpha^*(\mathfrak{C}-s)ds,$$

where $\mathfrak{B}^*$ and Q_α^* are the adjoint of $\mathfrak{B}$ and Q_α, respectively.

$$\aleph(\kappa,\Delta_0^{\mathfrak{C}})=(\kappa I+\Delta_0^{\mathfrak{C}})^{-1},\quad\kappa>0.$$

Lemma 2.4. *([37]) For any* $\tilde{x}_{\mathfrak{C}} \in \mathcal{L}^2(\Omega, \mathfrak{X})$, $\exists\ \tilde{\gamma}(s) \in \mathcal{L}^2(\Omega; \mathcal{L}^2(J, \mathfrak{L}^2_\Theta))$, *s.t.*

$$\tilde{x}_{\mathfrak{C}} = \mathbb{E}\tilde{x}_{\mathfrak{C}} + \int_0^{\mathfrak{C}} \tilde{\gamma}(s) dB^H(s).$$

3. Main result

Let us begin with some notations.

$$\|\mathfrak{B}\|^2 = \varsigma_{\mathfrak{B}}, \quad \mathfrak{C}^* = \left(\frac{1-q}{\alpha - q}\right)^{2(1-q)}, \quad \iota = \mathfrak{C}^* \mathfrak{C}^{2(\alpha - q)}.$$

To illustrate the main result, we introduce the following assumptions:

$(A1)$ $(Q_\alpha(t))_{t\geq 0}$ be compact and $\left\|\kappa \aleph(\kappa, \Delta_0^{\mathfrak{C}})\right\| \leq 1, \quad \forall \kappa > 0.$

$(A2)$ $\varrho : J \times \mathfrak{X} \to P_{b,cl,cv}(\mathfrak{X})$ satisfies:

(1) $\varrho(t, \cdot) : \mathfrak{X} \to \mathfrak{X}$ is u.s.c. $\forall\ t \in J$ and for each $x \in \mathfrak{X}$, the function $\varrho(\cdot, x) : J \to \mathfrak{X}$ is strongly measurable to t, and for each $x \in \mathfrak{X}$, the set $\mathbb{S}_{\varrho,x} := \left\{ \mathfrak{f} \in L^1(J, \mathfrak{X}) : \mathfrak{f}(t) \in \varrho(t, x(t)) \right\}$, for a.e. $t \in J$ is nonempty.

(2) $\exists$ a function $n(t) \in L^{1/q}, \quad q \in (0, \alpha)$ and a continuous nondecreasing function $\Psi : [0, \infty) \to (0, \infty)$, s.t. for any $(t, x) \in J \times \mathfrak{X}$, we have

$$\mathbb{E}\, \|\varrho(t, x)\|^2 = \sup\{\|\mathfrak{f}(t)\|^2 : \mathfrak{f}(t) \in \varrho(t, x)\} \leq n(t)\Psi(\|x\|^2), \quad \liminf_{r\to\infty} \frac{\Psi(r)}{r} = \Pi < \infty.$$

$(A3)$ $\sigma : J \times \mathfrak{X} \to \mathfrak{L}^2_\Theta(Y, \mathfrak{X})$ satisfies:

(1) $\sigma(\cdot, x)$ is measurable $\forall\ x \in \mathfrak{X}$, and $\sigma(t, \cdot) : \mathfrak{X} \to \mathfrak{L}^2_\Theta(Y, \mathfrak{X})$ is u.s.c. $\forall\ t \in J$ and $\forall\ x \in \mathfrak{X}$, the set $\mathbb{S}_{\sigma,x} := \left\{ \overline{h} \in \mathfrak{L}^2_\Theta(Y, \mathfrak{X}) : \overline{h}(t) \in \sigma(t, x) \right\}$, for a.e. $t \in J$ is nonempty.

(2) $g_r(t) : J \to \mathbb{R}^+, r \in \mathbb{N}, r > 0$ s.t.

$$\sup\left\{ \mathbb{E}\left\|\overline{h}\right\|^2 : \overline{h} \in \sigma(t, x) \right\} \leq g_r(t),$$

$\forall\ t \in J$ and $s \mapsto (t-s)^{2(\alpha-1)} g_r(t) \in L^1([0, t], \mathbb{R}^+)$ and $\exists\ \Lambda > 0$ s.t.

$$\liminf_{r\to 0} \frac{\int_0^t (t-s)^{2(\alpha-1)} g_r(s) ds}{r} = \Lambda < \infty.$$

$(A4)$ $\hbar : J \times \mathfrak{X} \times Z \to \mathfrak{X}$ satisfies:

(1) $\hbar(\cdot, x, \xi)$ is measurable $\forall\ (x, \xi) \in \mathfrak{X} \times Z$, and $\hbar(t, \cdot, \cdot) : \mathfrak{X} \times Z \to \mathfrak{X}$ is u.s.c. $\forall\ t \in J$.
For each $(x, \xi) \in \mathfrak{X} \times Z$, the set $\mathbb{S}_{\hbar,x} := \left\{ \overline{y} \in \mathfrak{L}^2(Y, \mathfrak{X}) : \overline{y}(t, \xi) \in \hbar(t, x, \xi) \right\}$, for a.e. $t \in J$ is nonempty.

(2) $C_r(t) : J \to \mathbb{R}^+$, $r \in \mathbb{N}, r > 0$ s.t.

$$\sup\left\{\int_Z \mathbb{E}\,\|\bar{y}\|^2\,\lambda d\xi : \bar{y}(t,\xi) \in \hbar(t,x,\xi)\right\} \leq C_r(t),$$

for a.e. $t \in J$ and $s \mapsto (t-s)^{2(\alpha-1)}C_r(t) \in L^1([0,t],\mathbb{R}^+)$ and $\exists\, \Lambda > 0$ s.t.

$$\liminf_{r\to 0} \frac{\int_0^t (t-s)^{2(\alpha-1)}C_r(s)ds}{r} = \Lambda < \infty.$$

$(A5)$ $\wp$ and $\mathfrak{R}$ are bounded linear operators, $\exists\, \theta$ and ψ s.t. $\|\mathfrak{R}\| \leq \theta$ and $\|\wp\| \leq \psi$.

$(A6)$

$$\left[\Lambda + \frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\,\Pi\iota + 2H\mathfrak{C}^{2H-1}\Lambda\right]\left\{9\left[\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right]^2 + 9\left[\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right]^2\right\}\tilde{\mathcal{K}} < 1,$$

where

$$\tilde{\mathcal{K}} = \left\{1 + 8\left[\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)\kappa}\right]^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2} + 8\left[\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\kappa}\right]^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\right\}.$$

$(A7)$ ϱ, σ and $\hbar$ are uniformly bounded $\forall\, t \in J$ and $x \in C$.

Theorem 3.1. *Assume that* $(A0)$ *through* $(A6)$ *are satisfied. Then, (1.1) has a mild solution on* $C(J, \mathcal{L}^2(\Omega, \mathfrak{X}))$.

Proof. Let $Q_r := \left\{x \in C(J, \mathcal{L}^2(\Omega, \mathfrak{X})) : \|x\|_C \leq r, r \geq 0, 0 \leq t \leq \mathfrak{C}\right\}$. Obviously, Q_r is a bounded, closed, convex set in $C(J, \mathcal{L}^2(\Omega, \mathfrak{X}))$.

For $\kappa > 0$, for all $x(\cdot) \in C(J, \mathcal{L}^2(\Omega, \mathfrak{X}))$, we take,

$$u(t) = \mathfrak{B}^* Q_\alpha^*(\mathfrak{C} - t)\aleph(\kappa, \Delta_0^{\mathfrak{C}})\mathfrak{P}(x(\cdot)),$$

where

$$\begin{aligned}
\mathfrak{P}(x(\cdot)) \;=\;& \mathbb{E}\tilde{x}_{\mathfrak{C}} + \int_0^{\mathfrak{C}} \tilde{\gamma}(s)dB^H(s) - \Phi^{-1}\mathfrak{R}S_\alpha(\mathfrak{C})\Phi x_0 - \frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}} \Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\mathfrak{f}(s)ds \\
& - \frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}} \Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\overline{h}(s)dB^H(s) - \frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}} \Phi^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{f}(s)ds \\
& - \frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}} \Phi^{-1}Q_\alpha(\mathfrak{C}-s)\overline{h}(s)dB^H(s) \\
& - \frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}} \Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z \bar{y}(s,\xi)\tilde{N}(ds,d\xi) \\
& - \frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}} \Phi^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z \bar{y}(s,\xi)\tilde{N}(ds,d\xi).
\end{aligned}$$

The operator $\mathfrak{T} : C \to P(C)$ is defined in terms of this control as follows:

$$\mathfrak{T}(x) \;=\; \left\{\mathfrak{U} \in C : \mathfrak{U}(t) = \Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0 + \frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\left\{\mathfrak{B}u(s) + \mathfrak{f}(s)\right\}ds\right.$$

$$+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\overline{h}(s)dB^H(s)+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}$$
$$\times\int_Z \overline{y}(s,\xi)\tilde{N}(ds,d\xi)+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\Big\{\mathfrak{B}u(s)+\mathfrak{f}(s)\Big\}ds$$
$$+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\overline{h}(s)dB^H(s)+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\int_Z \overline{y}(s,\xi)\tilde{N}(ds,d\xi)\Big\}.$$

We shall prove, $\mathfrak{T}: C\to P(C)$ has a fixed point for $\kappa>0$.
The proof is now divided into five steps.

Step 1. $\forall\ x\in Q_r$ the operator $\mathfrak{T}$ is convex.
Assume that $\mathfrak{U}_1,\mathfrak{U}_2\in\mathfrak{T}(x)$, then $\exists\ \mathfrak{f}_1,\mathfrak{f}_2\in\mathbb{S}_{\varrho,x}$, $\overline{h}_1,\overline{h}_2\in\mathbb{S}_{\sigma,x}$, and $\overline{y}_1,\overline{y}_2\in\mathbb{S}_{\hbar,x}$ s.t.

$$\mathfrak{U}_i(t)=\Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\mathfrak{f}_i(s)ds+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\mathfrak{B}\mathfrak{B}^*Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa,\Delta_0^{\mathfrak{C}})$$
$$\times\Big\{\mathbb{E}\tilde{x}_{\mathfrak{C}}+\int_0^{\mathfrak{C}}\tilde{\gamma}(s)dB^H(s)-\Phi^{-1}\mathfrak{R}S_\alpha(\mathfrak{C})\Phi x_0-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\mathfrak{f}_i(s)ds$$
$$-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\overline{h}_i(s)dB^H(s)-\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{f}_i(s)ds$$
$$-\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\overline{h}_i(s)dB^H(s)-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}$$
$$\times\int_Z\overline{y}_i(s,\xi)\tilde{N}(ds,d\xi)-\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\overline{y}_i(s,\xi)\tilde{N}(ds,d\xi)\Big\}ds$$
$$+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\overline{h}_i(s)dB^H(s)+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\int_Z\overline{y}_i(s,\xi)\tilde{N}(ds,d\xi)$$
$$+\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t\Phi^{-1}Q_\alpha(t-s)\mathfrak{f}_i(s)ds+\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t\Phi^{-1}Q_\alpha(t-s)\overline{h}_i(s)dB^H(s)$$
$$+\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t\Phi^{-1}Q_\alpha(t-s)\int_Z\overline{y}_i(s,\xi)\tilde{N}(ds,d\xi)+\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t\Phi^{-1}Q_\alpha(t-s)\mathfrak{B}\mathfrak{B}^*Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa,\Delta_0^{\mathfrak{C}})$$
$$\times\Big\{\mathbb{E}\tilde{x}_{\mathfrak{C}}+\int_0^{\mathfrak{C}}\tilde{\gamma}(s)dB^H(s)-\Phi^{-1}\mathfrak{R}S_\alpha(\mathfrak{C})\Phi x_0-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\mathfrak{f}_i(s)ds$$
$$-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\overline{h}_i(s)dB^H(s)-\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{f}_i(s)ds$$
$$-\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\overline{h}_i(s)dB^H(s)-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z\overline{y}_i(s,\xi)\tilde{N}(ds,d\xi)$$
$$-\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\overline{y}_i(s,\xi)\tilde{N}(ds,d\xi)\Big\}ds,\qquad i=1,2.$$

Let $\mathfrak{K}\in[0,1]$, then we get

$$\mathfrak{K}\mathfrak{U}_1+(1-\mathfrak{K})\mathfrak{U}_2=\Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\Big[\mathfrak{K}\mathfrak{f}_1(s)+(1-\mathfrak{K})\mathfrak{f}_2(s)\Big]ds$$
$$+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t\Phi^{-1}(t-s)^{\alpha-1}\mathfrak{B}\mathfrak{B}^*Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa,\Delta_0^{\mathfrak{C}})\times\Big\{\mathbb{E}\tilde{x}_{\mathfrak{C}}+\int_0^{\mathfrak{C}}\tilde{\gamma}(s)dB^H(s)$$
$$-\Phi^{-1}\mathfrak{R}S_\alpha(\mathfrak{C})\Phi x_0-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\Big[\mathfrak{K}\mathfrak{f}_1(s)+(1-\mathfrak{K})\mathfrak{f}_2(s)\Big]ds$$
$$-\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\Big[\mathfrak{K}\mathfrak{f}_1(s)+(1-\mathfrak{K})\mathfrak{f}_2(s)\Big]ds$$

$$
\begin{aligned}
&-\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\Big[\Re\overline{h}_1(s)+(1-\Re)\overline{h}_2(s)\Big]dB^H(s)\\
&-\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\Big[\Re\overline{h}_1(s)+(1-\Re)\overline{h}_2(s)\Big]dB^H(s)\\
&-\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z\Big[\Re\overline{y}_1(s,\xi)+(1-\Re)\overline{y}_2(s,\xi)\Big]\tilde{N}(ds,d\xi)\\
&-\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\Big[\Re\overline{y}_1(s,\xi)+(1-\Re)\overline{y}_2(s,\xi)\Big]\tilde{N}(ds,d\xi)\Big\}ds\\
&+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t}\Phi^{-1}(t-s)^{\alpha-1}\Big[\Re\overline{h}_1(s)+(1-\Re)\overline{h}_2(s)\Big]dB^H(s)\\
&+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t}\Phi^{-1}Q_\alpha(t-s)\Big[\Re\mathfrak{f}_1(s)+(1-\Re)\mathfrak{f}_2(s)\Big]ds\\
&+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t}\Phi^{-1}(t-s)^{\alpha-1}\int_Z\Big[\Re\overline{y}_1(s,\xi)+(1-\Re)\overline{y}_2(s,\xi)\Big]\tilde{N}(ds,d\xi)\\
&+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t}\Phi^{-1}Q_\alpha(t-s)\Big[\Re\overline{h}_1(s)+(1-\Re)\overline{h}_2(s)\Big]dB^H(s)\\
&+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t}\Phi^{-1}Q_\alpha(t-s)\int_Z\Big[\Re\overline{y}_1(s,\xi)+(1-\Re)\overline{y}_2(s,\xi)\Big]\tilde{N}(ds,d\xi)\\
&+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{B}\mathfrak{B}^*Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa,\Delta_0^{\mathfrak{C}})\times\Big\{\mathbb{E}\tilde{x}_{\mathfrak{C}}+\int_0^{\mathfrak{C}}\tilde{\gamma}(s)dB^H(s)\\
&-\Phi^{-1}\mathfrak{R}S_\alpha(\mathfrak{C})\Phi x_0-\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\Big[\Re\mathfrak{f}_1(s)+(1-\Re)\mathfrak{f}_2(s)\Big]ds\\
&-\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\Big[\Re\overline{h}_1(s)+(1-\Re)\overline{h}_2(s)\Big]dB^H(s)\\
&-\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\Big[\Re\mathfrak{f}_1(s)+(1-\Re)\mathfrak{f}_2(s)\Big]ds\\
&-\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\Big[\Re\overline{h}_1(s)+(1-\Re)\overline{h}_2(s)\Big]dB^H(s)\\
&-\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z\Big[\Re\overline{y}_1(s,\xi)+(1-\Re)\overline{y}_2(s,\xi)\Big]\tilde{N}(ds,d\xi)\\
&-\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\Big[\Re\overline{y}_1(s,\xi)+(1-\Re)\overline{y}_2(s,\xi)\Big]\tilde{N}(ds,d\xi)\Big\}ds.
\end{aligned}
$$

Since $\mathbb{S}_{\varrho,x}$, $\mathbb{S}_{\sigma,x}$, and $\mathbb{S}_{\hbar,x}$ are convex sets, $\Re\mathfrak{f}_1(s)+(1-\Re)\mathfrak{f}_2(s)\in\mathbb{S}_{\varrho,x}$, $\Re\overline{h}_1(s)+(1-\Re)\overline{h}_2(s)\in\mathbb{S}_{\sigma,x}$, and $\Re\overline{y}_1(s,\xi)+(1-\Re)\overline{y}_2(s,\xi)\in\mathbb{S}_{\hbar,x}$. Thus,

$$\Re\mathfrak{U}_1+(1-\Re)\mathfrak{U}_2\in\mathfrak{T}(x).$$

Step 2. $\forall\ \kappa>0$, $\exists$ a positive constant $r_0=r(\kappa)$, s.t. $\mathfrak{T}(Q_{r_0})\subset Q_{r_0}$.

If the opposite is true, then for any $r > 0\ \exists\ \overline{x} \in Q_r, \overline{u} \in \mathcal{L}_2(J, U)$ corresponding to $\overline{x}$, s.t. $\mathfrak{T}(\overline{x}) \nsubseteq Q_r$,

$$\mathbb{E}\left\|\mathfrak{T}(\overline{x})\right\|_C^2 = \sup\{\|\mathfrak{U}\|_C^2 : \mathfrak{U} \in \mathfrak{T}(\overline{x})\} \geq r.$$

$$\begin{aligned}
r \leq \mathbb{E}\left\|\mathfrak{T}(\overline{x})\right\|^2 &\leq 9\mathbb{E}\left\|\Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0\right\|^2 + 9\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\mathfrak{B}u(s)ds\right\|^2\\
&+9\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\mathfrak{f}(s)ds\right\|^2 + 9\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\overline{h}(s)dB^H(s)\right\|^2\\
&+9\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\int_Z \overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2 + 9\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\mathfrak{B}u(s)ds\right\|^2\\
&+9\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\mathfrak{f}(s)ds\right\|^2 \quad + 9\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\overline{h}(s)dB^H(s)\right\|^2\\
&+9\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\int_Z \overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2 = \sum_{n=1}^9 I_n.
\end{aligned}$$

By Using *Hölder's* inequality and assumptions $(A1)$–$(A5)$, for some $\mathfrak{f} \in \mathbb{S}_{\varrho,x}$, $\overline{h} \in \mathbb{S}_{\sigma,x}$ and $\overline{y} \in \mathbb{S}_{\hbar,x}$, we have

$$I_1 = 9\mathbb{E}\left\|\Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0\right\|^2 \leq 9\left\|\Phi\right\|^2\left\|\Phi^{-1}\right\|^2(\mathfrak{I}^*\theta)^2\mathbb{E}\left\|x_0\right\|^2,$$

$$\begin{aligned}
I_2 &= 9\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\mathfrak{B}u(s)ds\right\|^2\\
&\leq 9\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)\kappa}\right\}^2 \tfrac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\times 8\Bigg[\left\{\mathbb{E}\left\|\tilde{x}_{\mathfrak{C}}\right\|^2 + 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}\mathbb{E}\left\|\tilde{\gamma}(s)\right\|^2_{\mathfrak{L}^2_\Theta}ds\right\}\\
&+\left\|\Phi^{-1}\right\|^2\left\|\Phi\right\|^2(\theta\mathfrak{I}^*)^2\mathbb{E}\left\|x_0\right\|^2 + \left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\left\|n\right\|_{L^{1/q}}\iota\\
&+\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}g_r(s)ds\\
&+\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\left\|n\right\|_{L^{1/q}}\iota + \left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}g_r(s)ds\\
&+\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}C_r(s)ds + \left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}C_r(s)ds\Bigg],
\end{aligned}$$

$$I_3 = 9\mathbb{E}\left\|\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\mathfrak{f}(s)ds\right\|^2 \leq 9\left\{\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\left\|n\right\|_{L^{1/q}}\iota,$$

$$I_4 = 9\mathbb{E}\left\|\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\overline{h}(s)dB^H(s)\right\|^2 \leq 9\left\{\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^t (t-s)^{2(\alpha-1)}g_r(s)ds,$$

$$\begin{aligned}
I_5 &= 9\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\int_Z \overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2\\
&\leq 9\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\int_0^t (t-s)^{2(\alpha-1)}\times\int_Z\mathbb{E}\left\|\overline{y}\right\|^2\lambda(d\xi)ds \leq 9\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\int_0^t (t-s)^{2(\alpha-1)}C_r(s)ds,
\end{aligned}$$

$$I_6 = 9\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\mathfrak{B}u(s)ds\right\|^2$$

$$\leq 9\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\kappa}\right\}^2 \tfrac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\times 8\Bigg[\left\{\mathbb{E}\left\|\tilde{x}_{\mathfrak{C}}\right\|^2 + 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}\mathbb{E}\left\|\tilde{\gamma}(s)\right\|^2_{\mathfrak{L}^2_\Theta}ds\right\} + \left\|\Phi^{-1}\right\|^2\left\|\Phi\right\|^2(\theta\mathfrak{I}^*)^2\mathbb{E}\left\|x_0\right\|^2$$

$$+\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\|n\|_{L^{1/q}}\iota + \left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}g_r(s)ds$$

$$+\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\|n\|_{L^{1/q}}\iota + \left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}g_r(s)ds$$

$$+\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}C_r(s)ds + \left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}C_r(s)ds\Bigg],$$

$$I_7 = 9\mathbb{E}\left\|\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\mathfrak{f}(s)ds\right\|^2 \leq 9\left\{\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\|n\|_{L^{1/q}}\iota,$$

$$I_8 = 9\mathbb{E}\left\|\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\overline{h}(s)dB^H(s)\right\|^2 \leq 9\left\{\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^t(t-s)^{2(\alpha-1)}g_r(s)ds,$$

$$I_9 = 9\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\int_Z \overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2$$

$$\leq 9\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\int_0^t(t-s)^{2(\alpha-1)}\int_Z\mathbb{E}\|\overline{y}\|^2\lambda(d\xi)ds \leq 9\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\int_0^t(t-s)^{2(\alpha-1)}C_r(s)ds.$$

Combining these estimates, I_1–I_9 yields

$$r \leq O + 9\Bigg[\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\|n\|_{L^{1/q}}\iota + \left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\|n\|_{L^{1/q}}\iota$$

$$+\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^t(t-s)^{2(\alpha-1)}g_r(s)ds + \left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^t(t-s)^{2(\alpha-1)}g_r(s)ds$$

$$+\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\int_0^t(t-s)^{2(\alpha-1)}C_r(s)ds + \left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\int_0^t(t-s)^{2(\alpha-1)}C_r(s)ds\Bigg]$$

$$+72\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)\kappa}\right\}^2\tfrac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\Bigg[\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\|n\|_{L^{1/q}}\iota$$

$$+\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\|n\|_{L^{1/q}}\iota + \left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}g_r(s)ds$$

$$+\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}g_r(s)ds + \left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}C_r(s)ds$$

$$+\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}C_r(s)ds\Bigg]$$

$$+72\left\{\tfrac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\kappa}\right\}^2\tfrac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\Bigg[\left\{\tfrac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\tfrac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\|n\|_{L^{1/q}}\iota$$

$$+\left\{\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{J}_1^*}{V(\alpha)}\right\}^2 \frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\Psi(\|x\|^2)\,\|n\|_{L^{1/q}}\,\iota+\left\{\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}g_r(s)ds$$

$$+\left\{\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{J}_1^*}{V(\alpha)}\right\}^2 2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}g_r(s)ds+\left\{\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right\}^2\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}C_r(s)ds$$

$$+\left\{\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{J}_1^*}{V(\alpha)}\right\}^2\int_0^{\mathfrak{C}}(\mathfrak{C}-s)^{2(\alpha-1)}C_r(s)ds\Bigg], \tag{3.1}$$

where

$$\begin{aligned}
O &= 9\|\Phi\|^2\left\|\Phi^{-1}\right\|^2(\mathfrak{J}^*\theta)^2\mathbb{E}\|x_0\|^2\Bigg[1+8\Big\{\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|\mathfrak{J}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)\kappa}\Big\}^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2} \\
&\quad +8\Big\{\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{J}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\kappa}\Big\}^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\Bigg] \\
&\quad +72\Bigg[\Big\{\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|\mathfrak{J}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)\kappa}\Big\}^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\Big\{\mathbb{E}\|\tilde{x}_{\mathfrak{C}}\|^2+2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}\mathbb{E}\|\tilde{\gamma}(s)\|^2_{\mathfrak{L}^2_\Theta}\,ds\Big\} \\
&\quad +\Big\{\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{J}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\kappa}\Big\}^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\Big\{\mathbb{E}\|\tilde{x}_{\mathfrak{C}}\|^2+2H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}\mathbb{E}\|\tilde{\gamma}(s)\|^2_{\mathfrak{L}^2_\Theta}\,ds\Big\}\Bigg].
\end{aligned}$$

Dividing both sides of (3.1) by r and taking the lower limit $r\to+\infty$, we get

$$\begin{aligned}
1 \le 9\Bigg\{&\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2\Big[\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\,\Pi\iota+2H\mathfrak{C}^{2H-1}\Lambda+\Lambda\Big] \\
&+\left[\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{J}_1^*}{V(\alpha)}\right]^2\Big[\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\,\Pi\iota+2H\mathfrak{C}^{2H-1}\Lambda+\Lambda\Big]\Bigg\} \\
+72&\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|\mathfrak{J}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)\kappa}\right]^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\Bigg\{\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2\Big[\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\,\Pi\iota+2H\mathfrak{C}^{2H-1}\Lambda+\Lambda\Big] \\
&+\left[\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{J}_1^*}{V(\alpha)}\right]^2\Big[\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\,\Pi\iota+2H\mathfrak{C}^{2H-1}\Lambda+\Lambda\Big]\Bigg\} \\
+72&\left[\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{J}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\kappa}\right]^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\Bigg\{\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2\Big[\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\,\Pi\iota+2H\mathfrak{C}^{2H-1}\Lambda+\Lambda\Big] \\
&+\left[\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{J}_1^*}{V(\alpha)}\right]^2\Big[\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\,\Pi\iota+2H\mathfrak{C}^{2H-1}\Lambda+\Lambda\Big]\Bigg\},
\end{aligned}$$

We can then obtain this by performing some simplifications,

$$1\le\left[\Lambda+\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\,\Pi\iota+2H\mathfrak{C}^{2H-1}\Lambda\right]\left\{9\left[\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right]^2+9\left[\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{J}_1^*}{V(\alpha)}\right]^2\right\}\tilde{\mathcal{K}},$$

where

$$\tilde{\mathcal{K}}=\left\{1+8\left[\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|\mathfrak{J}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)\kappa}\right]^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}+8\left[\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{J}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\kappa}\right]^2\frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\right\},$$

which is a contradiction to $(A6)$. Thus, for every $\kappa>0$, there exists r_0, s.t. $\mathfrak{T}$ maps Q_{r_0} into itself.

Step 3. $\mathfrak{T}(Q_r)$ is equicontinuous.

Let $0 < t_1 < t_2 \leq \mathfrak{C}$. For each $x \in \overline{Q_r}$, there exist $\mathfrak{f} \in \mathbb{S}_{\varrho,x}$, $\overline{h} \in \mathbb{S}_{\sigma,x}$ and $\overline{y} \in \mathbb{S}_{\hbar,x}$, s.t.

$$
\begin{aligned}
&\mathbb{E}\|\mathfrak{U}(t_2)-\mathfrak{U}(t_1)\|^2 \leq 17\mathbb{E}\left\|\Phi^{-1}\mathfrak{R}(S_\alpha(t_2)-S_\alpha(t_1))\Phi x_0\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t_1}\Phi^{-1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\mathfrak{B}u(s)ds\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_{t_1}^{t_2}\Phi^{-1}(t_2-s)^{\alpha-1}\mathfrak{B}u(s)ds\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t_1}\Phi^{-1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\mathfrak{f}(s)ds\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_{t_1}^{t_2}\Phi^{-1}(t_2-s)^{\alpha-1}\mathfrak{f}(s)ds\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t_1}\Phi^{-1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\overline{h}(s)dB^H(s)\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_{t_1}^{t_2}\Phi^{-1}(t_2-s)^{\alpha-1}\overline{h}(s)dB^H(s)\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t_1}\Phi^{-1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_{t_1}^{t_2}\Phi^{-1}(t_2-s)^{\alpha-1}\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t_1}\Phi^{-1}\left[Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\right]\mathfrak{B}u(s)ds\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_{t_1}^{t_2}\Phi^{-1}Q_\alpha(t_2-s)\mathfrak{B}u(s)ds\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t_1}\Phi^{-1}\left[Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\right]\mathfrak{f}(s)ds\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_{t_1}^{t_2}\Phi^{-1}Q_\alpha(t_2-s)\mathfrak{f}(s)ds\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t_1}\Phi^{-1}\left[Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\right]\overline{h}(s)dB^H(s)\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_{t_1}^{t_2}\Phi^{-1}Q_\alpha(t_2-s)\overline{h}(s)dB^H(s)\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t_1}\Phi^{-1}\left[Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\right]\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2\\
&+17\mathbb{E}\left\|\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_{t_1}^{t_2}\Phi^{-1}Q_\alpha(t_2-s)\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2.
\end{aligned}
$$

Applying the *Hölder* inequality and conditions $(A0)$–$(A5)$, we get

$$
\begin{aligned}
\mathbb{E}\|\mathfrak{U}(t_2)-\mathfrak{U}(t_1)\|^2 &\leq 17\left[\tfrac{\theta\psi(1-\alpha)\|\Phi^{-1}\|\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)}\right]^2\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\times\\
&\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\mathbb{E}\|u(s)\|^2ds+17\theta^2\left\|\Phi^{-1}\right\|^2\|\Phi\|^2\mathbb{E}\|(S_\alpha(t_2)-S_\alpha(t_1))x_0\|^2
\end{aligned}
$$

$$+17\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)}\right]^2\left\{\frac{(t_2-t_1)^\alpha}{\alpha}\right\}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\|u(s)\|^2\,ds$$

$$+17\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\times\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\mathbb{E}\|\mathfrak{f}(s)\|^2\,ds$$

$$+17\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2\left\{\frac{(t_2-t_1)^\alpha}{\alpha}\right\}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\|\mathfrak{f}(s)\|^2\,ds$$

$$+17\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2 2H\mathfrak{C}^{2H-1}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]^2 g_r(s)ds$$

$$+17\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2 2H\mathfrak{C}^{2H-1}\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2}g_r(s)ds$$

$$+17\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]^2 C_r(s)ds$$

$$+17\left[\frac{\theta\psi(1-\alpha)\|\Phi^{-1}\|}{V(\alpha)\Gamma(\alpha)}\right]^2\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2}C_r(s)ds$$

$$+17\left[\frac{\theta^2\alpha\|\Phi^{-1}\|\varsigma_{\mathfrak{B}}}{V(\alpha)}\right]^2\int_0^{t_1}\|Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\|^2\,\mathbb{E}\|u(s)\|^2\,ds$$

$$+17\left[\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)}\right]^2\left\{\frac{(t_2-t_1)^\alpha}{\alpha}\right\}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\|u(s)\|^2\,ds$$

$$+17\left[\frac{\theta^2\alpha\|\Phi^{-1}\|}{V(\alpha)}\right]^2\int_0^{t_1}\|Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\|^2\,\mathbb{E}\|\mathfrak{f}(s)\|^2\,ds$$

$$+17\left[\frac{\theta^2\alpha\|\Phi^{-1}\|\mathfrak{I}_1^*}{V(\alpha)}\right]^2\left\{\frac{(t_2-t_1)^\alpha}{\alpha}\right\}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\|\mathfrak{f}(s)\|^2\,ds$$

$$+17\left[\frac{\theta^2\alpha\|\Phi^{-1}\|}{V(\alpha)}\right]^2 2H\mathfrak{C}^{2H-1}\int_0^{t_1}\|Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\|^2\,g_r(s)ds$$

$$+17\left[\frac{\theta^2\alpha\|\Phi^{-1}\|}{V(\alpha)}\right]^2 2H\mathfrak{C}^{2H-1}\int_{t_1}^{t_2}\|Q_\alpha(t_2-s)\|^2\,g_r(s)ds$$

$$+17\left[\frac{\theta^2\alpha\|\Phi^{-1}\|}{V(\alpha)}\right]^2\int_0^{t_1}\|Q_\alpha(t_2-s)-Q_\alpha(t_1-s)\|^2\,C_r(s)ds+17\left[\frac{\theta^2\alpha\|\Phi^{-1}\|}{V(\alpha)}\right]^2\int_{t_1}^{t_2}\|Q_\alpha(t_2-s)\|^2\,C_r(s)ds.$$

The right-hand side of the aforementioned inequality tends to zero as $t_2 \to t_1$ due to the strongly continuous operator $Q_\alpha(t)$. As a result, uniform operator topological continuity is required by $(A1)$. $\mathfrak{T}(Q_r)$ is hence equicontinuous.

Step 4. $E(t) = \left\{\mathfrak{U}(t), \mathfrak{U} \in \mathfrak{T}(\overline{Q_r})\right\}$ is a relatively compact on $\mathfrak{X}$.

The case $t = 0$ is trivial. Consider $0 < t \leq \mathfrak{C}$, $x \in \overline{Q_r}$. Then, for all $\mathfrak{K} \in (0, t)$, define an operator

$$\mathfrak{U}_{\mathfrak{K}}(t) = \Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0 + \frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t-\mathfrak{K}}\Phi^{-1}(t-s)^{\alpha-1}\left\{\mathfrak{B}u(s)+\mathfrak{f}(s)\right\}ds$$

$$+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t-\mathfrak{K}}\Phi^{-1}(t-s)^{\alpha-1}\overline{h}(s)dB^H(s)+\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{t-\mathfrak{K}}\Phi^{-1}(t-s)^{\alpha-1}\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)$$

$$+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t-\mathfrak{K}}\Phi^{-1}Q_\alpha(t-s)\Big\{\mathfrak{B}u(s)+\mathfrak{f}(s)\Big\}ds+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t-\mathfrak{K}}\Phi^{-1}Q_\alpha(t-s)\overline{h}(s)dB^H(s)$$

$$+\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_0^{t-\mathfrak{K}}\Phi^{-1}Q_\alpha(t-s)\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi),$$

where $\mathfrak{f}\in\mathbb{S}_{\varrho,x}$, $\overline{h}\in\mathbb{S}_{\sigma,x}$ and $\overline{y}\in\mathbb{S}_{\hbar,x}$. From $(A1)$, therefore, $E_{\mathfrak{K}}(t)=\Big\{\mathfrak{U}_{\mathfrak{K}}(t),\mathfrak{U}_{\mathfrak{K}}\in\mathfrak{T}_{\mathfrak{K}}(x),x\in\overline{Q_r}\Big\}$ is relatively compact in $\mathfrak{X}$ for all $\mathfrak{K}\in(0,t)$. In addition, for every $x\in\overline{Q_r}$, by using *Hölder* inequality, we have

$$\mathbb{E}\left\|\mathfrak{U}(t)-\mathfrak{U}_{\mathfrak{K}}(t)\right\|^2\leq 8\mathbb{E}\left\|\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_{t-\mathfrak{K}}^t\Phi^{-1}(t-s)^{\alpha-1}\mathfrak{B}u(s)ds\right\|^2$$
$$+8\mathbb{E}\left\|\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_{t-\mathfrak{K}}^t\Phi^{-1}(t-s)^{\alpha-1}\mathfrak{f}(s)ds\right\|^2+8\mathbb{E}\left\|\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_{t-\mathfrak{K}}^t\Phi^{-1}(t-s)^{\alpha-1}\overline{h}(s)dB^H(s)\right\|^2$$
$$+8\mathbb{E}\left\|\frac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_{t-\mathfrak{K}}^t\Phi^{-1}(t-s)^{\alpha-1}\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2+8\mathbb{E}\left\|\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_{t-\mathfrak{K}}^t\Phi^{-1}Q_\alpha(t-s)\mathfrak{B}u(s)ds\right\|^2$$
$$+8\mathbb{E}\left\|\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_{t-\mathfrak{K}}^t\Phi^{-1}Q_\alpha(t-s)\mathfrak{f}(s)ds\right\|^2+8\mathbb{E}\left\|\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_{t-\mathfrak{K}}^t\Phi^{-1}Q_\alpha(t-s)\overline{h}(s)dB^H(s)\right\|^2$$
$$+8\mathbb{E}\left\|\frac{\alpha\mathfrak{R}^2}{V(\alpha)}\int_{t-\mathfrak{K}}^t\Phi^{-1}Q_\alpha(t-s)\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2.$$

Hence,

$$\mathbb{E}\left\|\mathfrak{U}(t)-\mathfrak{U}_{\mathfrak{K}}(t)\right\|^2\leq 8\left[\left\{\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)}\right\}^2+\left\{\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)}\right\}^2\right]\frac{\mathfrak{K}^\alpha}{\alpha}\int_{t-\mathfrak{K}}^t(t-s)^{\alpha-1}\mathbb{E}\left\|u(s)\right\|^2ds$$
$$+8\left[\left\{\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2+\left\{\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\right]\frac{\mathfrak{K}^\alpha}{\alpha}\int_{t-\mathfrak{K}}^t(t-s)^{\alpha-1}\mathbb{E}\left\|\mathfrak{f}(s)\right\|^2ds$$
$$+8\left[\left\{\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2+\left\{\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\right]2H\mathfrak{C}^{2H-1}\int_{t-\mathfrak{K}}^t(t-s)^{2\alpha-2}\mathbb{E}\left\|\overline{h}(s)\right\|^2ds$$
$$+8\left[\left\{\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right\}^2+\left\{\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right\}^2\right]\frac{\mathfrak{K}^\alpha}{\alpha}\int_{t-\mathfrak{K}}^t(t-s)^{\alpha-1}\mathbb{E}\left\|\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|^2.$$

The above inequality gives us,

$$\mathbb{E}\left\|\mathfrak{U}(t)-\mathfrak{U}_{\mathfrak{K}}(t)\right\|^2\to 0,\quad when\mathfrak{K}\to 0^+.$$

Hence, there are relatively compact sets arbitrarily close to the set $E(t)=\Big\{\mathfrak{U}(t),\mathfrak{U}\in\mathfrak{T}(\overline{Q_r})\Big\}$ which implies $E(t)$ is also relatively compact in $\mathfrak{X}$.

Step 5. $\mathfrak{T}(x)$ has a closed graph.

Let $x^m\to x^*(m\to\infty)$, $\mathfrak{U}^m\to\mathfrak{U}^*(m\to\infty)$. We will prove that $\mathfrak{U}^*\in\mathfrak{T}(x^*)$. Since $\mathfrak{U}^m\in\mathfrak{T}(x^m)$, there exist $\mathfrak{f}^m \in \mathbb{S}_{\varrho,x^m}$, $\overline{h}^m \in \mathbb{S}_{\sigma,x^m}$ and $\overline{y}^m \in \mathbb{S}_{\hbar,x^m}$, s.t. for each $t \in (0, \mathfrak{C})$,

$$\mathfrak{U}^m(t) = \Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0 + \tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}(t-s)^{\alpha-1}\Big\{\mathfrak{B}u(s) + \mathfrak{f}^m(s)\Big\}ds$$
$$+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}(t-s)^{\alpha-1}\overline{h}^m(s)dB^H(s) + \tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}(t-s)^{\alpha-1} \int_Z \overline{y}^m(s,\xi)\tilde{N}(ds,d\xi)$$
$$+\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}Q_\alpha(t-s)\Big\{\mathfrak{B}u(s) + \mathfrak{f}^m(s)\Big\}ds + \tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}Q_\alpha(t-s)\overline{h}^m(s)dB^H(s)$$
$$+\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}Q_\alpha(t-s) \int_Z \overline{y}^m(s,\xi)\tilde{N}(ds,d\xi).$$

Finally, we will prove the existence of $\mathfrak{f}^* \in \mathbb{S}_{\varrho,x^*}$, $\overline{h}^* \in \mathbb{S}_{\sigma,x^*}$ and $\overline{y}^* \in \mathbb{S}_{\hbar,x^*}$ s.t. for each $t \in (0, \mathfrak{C}]$

$$\mathfrak{U}^*(t) = \Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0 + \tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}(t-s)^{\alpha-1}\Big\{\mathfrak{B}u(s) + \mathfrak{f}^*(s)\Big\}ds$$
$$+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}(t-s)^{\alpha-1}\overline{h}^*(s)dB^H(s) + \tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}(t-s)^{\alpha-1} \int_Z \overline{y}^*(s,\xi)\tilde{N}(ds,d\xi)$$
$$+\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}Q_\alpha(t-s)\Big\{\mathfrak{B}u(s) + \mathfrak{f}^*(s)\Big\}ds + \tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}Q_\alpha(t-s)\overline{h}^*(s)dB^H(s)$$
$$+\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{t-\mathfrak{K}} \Phi^{-1}Q_\alpha(t-s) \int_Z \overline{y}^*(s,\xi)\tilde{N}(ds,d\xi).$$

Now,

$$\mathbb{E}\left\|\Big\{\mathfrak{U}^m(t) - \Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0\Big\} - \Big\{\mathfrak{U}^*(t) - \Phi^{-1}\mathfrak{R}S_\alpha(t)\Phi x_0\Big\}\right\|^2 \to 0 \quad as \quad m \to \infty.$$

Consider

$$\Sigma : \mathcal{L}^2(J, \mathfrak{X}) \to C(J, \mathfrak{X}),$$

where

$$(\mathfrak{f}, \overline{h}, \overline{y}) \to \Sigma(\mathfrak{f}, \overline{h}, \overline{y})(t) = \tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^t \Phi^{-1}(t-s)^{\alpha-1}\mathfrak{f}(s)ds + \tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^t \Phi^{-1}Q_\alpha(t-s)\mathfrak{f}(s)ds$$
$$+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^t \Phi^{-1}(t-s)^{\alpha-1}\overline{h}(s)dB^H(s) + \tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^t \Phi^{-1}Q_\alpha(t-s)\overline{h}(s)dB^H(s)$$
$$+\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^t \Phi^{-1}(t-s)^{\alpha-1} \int_Z \overline{y}(s,\xi)\tilde{N}(ds,d\xi) + \tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^t \Phi^{-1}Q_\alpha(t-s) \int_Z \overline{y}(s,\xi)\tilde{N}(ds,d\xi)$$
$$-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^t \Phi^{-1}(t-s)^{\alpha-1}\mathfrak{B}\mathfrak{B}^*Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa, \Delta_0^{\mathfrak{C}}) \times$$
$$\left[\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{\mathfrak{C}} \Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\mathfrak{f}(s)ds + \tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{\mathfrak{C}} \Phi^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{f}(s)ds\right]ds$$
$$-\tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^t \Phi^{-1}Q_\alpha(t-s)\mathfrak{B}\mathfrak{B}^*Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa, \Delta_0^{\mathfrak{C}}) \times$$
$$\left[\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{\mathfrak{C}} \Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\mathfrak{f}(s)ds + \tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{\mathfrak{C}} \Phi^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{f}(s)ds\right]ds$$
$$-\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^t \Phi^{-1}(t-s)^{\alpha-1}\mathfrak{B}\mathfrak{B}^*Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa, \Delta_0^{\mathfrak{C}}) \times$$
$$\left[\tfrac{\wp\mathfrak{R}(1-\alpha)}{V(\alpha)\Gamma(\alpha)} \int_0^{\mathfrak{C}} \Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\overline{h}(s)dB^H(s) + \tfrac{\alpha\mathfrak{R}^2}{V(\alpha)} \int_0^{\mathfrak{C}} \Phi^{-1}Q_\alpha(\mathfrak{C}-s)\overline{h}(s)dB^H(s)\right]ds$$

$$-\frac{\alpha\Re^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\mathfrak{B}\mathfrak{B}^* Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa,\Delta_0^{\mathfrak{C}})\times$$
$$\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\overline{h}(s)dB^H(s)+\frac{\alpha\Re^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\overline{h}(s)dB^H(s)\right]ds$$
$$-\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^t \Phi^{-1}(t-s)^{\alpha-1}\mathfrak{B}\mathfrak{B}^* Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa,\Delta_0^{\mathfrak{C}})\times$$
$$\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)+\frac{\alpha\Re^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right]ds$$
$$-\frac{\alpha\Re^2}{V(\alpha)}\int_0^t \Phi^{-1}Q_\alpha(t-s)\mathfrak{B}\mathfrak{B}^* Q_\alpha^*(\mathfrak{C}-s)\aleph(\kappa,\Delta_0^{\mathfrak{C}})\times$$
$$\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)+\frac{\alpha\Re^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right]ds.$$

It is evident from Lemma 2.2 that $\Sigma\circ\mathbb{S}_{\varrho,\sigma,\hbar}$ is a closed graph operator, where $\mathbb{S}_{\varrho,\sigma,\hbar}=\left\{\mathfrak{f}\in\varrho(t,x(t))\right\}\times\left\{\overline{h}\in\sigma(t,x(t))\right\}\times\left\{\overline{y}\in\hbar(t,x(t))\right\}$. From the definition of Σ, we get

$$\left\{\mathfrak{U}^m(t)-\Phi^{-1}\Re S_\alpha(t)\Phi x_0\right\}\in\Sigma\left(\mathbb{S}_{\varrho,\sigma,\hbar,x^m}\right).$$

Since, x^m tends to x^*, as a result of Lemma 2.2,

$$\left\{\mathfrak{U}^*(t)-\Phi^{-1}\Re S_\alpha(t)\Phi x_0\right\}\in\Sigma\left(\mathbb{S}_{\varrho,\sigma,\hbar,x^*}\right).$$

It is clear from this that $\mathfrak{U}^*\in\mathfrak{T}(x^*)$. Hence, $\mathfrak{T}$ has a closed graph.
Since $\mathfrak{T}$ is a completely continuous multi-valued map with a compact value, we can infer that $\mathfrak{T}$ is u.s.c. from Proposition 2.1. According to Lemma 2.3, operator $\mathfrak{T}$ has a fixed point on Q_r, which is a mild solution of (1.1).

□

Theorem 3.2. *If* (A0)–(A7) *are satisfied, then (1.1) is approximately controllable on J.*

Proof. We can quickly demonstrate that the operator $\mathfrak{T}$ has a fixed point in Q_r, where $r=r(\kappa)$, for every $0<\kappa<1$ by using the method described in Theorem 3.1. A fixed point of $\mathfrak{T}$ in Q_r is defined as $x^\kappa(\cdot)$. Any fixed point of the operator $\mathfrak{T}$ is a mild solution of (1.1). This indicates that for each $t\in(0,\mathfrak{C}]$, by stochastic Fubini theorem, there exists $\mathfrak{f}^\kappa\in\mathbb{S}_{\varrho,x^\kappa}$, $\overline{h}^\kappa\in\mathbb{S}_{\sigma,x^\kappa}$ and $\overline{y}^\kappa\in\mathbb{S}_{\hbar,x^\kappa}$,

$$x^\kappa(\mathfrak{C})=\tilde{x}_{\mathfrak{C}}-\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}\left\{\mathbb{E}\tilde{x}_{\mathfrak{C}}+\int_0^{\mathfrak{C}}\tilde{\gamma}(s)dB^H(s)-\Phi^{-1}\Re S_\alpha(\mathfrak{C})\Phi x_0\right\}$$
$$+\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\mathfrak{f}^\kappa(s)ds$$
$$+\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\overline{h}^\kappa(s)dB^H(s)$$
$$+\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z\overline{y}^\kappa(s,\xi)\tilde{N}(ds,d\xi)$$
$$+\frac{\alpha\Re^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{f}^\kappa(s)ds+\frac{\alpha\Re^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\overline{h}^\kappa(s)dB^H(s)$$

$$+\frac{\alpha\Re^2}{V(\alpha)}\int_0^{\mathfrak{C}}\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\overline{y}^\kappa(s,\xi)\tilde{N}(ds,d\xi).$$

In addition, the Dunford-Pettis theorem and conditions on $\mathfrak{f},\overline{h}$ and $\overline{y}$, we have that $\mathfrak{f}^\kappa,\overline{h}^\kappa$ and $\overline{y}^\kappa$ are weakly compact in $\mathcal{L}^2(J,\mathfrak{X})$. Thus, there are subsequences determined by $\mathfrak{f}^\kappa,\overline{h}^\kappa$ and $\overline{y}^\kappa$ weakly converging to say $\mathfrak{f},\overline{h}$ and $\overline{y}$. Now, we have

$$\begin{aligned}
\mathbb{E}\|x^\kappa(\mathfrak{C})-\tilde{x}_{\mathfrak{C}}\|^2 \leq\ & 14\mathbb{E}\left\|\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}\left[\mathbb{E}\tilde{x}_{\mathfrak{C}}-\Phi^{-1}\Re S_\alpha(\mathfrak{C})\Phi x_0\right]\right\|^2+28H\mathfrak{C}^{2H-1}\int_0^{\mathfrak{C}}\mathbb{E}\|\tilde{\gamma}(s)\|^2_{\mathfrak{L}^2_\Theta}ds\\
&+14\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\right]^2\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\left\{\mathfrak{f}^\kappa(s)-\mathfrak{f}(s)\right\}\right\|ds\right\}^2\\
&+14\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\right]^2\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\mathfrak{f}(s)\right\|ds\right\}^2\\
&+28H\mathfrak{C}^{2H-1}\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\right]\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\left\{\overline{h}^\kappa(s)-\overline{h}(s)\right\}\right\|ds\right\}^2\\
&+28H\mathfrak{C}^{2H-1}\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\right]\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\overline{h}(s)\right\|ds\right\}^2\\
&+14\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\right]\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|\right\}^2\\
&+14\left[\frac{\wp\Re(1-\alpha)}{V(\alpha)\Gamma(\alpha)}\right]\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}(\mathfrak{C}-s)^{\alpha-1}\int_Z\left\{\overline{y}^\kappa(s,\xi)-\overline{y}(s,\xi)\right\}\tilde{N}(ds,d\xi)\right\|\right\}^2\\
&+14\left[\frac{\alpha\Re^2}{V(\alpha)}\right]^2\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\left\{\mathfrak{f}^\kappa(s)-\mathfrak{f}(s)\right\}\right\|ds\right\}^2\\
&+14\left[\frac{\alpha\Re^2}{V(\alpha)}\right]^2\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\mathfrak{f}(s)\right\|ds\right\}^2\\
&+28H\mathfrak{C}^{2H-1}\left[\frac{\alpha\Re^2}{V(\alpha)}\right]\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\left\{\overline{h}^\kappa(s)-\overline{h}(s)\right\}\right\|^2ds\right\}^2\\
&+28H\mathfrak{C}^{2H-1}\left[\frac{\alpha\Re^2}{V(\alpha)}\right]\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\overline{h}(s)\right\|ds\right\}^2\\
&+14\left[\frac{\alpha\Re^2}{V(\alpha)}\right]\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\left\{\overline{y}^\kappa(s,\xi)-\overline{y}(s,\xi)\right\}\tilde{N}(ds,d\xi)\right\|\right\}^2\\
&+14\left[\frac{\alpha\Re^2}{V(\alpha)}\right]\mathbb{E}\left\{\int_0^{\mathfrak{C}}\left\|\Phi^{-1}\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}Q_\alpha(\mathfrak{C}-s)\int_Z\overline{y}(s,\xi)\tilde{N}(ds,d\xi)\right\|\right\}^2.
\end{aligned}$$

According to the assumption $(A0)$, the operator $\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}\to 0$ strongly as $\kappa\to 0^+$ and also $\kappa(\kappa I-\Delta_0^{\mathfrak{C}})^{-1}\leq 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $Q_\alpha(t)$, it is implied that

$$\mathbb{E}\|x^\kappa(\mathfrak{C})-\tilde{x}_{\mathfrak{C}}\|^2\to 0 \quad as \quad \kappa\to 0^+.$$

Hence, we deduce the approximate controllability of the system (1.1). □

4. Application

We consider the stochastic partial differential inclusion with the AB fractional derivative:

$$
\begin{aligned}
{}^{ABC}D^{3/4}_{0+}\left[\left\{1-\tfrac{\partial^2}{\partial\zeta^2}\right\}x(t,\zeta)\right] &\in \tfrac{\partial^2}{\partial\zeta^2}x(t,\zeta)+\tilde{\varphi}(t,\zeta)+\tfrac{e^{-t}}{1+e^{-t}}sin(x(t,\zeta))+\sigma(t,x(t,\zeta))\tfrac{dB^H(t)}{dt} \\
&+\int_Z \hbar(t,x(t,\zeta),\xi)\tilde{N}(dt,d\xi), \quad t\in J:=(0,1], \quad \zeta\in[0,\pi], \\
&x(t,0)=x(t,\pi)=0, \quad t\in(0,1].
\end{aligned} \tag{4.1}
$$

To write the above system (4.1) into the abstract system (1.1), we choose the space $\mathfrak{X}=Y=U=\mathcal{L}^2([0,\pi],\mathbb{R})$ and define the operators $\mathfrak{A}:D(\mathfrak{A})\subset\mathfrak{X}\to\mathfrak{X}$ and $\Phi:D(\mathfrak{A})\subset\mathfrak{X}\to\mathfrak{X}$, $t\geq 0$ by $\mathfrak{A}=\frac{\partial^2}{\partial\zeta^2}$ and $\Phi=1-\mathfrak{A}$ with $D(\mathfrak{A})=D(\Phi)=\left\{x\in\mathfrak{X}; x,\frac{\partial x}{\partial\zeta} \quad be \quad absolutely \quad continuous, \frac{\partial^2 x}{\partial\zeta^2}\in\mathfrak{X}, x(0)=x(\pi)=0\right\}$. Then, $\mathfrak{A}$ and Φ can be written as

$$
\begin{aligned}
\mathfrak{A}x &= \sum_{k=1}^{\infty}k^2\langle x,x_k\rangle x_k, \quad x\in D(\mathfrak{A}), \\
\Phi x &= \sum_{k=1}^{\infty}(1+k^2)\langle x,x_k\rangle x_k, \quad x\in D(\Phi).
\end{aligned}
$$

Furthermore, for $x\in\mathfrak{X}$ we get

$$
\begin{aligned}
\mathfrak{A}\Phi^{-1}x &= \sum_{k=1}^{\infty}\frac{k^2}{1+k^2}\langle x,x_k\rangle x_k, \\
\Phi^{-1}x &= \sum_{k=1}^{\infty}\frac{1}{1+k^2}\langle x,x_k\rangle x_k.
\end{aligned}
$$

$\mathfrak{A}\Phi^{-1}$ is self-adjoint and $x_k=\sqrt{\frac{2}{\pi}}sin(kx), k=1,2,\cdots$ be the orthonormal basis of $\mathfrak{X}$. However, $\mathfrak{A}\Phi^{-1}$ forms a uniformly strongly continuous semigroup of bounded linear operators $S(t), t\geq 0$, on a separable Hilbert space $\mathfrak{X}$ which is in the form

$$
S(t)x=\sum_{k=1}^{\infty}e^{-k^2t}\langle x,x_k\rangle x_k, \quad x\in D(\mathfrak{A}).
$$

Assume that $\varpi(t)(\zeta)=x(t,\zeta), t\in J, \quad \zeta\in[0,\pi]$. Now, construct the bounded linear operator $\mathfrak{B}:U\to\mathfrak{X}$ and the function $\varrho:J\times\mathfrak{X}\to\mathfrak{X}$, respectively, for any $\varpi(t)\in\mathfrak{X}$.

$$
\begin{aligned}
\varrho(t,\varpi(t))(\zeta) &= \frac{e^{-t}}{1+e^{-t}}sin(x(t,\zeta)), \\
\mathfrak{B}u(t)(\zeta) &= \tilde{\varphi}(t,\zeta), \quad 0<\zeta<\pi,
\end{aligned}
$$

where $\tilde{\varphi}:J\times[0,\pi]\to[0,\pi]$ is continuous in t and $\mathfrak{B}=\mathfrak{B}^*=I$. Therefore, (4.1) can be reformulated as the abstract system (1.1). Clearly, all the assumptions of Theorem 3.1 are satisfied, and

$$
\left[\Lambda+\frac{\mathfrak{C}^{2\alpha-1}}{2\alpha-1}\|n\|_{L^{1/q}}\Pi\iota+2H\mathfrak{C}^{2H-1}\Lambda\right]\left\{9\left[\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|}{V(\alpha)\Gamma(\alpha)}\right]^2+9\left[\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*}{V(\alpha)}\right]^2\right\}\tilde{\mathcal{K}}<1,
$$

where

$$\tilde{\mathcal{K}} = \left\{1 + 8\left[\frac{\theta\psi(1-\alpha)\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\Gamma(\alpha)\kappa}\right]^2 \frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2} + 8\left[\frac{\theta^2\alpha\left\|\Phi^{-1}\right\|\mathfrak{I}_1^*\varsigma_{\mathfrak{B}}}{V(\alpha)\kappa}\right]^2 \frac{\mathfrak{C}^{4\alpha-2}}{(2\alpha-1)^2}\right\}.$$

As a result, the system (4.1) has a mild solution on J, in addition, it is approximately controllable on J, according to Theorem 3.2.

5. Conclusions

In this work, a new control model was presented with the Sobolev-type Atangana-Baleanu fractional stochastic differential inclusions including the fractional Brownian motion and Poisson jumps. We investigated the approximate controllability for the proposed problem (1.1). Our results were obtained with the aid of nonsmooth analysis, fractional calculus, stochastic analysis, and fixed-point theorems. Finally, we provided an example to illustrate the applicability of the results.

For future work, we can present neutral Atangana-Baleanu fractional stochastic differential inclusions with Clarke subdifferential.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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