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Research article

Triple correlation sums of coefficients of θ-series

Fei Hou[1,*] **and Bin Chen**[2]

[1] Department of Mathematics, College of Science, Xi'an University of Technology, Xi'an 710054, China

[2] Department of Mathematics, College of Mathematics and Physics,Weinan Normal University, Weinan 714099, China

* **Correspondence:** Email: fhou@xaut.edu.cn.

Abstract: We investigate the triple correlation sums of coefficients of θ-series and prove an asymptotic formula with power-saving error term. As a result, we present that this type of sum is non-trivial in the regime $H \geq X^{2/3+\varepsilon}$.

Keywords: Jacobi theta series; Fourier coefficients; triple correlation sums; asymptotic formula
Mathematics Subject Classification: 11F03, 11F30

1. Introduction

Let $a(n)$, $b(n)$ and $c(n)$ be three general arithmetic functions, $H, X \geq 2$ such that $H \leq X$ and $l_1, l_2 \in \mathbb{Z}$. Let U, V be two smooth weight functions supported on $[1/2, 5/2]$ with bounded derivatives, respectively; see, e.g., [5, Section 8] for detailed descriptions on various aspects of these functions. As a basic question in number theory, the triple correlation sums problem is conducted to seek the non-trivial bound for the sum

$$\Psi(U, V, H, X, a, b, c; l_1, l_2) = \sum_{h\geq 1} V\left(\frac{h}{H}\right) \sum_{n\geq 1} a(n)b(n + l_1 h)c(n + l_2 h)U\left(\frac{n}{X}\right), \tag{1.1}$$

which plays a tremendously vital rôle in many topics, such as the moments of L-functions (or zeta-functions), subconvexity and the Gauss circle problem, etc (see [2,6,8–11,13–15,19] and the references therein).

In the classic case of all the arithmetic functions being the divisor functions, in 2011, Browning [4] showed that, if $H \geq X^{3/4+\varepsilon}$,

$$\sum_{1\leq h\leq H} \sum_{1\leq n\leq X} \tau(n)\tau(n + h)\tau(n - h) = \frac{11}{8}\Upsilon(h) \prod_{p} \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right) HX \log^3 X + o\Big(HX \log^3 X\Big),$$

up to an explicit multiplicative function $\Upsilon(h)$. After that, Blomer [3] used the spectral decomposition for partially smoothed triple correlation sums to establish an asymptotic formula that

$$\sum_{h\geq 1} V\left(\frac{h}{H}\right) \sum_{1\leq n\leq X} \tau(n)\tau_\ell(n+h)\tau(n-h) = XH\widetilde{V}(1)P_{\ell+1}(\log X) + O_\varepsilon\left(X^\varepsilon\left(H^2 + HX^{1-\frac{1}{\ell+2}} + X\sqrt{H} + \frac{X^{\frac{3}{2}}}{\sqrt{H}}\right)\right),$$

for any $\ell \in \mathbb{N}$, where $\widetilde{V}$ denotes the normal Mellin transform of V, which is given by $\widetilde{V}(s) = \int_{\mathbb{R}^+} V(x)x^{s-1}\mathrm{d}x$ for any $s \in \mathbb{C}$, τ_ℓ is the ℓ-th fold divisor function, and P_ℓ is a polynomial of degree ℓ. Notice that Blomer improved the range of H substantially to $H \geq X^{1/3+\varepsilon}$, and produced a power saving error term. In addition, in [3], Blomer was able to attain a more general version that, for any complex sequence $\mathbf{a} = \{a(n)\}$,

$$\begin{aligned}\sum_{h\geq 1} V\left(\frac{h}{H}\right) \sum_{1\leq n\leq X} a(n)\tau(n-h)\tau(n+h) =& H\widetilde{V}(1) \sum_{1\leq n\leq X} a(n) \sum_{d\geq 1} \frac{S(2n,0;d)}{d^2}(\log n + 2\gamma - 2\log d)^2 \\ &+ O_\varepsilon\left(X^\varepsilon\left(\frac{H^2}{\sqrt{X}} + HX^{\frac{1}{4}} + \sqrt{XH} + \frac{X}{\sqrt{H}}\right)\|\mathbf{a}\|_2\right),\end{aligned} \tag{1.2}$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant, and $\|\mathbf{a}\|_2$ is the ℓ^2-norm of the sequence $\mathbf{a} = \{a(n)\}$. Let $k, k' \geq 2$ be any even integers. Let $f_1 \in \mathcal{B}_k^*(1)$ and $f_2 \in \mathcal{B}_{k'}^*(1)$ be two Hecke newforms on GL_2 with $\lambda_{f_1}(n)$ and $\lambda_{f_2}(n)$ being their n-th Hecke eigenvalues, respectively (see §2.1 for relevant descriptions). Subsequently, Lin [20] proved that

$$\sum_{h\geq 1} V\left(\frac{h}{H}\right) \sum_{1\leq n\leq X} a(n)\lambda_{f_1}(n+h)\lambda_{f_2}(n-h) \ll_{\varepsilon,k,k'} \frac{X^{1+\varepsilon}}{H}\left(\frac{X}{\sqrt{H}} + \sqrt{XH}\right)\|\mathbf{a}\|_2, \tag{1.3}$$

which beats the *trivial* bound barrier $O_{\varepsilon,k,k'}(X^\varepsilon H\sqrt{X}\|\mathbf{a}\|_2)$ for the correlation sum, provided that $H \geq X^{2/3+\varepsilon}$. Here and henceforth, the trivial bound means to take absolute value for each summand, followed by using the Rankin-Selbeg's estimate involving Fourier coefficients that

$$\sum_{1\leq n\leq X} |\lambda_f(n)|^2 \ll (XN)^\varepsilon X \tag{1.4}$$

uniformly for any $X \geq 2$ and newform f on the congruence subgroup $\Gamma_0(N)$, $N \in \mathbb{N}$, with trivial nebentypus, where the implied $\ll$-constant depends only on the weight parameter of the form f. As an immediate consequence of (1.3), one would see that

$$\sum_{h\geq 1} V\left(\frac{h}{H}\right) \sum_{1\leq n\leq X} \lambda_{f_1}(n)\lambda_{f_2}(n+h)\lambda_{f_3}(n-h) \ll_{\varepsilon,k,k',k''} X^\varepsilon \min\left(XH, \frac{X^2}{\sqrt{H}}\right)$$

for any $f_3 \in \mathcal{B}_{k''}^*(1)$ with $k'' \in 2\mathbb{N}$. In contrast to Lin's work, Singh [25] was able to attain

$$\sum_{h\geq 1} V\left(\frac{h}{H}\right) \sum_{n\geq 1} \lambda_{f_1}(n)\lambda_{f_2}(n+h)\lambda_{f_3}(n-h)U\left(\frac{n}{X}\right) \ll_{\varepsilon,k,k',k''} X^\varepsilon\left(X^{\frac{3}{2}} + \sqrt{X}H\right),$$

extending the range of H to $H \geq X^{1/2+\varepsilon}$. By now, the best result is due to Lü-Xi [21, 22] who achieved that

$$\sum_{h\geq 1} V\left(\frac{h}{H}\right) \sum_{1\leq n\leq X} a(n)b(n+h)\lambda_{f_1}(n-h) \ll_{\varepsilon,k} X^\varepsilon \Delta_1(X,H)\|\mathbf{a}\|_2\|\mathbf{b}\|_2,$$

for any complex sequence $\mathbf{b} = \{b(n)\}$, which allows one to take $H \geq X^{2/5+\varepsilon}$; the definition of $\Delta_1(X, H)$, however, can be referred to [22, Theorem 3.1].

In the present paper, we are more concerned about the Fourier coefficients $r_\ell(n)$ of theta series $\theta^\ell(z)$, $\ell \in \mathbb{N}$ (see §2.2 & §2.3 for definitions and relevant backgrounds). Here and thereafter, we denote by $r(n) := r_2(n)$ for any $n \in \mathbb{N}$ as a convention. Observe that the estimates of Blomer, Lin, Singh and Lü-Xi's presented above heavily rely on the deep analytic properties involving Fourier coefficients of cusp forms, such as the Kuznetsov's trace formulae, the Wilton-type bounds and the short exponential estimates, etc. However, all of these become *unreachable* for us in the time being. To the best of the author's knowledge, these crucial features have not been developed so far.

Note. Indeed, in the spirit of Lü-Xi's work, upon applying the relation that $\int_0^1 e(n\alpha)\mathrm{d}\alpha = 1$ if $n = 0$, and zero otherwise, the sum $\Psi(U, V, H, X, a, b, c; l_1, l_2)$ in (1.1) is boiled down to evaluating

$$\int_0^1 e\left(-(l_1 + l_2)h\alpha\right)\left(\sum_{n\geq 1} a(n)e(-2n\alpha)V\left(\frac{n}{X}\right)\right)\left(\sum_{m_1\geq 1} b(m_1)e(m_1\alpha)U\left(\frac{m_1}{X}\right)\right)\left(\sum_{m_2\geq 1} c(m_2)e(m_2\alpha)U\left(\frac{m_2}{X}\right)\right)\mathrm{d}\alpha,$$

which is $O_\varepsilon(X^{2+\varepsilon})$ by the Rankin-Selbeg's estimate (1.4), together with the Cauchy-Schwarz inequality, if the objects $a(n)$, $b(n)$ and $c(n)$ are taken as Fourier coefficients of theta series. However, this far exceeds what the trivial bound implies.

There is still a gap where the tools handling the triple correlation sums involving Fourier coefficients of cusp forms cannot be covered in some related topics, such as the study of certain sums involving Fourier coefficients of theta series. In the present paper, we are able to circumvent this kind of deadlock by proving the following main results:

Theorem 1.1. *Let $X, H \geq 2$, satisfying $H \leq X$. Let U, V be two smooth weight functions supported on* $[1/2, 5/2]$ *with bounded derivatives, respectively. Then, we have*

$$\begin{aligned}\sum_{h\geq 1} V\left(\frac{n}{H}\right)\sum_{n\geq 1} r(n)r(n+h)r(n-h)\, U\left(\frac{n}{X}\right) =& H\sum_{n\geq 1} r(n)\sum_{q\geq 1}\frac{S(-2n, 0; q)}{q^2}\,\mathcal{W}_{X,H}\left(\frac{n}{X}, \frac{q}{\sqrt{X}}\right)\\ &+ O_\varepsilon\left(\left(\frac{X^3}{H^2} + X^{\frac{3}{2}} + X^{\frac{3}{4}}H + \frac{X^2}{\sqrt{H}}\right)X^\varepsilon\right),\end{aligned} \tag{1.5}$$

where the weight function $\mathcal{W}_{X,H}$ is defined as in (3.5)*, and the implied $\ll$-constant depends only on ε.*

Observing that the trivial estimate is $O_\varepsilon(X^{1+\varepsilon}H)$, our asymptotic formula in (1.5) is effective, as long as H satisfies that $H \geq X^{2/3+\varepsilon}$. Particularly, as a direct application of Theorem 1.1, we obtain the following:

Corollary 1.1. *With the notation being as in Theorem 1.1, there holds that*

$$\sum_{h\geq 1}\sum_{n\geq 1} r(n)r(n+h)r(n-h)\, U\left(\frac{n}{X}\right)V\left(\frac{h}{X}\right) = X\sum_{n\geq 1} r(n)\sum_{q\geq 1}\frac{S(-2n, 0; q)}{q^2}\mathcal{W}_{X,X}\left(\frac{n}{X}, \frac{q}{\sqrt{X}}\right) + O_\varepsilon\left(X^{\frac{7}{4}+\varepsilon}\right). \tag{1.6}$$

Remark 1.1. *It easily to see that the main term on the right-hand side of* (1.6) *is $\asymp X^2$; we thus get a saving of roughly $X^{1/4-\varepsilon}$ in the error term.*

Remark 1.2. *One may wander if the corresponding results above hold for arithmetic functions r_ℓ with $\ell \geq 3$. This is indeed the case. One might prove an analog of Theorem 1.1 with one or several of the arithmetic functions in the summand replaced by r_ℓ, $\ell \geq 3$, with some more efforts. It is also natural to expect that the main result in Theorem 1.1 holds with the general function r replaced by the ℓ-fold divisor function τ_ℓ, $\ell \geq 3$, as a generalization of Blomer's work. We shall plan to further investigate these two topics on another occasion.*

Notations. *Throughout the paper, ε always denotes an arbitrarily small positive constant which might not be the same at each occurrence. $e(x) = \exp(2\pi i x)$ for any real number x, and $\tau_d(n) = \sum_{n_1 n_2 \cdots n_d | n} 1$ for any positive integer $d \geq 2$. We use Landau's $f = O(g)$ and Vinogradov's $f \ll g$ as synonyms. Thus, $f(x) \ll g(x)$ for $x \in \mathcal{X}$, where the set $\mathcal{X}$ must be specified either explicitly or implicitly, means that $|f(x)| \leq Cg(x)$ for all $x \in \mathcal{X}$ and some constant $C > 0$. We also use $f \asymp g$ to mean that both relations $f \ll g$ and $g \ll f$ hold with possibly different implied constants. As usual, the symbols $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ are respectively denote the real number field, the ring of integers and the ring of positive integers. Finally, we also follow the notational convention that, for any $m, n, c \in \mathbb{N}$, $S(m, n; c)$ denotes the classical Kloosterman sum, which is given by $S(m, n; c) = \sum^{*}_{\alpha \bmod c} e((m\alpha + n\overline{\alpha})/c)$, where $*$ indicates that the summation is restricted to $(x, c) = 1$, and $\overline{x}$ is the inverse of x modulo c.*

2. Preliminaries

2.1. GL_2-cusp forms and their Fourier coefficients

We will first give a recap of the theory of modular forms for SL_2. The good reference should be Iwaniec-Kowalski's book [16]. Let $k \geq 2$ be an even integer, and $N > 0$ an integer. Let χ be a primitive character to modulus q such that $N|q$, satisfying $\chi(-1) = (-1)^k$. We denote by $\mathcal{S}_k(N, \chi)$ the vector space of holomorphic cusp forms on $\Gamma_0(N)$ with nebentypus χ and weight k. For any $f \in \mathcal{S}_k(N, \chi)$, one has

$$f(z) = \sum_{n \geq 1} \psi_f(n) n^{\frac{k-1}{2}} e(nz)$$

for $z \in \mathfrak{h}$. Here, $\mathfrak{h}$ means the upper half-plane. Observe that $\mathcal{S}_k(N, \chi)$ is a finite dimensional Hilbert spaces which can be equipped with the Petersson inner products

$$\langle f_1, f_2 \rangle = \int_{\Gamma_0(N) \backslash \mathfrak{h}} f_1(z) \overline{f_2(z)} y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Let us recall the Hecke operators $\{T_n\}$ with $(n, N) = 1$, which satisfy the multiplicativity relation

$$T_n T_m = \sum_{d | (n,m)} \chi(d)\, T_{\frac{nm}{d^2}}. \tag{2.1}$$

Thus, it follows that, for any $f_1, f_2 \in \mathcal{S}_k(N, \chi)$, one has $\langle T_n f_1, f_2 \rangle = \chi(n) \langle f_1, T_n f_2 \rangle$ for all $(n, N) = 1$. One can also find an orthogonal basis $\mathcal{B}_k(N, \chi)$ of $\mathcal{S}_k(N, \chi)$ consisting of common eigenfunctions of all the Hecke operators T_n with $(n, N) = 1$. For each $f \in \mathcal{B}_k(N, \chi)$, denote by $\lambda_f(n)$ the n-th Hecke eigenvalue, which satisfies the relation $T_n f(z) = \lambda_f(n) f(z)$ for all $(n, N) = 1$. Thus, by (2.1),

$$\psi_f(m) \lambda_f(n) = \sum_{d | (n,m)} \chi(d)\, \psi_f\left(\frac{mn}{d^2}\right),$$

for any $m, n \in \mathbb{N}$ with $(n, N) = 1$. In particular, $\psi_f(1)\lambda_f(n) = \psi(n)$, if $(n, N) = 1$. Therefore,

$$\overline{\lambda_f(n)} = \overline{\chi(n)}\,\lambda_f(n), \quad \lambda_f(m)\lambda_f(n) = \sum_{d|(n,m)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right), \tag{2.2}$$

whenever $(mn, N) = 1$.

The Hecke eigenbasis $\mathcal{B}_k(N, \chi)$ also contains a subset of newforms $\mathcal{B}_k^*(N, \chi)$, those forms which are simultaneous eigenfunctions of all the Hecke operators T_n for any $n \geq 1$, and normalized to have first Fourier coefficient $\psi_f(1) = 1$. The elements of $\mathcal{B}_k^*(N, \chi)$ are usually called primitive forms (the symbol is simply abbreviated to $\mathcal{B}_k^*(N)$, if χ is trivial). In particular, for any primitive form $f \in \mathcal{B}_k^*(N, \chi)$, the relations in (2.2) holds for any $m, n \in \mathbb{N}$, from which one may have the exact factorization that $\lambda_f(dm) = \lambda_f(d)\lambda_f(m)$ for $d|N$. On the other hand, it is worthwhile to note that Deligne's bound asserts that $|\lambda_f(n)| \leq \tau(n)$ for any primitive form $f \in \mathcal{B}_k^*(1)$ and general $n \in \mathbb{N}$.

2.2. *Jacobi theta series and Fourier coefficients*

For any $z \in \mathfrak{h}$, let

$$\theta(z) = \sum_{n\in\mathbb{Z}} e(nz^2)$$

be the classical Jacobi theta series, which is a modular form (but not a cuspidal form) of weight 1/2 for $\Gamma_0(4)$. Then, the modular form $\theta^\ell(z)$, for any $\ell \in \mathbb{N}$, admits a Fourier expansion

$$\theta^\ell(z) = \sum_{n=0}^{\infty} r_\ell(n)e(nz).$$

In particular,

$$r_\ell(n) = \sharp\left\{(n_1, n_2, \cdots, n_\ell) \in \mathbb{Z}^\ell : n_1^2 + n_2^2 + \cdots + n_\ell^2 = n\right\},$$

which is $O_\varepsilon(n^{\ell/2-1+\varepsilon})$ for any $\varepsilon > 0$; see, e.g., [23].

2.3. *Voronoĭ-type summation formula*

As usual, we will need the following Voronoĭ-type summation formula for Fourier coefficients of theta series; see, e.g., [23, Lemma 2].

Lemma 2.1. *Let $X \geq 2$. Let $a \geq 2$ be an integer co-prime with q, $aq \equiv 1 \bmod q$ and*

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \bmod 4, \\ i, & \text{if } d \equiv -1 \bmod 4. \end{cases} \tag{2.3}$$

If $h \in \mathbb{C}^\infty(\mathbb{R}^{\times,+})$ is a Schwartz class function vanishing in a neighborhood of zero, then we have

$$\begin{aligned} &\sum_{n\geq1} r_\ell(n)e\left(\frac{an}{q}\right)h\left(\frac{n}{X}\right) \\ &= \left(\frac{2\pi i}{q}\right)^{\frac{\ell}{2}}\Gamma\left(\frac{\ell}{2}\right)^{-1}\left(\left(\frac{q}{d}\right)\epsilon_d^{-1}\right)^\ell \check{h}\left(\frac{\ell}{2}\right) + \frac{2\pi i^{\frac{\ell}{2}}}{q}\left(\left(\frac{q}{d}\right)\epsilon_d^{-1}\right)^\ell \sum_{n\geq1} r_\ell(n)\,e\left(-\frac{dn}{q}\right)n^{\frac{1-\frac{\ell}{2}}{2}}\mathcal{R}\left(\frac{nX}{q^2}; h\right), \end{aligned} \tag{2.4}$$

where $\check{h}(\xi) = X^\xi \int_{\mathbb{R}^+} h(x)x^{\xi-1}\mathrm{d}x$ for any $\xi \in \mathbb{R}^+$, and

$$\mathcal{R}(x; h) = X^{\frac{\frac{\ell}{2}+1}{2}} \int_{\mathbb{R}^+} h(y)y^{\frac{\frac{\ell}{2}-1}{2}} J_{\frac{\ell}{2}-1}(4\pi\sqrt{xy})\mathrm{d}y.$$

2.4. Bessel functions

For any $s \geq 1$, one may write

$$J_{s-1}(x) = x^{-\frac{1}{2}} \left(F_s^+(x) e(x) + F_s^-(x) e(-x) \right) \tag{2.5}$$

for some smooth functions $F^{\pm}$ satisfying that

$$x^j F_s^{\pm(j)}(x) \ll_{s,j} \frac{x}{(1+x)^{\frac{3}{2}}} \tag{2.6}$$

for any $j \in \mathbb{N}$. The existence of such functions $F^{\pm}$ is guaranteed, e.g., by [26, Section 6.5] if $x < 1$ and [26, Section 3.4] if $x \geq 1$.

2.5. The delta method

The δ-symbol method was developed in [6, 7] as variant of the circle method. Further development and applications can be found in Jutila [17, 18], Heath-Brown [12], Munshi [24], and more recently [1] to name a few. The main purpose is to express $\delta(n, 0)$ the Dirac symbol at 0 (restricted to the integers n in some given range: $|n| \leq X$), in terms of the *harmonics* $e(an/q)$ for some integers a, q satisfying $(a, q) = 1$ and $q \leq Q$, with Q being any fixed positive real number. In order to be of practical use, one expects the δ-symbol method should be capable of providing an expression for $\delta(n, 0)$ in terms of harmonics of a small moduli. Nevertheless, the modulus in the circle method cannot be less than $\sqrt{X}$, which corresponds to using Dirichlet's approximation theorem to produce values $q \leq Q$ (see [12]).

We will now briefly recall a version of the circle method which is due to Heath-Brown [12].

Lemma 2.2. *For any $Q > 1$, there exist a positive c_Q and an infinitely differentiable function $h(x, y)$ defined on the set $(0, \infty) \times \mathbb{R}$ such that*

$$\delta(n, 0) = \frac{c_Q}{Q^2} \sum_{q \geq 1} \sideset{}{^*}\sum_{a \bmod q} e\left(\frac{an}{q}\right) h\left(\frac{q}{Q}, \frac{n}{Q^2}\right).$$

The constant c_Q satisfies $c_Q = 1 + O(Q^{-A})$ for any $A > 0$. $h(x, y)$ is non-zero only for $x \leq \max(1, 2|y|)$ and $h(x, y) \ll x^{-1}$ for all y. Moreover,

$$x^i \frac{\partial}{\partial x^i} h(x, y) \ll_i x^{-1} \quad \textit{and} \quad \frac{\partial}{\partial y} h(x, y) = 0,$$

for $|x| \leq 1$ and $|y| \leq x/2$. For $|y| > x/2$, we also have

$$x^i y^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} h(x, y) \ll_{i,j} x^{-1},$$

and for any $Y \geq 2$,

$$\int_{-Y}^{Y} |h(x, y)| \mathrm{d}y \ll 1 + Y \log Y. \tag{2.7}$$

3. Proof of Theorem 1.1

In this section, we shall devote to the proof of the main result in Theorem 1.1. Let

$$\mathcal{T}(X,H)=\sum_{h\geq 1}V\Big(\frac{h}{H}\Big)\sum_{n\geq 1}r(n)\,r(n+h)\,r(n-h)\,U\Big(\frac{n}{X}\Big).$$

The sum we are concerned about is

$$\sum_{n\geq 1}r(n)\sum_{H/3\leq h\leq 3H}r(n+h)\,r(n-h)\,U\Big(\frac{n}{X}\Big)V\Big(\frac{h}{H}\Big). \tag{3.1}$$

Let $m_1=n+h$ and $m_2=n-h$. The inner-sum over h becomes

$$\sum_{m_1+m_2=2n}r(m_1)\,r(m_2)\,V\Big(\frac{m_1-n}{H}\Big)V\Big(\frac{n-m_2}{X}\Big)$$

(see, e.g., [20] for a comparison). Thus,

$$\mathcal{T}(X,H)=\sum_{n\geq 1}r(n)\,U\Big(\frac{n}{X}\Big)\sum_{m_1+m_2=2n}r(m_1)\,r(m_2)\,V\Big(\frac{m_1-n}{H}\Big)V\Big(\frac{n-m_2}{X}\Big).$$

We shall manage to separate the variables in the shift $m_1+m_2=2n$ by invoking Lemma 2.2. It follows that actually there holds the following alternative form for the sum $\mathcal{T}$:

$$\begin{aligned}&\frac{c_Q}{Q^2}\sum_{q\geq 1}\ \sideset{}{^*}\sum_{\gamma \bmod q}\ \sum_{n\geq 1}r(n)\,e\left(-\frac{2\gamma n}{q}\right)\sum_{m_1\geq 1}r(m_1)\,e\left(\frac{m_1\gamma}{q}\right)\sum_{m_2\geq 1}r(m_2)\\&\times e\left(\frac{m_2\gamma}{q}\right)U\left(\frac{n}{X}\right)V\Big(\frac{m_1-n}{H}\Big)V\Big(\frac{n-m_2}{X}\Big)h\left(\frac{q}{Q},\frac{m_1+m_2-2n}{Q^2}\right).\end{aligned} \tag{3.2}$$

Here and thereafter, the parameter Q is taken as

$$Q=\sqrt{X}$$

and the constant c_Q is as in Lemma 2.2. We now intend to apply the Voronoĭ-type summation formula, Lemma 2.1, to the sums over m_1,m_2, respectively. Recall (2.4). One finds that two sub-sums arise every time that the Voronoĭ-type formula is put into use. We are thus led to four parts, i.e., the two degenerate terms $\mathcal{T}^{\text{Deg.},1}$, $\mathcal{T}^{\text{Deg.},2}$, the non-degenerate term $\mathcal{T}^{\text{Non-de.}}$ and the main term $\mathcal{T}^{\text{Main}}$, so that

$$\mathcal{T}(X,H)=\mathcal{T}^{\text{Deg.},1}(X,H)+\mathcal{T}^{\text{Deg.},2}(X,H)+\mathcal{T}^{\text{Non-de.}}(X,H)+\mathcal{T}^{\text{Main}}(X,H).$$

Here, these four terms are respectively given by the following

$$\begin{aligned}\mathcal{T}^{\text{Deg.},1}(X,H)=&\frac{(2\pi i)^2c_Q}{Q^2}\sum_{q\geq 1}\frac{1}{q^2}\sum_{n\geq 1}r(n)\sum_{m\geq 1}r(m)\,S(-2n,-m;q)\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}U\left(\frac{n}{X}\right)\\&\times V\Big(\frac{x-n}{H}\Big)V\Big(\frac{n-y}{X}\Big)h\left(\frac{q}{Q},\frac{x+y-2n}{Q^2}\right)J_0\left(\frac{4\pi\sqrt{mx}}{q}\right)\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{3.3}$$

$$\mathcal{T}^{\text{Deg.},2}(X,H) = \frac{(2\pi i)^2 c_Q}{Q^2}\sum_{q\geq 1}\frac{1}{q^2}\sum_{n\geq 1} r(n)\sum_{m\geq 1} r(m)\, S(-2n,-m;q)\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} U\left(\frac{n}{X}\right)$$
$$\times V\left(\frac{x-n}{H}\right)V\left(\frac{n-y}{X}\right)h\left(\frac{q}{Q},\frac{x+y-2n}{Q^2}\right)J_0\left(\frac{4\pi\sqrt{my}}{q}\right)\mathrm{d}x\mathrm{d}y,$$

$$\begin{aligned}\mathcal{T}^{\text{Non-de.}}(X,H) = &\frac{(2\pi i)^2 c_Q}{Q^2}\sum_{q\geq 1}\frac{1}{q^2}\sum_{n\geq 1} r(n)\sum_{m_1,m_2\geq 1} r(m_1)\, r(m_2)\, S(-2n,-(m_1+m_2);q)\\ &\times\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} U\left(\frac{n}{X}\right)V\left(\frac{x-n}{H}\right)V\left(\frac{n-y}{X}\right)h\left(\frac{q}{Q},\frac{x+y-2n}{Q^2}\right)\\ &\times J_0\left(\frac{4\pi\sqrt{m_1x}}{q}\right)J_0\left(\frac{4\pi\sqrt{m_2y}}{q}\right)\mathrm{d}x\mathrm{d}y,\end{aligned}\tag{3.4}$$

and

$$\mathcal{T}^{\text{Main}}(X,H) = \frac{(2\pi i)^2 c_Q}{Q^2}\sum_{q\geq 1}\frac{1}{q^2}\sum_{n\geq 1} r(n)\, S(-2n,0;q)\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} U\left(\frac{x}{X}\right)$$
$$\times V\left(\frac{x-n}{H}\right)V\left(\frac{n-y}{X}\right)h\left(\frac{q}{Q},\frac{x+y-2n}{Q^2}\right)\mathrm{d}x\mathrm{d}y.$$

Meanwhile, it is remarkable that

$$\mathcal{T}^{\text{Main}}(X,H) = H\sum_{n\geq 1} r(n)\sum_{q\geq 1}\frac{S(-2n,0;q)}{q^2}\mathcal{W}_{X,H}\left(\frac{n}{X},\frac{q}{\sqrt{X}}\right) + O\left(X^{-A}\right),$$

with

$$\mathcal{W}_{X,H}(z,w) = (2\pi i)^2\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} U\left(\frac{xH}{X}\right)V(z-y)\, V\left(x-\frac{zX}{H}\right)h\left(w, y-2z+\frac{xH}{X}\right)\mathrm{d}x\mathrm{d}y,\tag{3.5}$$

for any $z, w \in \mathbb{R}^+$ and sufficiently large $A \in \mathbb{R}^+$. However, this matches the main term in (1.2). Our task now is estimating these sub-sums above. One verifies that it suffices to consider the non-degenerate term $\mathcal{T}^{\text{Non-de.}}$ and the degenerate term $\mathcal{T}^{\text{Deg.},1}$. The same argument of $\mathcal{T}^{\text{Deg.},1}$ works for the term $\mathcal{T}^{\text{Deg.},2}$, which will indicate less importance of the latter case, as far as the contribution is concerned. In what follows, we will be dedicated to estimating the two crucial terms $\mathcal{T}^{\text{Non-de.}}$ and $\mathcal{T}^{\text{Deg.},1}$, one after another.

3.1. Bounding $\mathcal{T}^{\text{Non-de.}}$

In this part, let us have a look at the multiple sum $\mathcal{T}^{\text{Non-de.}}$ in (3.4). We are now ready to apply Lemma 2.1 to the sum over n. In view of the fact that

$$\left(\left(\frac{q}{d}\right)\epsilon_d^{-1}\right)^2 = \chi_4(a)$$

in (2.4). Here, $ad \equiv 1 \bmod q$, and χ_4 denotes the character modulo 4 which is given by $\chi_4(a) = 1$ if $a \equiv 1 \bmod 4$, and $\chi_4(a) = -1$ if $a \equiv 3 \bmod 4$. The sum $\mathcal{T}^{\text{Non-de.}}$ is thus converted into the following two sub-sums

$$\begin{aligned}\mathfrak{T}_1^{\text{Non-de.}}(X,H) =&\frac{(2\pi i)^3}{Q^2}\sum_{q\ll Q}\frac{1}{q^3}\sum_{n\geq 1} r(n)\sum_{m_1,m_2\geq 1} r(m_1)\, r(m_2)\,\widehat{S}(2n-(m_1+m_2),0;q)\\ &\times\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} U\left(\frac{z}{X}\right)V\left(\frac{x-z}{H}\right)V\left(\frac{z-y}{X}\right)h\left(\frac{q}{Q},\frac{x+y-2z}{Q^2}\right)\\ &\times J_0\left(\frac{4\pi\sqrt{m_1x}}{q}\right)J_0\left(\frac{4\pi\sqrt{m_2y}}{q}\right)J_0\left(\frac{4\pi\sqrt{nz}}{q}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z,\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\mathfrak{T}_2^{\text{Non-de.}}(X,H) =&\frac{(2\pi i)^3}{Q^2}\sum_{q\ll Q}\frac{1}{q^3}\sum_{m_1,m_2\geq 1} r(m_1)\, r(m_2)\,\widehat{S}(-(m_1+m_2),0;q)\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} U\left(\frac{z}{X}\right)\\ &\times V\left(\frac{x-z}{H}\right)V\left(\frac{z-y}{X}\right)h\left(\frac{q}{Q},\frac{x+y-2z}{Q^2}\right)J_0\left(\frac{4\pi\sqrt{m_1x}}{q}\right)J_0\left(\frac{4\pi\sqrt{m_2y}}{q}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.\end{aligned}\tag{3.7}$$

Here, the exponential sum $\widehat{S}$ is given by the following

$$\widehat{S}(n,0;q) = \sideset{}{^*}\sum_{a \bmod q}\chi_4(a)e\left(\frac{an}{q}\right)$$

for any $n, q \in \mathbb{N}$.

We now evaluate these two resulting terms above to achieve a satisfactory upper-bound estimate for $\mathfrak{T}^{\text{Non-de.}}$. First, one claims that the sum $\widehat{S}(n,0;q)$ can be replaced by $S(4n+\nu,0;q)$ for some $0\leq\nu\leq 4q$ in the following sense:

$$\begin{aligned}\widehat{S}(n,0;q) =& \sideset{}{^*}\sum_{\substack{a \bmod q\\ a\equiv 1 \bmod q}} e\left(\frac{na}{q}\right)-\sideset{}{^*}\sum_{\substack{a \bmod q\\ a\equiv -1 \bmod q}} e\left(\frac{na}{q}\right)\\ \asymp&\sum_{b \bmod 4} e\left(-\frac{b}{4}\right)\sum_{0\leq\nu\leq 4q}\frac{1}{1+\nu}\sideset{}{^*}\sum_{\alpha \bmod 4q} e\left(\frac{\alpha(4n+bq+\nu)}{4q}\right)\\ \asymp&\sum_{0\leq\nu\leq 4q}\frac{1}{1+\nu}S(4n+\nu,0;q)\end{aligned}\tag{3.8}$$

by the completing method (see [16, Chapter 12]). With these preparations, we now come to evaluating $\mathfrak{T}_1^{\text{Non-de.}}$ in (3.6). By invoking the features of Bessel functions as shown in (2.5) and (2.6), the variables n, m_1, m_2 essentially are truncated at

$$n, m_1 \ll_\varepsilon X^\varepsilon\left(\frac{q^2X}{H^2}+1\right),\quad m_2\ll_\varepsilon\frac{X^{1+\varepsilon}}{Q^2}\ll_\varepsilon X^\varepsilon,\tag{3.9}$$

respectively, by repeated integration by parts for many times (the contributions in the complementary ranges of theses variables, however, are all negligibly small). In the mean time, one infers that the triple integral in (3.6) is

$$\ll_\varepsilon \frac{X^{\frac{5}{4}+\varepsilon}q^{\frac{3}{2}}H}{(nm_1m_2)^{\frac{1}{4}}}\int_{\frac{1}{2}}^{\frac{5}{2}}\int_{z-\frac{5}{2}}^{z-\frac{1}{2}}\int_{\frac{zX}{H}+\frac{1}{2}}^{\frac{zX}{H}+\frac{5}{2}} U(z)\,V(z-y)\,V\left(x-\frac{zX}{H}\right)\left|h\left(\frac{q}{Q},\frac{xH+X(y-2z)}{Q^2}\right)\right|\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\ll_\varepsilon \frac{X^{\frac{1}{4}+\varepsilon}q^{\frac{3}{2}}HQ^2}{(nm_1m_2)^{\frac{1}{4}}}\int_{-\frac{X^{1+\varepsilon}}{Q^2}}^{\frac{X^{1+\varepsilon}}{Q^2}}\left|h\left(\frac{q}{Q},\xi\right)\right|\mathrm{d}\xi \ll_\varepsilon \frac{X^{\frac{5}{4}+\varepsilon}q^{\frac{3}{2}}H}{(nm_1m_2)^{\frac{1}{4}}}$$

by invoking (2.5) and (2.7). This implies that the right-hand side of (3.6) is dominated by

$$\begin{aligned}
&\ll_\varepsilon \frac{X^{\frac{5}{4}+\varepsilon}H}{Q^2}\sum_{q\ll Q}\frac{1}{q^{\frac{1}{2}-\varepsilon}}\sum_{n\ll_\varepsilon X^\varepsilon\left(\frac{q^2X}{H^2}+1\right)}\sup_{0\le\nu\le 4q}\sum_{\substack{m_1\ll_\varepsilon X^\varepsilon\left(\frac{q^2X}{H^2}+1\right)\\ m_2\ll_\varepsilon X^\varepsilon\\ 4(m_1+m_2)\equiv 8n+\nu \bmod q}}\frac{r(n)r(m_1)r(m_2)}{(nm_1m_2)^{\frac{1}{4}}}\\
&\ll_\varepsilon \frac{X^{\frac{5}{4}+\varepsilon}H}{Q^2}\sum_{q\ll Q}\frac{1}{q^{\frac{1}{2}-\varepsilon}}\left[\left(\frac{q^2X}{H^2}+1\right)^{\frac{3}{4}}+\left(\frac{q^2X}{H^2}+1\right)^{\frac{3}{2}}q^{-1}\right]\\
&\ll_\varepsilon X^{\frac{1}{4}+\varepsilon}H+\frac{X^{2+\varepsilon}}{\sqrt{H}}+\frac{X^{3+\varepsilon}}{H^2},
\end{aligned}\tag{3.10}$$

upon recalling (3.8), where we have have applied the estimate involving Ramanujan sum that

$$S(n,0;q)=\sum_{ab=q}\mu(a)\sum_{\beta \bmod q}e\left(\frac{\beta n}{q}\right).$$

Here, we have also followed the notational convention that μ is the Möbius function. One finds that the final bound of (3.10) beats the trivial estimate $O_\varepsilon(X^{1+\varepsilon}H)$ for $\mathcal{T}$, if it satisfies that $H \gg X^{2/3+\varepsilon}$.

Next, we turn to the estimation of $\mathcal{T}_2^{\text{Non-de.}}$ in (3.7). Note that the variables m_1, m_2 still satisfy the truncations as shown in (3.9). Meanwhile, one sees that the triple integral in (3.7) can be estimated as

$$\begin{aligned}
&\ll_\varepsilon \frac{X^{\frac{3}{2}+\varepsilon}Hq}{(m_1m_2)^{\frac{1}{4}}}\int_{\frac{1}{2}}^{\frac{5}{2}}\int_{z-\frac{5}{2}}^{z-\frac{1}{2}}\int_{\frac{zX}{H}+\frac{1}{2}}^{\frac{zX}{H}+\frac{5}{2}}U(z)V(z-y)V\left(x-\frac{zX}{H}\right)\left|h\left(\frac{q}{Q},\frac{xH+X(y-2z)}{Q^2}\right)\right|\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\ll_\varepsilon \frac{X^{\frac{3}{2}+\varepsilon}Hq}{(m_1m_2)^{\frac{1}{4}}}.
\end{aligned}\tag{3.11}$$

Thus, we infer that

$$\begin{aligned}
\mathcal{T}_2^{\text{Non-de.}}(X,H) &\ll_\varepsilon \frac{X^{\frac{3}{2}+\varepsilon}H}{Q^2}\sum_{q\ll Q}\frac{1}{q^{1-\varepsilon}}\sup_{0\le\nu\le 4q}\sum_{\substack{m_1\ll_\varepsilon X^\varepsilon\left(\frac{Q^2X}{H^2}+1\right)\\ m_2\ll_\varepsilon X^\varepsilon\left(\frac{Q^2}{X}+1\right)\\ 4(m_1+m_2)\equiv\nu \bmod q}}\frac{r(m_1)r(m_2)}{(m_1m_2)^{\frac{1}{4}}}\\
&\ll_\varepsilon \frac{X^{\frac{3}{2}+\varepsilon}H}{Q^2}\sum_{q\ll Q}\frac{1}{q^{1-\varepsilon}}\left(1+\frac{X^{\frac{3}{2}}}{qH^{\frac{3}{2}}}\right)\ll_\varepsilon X^{\frac{1}{2}+\varepsilon}H+\frac{X^{2+\varepsilon}}{\sqrt{H}}.
\end{aligned}\tag{3.12}$$

3.2. Bounding $\mathcal{T}^{Deg.,1}$

In this section, we proceed to considering another object $\mathcal{T}^{\text{Deg.},1}$, which is given as in (3.3). We will now follow closely the argument in §3.1. Upon repeating the procedure as in §3.1, it follows that, instead of (3.6) & (3.7), one has the following transformations for $\mathfrak{T}^{\mathrm{Deg.},1}$ (i.e., $\mathfrak{T}^{\mathrm{Deg.},1} = \mathfrak{T}_1^{\mathrm{Deg.},1} + \mathfrak{T}_2^{\mathrm{Deg.},1}$):

$$\begin{aligned}\mathfrak{T}_1^{\mathrm{Deg.},1}(X,H) = &\frac{(2\pi i)^3}{Q^2}\sum_{q\ll Q}\frac{1}{q^3}\sum_{n\geq 1}r(n)\sum_{m\geq 1}r(m)\,\widehat{S}(2n-m,0;q)\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}U\left(\frac{z}{X}\right)\\ &\times V\left(\frac{x-z}{H}\right)V\left(\frac{z-y}{X}\right)h\left(\frac{q}{Q},\frac{x+y-2z}{Q^2}\right)J_0\left(\frac{4\pi\sqrt{mx}}{q}\right)J_0\left(\frac{4\pi\sqrt{nz}}{q}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z,\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\mathfrak{T}_2^{\mathrm{Deg.},1}(X,H) = &\frac{(2\pi i)^3}{Q^2}\sum_{q\ll Q}\frac{1}{q^3}\sum_{m\geq 1}r(m)\,\widehat{S}(-m,0;q)\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}U\left(\frac{z}{X}\right)\\ &\times V\left(\frac{x-z}{H}\right)V\left(\frac{z-y}{X}\right)h\left(\frac{q}{Q},\frac{x+y-2z}{Q^2}\right)J_0\left(\frac{4\pi\sqrt{mx}}{q}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.\end{aligned} \tag{3.14}$$

Note that the triple integral in (3.13) has already been controlled by $O_\varepsilon\left(\frac{X^{3/2+\varepsilon}qH}{(mn)^{1/4}}\right)$, as in (3.11). Recall (3.9). Thus,

$$\begin{aligned}\mathfrak{T}_1^{\mathrm{Deg.},1}(X,H) &\ll_\varepsilon \frac{X^{\frac{3}{2}+\varepsilon}H}{Q^2}\sum_{q\ll Q}\frac{1}{q^{1-\varepsilon}}\sup_{0\leq\nu\leq 4q}\sum_{\substack{m\ll_\varepsilon X^\varepsilon\left(\frac{q^2X}{H^2}+1\right)\\ n\ll_\varepsilon X^\varepsilon\left(\frac{q^2X}{H^2}+1\right)\\ 4m\equiv 8n+\nu \bmod q}}\frac{r(m)r(n)}{(mn)^{\frac{1}{4}}}\\ &\ll_\varepsilon \frac{X^{\frac{3}{2}+\varepsilon}H}{Q^2}\sum_{q\ll Q}\frac{1}{q^{1-\varepsilon}}\left[\left(\frac{q^2X}{H^2}+1\right)^{\frac{1}{2}}+\left(\frac{q^2X}{H^2}+1\right)^{\frac{3}{2}}q^{-1}\right]\\ &\ll_\varepsilon X^{\frac{3}{2}+\varepsilon}+\frac{X^{3+\varepsilon}}{H^2}.\end{aligned} \tag{3.15}$$

In the same vein, it is inferrable that, in (3.14), the triple integral satisfies that

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\cdots \ll_\varepsilon \frac{X^{\frac{7}{4}+\varepsilon}\sqrt{q}H}{m^{\frac{1}{4}}},$$

whereby we are able to find

$$\begin{aligned}\mathfrak{T}_2^{\mathrm{Deg.},1}(X,H) &\ll_\varepsilon \frac{X^{\frac{7}{4}+\varepsilon}H}{Q^2}\sum_{q\ll Q}\frac{1}{q^{\frac{3}{2}-\varepsilon}}\sup_{0\leq\nu\leq 4q}\sum_{\substack{m\ll_\varepsilon X^\varepsilon\left(\frac{q^2X}{H^2}+1\right)\\ 4m\equiv\nu \bmod q}}\frac{r(m)}{m^{\frac{1}{4}}}\\ &\ll_\varepsilon \frac{X^{\frac{7}{4}+\varepsilon}H}{Q^2}\sum_{q\ll Q}\frac{1}{q^{\frac{3}{2}-\varepsilon}}\left[1+\left(\frac{q^2X}{H^2}+1\right)^{\frac{3}{4}}q^{-1}\right]\ll_\varepsilon X^{\frac{3}{4}+\varepsilon}H+\frac{X^{\frac{3}{2}+\varepsilon}}{\sqrt{H}}.\end{aligned} \tag{3.16}$$

Next, it remains to prove Theorem 1.1. Indeed, upon combining with (3.10), (3.12), (3.15) and (3.16), the desired estimate in (1.5) follows immediately.

4. Conclusions

In this paper, we study the correlation sums involving the Fourier coefficients of theta series. We try to develop the method to establish the asymptotic formula with power-saving error term. As a result, we show that whenever the parameter $H \geq X^{2/3+\varepsilon}$, this becomes possible. As remarked before, one might cover the case where the coefficients $r_\ell(n)$, $\ell \geq 3$, are taken into account. One may expect if the non-trivial results can be established for the sum in (1.5), with some of arithmetic functions therein replaced by Fourier coefficients of cusp forms of higher rank like GL_4-Hecke Maaß forms. However, these are all the programs we will pursue in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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