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Research article

A direct method for updating piezoelectric smart structural models based on measured modal data

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Abstract: A direct method for simultaneously updating mass and stiffness matrices of the undamped piezoelectric smart structural models based on incomplete modal measured data is presented. By applying the generalized singular value decomposition and some matrix derivatives, the optimal approximate mass and stiffness matrices which satisfy the required eigenvalue equation and the orthogonality relation are found under the Frobenius norm sense. The method is computationally efficient as neither iteration nor eigenanalysis is required. Numerical results are included to illustrate the effectiveness of the proposed method.

Keywords: piezoelectric smart structure; generalized singular value decomposition; model updating
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1. Introduction

Throughout this paper, $\mathbb{R}^{m\times n}$, $\mathbb{OR}^{n\times n}$ and $\mathbb{SR}^{n\times n}$ denote the sets of all $m\times n$ real matrices, all $n\times n$ orthogonal matrices and all $n\times n$ symmetric matrices, respectively. $A^\top$ and $\mathrm{tr}(A)$ stand for the transpose and the trace of a matrix A, respectively. I_n denotes the identity matrix of size n.

Piezoelectric smart materials are a class of materials with piezoelectric effect. Due to the development of smart materials and structures, these materials are endowed with strong vitality. Piezoelectric smart materials can rapidly transform pressure, vibration into electrical signals, or electrical signals into vibration signals, that is, the piezoelectric elements can be used as both sensors and actuators, which realizes the unity of the sensing elements and the action elements, and make them widely used in engineering. For example, piezoelectric materials are applied in active vibration control [1, 2], distributed dynamic measurement [3] and structural health monitoring [4–6], etc.

By using the finite element techniques, the global equation of motion for the undamped piezoelectric smart structure system with n degrees of freedom can be written as [7]:

$$M_A\ddot{\boldsymbol{v}} + K_A\boldsymbol{v} = \boldsymbol{f}. \tag{1.1}$$

The matrices M_A, K_A, and the vectors $\boldsymbol{v}, \boldsymbol{f}$ are of the form

$$M_A = \begin{bmatrix} \tilde{M}_{uu} & 0 \\ 0 & 0 \end{bmatrix},\ K_A = \begin{bmatrix} \tilde{K}_{uu} & \tilde{K}_{u\phi} \\ \tilde{K}_{u\phi}^\top & \tilde{K}_{\phi\phi} \end{bmatrix},\ \boldsymbol{v} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\phi} \end{bmatrix},\ \boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}_u \\ \boldsymbol{f}_\phi \end{bmatrix}, \tag{1.2}$$

where $\tilde{M}_{uu} \in \mathbb{SR}^{n_u\times n_u}$ is the structural mass matrix, $\tilde{K}_{uu} \in \mathbb{SR}^{n_u\times n_u}$ is the structural stiffness matrix, $\tilde{K}_{u\phi} \in \mathbb{R}^{n_u\times n_\phi}$ ($n_u + n_\phi = n$) is the piezoelectric coupling matrix, $\tilde{K}_{\phi\phi} \in \mathbb{SR}^{n_\phi\times n_\phi}$ is the dielectric stiffness matrix, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ denotes the mechanical displacement vector, $\boldsymbol{\phi} \in \mathbb{R}^{n_\phi}$ denotes the electrical potential vector, $\boldsymbol{f}_u \in \mathbb{R}^{n_u}$ denotes the mechanical external force vector and $\boldsymbol{f}_\phi \in \mathbb{R}^{n_\phi}$ denotes the external electric charge vector. It is known that the vibration of the mathematical model (1.1) is characterized by eigenvalues and eigenvectors of the following generalized inverse eigenvalue problem:

$$\omega \begin{bmatrix} \tilde{M}_{uu} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z_1} \\ \boldsymbol{z_2} \end{bmatrix} = \begin{bmatrix} \tilde{K}_{uu} & \tilde{K}_{u\phi} \\ \tilde{K}_{u\phi}^\top & \tilde{K}_{\phi\phi} \end{bmatrix} \begin{bmatrix} \boldsymbol{z_1} \\ \boldsymbol{z_2} \end{bmatrix}. \tag{1.3}$$

In general, owning to the difficulty in accurately determining some structural parameters, the unreasonable coupling simplification [8] and the mathematical description error of geometry and boundary conditions [9], the piezoelectric smart structure model established by finite element techniques may not truly describe the actual characteristics of the structure. Therefore, we need to update the model by applying the measured modal data such that the updated model can better reflect the physical structure and the measured results. Mathematically, the problem of updating piezoelectric smart structure model can be formulated as the following problem.

Problem IEP. Let $\Omega = \mathrm{diag}(\omega_1, ..., \omega_p) \in \mathbb{R}^{p\times p}$ and $Z = [Z_1^\top, Z_2^\top]^\top \in \mathbb{R}^{n\times p}$ be the measured eigenvalue and eigenvector matrices, where $Z_1 \in \mathbb{R}^{n_u\times p}$, $Z_2 \in \mathbb{R}^{n_\phi\times p}$ and $\mathrm{rank}(Z_1) = p$. Find matrices $M = \begin{bmatrix} M_{uu} & 0 \\ 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} K_{uu} & K_{u\phi} \\ K_{u\phi}^\top & K_{\phi\phi} \end{bmatrix} \in \mathbb{SR}^{n\times n}$ such that

$$MZ\Omega = KZ, \quad Z_1^\top M_{uu} Z_1 = I_p. \tag{1.4}$$

It is well known that the numerical model is a "good" representation of the structure, we hope to find a model that is closest to the original model. Thus, we should further consider the following best approximate problem:

Problem BAP. Given matrices $M_A, K_A \in \mathbb{SR}^{n\times n}$. Find $(\hat{M}, \hat{K}) \in \kappa_{\mathbb{S}}$ such that

$$\|\hat{M} - M_A\|^2 + \|\hat{K} - K_A\|^2 = \min_{(M,K)\in\kappa_{\mathbb{S}}} \left(\|M - M_A\|^2 + \|K - K_A\|^2\right), \tag{1.5}$$

where $\|\cdot\|$ is the Frobenius norm and $\kappa_{\mathbb{S}}$ is the solution set of Problem IEP.

Fish and Chen [10] developed the solution procedures for large-scale transient analysis of piezocomposites by using the representative volume element-based multilevel method. Xu and Koko [11] presented a general purpose design scheme of actively controlled piezoelectric smart structures by finite element modal analysis. More recently, Zhao and Liao [12] solved the updating problem of undamped piezoelectric smart structure systems with no-spillover and derived a set of parametric solutions. Nevertheless, the problem of BAP seems rarely to be discussed in the literatures, which motivates us to provide a numerical method to solve problems IEP and BAP. By applying the generalized singular value decomposition(GSVD) of a matrix pair, the expression of the general solution of Problem IEP is derived when the solvability conditions are satisfied and the best approximate solution of Problem BAP is obtained. Finally, two numerical examples are given to verify the correctness of the results.

2. The solution to Problem IEP

Note that $\operatorname{rank}(Z_1) = p$, then the GSVD [13,14] of the matrix pair $[Z_1^\top, Z_2^\top]$ is of the following form:

$$Z_1 = U\Sigma_1 N, \quad Z_2 = V\Sigma_2 N, \tag{2.1}$$

where $N \in \mathbb{R}^{p\times p}$ is nonsingular, and

$$\Sigma_1 = \begin{bmatrix} I & 0 \\ 0 & \Theta \\ 0 & 0 \end{bmatrix} \begin{matrix} p-s \\ s \\ n_u - p \end{matrix}, \quad \Sigma_2 = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \\ 0 & 0 \end{bmatrix} \begin{matrix} p-s \\ s \\ n_\phi - p \end{matrix},$$
$$\qquad\qquad p-s \quad s \qquad\qquad\qquad\qquad p-s \quad s$$

$$U = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix} \in \mathbb{OR}^{n_u\times n_u}, \quad V = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \in \mathbb{OR}^{n_\phi\times n_\phi},$$

and

$$\Theta = \operatorname{diag}(\theta_1, \cdots, \theta_s), \quad \Delta = \operatorname{diag}(\delta_1, \cdots, \delta_s)$$

with

$$1 > \theta_1 \geq \theta_2 \geq \cdots \geq \theta_s > 0, \quad 0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_s < 1,$$
$$\theta_i^2 + \delta_i^2 = 1, \ i = 1, \cdots, s.$$

By (2.1), Eq (1.4) can be equivalently written as

$$\begin{bmatrix} M_{uu} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U\Sigma_1 N \\ V\Sigma_2 N \end{bmatrix} \Omega = \begin{bmatrix} K_{uu} & K_{u\phi} \\ K_{u\phi}^\top & K_{\phi\phi} \end{bmatrix} \begin{bmatrix} U\Sigma_1 N \\ V\Sigma_2 N \end{bmatrix}, \tag{2.2}$$

$$N^\top \Sigma_1^\top U^\top M_{uu} U\Sigma_1 N = I_p, \tag{2.3}$$

that is,

$$M_{uu} U\Sigma_1 N\Omega = K_{uu} U\Sigma_1 N + K_{u\phi} V\Sigma_2 N, \tag{2.4}$$

$$K_{u\phi}^\top U\Sigma_1 N + K_{\phi\phi} V\Sigma_2 N = 0, \tag{2.5}$$

$$\Sigma_1^\top U^\top M_{uu} U\Sigma_1 = N^{-\top} N^{-1}. \tag{2.6}$$

Let

$$N\Omega N^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{matrix} p-s \\ s \end{matrix}, \quad N^{-\top} N^{-1} = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^\top & N_{22} \end{bmatrix} \begin{matrix} p-s \\ s \end{matrix}, \tag{2.7}$$
$$\qquad p-s \quad s \qquad\qquad\qquad\qquad p-s \quad s$$

$$U^{\top}M_{uu}U=\begin{array}{c}\left[\begin{array}{ccc} M_{11} & M_{12} & M_{13} \\ M_{12}^{\top} & M_{22} & M_{23} \\ M_{13}^{\top} & M_{23}^{\top} & M_{33} \end{array}\right]\begin{array}{c} p-s \\ s \\ n_u-p \end{array} \\ \begin{array}{ccc} p-s & s & n_u-p \end{array}\end{array}, \tag{2.8}$$

$$U^{\top}K_{uu}U=\begin{array}{c}\left[\begin{array}{ccc} F_{11} & F_{12} & F_{13} \\ F_{12}^{\top} & F_{22} & F_{23} \\ F_{13}^{\top} & F_{23}^{\top} & F_{33} \end{array}\right]\begin{array}{c} p-s \\ s \\ n_u-p \end{array} \\ \begin{array}{ccc} p-s & s & n_u-p \end{array}\end{array}, \tag{2.9}$$

$$V^{\top}K_{\phi\phi}V=\begin{array}{c}\left[\begin{array}{ccc} G_{11} & G_{12} & G_{13} \\ G_{12}^{\top} & G_{22} & G_{23} \\ G_{13}^{\top} & G_{23}^{\top} & G_{33} \end{array}\right]\begin{array}{c} p-s \\ s \\ n_{\phi}-p \end{array} \\ \begin{array}{ccc} p-s & s & n_{\phi}-p \end{array}\end{array}, \tag{2.10}$$

$$U^{\top}K_{u\phi}V=\begin{array}{c}\left[\begin{array}{ccc} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{array}\right]\begin{array}{c} p-s \\ s \\ n_u-p \end{array} \\ \begin{array}{ccc} p-s & s & n_{\phi}-p \end{array}\end{array}. \tag{2.11}$$

Thus, Eqs (2.4) – (2.6) are equivalent to

$$\begin{aligned}&\left[\begin{array}{ccc} F_{11} & F_{12} & F_{13} \\ F_{12}^{\top} & F_{22} & F_{23} \\ F_{13}^{\top} & F_{23}^{\top} & F_{33} \end{array}\right]\left[\begin{array}{cc} I & 0 \\ 0 & \Theta \\ 0 & 0 \end{array}\right]+\left[\begin{array}{ccc} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{array}\right]\left[\begin{array}{cc} 0 & 0 \\ 0 & \Delta \\ 0 & 0 \end{array}\right]= \\ &\left[\begin{array}{ccc} M_{11} & M_{12} & M_{13} \\ M_{12}^{\top} & M_{22} & M_{23} \\ M_{13}^{\top} & M_{23}^{\top} & M_{33} \end{array}\right]\left[\begin{array}{cc} I & 0 \\ 0 & \Theta \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} S_{11} & S_{12} \\ S_{21} & S_{22} \end{array}\right],\end{aligned} \tag{2.12}$$

$$\left[\begin{array}{ccc} L_{11}^{\top} & L_{21}^{\top} & L_{31}^{\top} \\ L_{12}^{\top} & L_{22}^{\top} & L_{32}^{\top} \\ L_{13}^{\top} & L_{23}^{\top} & L_{33}^{\top} \end{array}\right]\left[\begin{array}{cc} I & 0 \\ 0 & \Theta \\ 0 & 0 \end{array}\right]+\left[\begin{array}{ccc} G_{11} & G_{12} & G_{13} \\ G_{12}^{\top} & G_{22} & G_{23} \\ G_{13}^{\top} & G_{23}^{\top} & G_{33} \end{array}\right]\left[\begin{array}{cc} 0 & 0 \\ 0 & \Delta \\ 0 & 0 \end{array}\right]=0, \tag{2.13}$$

$$\left[\begin{array}{ccc} I & 0 & 0 \\ 0 & \Theta & 0 \end{array}\right]\left[\begin{array}{ccc} M_{11} & M_{12} & M_{13} \\ M_{12}^{\top} & M_{22} & M_{23} \\ M_{13}^{\top} & M_{23}^{\top} & M_{33} \end{array}\right]\left[\begin{array}{cc} I & 0 \\ 0 & \Theta \\ 0 & 0 \end{array}\right]=\left[\begin{array}{cc} N_{11} & N_{12} \\ N_{12}^{\top} & N_{22} \end{array}\right]. \tag{2.14}$$

Comparing two sides of (2.12) – (2.14), we have

$$M_{11}=N_{11},\ M_{12}=N_{12}\Theta^{-1},\ M_{22}=\Theta^{-1}N_{22}\Theta^{-1}, \tag{2.15}$$

$$F_{11}=N_{11}S_{11}+N_{12}S_{21}, \tag{2.16}$$

$$F_{12}=(N_{11}S_{12}+N_{12}S_{22})\Theta^{-1},\ F_{12}^{\top}=\Theta^{-1}(N_{12}^{\top}S_{11}+N_{22}S_{21}), \tag{2.17}$$

$$F_{13}=S_{11}^{\top}M_{13}+S_{21}^{\top}\Theta M_{23},\ F_{22}=\Theta^{-1}(N_{12}^{\top}S_{12}+N_{22}S_{22}+\Delta G_{22}\Delta)\Theta^{-1}, \tag{2.18}$$

$$F_{23}=\Theta^{-1}(S_{12}^{\top}M_{13}+S_{22}^{\top}\Theta M_{23}-\Delta L_{32}^{\top}), \tag{2.19}$$

$$L_{11}=0,\ L_{12}=0,\ L_{13}=0, \tag{2.20}$$

$$L_{21}=-\Theta^{-1}\Delta G_{12}^{\top},\ L_{22}=-\Theta^{-1}\Delta G_{22},\ L_{23}=-\Theta^{-1}\Delta G_{23}. \tag{2.21}$$

In summary, we can state the follwing theorem.

Theorem 2.1. *Suppose that* $\Omega = \mathrm{diag}(\omega_1, ..., \omega_\mathrm{p}) \in \mathbb{R}^{\mathrm{p}\times\mathrm{p}}$ *and* $Z = [Z_1^\top, Z_2^\top]^\top \in \mathbb{R}^{n\times p}$ *are the measured eigenvalue and eigenvector matrices, where* $Z_1 \in \mathbb{R}^{n_u\times p}$, $Z_2 \in \mathbb{R}^{n_\phi\times p}$ *and* $\mathrm{rank}(\mathrm{Z}_1) = \mathrm{p}$. *Let the GSVD of the matrix pair* $[Z_1^\top, Z_2^\top]$ *be given by* (2.1). *Then Problem IEP is solvable if and only if*

$$\begin{aligned} N_{11}S_{11} + N_{12}S_{21} &= S_{11}^\top N_{11} + S_{21}^\top N_{12}^\top, \\ S_{12}^\top N_{11} + S_{22}^\top N_{12}^\top &= N_{12}^\top S_{11} + N_{22}S_{21}, \\ N_{12}^\top S_{12} + N_{22}S_{22} &= S_{12}^\top N_{12} + S_{22}^\top N_{22}. \end{aligned} \tag{2.22}$$

In this case, the solution set of Problem IEP can be expressed as

$$\kappa_{\mathbb{S}} = \left\{ (M, K): \quad M = \begin{bmatrix} M_{uu} & 0 \\ 0 & 0 \end{bmatrix}, \ K = \begin{bmatrix} K_{uu} & K_{u\phi} \\ K_{u\phi}^\top & K_{\phi\phi} \end{bmatrix} \right\}, \tag{2.23}$$

where

$$M_{uu} = U \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12}^\top & M_{22} & M_{23} \\ M_{13}^\top & M_{23}^\top & M_{33} \end{bmatrix} U^\top, \quad K_{uu} = U \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{12}^\top & F_{22} & F_{23} \\ F_{13}^\top & F_{23}^\top & F_{33} \end{bmatrix} U^\top \ , \tag{2.24}$$

$$K_{u\phi} = U \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} V^\top, \quad K_{\phi\phi} = V \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}^\top & G_{22} & G_{23} \\ G_{13}^\top & G_{23}^\top & G_{33} \end{bmatrix} V^\top \ , \tag{2.25}$$

$M_{h3}(h = 1, 2)$, $L_{3k}(k = 1, 2, 3)$, G_{12}, G_{13}, G_{23} are arbitrary matrices, and M_{33}, F_{33}, $G_{kk}(k = 1, 2, 3)$ are arbitrary symmetric matrices; and M_{21}, F_{12}, F_{13}, F_{23}, $L_{hk}(h = 1, 2, k = 1, 2, 3)$ and $M_{kk}, F_{kk}(k = 1, 2)$ are given by (2.15)–(2.21).

3. The solution to Problem BAP

In order to solve Problem BAP, the following lemma is needed.

Lemma 3.1. *Let* $A,\ B \in \mathbb{SR}^{s\times s}$, $C \in \mathbb{R}^{s\times s}$, *and* $\Theta = \mathrm{diag}(\theta_1, \cdots, \theta_\mathrm{s}) \in \mathbb{R}^{\mathrm{s}\times\mathrm{s}}$, $\Delta = \mathrm{diag}(\delta_1, \cdots, \delta_\mathrm{s}) \in \mathbb{R}^{\mathrm{s}\times\mathrm{s}}$ *satisfy* $\theta_i^2 + \delta_i^2 = 1,\ i = 1, \cdots, s$. *Then*

$$\Psi(G_{22}) = \|\Theta^{-1}\Delta G_{22}\Delta\Theta^{-1} + A\|^2 + \|G_{22} - B\|^2 + 2\|\Theta^{-1}\Delta G_{22} + C\|^2 = \min,$$

$$\text{s. t. } G_{22} = G_{22}^\top$$

if and only if

$$G_{22} = -\Delta\Theta A\Theta\Delta + \Theta^2 B\Theta^2 - \Delta\Theta C\Theta^2 - \Theta^2 C^\top \Theta\Delta. \tag{3.1}$$

Proof. Let $A = [a_{ij}],\ B = [b_{ij}],\ C = [c_{ij}] \in \mathbb{R}^{s\times s}$, and $G_{22} = [g_{ij}] \in \mathbb{R}^{s\times s}$. Then

$$\Psi(G_{22}) := \sum_{i=1}^{s}\sum_{j=1}^{s}\left((\frac{\delta_i}{\theta_i} g_{ij} \frac{\delta_j}{\theta_j} + a_{ij})^2 + (g_{ij} - b_{ij})^2 + 2(\frac{\delta_i}{\theta_i} g_{ij} + c_{ij})^2 \right).$$

Now we minimize the quantities

$$\begin{aligned} \psi_{ij} &= (\tfrac{\delta_i}{\theta_i} g_{ij} \tfrac{\delta_j}{\theta_j} + a_{ij})^2 + (g_{ij} - b_{ij})^2 + 2(\tfrac{\delta_i}{\theta_i} g_{ij} + c_{ij})^2 \\ &+ (\tfrac{\delta_j}{\theta_j} g_{ji} \tfrac{\delta_i}{\theta_i} + a_{ji})^2 + (g_{ji} - b_{ji})^2 + 2(\tfrac{\delta_j}{\theta_j} g_{ji} + c_{ji})^2, \ 1 \le i, j \le s. \end{aligned}$$

By direct calculation, we have the minimizers

$$g_{ij} = -\delta_i\theta_i a_{ij}\theta_j\delta_j + \theta_i^2 b_{ij}\theta_j^2 - \delta_i\theta_i c_{ij}\theta_j^2 - \theta_j^2 c_{ji}\theta_i\delta_i,\ 1 \le i, j \le s. \tag{3.2}$$

By rewriting (3.2) in matrix form, we can get (3.1). □

It is easy to verify that $\kappa_{\mathbb{S}}$ is a closed convex subset of $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$. From the best approximate theorem [15], we know that there exists a unique solution $(\hat{M}, \hat{K}) \in \kappa_{\mathbb{S}}$ to Problem BAP. For the given matrices $M_A, K_A \in \mathbb{SR}^{n\times n}$, write

$$U^\top \tilde{M}_{uu} U = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} & \tilde{M}_{13} \\ \tilde{M}_{12}^\top & \tilde{M}_{22} & \tilde{M}_{23} \\ \tilde{M}_{13}^\top & \tilde{M}_{23}^\top & \tilde{M}_{33} \end{bmatrix} \begin{matrix} p-s \\ s \\ n_u - p \end{matrix}, \quad \begin{matrix} p-s & s & n_u-p \end{matrix} \tag{3.3}$$

$$U^\top \tilde{K}_{uu} U = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} & \tilde{F}_{13} \\ \tilde{F}_{12}^\top & \tilde{F}_{22} & \tilde{F}_{23} \\ \tilde{F}_{13}^\top & \tilde{F}_{23}^\top & \tilde{F}_{33} \end{bmatrix} \begin{matrix} p-s \\ s \\ n_u - p \end{matrix}, \quad \begin{matrix} p-s & s & n_u-p \end{matrix} \tag{3.4}$$

$$V^\top \tilde{K}_{\phi\phi} V = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} & \tilde{G}_{13} \\ \tilde{G}_{12}^\top & \tilde{G}_{22} & \tilde{G}_{23} \\ \tilde{G}_{13}^\top & \tilde{G}_{23}^\top & \tilde{G}_{33} \end{bmatrix} \begin{matrix} p-s \\ s \\ n_\phi - p \end{matrix}, \quad \begin{matrix} p-s & s & n_\phi-p \end{matrix} \tag{3.5}$$

$$U^\top \tilde{K}_{u\phi} V = \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} \\ \tilde{L}_{21} & \tilde{L}_{22} & \tilde{L}_{23} \\ \tilde{L}_{31} & \tilde{L}_{32} & \tilde{L}_{33} \end{bmatrix} \begin{matrix} p-s \\ s \\ n_u - p \end{matrix}. \quad \begin{matrix} p-s & s & n_\phi-p \end{matrix} \tag{3.6}$$

Then

$$\begin{aligned} & \|M - M_A\|^2 + \|K - K_A\|^2 \\ = \ & \|M_{uu} - \tilde{M}_{uu}\|^2 + \|K_{uu} - \tilde{K}_{uu}\|^2 + 2\|K_{u\phi} - \tilde{K}_{u\phi}\|^2 + \|K_{\phi\phi} - \tilde{K}_{\phi\phi}\|^2 \\ = \ & \left\| \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12}^\top & M_{22} & M_{23} \\ M_{13}^\top & M_{23}^\top & M_{33} \end{bmatrix} - U^\top \tilde{M}_{uu} U \right\|^2 + \left\| \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{12}^\top & F_{22} & F_{23} \\ F_{13}^\top & F_{23}^\top & F_{33} \end{bmatrix} - U^\top \tilde{K}_{uu} U \right\|^2 \\ + \ & 2\left\| \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} - U^\top \tilde{K}_{u\phi} V \right\|^2 + \left\| \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}^\top & G_{22} & G_{23} \\ G_{13}^\top & G_{23}^\top & G_{33} \end{bmatrix} - V^\top \tilde{K}_{\phi\phi} V^\top \right\|^2. \end{aligned}$$

Therefore, $\|M - M_A\|^2 + \|K - K_A\|^2 = \min$ if and only if

$$M_{33} = \tilde{M}_{33}, F_{33} = \tilde{F}_{33}, L_{31} = \tilde{L}_{31}, L_{33} = \tilde{L}_{33},$$

$$G_{11} = \tilde{G}_{11}, G_{13} = \tilde{G}_{13}, G_{33} = \tilde{G}_{33},$$

$$\begin{aligned} f(M_{13}, M_{23}, L_{32}) \ &= 2\|M_{13} - \tilde{M}_{13}\|^2 + 2\|M_{23} - \tilde{M}_{23}\|^2 + 2\|S_{11}^\top M_{13} + S_{21}^\top \Theta M_{23} - \tilde{F}_{13}\|^2 \\ & +2\|\Theta^{-1}(S_{12}^\top M_{13} + S_{22}^\top \Theta M_{23} - \Delta L_{32}^\top) - \tilde{F}_{23}\|^2 + 2\|L_{32} - \tilde{L}_{32}\|^2 = \min, \end{aligned} \tag{3.7}$$

$$f(G_{12}) = 2\|\Theta^{-1}\Delta G_{12}^{\top} + \tilde{L}_{21}\|^2 + 2\|G_{12} - \tilde{G}_{12}\|^2 = \min, \tag{3.8}$$

$$\begin{aligned} f(G_{22}) &= \|\Theta^{-1}(N_{12}^{\top}S_{12} + N_{22}S_{22} + \Delta G_{22}\Delta)\Theta^{-1} - \tilde{F}_{22}\|^2 \\ &+\|G_{22} - \tilde{G}_{22}\|^2 + 2\|\Theta^{-1}\Delta G_{22} + \tilde{L}_{22}\|^2 = \min, \end{aligned} \tag{3.9}$$

$$f(G_{23}) = 2\|\Theta^{-1}\Delta G_{23} + \tilde{L}_{23}\|^2 + 2\|G_{23} - \tilde{G}_{23}\|^2 = \min. \tag{3.10}$$

From (3.7), we have

$$\begin{aligned} f(M_{13}, M_{23}, L_{32}) &= 2\mathrm{tr}[(M_{13} - \tilde{M}_{13})^{\top}(M_{13} - \tilde{M}_{13}) + (M_{23} - \tilde{M}_{23})^{\top}(M_{23} - \tilde{M}_{23}) \\ &+(S_{11}^{\top}M_{13} + S_{21}^{\top}\Theta M_{23} - \tilde{F}_{13})^{\top}(S_{11}^{\top}M_{13} + S_{21}^{\top}\Theta M_{23} - \tilde{F}_{13}) \\ &+(\Theta^{-1}(S_{12}^{\top}M_{13} + S_{22}^{\top}\Theta M_{23} - \Delta L_{32}^{\top}) - \tilde{F}_{23})^{\top} \\ &(\Theta^{-1}(S_{12}^{\top}M_{13} + S_{22}^{\top}\Theta M_{23} - \Delta L_{32}^{\top}) - \tilde{F}_{23}) + (L_{32} - \tilde{L}_{32})^{\top}(L_{32} - \tilde{L}_{32})]. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial f(M_{13},M_{23},L_{32})}{\partial M_{13}} &= 4(M_{13} - \tilde{M}_{13} + S_{11}S_{11}^{\top}M_{13} + S_{11}S_{21}^{\top}\Theta M_{23} - S_{11}\tilde{F}_{13} \\ &+S_{12}\Theta^{-2}S_{12}^{\top}M_{13} + S_{12}\Theta^{-2}S_{22}^{\top}\Theta M_{23} - S_{12}\Theta^{-2}\Delta L_{32}^{\top} - S_{12}\Theta^{-1}\tilde{F}_{23}), \\ \frac{\partial f(M_{13},M_{23},L_{32})}{\partial M_{23}} &= 4(M_{23} - \tilde{M}_{23} + \Theta S_{21}S_{11}^{\top}M_{13} + \Theta S_{21}S_{21}^{\top}\Theta M_{23} - \Theta S_{21}\tilde{F}_{13} \\ &+\Theta S_{22}\Theta^{-2}S_{12}^{\top}M_{13} + \Theta S_{22}\Theta^{-2}S_{22}^{\top}\Theta M_{23} - \Theta S_{22}\Theta^{-2}\Delta L_{32}^{\top} - \Theta S_{22}\Theta^{-1}\tilde{F}_{23}), \\ \frac{\partial f(M_{13},M_{23},L_{32})}{\partial L_{32}} &= 4(L_{32} - \tilde{L}_{32} - M_{13}^{\top}S_{12}\Theta^{-2}\Delta - M_{23}^{\top}\Theta S_{22}\Theta^{-2}\Delta + L_{32}\Delta\Theta^{-2}\Delta + \tilde{F}_{23}^{\top}\Theta^{-1}\Delta). \end{aligned}$$

Clearly, $f(M_{13}, M_{23}, L_{32}) = \min$ if and only if

$$\frac{\partial f(M_{13}, M_{23}, L_{32})}{\partial M_{13}} = 0,\ \frac{\partial f(M_{13}, M_{23}, L_{32})}{\partial M_{23}} = 0,\ \frac{\partial f(M_{13}, M_{23}, L_{32})}{\partial L_{32}} = 0.$$

When $\frac{\partial f(M_{13},M_{23},L_{32})}{\partial M_{13}} = 0$, we arrive at

$$M_{13} = P_2M_{23} + P_3L_{32}^{\top} + P_4, \tag{3.11}$$

where

$$\begin{aligned} P_1 &= (I_{p-s} + S_{11}S_{11}^{\top} + S_{12}\Theta^{-2}S_{12}^{\top})^{-1}, & P_2 &= -P_1(S_{11}S_{21}^{\top}\Theta + S_{12}\Theta^{-2}S_{22}^{\top}\Theta), \\ P_3 &= P_1S_{12}\Theta^{-2}\Delta, & P_4 &= P_1(\tilde{M}_{13} + S_{11}\tilde{F}_{13} + S_{12}\Theta^{-1}\tilde{F}_{23}). \end{aligned}$$

When $\frac{\partial f(M_{13},M_{23},L_{32})}{\partial M_{23}} = 0$, we obtain

$$M_{23} = P_6M_{13} + P_7L_{32}^{\top} + P_8, \tag{3.12}$$

where

$$\begin{aligned} P_5 &= (I_s + \Theta S_{21}S_{21}^{\top}\Theta + \Theta S_{22}\Theta^{-2}S_{22}^{\top}\Theta)^{-1}, \\ P_6 &= -P_5(\Theta S_{21}S_{11}^{\top} + \Theta S_{22}\Theta^{-2}S_{12}^{\top}), \\ P_7 &= P_5\Theta S_{22}\Theta^{-2}\Delta, \\ P_8 &= P_5(\tilde{M}_{23} + \Theta S_{21}\tilde{F}_{13} + \Theta S_{22}\Theta^{-1}\tilde{F}_{23}). \end{aligned}$$

When $\frac{\partial f(M_{13},M_{23},L_{32})}{\partial L_{32}} = 0$, we get

$$L_{32} = M_{13}^{\top}P_{10} + M_{23}^{\top}P_{11} + P_{12}, \tag{3.13}$$

where

$$P_9 = (I_s + \Delta\Theta^{-2}\Delta)^{-1}, \quad P_{10} = S_{12}\Theta^{-2}\Delta P_9,$$
$$P_{11} = \Theta S_{22}\Theta^{-2}\Delta P_9, \quad P_{12} = (\tilde{L}_{32} - \tilde{F}_{23}^{\top}\Theta^{-1}\Delta)P_9.$$

Substituting (3.13) into (3.11) leads to

$$M_{13} = P_{14}M_{23} + P_{15}, \tag{3.14}$$

where

$$P_{13} = (I_{p-s} - P_3P_{10}^{\top})^{-1}, \quad P_{14} = P_{13}(P_2 + P_3P_{11}^{\top}), \quad P_{15} = P_{13}(P_3P_{12}^{\top} + P4).$$

Substituting (3.13) and (3.14) into (3.12), we have

$$M_{23} = (I_s - P_6P_{14} - P_7P_{10}^{\top}P_{14} - P_7P_{11}^{\top})^{-1}(P_6P_{15} + P_7P_{10}^{\top}P_{15} + P_7P_{12}^{\top} + P_8). \tag{3.15}$$

From (3.8), we have

$$f(G_{12}) = 2\mathrm{tr}[(\Theta^{-1}\Delta G_{12}^{\top} + \tilde{L}_{21})^{\top}(\Theta^{-1}\Delta G_{12}^{\top} + \tilde{L}_{21}) + (G_{12} - \tilde{G}_{12})^{\top}(G_{12} - \tilde{G}_{12})].$$

Consequently,

$$\frac{\partial f(G_{12})}{\partial G_{12}} = 4(G_{12}\Delta\Theta^{-2}\Delta + \tilde{L}_{21}^{\top}\Theta^{-1}\Delta + G_{12} - \tilde{G}_{12}),$$

when $\frac{\partial f(G_{12})}{\partial G_{12}} = 0$, we conclude that

$$G_{12} = (\tilde{G}_{12} - \tilde{L}_{21}^{\top}\Theta^{-1}\Delta)(I_s + \Delta\Theta^{-2}\Delta)^{-1}. \tag{3.16}$$

From (3.10), we have

$$f(G_{23}) = 2\mathrm{tr}[(\Theta^{-1}\Delta G_{23} + \tilde{L}_{23})^{\top}(\Theta^{-1}\Delta G_{23} + \tilde{L}_{23}) + (G_{23} - \tilde{G}_{23})^{\top}(G_{23} - \tilde{G}_{23})].$$

Thus,

$$\frac{\partial f(G_{23})}{\partial G_{23}} = 4(\Delta\Theta^{-2}\Delta G_{23} + \Delta\Theta^{-1}\tilde{L}_{23} + G_{23} - \tilde{G}_{23}),$$

when $\frac{\partial f(G_{23})}{\partial G_{23}} = 0$, we can get

$$G_{23} = (I_s + \Delta\Theta^{-2}\Delta)^{-1}(\tilde{G}_{23} - \Delta\Theta^{-1}\tilde{L}_{23}). \tag{3.17}$$

Solving the minimization problem $f(G_{22})$ by using Lemma 3.1, we have

$$G_{22} = -\Delta(N_{12}^{\top}S_{12} + N_{22}S_{22} - \Theta\tilde{F}_{22}\Theta)\Delta + \Theta^2\tilde{G}_{22}\Theta^2 - \Delta\Theta\tilde{L}_{22}\Theta^2 - \Theta^2\tilde{L}_{22}^{\top}\Theta\Delta. \tag{3.18}$$

Theorem 3.1. *Given matrices $M_A, K_A \in \mathbb{SR}^{n\times n}$. If the solvability conditions of* (2.22) *are satisfied, then the solution of Problem BAP is*

$$\hat{M} = \begin{bmatrix} M_{uu} & 0 \\ 0 & 0 \end{bmatrix}, \ \hat{K} = \begin{bmatrix} K_{uu} & K_{u\phi} \\ K_{u\phi}^{\top} & K_{\phi\phi} \end{bmatrix}, \tag{3.19}$$

where

$$M_{uu} = U \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12}^\top & M_{22} & M_{23} \\ M_{13}^\top & M_{23}^\top & M_{33} \end{bmatrix} U^\top, \quad K_{uu} = U \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{12}^\top & F_{22} & F_{23} \\ F_{13}^\top & F_{23}^\top & F_{33} \end{bmatrix} U^\top, \tag{3.20}$$

$$K_{u\phi} = U \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} V^\top, \quad K_{\phi\phi} = V \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}^\top & G_{22} & G_{23} \\ G_{13}^\top & G_{23}^\top & G_{33} \end{bmatrix} V^\top, \tag{3.21}$$

and $M_{13},\ M_{23},\ L_{32},\ G_{12},\ G_{22}, G_{23}$ *are given by* (3.14), (3.15), (3.13), (3.16), (3.18), (3.17), *respectively.*

4. Numerical examples

According to Theorems 2.1 and 3.1, we can describe a numerical algorithm to solve Problem BAP.

Algorithm 1

1: Input Ω, Z, M_A, K_A.
2: Compute the GSVD of the matrix pair $[Z_1^\top, Z_2^\top]$ by (2.1).
3: Compute $N_{ij}, S_{ij}, i, j = 1, 2$ by (2.7).
4: If the conditions (2.22) are satisfied, go to Step 5; otherwise, Problem IEP has no solution, and stop.
5: Compute $\tilde{M}_{hk}, \tilde{F}_{hk}, \tilde{G}_{hk}, \tilde{L}_{hk}, h, k = 1, 2, 3$ by (3.3)–(3.6).
6: Compute $M_{13},\ M_{23},\ L_{32},\ G_{12},\ G_{22}$ and G_{23} by (3.14), (3.15), (3.13), (3.16), (3.18) and (3.17), respectively.
7: Compute $M_{uu},\ K_{uu}$ and $K_{u\phi},\ K_{\phi\phi}$ by (3.20) and (3.21), respectively.
8: Compute $\hat{M}, \hat{K}$ by (3.19).

Example 4.1. Let $n = 10,\ p = 3$, and the matrices $\Omega,\ Z,\ M_A$ and K_A be given by

$$\Omega = \text{diag}\{0.5339,\ 4.5445,\ 179.7010\},$$

$$Z = \begin{bmatrix} -0.5315 & 2.5021 & -15.6029 \\ 0.5559 & -0.9288 & 1.4348 \\ 0.3428 & -0.0872 & -11.3065 \\ -0.0258 & -0.7476 & 1.4203 \\ -0.3691 & -1.5875 & -7.1639 \\ -0.3037 & -0.2276 & 1.3191 \\ -0.6784 & -1.4794 & -3.5477 \\ 0.0547 & 0.2352 & 1.0613 \\ -0.2982 & -0.5633 & -0.9779 \\ 0.1559 & 0.3111 & 0.6216 \end{bmatrix},$$

$$M_A = \begin{bmatrix} 1/3 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 2/3 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/6 & 2/3 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_A = \begin{bmatrix} 2 & 3 & -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -3 & 4 & 0 & -2 & 3 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 14 & -3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 & 4 & 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 14 & -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -3 & 4 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 3 & 3 & 0 & 14 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 14 \end{bmatrix}.$$

It is easy to verify that the conditions (2.22) hold. By applying Algorithm 1, we can obtain the unique solution $(\hat{M}, \hat{K})$ of Problem BAP as follows:

$$\hat{M} = \begin{bmatrix} 0.1200 & 0.0568 & -0.1942 & 0.0136 & 0.0804 & 0.1002 & 0 & 0 & 0 & 0 \\ 0.0568 & 0.7758 & 0.1625 & -0.0581 & -0.2730 & -0.2024 & 0 & 0 & 0 & 0 \\ -0.1942 & 0.1625 & 0.4989 & 0.0412 & -0.3591 & -0.2163 & 0 & 0 & 0 & 0 \\ 0.0136 & -0.0581 & 0.0412 & 0.6294 & 0.0200 & 0.0177 & 0 & 0 & 0 & 0 \\ 0.0804 & -0.2730 & -0.3591 & 0.0200 & 0.3903 & 0.2391 & 0 & 0 & 0 & 0 \\ 0.1002 & -0.2024 & -0.2163 & 0.0177 & 0.2391 & 0.8587 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{K} = \begin{bmatrix} 2.1139 & 3.0315 & -1.8744 & 2.9638 & -0.0026 & -0.2516 & 0.0048 & -0.0027 & 0.0024 & -0.0061 \\ 3.0315 & 5.8719 & -3.0924 & 3.0567 & 0.2304 & 0.3243 & -0.0129 & 0.0270 & -0.0020 & -0.0650 \\ -1.8744 & -3.0924 & 4.0913 & 0.0762 & -1.5621 & 3.2344 & -0.0396 & 0.0776 & -0.0151 & -0.0478 \\ 2.9638 & 3.0567 & 0.0762 & 14.0194 & -2.9382 & 2.9753 & 0.0080 & 0.0220 & 0.0000 & 0.0401 \\ -0.0026 & 0.2304 & -1.5621 & -2.9382 & 4.3324 & -0.2213 & -2.0242 & 3.1203 & -0.0227 & 0.1663 \\ -0.2516 & 0.3243 & 3.2344 & 2.9753 & -0.2213 & 13.6972 & -3.2419 & 3.0230 & -0.1112 & 0.1281 \\ 0.0048 & -0.0129 & -0.0396 & 0.0080 & -2.0242 & -3.2419 & 4.1122 & 0.1160 & -1.9415 & 3.1818 \\ -0.0027 & 0.0270 & 0.0776 & 0.0220 & 3.1203 & 3.0230 & 0.1160 & 13.9571 & -2.9592 & 2.9535 \\ 0.0024 & -0.0020 & -0.0151 & 0.0000 & -0.0227 & -0.1112 & -1.9415 & -2.9592 & 4.0287 & 0.0735 \\ -0.0061 & -0.0650 & -0.0478 & 0.0401 & 0.1663 & 0.1281 & 3.1818 & 2.9535 & 0.0735 & 13.9135 \end{bmatrix},$$

and

$$\|\hat{M}Z\Omega - \hat{K}Z\| = 3.8215 \times 10^{-13},$$

which implies that $\hat{M}Z\Omega = \hat{K}Z$ reproduces the desired eigenvalues and eigenvectors.

Example 4.2. Let $n = 8,\ p = 3$, and the matrices $\Omega,\ Z,\ M_A$ and K_A be given by

$$\Omega = \mathrm{diag}\{1.5206,\ 2.5270,\ 91.3913\},$$

$$Z = \begin{bmatrix} -0.5355 & -0.3712 & -0.6789 \\ 1.0882 & -0.8303 & 0.3868 \\ -0.9698 & -1.2271 & 2.1975 \\ 1.3025 & 0.0950 & -1.5919 \\ 0.7994 & 0.7872 & -9.5128 \\ -1.0684 & 0.8012 & 2.0846 \\ -1.0579 & 0.6595 & 1.8357 \\ 0.5541 & 0.3212 & 0.7364 \end{bmatrix},$$

$$M_A = \begin{bmatrix} 1.3402 & 0.5146 & 0.5814 & 0.6866 & 0.7620 & 0.9819 & 0 & 0 \\ 0.5146 & 0.4961 & 0.2583 & 0.5576 & 0.4578 & 0.4649 & 0 & 0 \\ 0.5814 & 0.2583 & 0.6344 & 0.3307 & 0.3011 & 0.5966 & 0 & 0 \\ 0.6866 & 0.5576 & 0.3307 & 1.0126 & 0.8409 & 0.6567 & 0 & 0 \\ 0.7620 & 0.4578 & 0.3011 & 0.8409 & 1.0829 & 0.6275 & 0 & 0 \\ 0.9819 & 0.4649 & 0.5966 & 0.6567 & 0.6275 & 1.0743 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_A = \begin{bmatrix} 0.7158 & 0.6563 & 0.6049 & 0.6571 & 0.5776 & 1.1711 & 0.6820 & 0.8292 \\ 0.6563 & 0.8111 & 0.7171 & 0.7692 & 0.7529 & 1.2061 & 0.7816 & 0.9808 \\ 0.6049 & 0.7171 & 0.7705 & 0.8419 & 0.6858 & 1.2360 & 0.8461 & 1.0336 \\ 0.6571 & 0.7692 & 0.8419 & 1.1459 & 0.6178 & 1.2566 & 0.8430 & 1.1389 \\ 0.5776 & 0.7529 & 0.6858 & 0.6178 & 1.0802 & 1.1265 & 0.6929 & 1.0651 \\ 1.1711 & 1.2061 & 1.2360 & 1.2566 & 1.1265 & 2.3207 & 1.5727 & 1.5929 \\ 0.6820 & 0.7816 & 0.8461 & 0.8430 & 0.6929 & 1.5727 & 1.2934 & 1.1164 \\ 0.8292 & 0.9808 & 1.0336 & 1.1389 & 1.0651 & 1.5929 & 1.1164 & 1.5667 \end{bmatrix}.$$

It can easily be seen that the conditions (2.22) hold. By applying Algorithm 1, we can obtain the unique solution $(\hat{M}, \hat{K})$ of Problem BAP as follows:

$$\hat{M} = \begin{bmatrix} 0.9842 & 0.3968 & 0.3241 & 0.5426 & 0.1421 & 0.9505 & 0 & 0 \\ 0.3968 & 0.2152 & 0.1395 & 0.3217 & 0.0722 & 0.5171 & 0 & 0 \\ 0.3241 & 0.1395 & 0.3701 & 0.2459 & 0.1110 & 0.3585 & 0 & 0 \\ 0.5426 & 0.3217 & 0.2459 & 0.8048 & 0.1161 & 1.0165 & 0 & 0 \\ 0.1421 & 0.0722 & 0.1110 & 0.1161 & 0.0745 & 0.2885 & 0 & 0 \\ 0.9505 & 0.5171 & 0.3585 & 1.0165 & 0.2885 & 1.9346 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{K} = \begin{bmatrix} 1.1470 & 0.7694 & 1.2597 & 0.4815 & 0.9014 & 0.9480 & 0.6446 & 0.7689 \\ 0.7694 & 1.0790 & 1.0498 & 0.8036 & 0.6304 & 1.0632 & 1.0341 & 1.1680 \\ 1.2597 & 1.0498 & 1.7305 & 0.7554 & 1.1954 & 0.9108 & 0.9058 & 1.1959 \\ 0.4815 & 0.8036 & 0.7554 & 1.1348 & 0.2496 & 1.2645 & 0.9608 & 1.1139 \\ 0.9014 & 0.6304 & 1.1954 & 0.2496 & 1.6305 & 0.7494 & 0.5127 & 0.6545 \\ 0.9480 & 1.0632 & 0.9108 & 1.2645 & 0.7494 & 2.4984 & 1.2702 & 1.3983 \\ 0.6446 & 1.0341 & 0.9058 & 0.9608 & 0.5127 & 1.2702 & 0.6414 & 0.8531 \\ 0.7689 & 1.1680 & 1.1959 & 1.1139 & 0.6545 & 1.3983 & 0.8531 & 1.3044 \end{bmatrix},$$

and

$$||\hat{M}Z\Omega - \hat{K}Z|| = 8.0674 \times 10^{-14}.$$

Observe that the prescribed eigenvalues and eigenvectors have been embedded in the new model $\hat{M}Z\Omega = \hat{K}Z$.

5. Conclusions

A direct updating method for the piezoelectric smart structural models has been established by applying the generalized singular value decomposition. This method makes use of the constrained minimization theory to formulate the minimization error function such that the resulting changes to mass and stiffness matrices are a minimum. The updated model can accurately reproduce the measured eigenstructure data. The approach was verified by two numerical examples and the reasonable results were obtained.

Use of AI tools declaration

The authors declare that we have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest.

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