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Research article

Multiple periodic solutions of second order parameter-dependent equations via rotation numbers

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Abstract: We investigate the existence of multiple periodic solutions for a class of second order parameter-dependent equations of the form $x'' + f(t,x) = sp(t)$. We compare the behavior of its solutions with suitable linear and piecewise linear equations near positive infinity and infinity. Furthermore, in this context, the nonlinearity f does not satisfy the usual sign condition, and the global existence of solutions for the Cauchy problem associated to the equation is not guaranteed. Our approach is based on the Poincaré-Birkhoff twist theorem, a rotation number approach and the phase-plane analysis. Our result generalizes the result in Fonda and Ghirardelli [1] for second order parameter-dependent equations.

Keywords: periodic solutions; indefinite term; Poincaré-Birkhoff twist theorem; rotation number; parameter-dependent
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1. Introduction

In this paper, we focus on studying the multiplicity of periodic solutions of the second order parameter-dependent equation

$$x'' + f(t,x) = sp(t). \tag{1.1}$$

We assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic in the first variable and locally Lipschitz-continuous in the second variable. Moreover, we assume that $p : \mathbb{R} \to \mathbb{R}$ is locally integrable and 2π-periodic, and s is a positive parameter. Similar results can be obtained for a negative s. We investigate the case in which $xf(t,x)$ is a sign-changing function (named "indefinite term"). Moreover, this framework can be extended to the Carathéodory settings.

Interest in such kind of parameter-dependent differential equations can be found in connection to the celebrated Ambrosetti-Prodi problem, which originates from the seminal work of Ambrosetti and Prodi [2]. The Ambrosetti-Prodi problem was initially investigated within the framework of the Dirichlet problem for elliptic PDEs. More precisely, Berger and Podolak [3] investigated a second order differential equation

$$x'' + g(x) = sw(t), \tag{1.2}$$

or a more general elliptic PDE. They assumed that $g : \mathbb{R} \to \mathbb{R}$ is of class C^2 such that $g'' > 0$ and $g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2$, and $w(t) = \sin(\frac{\pi}{T}t)$, which is the eigenfunction corresponding to the first eigenvalue $\lambda_1 = (\pi/T)^2$ for the Dirichlet problem on the interval $[0, T]$. A distinctive feature of this kind of problem is assuming the so-called "jumping or asymmetric conditions" (see [4, 5]) on g, i.e., $g(x)/x$ or $g'(x)$ having different behaviors at $\pm\infty$.

Since then, significant progress has been achieved in the investigation of the Ambrosetti-Prodi problem. Particularly, Fabry, Mawhin and Nkashama [6] initiated the investigation of the Ambrosetti-Prodi problem with periodic boundary conditions. Ortega [7] discussed the Ambrosetti-Prodi periodic problem for a damped Duffing equation from the point of view of the stability of the solutions. The significant works in [6, 7], as well as [8–10], have motivated a great deal of studies in this area, such as the work of Sovrano and Zanolin [11–13] and the references therein. For additional contributions in this field, and other works concerning the almost periodic solutions problems, please refer to [1, 14–29] and the references therein.

In particular, in 1987, Lazer and McKenna [14] provided a multiplicity result for the periodic problem

$$\begin{cases} x'' + g(x) = s(1 + w(t)), \\ x(0) = x(T), \quad x'(0) = x'(T). \end{cases} \tag{1.3}$$

They assumed that $g : \mathbb{R} \to \mathbb{R}$ is a C^1-function, and asymptotically asymmetric, that is, $g'(-\infty) \neq g'(+\infty)$; $w : \mathbb{R} \to \mathbb{R}$ is a "small" continuous and T-periodic function. In 1992, their result was slightly generalized by Del Pino, Manasevich and Murua [20]. They assumed that the limits

$$\lim_{x\to-\infty} g'(x) = \nu, \quad \lim_{x\to+\infty} g'(x) = \mu$$

exist and there are two positive integers k, m such that

$$\left(\frac{2\pi(k-1)}{T}\right)^2 < \nu < \left(\frac{2\pi k}{T}\right)^2 \leq \left(\frac{2\pi m}{T}\right)^2 < \mu < \left(\frac{2\pi(m+1)}{T}\right)^2. \tag{1.4}$$

Furthermore, they let $n \geq 0$ be an integer such that

$$\frac{T}{n+1} < \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} < \frac{T}{n}.$$

For convenience, if $n = 0$, they agreed that T/n is $+\infty$. Later on, further generalizations were given in [17, 18, 21]. In those papers, the differentiability of g was always required.

In 2005, Zanini and Zanolin [22] investigated the existence of multiple solutions for a periodic boundary value problem

$$\begin{cases} x'' + g(t, x) = sw(t), \\ x(0) = x(T), \quad x'(0) = x'(T). \end{cases} \tag{1.5}$$

They assumed the existence of the limits

$$\lim_{x\to-\infty} \frac{g(t,x)}{x} = b(t), \quad \lim_{x\to+\infty} \frac{\partial g}{\partial x}(t, x) = a(t). \tag{1.6}$$

Furthermore, they required the only solution of the problem

$$\begin{cases} x'' + a(t)x = w(t), \\ x(0) = x(T), \quad x'(0) = x'(T) \end{cases} \tag{1.7}$$

to be strictly positive. Fonda and Ghirardelli [1] removed the differentiability of g in x and extended the results in [22] under the following assumptions.

(i) There are two positive numbers ν_1, ν_2 such that

$$\nu_1 \leq \liminf_{x\to-\infty} \frac{g(t,x)}{x} \leq \limsup_{x\to-\infty} \frac{g(t,x)}{x} \leq \nu_2,$$

uniformly for almost every $t \in [0, T]$.

(ii) There is a function $a(t)$ such that

$$\lim_{x\to+\infty} \frac{g(t,x)}{x} = a(t),$$

uniformly for almost every $t \in [0, T]$.

(iii) There are two positive numbers $\mu_1,\ \mu_2$ and an ingeter $m \geq 0$ such that, for almost every $t \in [0, T]$,

$$\left(\frac{2\pi m}{T}\right)^2 < \mu_1 \leq a(t) \leq \mu_2 < \left(\frac{2\pi(m+1)}{T}\right)^2. \tag{1.8}$$

Moreover, the only solution of (1.7) is strictly positive.

(iv) There is an integer $n \geq 0$ such that

$$\frac{T}{n+1} < \frac{\pi}{\sqrt{\mu_2}} + \frac{\pi}{\sqrt{\nu_2}} \leq \frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}} < \frac{T}{n}.$$

It is worth noticing that the sign condition

$$\operatorname{sgn}(x)g(t,x) > 0, \quad \text{for } |x| \gg 1$$

is implicit in the aforementioned conditions (i)–(iii), and we can notice the presence of a similar sign condition in (1.4). The sign condition is not trivial in the phase-plane analysis. In the process of applying the Poincaré-Birkhoff twist theorem, the introduction of sign conditions can simplify the problem. To illustrate this, let us consider the second order equation

$$x'' + f(t,x) = 0. \tag{1.9}$$

If $(x(t), x'(t)) \neq (0,0)$ and $xf(t,x) > 0$, pass to polar coordinates $x(t) = r(t)\cos\theta(t)$, $x'(t) = r(t)\sin\theta(t)$, then

$$\theta'(t) = -\frac{(x'(t))^2 + x(t)f(t,x(t))}{x^2(t) + (x'(t))^2} < 0,$$

where $\theta(t)$ is the argument function of the solution $x(t)$ to Eq (1.9). Therefore, it is convenient to estimate the rotation angle difference $\Delta\theta(t_1,t_0) = \theta(t_1) - \theta(t_0)$, which is a crucial technical step in the application of the Poincaré-Birkhoff twist theorem. If the sign conditions are removed, the complexity of the problem significantly increases, thereby constraining the applicability of the original ideas and methods.

On the other hand, from the above conditions (i) and (ii), as well as (1.4) and (1.6), the global existence of solutions for the Cauchy problem associated with Eq (1.1) is guaranteed (see page 4010 in [1] for detailed proof). It is well known that global existence of solutions is important for applying the Poincaré-Birkhoff twist theorem. Without the global existence of solutions, the corresponding Poincaré map may not be well-defined. To overcome this difficulty, it is necessary to utilize some a priori estimates for the solutions that have a prescribed number of rotations in the phase plane, as shown in Fonda and Sfecci [30].

In recent years, there has been growing interest in studying periodic solution problems with indefinite terms. Several authors have contributed to this field, including works in [31–38]. Most of these studies have focused on indefinite problems without parameters. This has inspired us to investigate periodic solution problems for parameter-dependent differential equations with indefinite terms.

In this paper, we prove the existence of multiple periodic solutions for the second order parameter-dependent equation (1.1). We investigate the case in which $xf(t,x)$ is an indefinite term, and the asymmetric conditions are in the sense of rotation numbers. Therefore, our results generalize the results in Fonda and Ghirardelli [1], as well as the results in [22, 29].

Here and throughout the paper, the standard notations $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$ are used. Additionally, we will use $\rho(q)$ and $\rho(\nu)$ to denote rotation numbers, the precise definitions of which will be provided in Section 2. We assume the following conditions.

(H_0) $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, 2π-periodic in the first variable, and locally Lipschitz-continuous in the second variable, and $p : \mathbb{R} \to \mathbb{R}$ is locally integrable and 2π-periodic.

(H_1) There is a function $\nu(t) \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ such that

$$\liminf_{x\to-\infty} \frac{f(t,x)}{x} \geq \nu(t), \tag{1.10}$$

uniformly for almost every $t \in [t_0, t_0 + 2\pi]$.

(H_2) There is a function $q(t) \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ such that

$$\lim_{x\to+\infty} \frac{f(t,x)}{x} = q(t),$$

uniformly for almost every $t \in [t_0, t_0 + 2\pi]$.

(H_3) There is an integer $m \geq 0$ such that

$$m < \rho(q) < m + 1, \tag{1.11}$$

where $\rho(q)$ denotes the rotation number of the equation $x'' + q(t)x = 0$. Moreover, the only 2π-periodic solution of

$$x'' + q(t)x = p(t) \tag{1.12}$$

is strictly positive.

(H_4) There is an integer $n \geq 0$ such that

$$\rho(\nu) > n,$$

where $\rho(v)$ denotes the rotation number of the equation

$$x'' + q(t)x^+ - v(t)x^- = 0. \tag{1.13}$$

Our main result is as follows.

Theorem 1.1. *Assume that* $(H_0) - (H_4)$ *hold with* $n > m$. *Then, there exists a* $s_0 \geq 0$ *such that, for every* $s \geq s_0$, *Eq (1.1) has at least* $2(n-m)+1$ *distinct* 2π*-periodic solutions.*

Remark 1.1. *The nonresonance conditions outlined in* $(H_1) - (H_4)$ *represent generalizations of the classical nonresonance conditions outlined in (i)–(iv), where* v_1, v_2 *and* $a(t)$ *can only be positive. However,* $v(t)$ *and* $q(t)$ *are allowed to be sign-changing in* (H_1) *and* (H_2), *which implies that* $\mathrm{sgn}(x)f(t,x)$ *could be an indefinite term. The following is an interesting example regarding the indefinite terms. Define two sign-changing functions*

$$q(t) = \begin{cases} (2m+1)^2, & t \in [0,\pi], \\ -\lambda^2, & t \in [\pi, 2\pi], \end{cases} \qquad v(t) = \begin{cases} (2\alpha+\varrho)^2, & t \in [0,\pi], \\ -\mu^2, & t \in [\pi, 2\pi], \end{cases}$$

where $m \in \mathbb{N}^+$, $\alpha, \varrho \in \mathbb{R}^+$, $\arctan|\lambda| \leq \pi/(2(2m+1))$ *and* $1 - n/(2\alpha+\varrho) - n/(2m+1) \geq 2\max\{\arctan|\lambda|,\ \arctan|\mu|\}/\pi > 0$. *Moreover, assume that there exists an integer* $n > 0$ *satisfying*

$$\frac{\pi}{m} + \frac{\pi}{\alpha} < \frac{2\pi}{n}. \tag{1.14}$$

Then it can be proved (see the details in Section 5)

$$m < \rho(q) < m+1, \quad \rho(v) > n. \tag{1.15}$$

Therefore, the nonresonant conditions described in the sense of rotation numbers are different from the previous nonresonant conditions.

Remark 1.2. *When condition* (H_1) *holds, the nonlinearity* $f(t,x)$ *may grow superlinearly in the left half-plane, which will destroy the global existence of solutions of the Cauchy problem associated to Eq* (1.1). *From the perspective of whether the global existence of solutions is absent, this implies that our result is a generalization of the results in [1, 22, 29].*

On the other hand, with respect to condition (H_1), *it is sufficient to fulfill the twist condition when applying the Poincaré-Birkhoff twist theorem, as outlined in the proof of Theorem 1.1 in Section 4. This condition represents an improvement compared to condition (i) presented in [1]. Additionally, if we replace* (1.10) *with*

$$\limsup_{x \to -\infty} \frac{f(t,x)}{x} \leq v(t),$$

the result stated in Theorem 1.1 can still be proved while $m > n$.

The rest of the paper is organized as follows. In Section 2, we introduce the definitions and properties of rotation numbers and 2π-rotation numbers, along with some auxiliary lemmas. In Section 3, we present some preliminary lemmas. In Section 4, we provide the proof of Theorem 1.1, and an application of it. Finally, we give the detailed proofs of some technical lemmas in Section 5, including the proof of (1.15).

2. Definitions and properties of rotation numbers

In this section, we introduce the definitions of 2π-rotation numbers of solutions of the first order planar systems, as well as the definitions of the rotation numbers of the first order piecewise linear systems. Additionally, we explore the relationship between the rotation numbers and the 2π-rotation numbers. The discussion is primarily based on the works of [31, 33] and the references therein.

Consider the first order planar system

$$x' = y, \quad y' = -h(t, x), \tag{2.1}$$

where $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, 2π-periodic in the first variable. Let $z = (x, y) \in \mathbb{R}^2$, and suppose that the Cauchy problem of system (2.1) has a unique solution, then a solution $z(t; z_0)$ with the initial value $z_0 \neq 0$ can be written in the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then we have

$$\begin{cases} \theta' = -\sin^2\theta - \dfrac{h(t, x)}{r}\cos\theta, \\ r' = r\sin\theta\cos\theta - h(t, x)\sin\theta. \end{cases} \tag{2.2}$$

If $z(t; z_0)$ exists in $[t_0, t_0 + 2\pi]$, we can define the 2π-rotation number associated to $z(t; z_0)$ as

$$\mathrm{Rot}_h(z_0) = \frac{\theta(t_0; z_0) - \theta(t_0 + 2\pi; z_0)}{2\pi} = \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} \frac{xh(t, x) + y^2}{x^2 + y^2} dt,$$

where $\theta(t; z_0)$ is the argument function of $z(t; z_0)$. Accordingly, $\mathrm{Rot}_h(z_0)$ represents the total algebraic count of the clockwise rotations of the solution $z(t; z_0)$ around the origin during the time interval $[t_0, t_0 + 2\pi]$.

When (2.1) is a piecewise linear system

$$x' = y, \quad y' = -a_+(t)x^+ + a_-(t)x^-, \tag{2.3}$$

where $a_\pm(t) \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$, the argument function $\theta(t; z_0)$ satisfies

$$\theta' = -a_+(t)((\cos\theta)^+)^2 - a_-(t)((\cos\theta)^-)^2 - \sin^2\theta. \tag{2.4}$$

Therefore, $\theta(t; z_0)$ only depends on the initial time t_0 and the initial argument value $\theta_0 \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. In this case, we can denote the 2π-rotation number of $z(t; z_0)$ as $\mathrm{Rot}_a(\omega_0)$, where $\omega_0 = z_0/|z_0|$.

In addition, the function $-a_+(t)((\cos\theta)^+)^2 - a_-(t)((\cos\theta)^-)^2 - \sin^2\theta$ is 2π-periodic in t and 2π-periodic in θ, Eq (2.4) is therefore a differential equation on a torus. Consequently, we can define the rotation number of (2.4) as

$$\rho(a) = \lim_{t\to\infty} \frac{\theta_0 - \theta(t_0 + t; \theta_0)}{t} = \lim_{k\to\infty} \frac{\theta_0 - \theta(t_0 + 2k\pi; \theta_0)}{2k\pi}, \tag{2.5}$$

which exists independently of (t_0, θ_0), see Theorem 2.1 in Chapter 2 of Hale [39]. By extension, we refer to $\rho(a)$ as the rotation number of the system (2.3).

Next, similar to propositions or lemmas outlined in [31, 33], we present the relationship between the rotation number $\rho(a)$ and the 2π-rotation number $\mathrm{Rot}_a(\omega_0)$ of system (2.3) in Lemma 2.1, the comparison result concerning the 2π-rotation numbers between system (2.1) and system (2.3) in Lemma 2.2, and the relationship between the 2π-rotation number of system (2.1) and the rotation number of system (2.3) in Lemma 2.3. The proofs of Lemmas 2.1–2.3 are similar to those in [31, 33]. Therefore, we omit them in this discussion.

Lemma 2.1. *For an arbitrary integer n, we have*

(i) $\rho(a) > n \Leftrightarrow \mathrm{Rot}_a(\omega_0) > n$, $\forall\ \omega_0 \in \mathbb{S}^1$;

(ii) $\rho(a) < n \Leftrightarrow \mathrm{Rot}_a(\omega_0) < n$, $\forall\ \omega_0 \in \mathbb{S}^1$;

(iii) $\rho(a) = n$ *if and only if there is at least one nontrivial 2π-periodic solution $\theta(t; t_0, \theta_0)$ of system* (2.3) *with* $\theta(t_0 + 2\pi; t_0, \theta_0) - \theta_0 = 2n\pi$.

Lemma 2.2. *(Comparison lemma) Let $h : [t_0, t_0 + 2\pi] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, and $a_\pm \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$.*

(i) If

$$\liminf_{x\to\pm\infty} \frac{h(t, x)}{x} \geq a_\pm(t) \tag{2.6}$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$. Then, for each $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\mathrm{Rot}_h(z_0) \geq \mathrm{Rot}_a(\omega_0) - \varepsilon, \quad \forall\ t \in [t_0, t_0 + 2\pi],\ \omega_0 = z_0/|z_0| \tag{2.7}$$

holds for every solution of system (2.1) satisfying $|z(t)| \geq R_\varepsilon$, $\forall\ t \in [t_0, t_0 + 2\pi]$.

(ii) If

$$\limsup_{x\to\pm\infty} \frac{h(t, x)}{x} \leq a_\pm(t) \tag{2.8}$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$. Then, for each $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\mathrm{Rot}_h(z_0) \leq \mathrm{Rot}_a(\omega_0) + \varepsilon, \quad \forall\ t \in [t_0, t_0 + 2\pi],\ \omega_0 = z_0/|z_0| \tag{2.9}$$

holds for every solution of system (2.1) satisfying $|z(t)| \geq R_\varepsilon$, $\forall\ t \in [t_0, t_0 + 2\pi]$.

Lemma 2.3. *Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, 2π-periodic in the first variable, and $a_\pm \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$, then*

(i) If $\rho(a) > n$ and (2.6) *holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$, then there exists $\tilde{R} > 0$ such that* $\mathrm{Rot}_h(z_0) > n$ *holds for every solution $z(t; z_0)$ of Eq (2.1) satisfying $|z(t)| \geq \tilde{R}$, $\forall\ t \in [t_0, t_0 + 2\pi]$.*

(ii) If $\rho(a) < n$ and (2.8) *holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$, then there exists $\tilde{R} > 0$ such that* $\mathrm{Rot}_h(z_0) < n$ *holds for every solution $z(t; z_0)$ of Eq (2.1) satisfying $|z(t)| \geq \tilde{R}$, $\forall\ t \in [t_0, t_0 + 2\pi]$.*

3. Some preliminary lemmas

In this section, we prepare some preliminary lemmas that will be used in the proof of Theorem 1.1. It is worth noticing that our discussion is based on the indefinite terms, which distinguishes it from the approach followed in [1]. We assume that $s \geq 1$ and we denote the standard norm in $L^p([t_0, t_0 + 2\pi], \mathbb{R})$ by $|\cdot|_p$.

Lemma 3.1. *There exist three positive constants ε_0, c_0 and C_0 such that, if β and γ are 2π-periodic functions, and β, $\gamma \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ satisfy*

$$||\beta||_1 \leq \varepsilon_0, \quad ||\gamma - q||_1 \leq \varepsilon_0,$$

then the linear equation

$$u'' + \gamma(t)u = p(t) + \beta(t)$$

has a unique 2π-periodic solution u, and $c_0 \leq u(t) \leq C_0$, for every $t \in [t_0, t_0 + 2\pi]$.

The proofs of Lemma 3.1 and the subsequent Lemma 3.4 are similar as those presented in [1], so we omit them here. The detailed proofs for Lemmas 3.2 and 3.3 will be provided in Section 6.

Lemma 3.2. *Assume that* $(H_0)-(H_2)$ *hold. Let* ε_0 *be a positive value satisfying* $\varepsilon_0 < \min\{\rho(q)-m,\ m+1-\rho(q),\ \rho(v)-n\}$. *Then we can write the function* f *as*

$$f(t,x) = \tilde{a}(t,x)x^+ - \tilde{b}(t,x)x^- + r(t,x),$$

where $\tilde{a},\tilde{b},r : \mathbb{R}^2 \to \mathbb{R}$ *are Carathéodory functions such that, for almost every* $t \in [t_0, t_0+2\pi]$ *and every* $x \in \mathbb{R}$,

$$q(t) - \varepsilon_0 \le \tilde{a}(t,x) \le q(t) + \varepsilon_0, \tag{3.1}$$

$$\tilde{b}(t,x) \ge v(t) - \varepsilon_0, \tag{3.2}$$

and $r(t,x)$ *is bounded: there is a* 2π*-periodic function* $\tilde{r}(t)$ *with* $\tilde{r} \in L^1([t_0,t_0+2\pi],\mathbb{R}^+)$, *such that, for almost every* $t \in [t_0,t_0+2\pi]$ *and every* $x \in \mathbb{R}$,

$$|r(t,x)| \le \tilde{r}(t). \tag{3.3}$$

We now proceed with a change of variable. In (1.1), set

$$u(t) = \frac{1}{s}x(t).$$

Then Eq (1.1) is transformed into

$$u'' + \frac{f(t,su)}{s} = p(t). \tag{3.4}$$

Lemma 3.3. *Assume that* $(H_0)-(H_3)$ *hold. Then there exists a* $s_1 \ge 1$ *such that for every* $s \ge s_1$, *Eq* (3.4) *has a* 2π*-periodic solution* $\tilde{u}_s$ *satisfying*

$$c_0 \le \tilde{u}_s(t) \le C_0,$$

for every $t \in [t_0,t_0+2\pi]$, *where* c_0 *and* C_0 *are two positive constants given in Lemma 3.1.*

Next, we perform another change of variable, in Eq (3.4), set

$$v(t) = u(t) - \tilde{u}_s(t).$$

As a result, Eq (3.4) is transformed into

$$v'' + \frac{f(t,s(v+\tilde{u}_s(t))) - f(t,s\tilde{u}_s(t))}{s} = 0. \tag{3.5}$$

Notice that $v = 0$ is a solution of (3.5).

Lemma 3.4. *Assume that* (H_0), (H_2) *and* (H_3) *hold. Then*

$$\lim_{s\to+\infty} \frac{f(t,s(v+\tilde{u}_s(t))) - f(t,s\tilde{u}_s(t))}{s} = q(t)v$$

holds uniformly for a.e. $t \in [t_0,t_0+2\pi]$ *and every* $v \in [-\frac{1}{2}c_0, \frac{1}{2}c_0]$.

4. Spiral property and the multiplicity of periodic solutions

Set
$$\tilde{f}_s(t, v) = \frac{f(t, s(v + \tilde{u}_s(t))) - f(t, s\tilde{u}_s(t))}{s},$$
then Eq (3.5) is changed into
$$v'' + \tilde{f}_s(t, v) = 0. \tag{4.1}$$

Now, regarding Eq (4.1), notice that the global existence of solutions may be destroyed under the conditions of Theorem 1.1. It is well known that global existence is a crucial requirement for the application of the Poincaré-Birkhoff twist theorem. Consequently, we try to give a spiral property of solutions of Eq (4.1) under the conditions $(H_0) - (H_2)$. Subsequently, we modify the system associated to Eq (4.1) using the spiral property. This modification enables us to establish the global existence of solutions for the modified system.

4.1. Spiral property

Let us consider the first-order planar system given by
$$v' = w, \quad w' = -\tilde{f}_s(t, v), \tag{4.2}$$
which is associated to Eq (4.1), where $\tilde{f}_s(t, v)$ satisfies the following conditions.

$(H_0)'$ $\tilde{f}_s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic in the first variable, and locally Lipschitz-continuous in the second variable, $\tilde{f}_s(t, 0) = 0$.

$(H_1)'$ For the function $\nu(t) \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ in (H_1), we have
$$\liminf_{v \to -\infty} \frac{\tilde{f}_s(t, v)}{v} \geq \nu(t), \tag{4.3}$$
uniformly for almost every $t \in [t_0, t_0 + 2\pi]$.

$(H_2)'$ For the function $q(t) \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ in (H_2), we have
$$\lim_{v \to +\infty} \frac{\tilde{f}_s(t, v)}{v} = q(t),$$
uniformly for almost every $t \in [t_0, t_0 + 2\pi]$.

Now, we will outline the deduction process for the aforementioned conditions based on $(H_0) - (H_2)$.

First, by assumption (H_0), we can conclude that $\tilde{f}_s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, 2π-periodic with respect to the first variable, and $\tilde{f}_s(t, 0) = 0$. Moreover, using the Lipschitz-continuity of f with respect to the second variable as presented in (H_0), we can deduce that, for arbitrary $t \in \mathbb{R}$ and for $v_1,\ v_2 \in U(v_s(t_0))$, where $U(v_s(t_0))$ represents an arbitrary neighborhood of $v_s(t_0)$, there exists a positive constant L such that
$$\left|\tilde{f}_s(t, v_1) - \tilde{f}_s(t, v_2)\right| = \frac{1}{s}\left|f(t, s(v_1 + \tilde{u}_s(t))) - f(t, s(v_2 + \tilde{u}_s(t)))\right| \leq L|v_1 - v_2|.$$

Second, by (H_1), we can conclude that
$$\begin{aligned}\liminf_{v \to -\infty} \frac{\tilde{f}_s(t, v)}{v} &= \liminf_{v \to -\infty} \frac{f(t, s(v + \tilde{u}_s(t))) - f(t, s\tilde{u}_s(t))}{sv} = \liminf_{v \to -\infty} \frac{f(t, s(v + \tilde{u}_s(t)))}{sv} \\ &= \liminf_{v \to -\infty} \frac{f(t, s(v + \tilde{u}_s(t)))}{s(v + \tilde{u}_s(t))} \cdot \frac{s(v + \tilde{u}_s(t))}{sv} = \liminf_{v \to -\infty} \frac{f(t, s(v + \tilde{u}_s(t)))}{s(v + \tilde{u}_s(t))} \geq \nu(t).\end{aligned}$$

Similarly, by (H_2) we have

$$\lim_{v\to+\infty}\frac{\tilde{f}_s(t,v)}{v}=\lim_{v\to+\infty}\frac{f(t,s(v+\tilde{u}_s(t)))-f(t,s\tilde{u}_s(t))}{sv}=\lim_{v\to+\infty}\frac{f(t,s(v+\tilde{u}_s(t)))}{sv}$$
$$=\lim_{v\to+\infty}\frac{f(t,s(v+\tilde{u}_s(t)))}{s(v+\tilde{u}_s(t))}\cdot\frac{s(v+\tilde{u}_s(t))}{sv}=\lim_{v\to+\infty}\frac{f(t,s(v+\tilde{u}_s(t)))}{s(v+\tilde{u}_s(t))}=q(t).$$

Denote by $\tilde{z}_s(t) := (v_s(t), w_s(t))$ a solution of system (4.2) with an initial value $\tilde{z}_0 := \tilde{z}_s(t_0) = (v_s(t_0), w_s(t_0))$. Since $\tilde{z}_s(t) = 0$ is a solution of system (4.2), we know that $\tilde{z}_s(t) \neq 0$ if $\tilde{z}_0 \neq 0$ by uniqueness. Passing to the polar coordinates $v = r\cos\theta$, $w = r\sin\theta$, we have

$$\begin{cases} \theta' = -\sin^2\theta - \dfrac{\tilde{f}_s(t,v)}{r}\cos\theta, \\ r' = r\sin\theta\cos\theta - \tilde{f}_s(t,v)\sin\theta. \end{cases} \tag{4.4}$$

Denote by $(\tilde{\theta}_s(t), \tilde{r}_s(t)) = (\tilde{\theta}_s(t;\tilde{z}_0), \tilde{r}_s(t;\tilde{z}_0))$ a solution of system (4.4) with an initial value $(\tilde{\theta}_s(t_0), \tilde{r}_s(t_0)) = (\theta_0, r_0)$.

Lemma 4.1. *Assume $(H_0)' - (H_2)'$, and $s \geq s_1$. Then, for any fixed l, $N_0 \in \mathbb{N}$ and a sufficiently large r^*, there are two strictly monotonically increasing functions $\xi^-_{N_0}, \xi^+_{N_0} : [r^*, +\infty) \to \mathbb{R}$, such that*

$$\xi^{\pm}_{N_0}(r) \to +\infty \iff r \to +\infty. \tag{4.5}$$

Moreover, for any $r_0 \geq r^*$, the solution $(\tilde{\theta}_s(t), \tilde{r}_s(t))$ of system (4.4) satisfies that either

$$\xi^-_{N_0}(r_0) \leq \tilde{r}_s(t) \leq \xi^+_{N_0}(r_0), \quad t \in [t_0, t_0 + 2l\pi],$$

or there exists $t_{N_0} \in (t_0, t_0 + 2l\pi)$ such that

$$\theta_0 - \tilde{\theta}_s(t_{N_0};\tilde{z}_0) = 2N_0\pi,$$

and

$$\xi^-_{N_0}(r_0) \leq \tilde{r}_s(t) \leq \xi^+_{N_0}(r_0), \quad t \in [t_0, t_{N_0}].$$

Proof. Divide $\mathbb{R}^2$ into four regions

$$\mathcal{D}_1 = \{(v,w) : v \geq 0,\ w > 0\}, \quad \mathcal{D}_2 = \{(v,w) : v > 0,\ w \leq 0\},$$

$$\mathcal{D}_3 = \{(v,w) : v \leq 0,\ w < 0\}, \quad \mathcal{D}_4 = \{(v,w) : v < 0,\ w \geq 0\}.$$

Now, we define two functions. The first is

$$S(v,w) = v^2 + w^2, \quad (v,w) \in \mathbb{R}^2,$$

it follows that

$$S(v,w) \to +\infty \iff v^2 + w^2 \to +\infty.$$

The second is

$$T(v,w) = \frac{1}{2}w^2 + F_+(v), \quad (v,w) \in \mathbb{R}^2,$$

where $\tilde{F}_s^+(v) = \int_0^v \tilde{f}_s^+(\tau)d\tau$, $\tilde{f}_s^+(v) = \operatorname{sgn}(v)\max\{|v|, \max_{t\in[0,T]} |\tilde{f}_s(t,v)| + 1\}$. It follows that $\tilde{F}_s^+(v) \to +\infty$ as $|v| \to +\infty$. Therefore

$$T(v,w) \to +\infty \iff v^2 + w^2 \to +\infty.$$

Step 1. The case $(v(t), w(t)) \in \mathcal{D}_1 \cup \mathcal{D}_3$. By $(H_l)'$ and $(H_2)'$, there exist $\varepsilon_0 \le 1$ and $M_{\varepsilon_0} > 0$ such that

$$\frac{\tilde{f}_s(t,v)}{v} \ge v(t) - \varepsilon_0, \quad \text{for } v \le -M_{\varepsilon_0}, \tag{4.6}$$

and

$$q(t) - \varepsilon_0 \le \frac{\tilde{f}_s(t,v)}{v} \le q(t) + \varepsilon_0, \quad \text{for } v \ge M_{\varepsilon_0}. \tag{4.7}$$

When $|v| \le M_{\varepsilon_0}$, it follows that $|w|$ is large enough for sufficiently large r. Take

$$K_{\varepsilon_0} := \max\{|\tilde{f}_s(t,v)| : t \in [t_0, t_0 + 2\pi], |v| \le M_{\varepsilon_0}\}.$$

Thereby, we have

$$\begin{aligned} \frac{dS(v(t),w(t))}{dt} &= 2vw - 2w\tilde{f}_s(t,v) \le v^2 + w^2 + 2K_{\varepsilon_0}|w| \\ &\le v^2 + w^2 + 2K_{\varepsilon_0}w^2 \le (1 + 2K_{\varepsilon_0})S(v(t),w(t)). \end{aligned} \tag{4.8}$$

When $v < -M_{\varepsilon_0}$, by (4.6) we have

$$\begin{aligned} \frac{dS(v(t),w(t))}{dt} &= 2vw - 2w\tilde{f}_s(t,v) \le 2vw - 2vw(v(t) - \varepsilon_0) \\ &\le (|v(t)| + 2)S(v(t),w(t)). \end{aligned} \tag{4.9}$$

When $v > M_{\varepsilon_0}$, by (4.7) we have

$$\begin{aligned} \frac{dS(v(t),w(t))}{dt} &= 2vw - 2w\tilde{f}_s(t,v) \le 2vw - 2vw(q(t) - \varepsilon_0) \\ &\le (|q(t)| + 2)S(v(t),w(t)). \end{aligned} \tag{4.10}$$

Therefore, by (4.8)–(4.10), we have

$$\frac{dS(v(t),w(t))}{dt} \le c(t)S(v(t),w(t)), \tag{4.11}$$

where $c(t) = \max\{1 + 2K_{\varepsilon_0},\ |v(t)| + 2,\ |q(t)| + 2\}$.

On the other hand, we have

$$\frac{d}{dt}T(v(t),w(t)) = ww' + \tilde{f}_s^+(v)v' = w(\tilde{f}_s^+(v) - \tilde{f}_s(t,v)) \ge 0.$$

Step 2. The case $(v(t), w(t)) \in \mathcal{D}_2 \cup \mathcal{D}_4$. When $|v| \le M_{\varepsilon_0}$, it also follows that $|w|$ is large enough for sufficiently large r. Then, we have

$$\begin{aligned} \frac{dS(v(t),w(t))}{dt} &= 2vw - 2w\tilde{f}_s(t,v) \ge -(v^2 + w^2 + 2K_{\varepsilon_0}|w|) \\ &\ge -(v^2 + w^2 + 2K_{\varepsilon_0}w^2) \ge -(1 + 2K_{\varepsilon_0})S(v(t),w(t)). \end{aligned} \tag{4.12}$$

When $v < -M_{\varepsilon_0}$, we have

$$\begin{aligned}\frac{dS(v(t),w(t))}{dt} &= 2vw - 2w\tilde{f}_s(t,v) \geq 2vw - 2vw(v(t)-\varepsilon_0) \\ &\geq -(|v(t)|+2)S(v(t),w(t)).\end{aligned} \tag{4.13}$$

When $v > M_{\varepsilon_0}$, we have

$$\begin{aligned}\frac{dS(v(t),w(t))}{dt} &= 2vw - 2w\tilde{f}_s(t,v) \geq 2vw - 2vw(q(t)-\varepsilon_0) \\ &\geq -(|q(t)|+2)S(v(t),w(t)).\end{aligned} \tag{4.14}$$

Therefore, by (4.12)–(4.14), we have

$$\frac{dS(v(t),w(t))}{dt} \geq -c(t)S(v(t),w(t)).$$

On the other hand, we have

$$\frac{d}{dt}T(v(t),w(t)) = ww' + \tilde{f}_s^+(v)v' = w(\tilde{f}_s^+(v) - \tilde{f}_s(t,v)) \leq 0. \tag{4.15}$$

Step 3. In Lemma 4.1 of [34], replace (x,y) with (v,w), and let $\varphi(v) = 0$, $\frac{\partial H}{\partial w} = w$, and define suitable $V(v,w)$ and $U(v,w)$ as follows

$$V(v,w) = \begin{cases} S(v,w), & (v(t),w(t)) \in \mathcal{D}_1 \cup \mathcal{D}_3, \\ T(v,w), & (v(t),w(t)) \in \mathcal{D}_2 \cup \mathcal{D}_4, \end{cases}$$

$$U(v,w) = \begin{cases} T(v,w), & (v(t),w(t)) \in \mathcal{D}_1 \cup \mathcal{D}_3, \\ S(v,w), & (v(t),w(t)) \in \mathcal{D}_2 \cup \mathcal{D}_4. \end{cases}$$

By (4.11) and (4.15), we have

$$\frac{dV(v(t),w(t))}{dt} \leq c(t)V(v(t),w(t)), \quad (v(t),w(t)) \in \mathcal{D}_1 \cup \mathcal{D}_3,$$

and

$$\frac{dV(v(t),w(t))}{dt} \leq 0, \quad (v(t),w(t)) \in \mathcal{D}_2 \cup \mathcal{D}_4.$$

Furthermore, it is evident that

(i) $V : \mathbb{R}^2 \to \mathbb{R}^+$ is a piecewise continuously differentiable function.

(ii) $V(v,w) \to +\infty \iff v^2 + w^2 \to +\infty$, for $(v(t),w(t)) \in \mathcal{D}_i,\ i = 1,2,3,4$.

(iii) $V(v,0)$ is a monotone function with respect to v, for $(v(t),w(t)) \in \mathcal{D}_i$, $i = 1,2,3,4$, respectively.

We can also verify that $U(v,w)$ has the similar property to $V(v,w)$. Using Lemma 4.1, Lemma 3.3, and the definitions of upper and lower spiral functions $\xi^{\pm}(\cdot)$ in [34], it follows that there exist two strictly monotonically increasing functions $\xi^{\pm}_{N_0}(r)$ for system (4.2). Therefore, the proof is completed. $\square$

4.2. The modified system and proof of Theorem 1.1

Now, we define a modified Hamiltonian system. To ensure the existence of global solutions for the associated Cauchy problems, we define the following truncated function

$$\tilde{g}_s(t,v)=\begin{cases}\tilde{f}_s(t,-R), & v<-R,\\ \tilde{f}_s(t,v), & |v|\le R,\\ \tilde{f}_s(t,R), & v>R,\end{cases}$$

where $R>c_0/2$ is a positive parameter, and its specific value will be given in the proof of Theorem 1.1. Then, the Hamiltonian system associated to $\tilde{g}_s(t,v)$ is given by

$$v'=w,\quad w'=-\tilde{g}_s(t,v). \tag{4.16}$$

We will now discuss several general properties of solutions for the modified system (4.16), including uniqueness, global existence, elastic property and rotational property. For simplicity, we will continue to denote by $\tilde{z}_s(t)$ a solution of system (4.16) with an initial value $\tilde{z}_0$, and $\tilde{\theta}_s(t)$ the argument function of $\tilde{z}_s(t)$ with $\tilde{\theta}_s(t_0)=\theta_0$.

Lemma 4.2. *Assume* (H_0)*. Then every solution of the Cauchy problem associated to system* (4.16) *exists uniquely and globally. If* $|\tilde{z}_s(t)|\le R$*,* $\tilde{z}_s(t)$ *is also a solution of system* (4.2)*.*

Based on the global existence of solutions presented above, we have the following elastic property.

Lemma 4.3. *The solution* $\tilde{z}_s(t)$ *of system* (4.16) *satisfies the elastic property on the phase plane, which can be described as follows.*

(i) $\forall\, R_0>0$*, there exists* $R_0'\ge R_0$ *such that if* $|\tilde{z}_0|\ge R_0'$ *and* $s\ge s_1$*, then we have*

$$|\tilde{z}_s(t)|\ge R_0,\quad t\in[t_0,t_0+2\pi];$$

(ii) $\forall\, R_0>0$*, there exists* $R_0''\ge R_0$ *such that if* $|\tilde{z}_0|\le R_0$ *and* $s\ge s_1$*, then we have*

$$|\tilde{z}_s(t)|\le R_0'',\quad t\in[t_0,t_0+2\pi].$$

Next, we present the rotational property as follows. The proof is similar to Lemma 4.1 in [36], so we will omit it here.

Lemma 4.4. *Nonzero solutions of* (4.16) *satisfy the rotational property. More precisely, if* $\tilde{z}_s(t)$ *is a nontrivial solution of system* (4.16)*, then,*

$$\tilde{\theta}_s(t_2)-\tilde{\theta}_s(t_1)<\pi,\quad \text{for any }\ t_2>t_1.$$

Before starting the proof of Theorem 1.1, in order to find the inner boundary of a suitable annulus for applying the Poincaré-Birkhoff twist theorem, we present the following lemma.

Lemma 4.5. *There are three positive constants* δ*,* $\tilde{r}$ *and* s_2*, with* $\delta<\tilde{r}<\frac{1}{2}c_0$ *and* $s_2\ge s_1$*, such that, for every* $s\ge s_2$*, if* $r(t_0)=\tilde{r}$*, then the solution to* (4.1) *satisfies*

$$\delta<r(t)<\frac{1}{2}c_0,$$

for every $t\in[t_0,t_0+2\pi]$*.*

Proof. We first prove that $r(t) < \frac{1}{2}c_0$ for every $t \in [t_0, t_0 + 2\pi]$. We assume by contradiction that there exists a $\bar{t} \in [t_0, t_0 + 2\pi]$ such that

$$r(t) < \frac{1}{2}c_0 \quad \text{for every } \ t \in [t_0, \bar{t}), \quad \text{and} \quad r(\bar{t}) = \frac{1}{2}c_0. \tag{4.17}$$

Set

$$\tilde{r} = \frac{1}{8}c_0 e^{-2(1+\|q\|_\infty)\pi}, \quad \delta = \frac{1}{4}\tilde{r}e^{-2(1+\|q\|_\infty)\pi} \quad \text{and} \quad \varepsilon = \frac{\tilde{r}}{2\pi}.$$

It is clear that $0 < \delta < \tilde{r} < \frac{1}{2}c_0$. By Lemma 3.4, there exists a $s_2 \geq s_1$ such that, for every $s \geq s_2$, almost every $t \in [t_0, \bar{t}]$ and every $v \in [-\frac{1}{2}c_0, \frac{1}{2}c_0]$,

$$|\tilde{f}_s(t, v) - q(t)v| \leq \varepsilon.$$

Then by (4.4) we have

$$|r'(t)| = |r \sin\theta \cos\theta - \tilde{f}_s(t, v) \sin\theta| \leq r(t) + |\tilde{f}_s(t, v)| \leq (1 + \|q\|_\infty)r(t) + \varepsilon.$$

By Gronwall inequality, we have

$$r(t) \leq (\tilde{r} + \varepsilon\bar{t})e^{(1+\|q\|_\infty)t},$$

thus for the solution of (4.1), we have

$$r(\bar{t}) \leq (\tilde{r} + 2\varepsilon\pi)e^{2(1+\|q\|_\infty)\pi} = \frac{1}{4}c_0,$$

which contradicts (4.17). By a similar discussion, we can prove that $r(t) > \delta$ for every $t \in [t_0, t_0 + 2\pi]$. □

Proof of Theorem 1.1. We will divide the proof into four steps.

Step 1. Define a set

$$\Omega := \{z \in \mathbb{R}^2 : \delta < |z| < \frac{1}{2}c_0\},$$

and let

$$\Gamma_- := \{z : |z| = \tilde{r}\}.$$

Now, consider a solution $\tilde{z}_s(t)$ of (4.16) with $\tilde{z}_0 \in \Gamma_-$. By Lemma 4.5, there is a positive constant s_2 with $s_2 \geq s_1$, such that $\tilde{z}_s(t) \in \Omega$ when $s \geq s_2$, that is

$$\delta < |\tilde{z}_s(t)| < \frac{1}{2}c_0, \quad t \in [t_0, t_0 + 2\pi].$$

Therefore, it follows that the component v satisfies $0 < |v| \leq \frac{1}{2}c_0$. Since $\frac{1}{2}c_0 < R$, then system (4.16) is equivalent to system (4.2). Therefore, by Lemma 3.4, it follows that

$$\lim_{s\to+\infty} \tilde{f}_s(t, v) = q(t)v$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $z = (v, w) \in \Omega$. So we have

$$\lim_{s\to+\infty} \frac{\tilde{f}_s(t, v)}{v} = q(t) \tag{4.18}$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $z = (v, w) \in \Omega$. Furthermore, we observe that

$$s \to +\infty \iff s(v + \tilde{u}_s) \to +\infty,$$

where $v + \tilde{u}_s \in [\frac{1}{2}c_0, \frac{1}{2}c_0 + C_0]$. Then by (H_3) and Lemma 2.3, we have

$$m < \mathrm{Rot}_{\tilde{f}_s}(\tilde{z}_0) < m + 1.$$

Therefore,

$$m < \mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) < m + 1, \quad \text{for} \quad \tilde{z}_0 \in \Gamma_-. \tag{4.19}$$

Step 2. By $(H_1)'$ and $(H_2)'$, we have

$$\liminf_{v\to-\infty} \frac{\tilde{f}_s(t, v)}{v} \geq \nu(t), \qquad \lim_{v\to+\infty} \frac{\tilde{f}_s(t, v)}{v} = q(t).$$

Therefore, by (H_4) and (i) in Lemma 2.3, for arbitrary $\varepsilon > 0$ with $\varepsilon < \min\{\rho(q)-m, m+1-\rho(q), \rho(\nu)-n\}$, and $s \geq s_2$, there exists a $R_\varepsilon > 0$ such that if the solution $\tilde{z}_s(t)$ of system (4.2) satisfying $|\tilde{z}_s(t)| \geq R_\varepsilon$, $t \in [t_0, t_0 + 2\pi]$, it follows that

$$\mathrm{Rot}_{\tilde{f}_s}(\tilde{z}_0) > n. \tag{4.20}$$

For the above $R_\varepsilon > 0$, by (i) in Lemma 4.3, we can find a R_∞ with $R_\infty \geq R_\varepsilon$ such that, if $|\tilde{z}_0| \geq R_\infty$, then $|\tilde{z}_s(t)| \geq R_\varepsilon$, for $t \in [t_0, t_0 + 2\pi]$. Therefore, let

$$\Gamma_+ := \{z : |z| = R_\infty\},$$

and choose $R = R'_\infty$, where

$$R_\infty > (\xi^-_{n+1})^{-1}(R_\varepsilon), \quad R'_\infty > \xi^+_{n+1}(R_\infty),$$

then system (4.16) is equivalent to system (4.2) when $|\tilde{z}_s(t)| \leq R$.

Now, consider the solution of system (4.16) starting from $\tilde{z}_0 \in \Gamma_+$. If $R_\varepsilon \leq |\tilde{z}_s(t)| \leq R'_\infty$, for all $t \in [t_0, t_0 + 2\pi]$, then from (4.20) we have

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > n, \quad \text{for} \quad \tilde{z}_0 \in \Gamma_+. \tag{4.21}$$

If there is $t_* \in [t_0, t_0 + 2\pi]$ such that $|\tilde{z}_s(t_*)| \geq R'_\infty > \xi^+_{n+1}(R_\infty)$, then

$$\xi^-_{n+1}(|\tilde{z}_0|) \leq |\tilde{z}_s(t)| \leq \xi^+_{n+1}(|\tilde{z}_0|)$$

does not hold for all $t \in (t_0, t_0 + 2\pi)$. Therefore, by Lemma 4.1, there is $t'_* \in (t_0, t_*]$ such that

$$\theta_0 - \tilde{\theta}_s(t'_*) = 2(n + 1)\pi.$$

Moreover, by Lemma 4.4 we have

$$\theta_0 - \tilde{\theta}_s(t_0 + 2\pi) = \theta_0 - \tilde{\theta}_s(t'_*) + \tilde{\theta}_s(t'_*) - \tilde{\theta}_s(t_0 + 2\pi) \geq -\pi + 2(n+1)\pi > 2n\pi.$$

Then

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > n, \quad \text{for} \quad \tilde{z}_0 \in \Gamma_+. \tag{4.22}$$

Finally, if there is $t''_* \in (t_0, t_0 + 2\pi)$ such that $|\tilde{z}_s(t''_*)| \leq R_\varepsilon < \xi^-_{n+1}(R_\infty)$, then

$$\xi^-_{n+1}(|\tilde{z}_0|) \leq |\tilde{z}_s(t)| \leq \xi^+_{n+1}(|\tilde{z}_0|)$$

does not hold for all $t \in (t_0, t_0 + 2\pi)$. By the same discussion as above we can prove that

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > n. \tag{4.23}$$

Combined (4.21), (4.22) with (4.23), we can conclude that if the solution of system (4.16) starts from $\tilde{z}_0 \in \Gamma_+$, then we have

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > n. \tag{4.24}$$

Step 3. Define the Poincaré map

$$\begin{aligned} \mathcal{P}: \quad \mathbb{R}^2 &\to \mathbb{R}^2, \\ \tilde{z}_0 &\mapsto \tilde{z}_s(t_0 + 2\pi). \end{aligned}$$

From the global existence of solutions in Lemma 4.2 we know that $\mathcal{P}$ is well-defined. Additionally, the uniqueness of solutions implies that $\mathcal{P}$ is a homeomorphism, and (4.16) has a Hamiltonian structure, $\mathcal{P}$ is therefore an area-preserving homeomorphism.

Take $k = m+1,\ m+2, \cdots, n$, by (4.19) and (4.24) we have

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) < k, \quad \tilde{z}_0 \in \Gamma_-, \qquad \mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > k, \quad \tilde{z}_0 \in \Gamma_+.$$

Therefore, by the Poincaré-Birkhoff twist theorem (see [40,41]), we conclude that $\mathcal{P}$ has at least $n-m$ pairs of geometrically distinct fixed points $\tilde{z}_{i,j}$, $i = 1, \cdots, n-m,\ j = 1, 2$, which correspond to $n-m$ pairs of 2π-periodic solutions

$$\tilde{z}_s(t; \tilde{z}_{i,j}), \quad i = 1, \cdots, n-m,\ j = 1, 2$$

of system (4.16) with

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_{i,j}) = k, \quad i = 1, \cdots, n-m,\ j = 1, 2. \tag{4.25}$$

Step 4. We will prove that $\tilde{z}_s(t; \tilde{z}_{i,j})$, $i = 1, \cdots, n-m,\ j = 1, 2$ are in fact 2π-periodic solutions of system (4.2). That is, we will prove that $|\tilde{z}_s(t; \tilde{z}_{i,j})| \leq R$, for $t \in [t_0, t_0 + 2\pi]$, $i = 1, \cdots, n-m,\ j = 1, 2$. Note that

$$0 < |\tilde{z}_{i,j}| < R_\infty, \quad i = 1, \cdots, n-m,\ j = 1, 2.$$

Take $\tilde{z}_s(t; \tilde{z}_{1,1})$ as an example. By contradiction, assume that there exists $t_1 \in (t_0, t_0 + 2\pi)$ such that $|\tilde{z}_s(t_1; \tilde{z}_{1,1})| > R = R'_\infty$, and

$$|\tilde{z}_s(t; \tilde{z}_{1,1})| \leq R'_\infty, \quad \text{for } t \in [t_0, t_1].$$

Then by Lemma 4.1, we have

$$\tilde{\theta}_s(t_0;\tilde{z}_{1,1})-\tilde{\theta}_s(t_1;\tilde{z}_{1,1})=2(k+1)\pi.$$

Furthermore, by means of Lemma 4.4, we have

$$\begin{aligned}\tilde{\theta}_s(t_0;\tilde{z}_{1,1})-\tilde{\theta}_s(t_0+2\pi;\tilde{z}_{1,1})&=\tilde{\theta}_s(t_0;\tilde{z}_{1,1})-\tilde{\theta}_s(t_1;\tilde{z}_{1,1})\\&+\tilde{\theta}_s(t_1;\tilde{z}_{1,1})-\tilde{\theta}_s(t_0+2\pi;\tilde{z}_{1,1})\geq-\pi+2(k+1)\pi>2k\pi.\end{aligned}$$

This implies

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_{1,1})>k,$$

which contradicts (4.25). Therefore, we have

$$|\tilde{z}_s(t;\tilde{z}_{1,1})|\leq R,\quad \text{for } t\in[t_0,t_0+2\pi],$$

and we can conclude that $\tilde{z}_s(t;\tilde{z}_{1,1})$ is a 2π-periodic solution of system (4.2). The aforementioned discussions hold true for all other solutions as well.

Recalling the zero solution of Eq (4.1) which corresponds to the solution $\tilde{u}_s(t)$ of (3.4), we get $2(n-m)+1$ distinct 2π-periodic solutions of Eq (3.4), which means that Eq (1.1) has $2(n-m)+1$ distinct 2π-periodic solutions. The proof is completed. □

Remark 4.1. *In order to illustrate the application of the main result, we present an example in the end of this section. Define a function* $f(t,x)$ *as*

$$f(t,x)=\begin{cases}|\nu(t)|x^3, & x\leq 0,\\ q(t)x, & x>0,\end{cases}$$

where $\nu(t)$ *and* $q(t)$ *are the functions defined in Remark 1.1. Then we can deduce that*

$$\liminf_{x\to-\infty}\frac{f(t,x)}{x}\geq\nu(t),\quad \lim_{x\to+\infty}\frac{f(t,x)}{x}=q(t)\tag{4.26}$$

hold uniformly for a.e. $t\in[0,2\pi]$. *Therefore* (H_1) *and* (H_2) *hold.*

Next, we verify the validity of (H_3). *Consider the equation*

$$x''+q(t)x=0,\tag{4.27}$$

from Remark 1.1 we have $m<\rho(q)<m+1$, *which is a nonresonance condition. As discussed in [22], it is known that, from the Fredholm alternative, if* (4.27) *is nonresonant, then for each* $p(t)\in L^1(\mathbb{R}/2\pi\mathbb{Z})$ *there is a unique* 2π*-periodic solution* $\tilde{x}_p\in W^{2,1}(\mathbb{R}/2\pi\mathbb{Z})$ *to the nonhomogeneous equation*

$$x''+q(t)x=p(t).\tag{4.28}$$

Moreover, from the discussion of Remark 6 in [1], the only 2π*-periodic solution of Eq* (4.28) *may have no definite sign. We would like to give an example of* $p(t)$ *for Eq* (4.28) *to verify the validity of the assumption* (H_3). *Let* $p:\mathbb{R}\to\mathbb{R}$ *be a* 2π*-periodic function, and define it for* $t\in[0,2\pi]$ *as follows,*

$$p(t)=\begin{cases}1, & t\in[0,\pi],\\ 0, & t\in[\pi,2\pi].\end{cases}$$

Then with the initial values $x(0) = 1/10$, $x'(0) = 0$, $x(\pi) = e^{\lambda\pi}$, $x'(\pi) = \lambda e^{\lambda\pi}$, *we have*

$$x(t) = \begin{cases} (\frac{1}{10} - \frac{1}{(2m+1)^2})\cos(2m+1)t + \frac{1}{(2m+1)^2}, & t \in [0, \pi], \\ e^{\lambda t}, & t \in [\pi, 2\pi], \end{cases}$$

which is the expression of the solution on a period. Then, $x(t) > 0$ *with suitable* $m \in \mathbb{N}$*. Hence,* (H_3) *holds.*

Finally, from Remark 1.1, we can deduce that (H_4) *holds. Furthermore, we suppose* $n > m$*. Then for the equation* $x'' + f(t, x) = sp(t)$*, there is a* $s_0 \geq 0$ *such that, for every* $s \geq s_0$*, it has at least* $2(n-m)+1$ 2π*-periodic solutions by Theorem 1.1.*

5. Proofs of some technical lemmas and (1.15)

Proof of Lemma 3.2. By (H_2), there exists a constant $M_+ > 0$ such that, when $x \geq M_+$,

$$q(t) - \varepsilon_0 \leq \frac{f(t, x)}{x} \leq q(t) + \varepsilon_0$$

holds for a.e. $t \in [t_0, t_0 + 2\pi]$. We define

$$\tilde{a}(t, x) = \begin{cases} \frac{f(t, x)}{x}, & x > M_+, \\ \frac{f(t, M_+)}{M_+}, & x \leq M_+. \end{cases}$$

Similarly, by (H_1), there exists a constant $M_- < 0$ such that, when $x \leq M_-$,

$$\frac{f(t, x)}{x} \geq \nu(t) - \varepsilon_0$$

holds for a.e. $t \in [t_0, t_0 + 2\pi]$. We define

$$\tilde{b}(t, x) = \begin{cases} \frac{f(t, x)}{x}, & x < M_-, \\ \frac{f(t, M_-)}{M_-}, & x \geq M_-. \end{cases}$$

Now, take

$$r(t, x) = f(t, x) - \tilde{a}(t, x)x^+ + \tilde{b}(t, x)x^-.$$

By the definitions of $\tilde{a}(t, x)$ and $\tilde{b}(t, x)$, for almost every $t \in [t_0, t_0 + 2\pi]$, we have

$$r(t, x) = \begin{cases} f(t, x) - \frac{f(t, M_-)}{M_-}x, & M_- \leq x \leq 0, \\ f(t, x) - \frac{f(t, M_+)}{M_+}x, & 0 < x \leq M_+, \\ 0, & x \notin [M_-, M_+]. \end{cases}$$

Hence, $r(t, x) = 0$, for $x \notin [M_-, M_+]$. Furthermore, by (H_0), for almost every $t \in [t_0, t_0 + 2\pi]$, $f(t, x) - \frac{f(t, M_-)}{M_-}x$ and $f(t, x) - \frac{f(t, M_+)}{M_+}x$ are continuous functions for $x \in [M_-, 0)$ and $(0, M_+]$, respectively; and

$r(t,x)$ is also continuous at $x = 0$. Consequently, for almost every $t \in [t_0, t_0 + 2\pi]$, by the properties of continuous functions on closed intervals and the fact that $r(t, x) = 0$ for $x \notin [M_-, M_+]$, we can conclude that $r(t, x)$ is bounded for $x \in \mathbb{R}$. Therefore, there is a 2π-periodic function $\tilde{r}(t)$ with $\tilde{r} \in L^1([t_0, t_0 + 2\pi], \mathbb{R}^+)$ such that, for almost every $t \in [t_0, t_0 + 2\pi]$ and every $x \in \mathbb{R}$, we have

$$|r(t, x)| \leq \tilde{r}(t).$$

□

Due to the fact that $v(t)$ in (H_1) and $q(t)$ in (H_2) are permitted to change signs, we cannot prove Lemma 3.3 by the nonresonance result utilized in the proof of Lemma 3 in [1]. Therefore, we provide a new proof of Lemma 3.3 as follows.

Proof of Lemma 3.3. By Lemma 3.2, we can rewrite (3.4) as

$$u'' + \tilde{a}(t, su)u^+ - \tilde{b}(t, su)u^- = p(t) - \frac{r(t, su)}{s}. \tag{5.1}$$

Next we will prove that (5.1) has a positive 2π-periodic solution. If such a positive solution exists, it satisfies

$$u'' + \tilde{a}(t, su)u = p(t) - \frac{r(t, su)}{s}. \tag{5.2}$$

The converse is also true. Now, set

$$G(t, su) = \tilde{a}(t, su)u - p(t) + \frac{r(t, su)}{s},$$

then by (3.1) and (3.3) we have

$$\lim_{u \to +\infty} \frac{G(t, su)}{u} = \lim_{x \to +\infty} \frac{f(t, x)}{x} = q(t). \tag{5.3}$$

We define a function

$$\tilde{G}(t, su) = \begin{cases} G(t, su), & u > 0, \\ 0, & u \leq 0, \end{cases}$$

then Eq (5.2) is changed into

$$u'' + \tilde{G}(t, su) = 0. \tag{5.4}$$

Next, we prove that Eq (5.4) has a 2π-periodic solution by Theorem 1.1 in [33].

First, we prove that every solution of the Cauchy problem associated to Eq (5.4) exists uniquely and globally. Indeed, on the one hand, let $u(t; u_0)$ be a solution of Eq (5.4) with an initial value $u(t_0; u_0) = u_0$. By using the Lipschitz-continuity of f in the second variable as stated in (H_0), we can deduce that for arbitrary $t \in \mathbb{R}$ and for $u_1, u_2 \in U(u_0)$ (either $u_1 > 0,\ u_2 > 0$, or $u_1 \leq 0,\ u_2 \leq 0$), where $U(u_0)$ represents an arbitrary neighborhood of u_0, there exists a positive constant L such that

$$\begin{aligned} |\tilde{G}(t, su_1) - \tilde{G}(t, su_2)| &\leq |G(t, su_1) - G(t, su_2)| \\ &= \Big|(\tilde{a}(t, su_1)u_1 + \frac{r(t, su_1)}{s}) - (\tilde{a}(t, su_2)u_2 + \frac{r(t, su_2)}{s})\Big| \\ &= \Big|\frac{f(t, su_1)}{s} - \frac{f(t, su_2)}{s}\Big| \leq L|u_1 - u_2|. \end{aligned}$$

Therefore, every solution of the Cauchy problem associated to Eq (5.4) exists uniquely. On the other hand, by (3.1) and (3.3), we have

$$\begin{aligned}|\tilde{G}(t,su)| &\leq |G(t,su)| \leq |\tilde{a}(t,su)||u| + |p(t)| + \frac{1}{s}|r(t,su)| \\ &\leq (|q(t)| + \varepsilon_0)|u| + |p(t)| + \frac{1}{s}\tilde{r}(t).\end{aligned}$$

This implies that $\tilde{G}$ has an at most linear growth of u, so every solution of the Cauchy problem associated to Eq (5.4) exists globally.

Second, we verify the nonresonance conditions. By (5.3) and the definition of $\tilde{G}(t,su)$, we have

$$\lim_{u\to+\infty}\frac{\tilde{G}(t,su)}{u} = q(t), \quad \lim_{u\to-\infty}\frac{\tilde{G}(t,su)}{u} = 0.$$

Consider the piecewise linear equation

$$u'' + q(t)u^+ - 0u^- = 0,$$

that is

$$u'' + q(t)u = 0,$$

then by $m < \rho(q) < m+1$ in (H_3) and Theorem 1.1 in [33], Eq (5.2) has a 2π-periodic solution $\tilde{u}_s(t)$, for any $s \geq 1$.

Finally, we will prove that such a solution is a positive solution for s large enough. Notice that $\tilde{u}_s(t)$ solves the linear equation

$$u'' + \tilde{a}(t, s\tilde{u}_s(t))u = p(t) - \frac{r(t, s\tilde{u}_s(t))}{s}. \tag{5.5}$$

Now set $s_1 = \frac{1}{\varepsilon_0}\|\tilde{r}\|_1$, for every $s \geq s_1$, by (3.1) and (3.3), we have

$$\|\tilde{a}(\cdot, s\tilde{u}_s(\cdot)) - q(\cdot)\|_1 \leq \varepsilon_0, \quad \Big\|\frac{r(\cdot, s\tilde{u}_s(\cdot))}{s}\Big\|_1 \leq \varepsilon_0.$$

Thus by Lemma 3.1, for $s \geq s_1$, Eq (5.5) has a unique 2π-periodic solution, which must coincide with $\tilde{u}_s(t)$ satisfying $c_0 \leq \tilde{u}_s(t) \leq C_0$. □

In order to facilitate the proof of (1.15), we prepare the following two lemmas. These lemmas can be seen as slight generalizations of Proposition 1 in [31] and Lemma 4.5 in [33], respectively.

Lemma 5.1. *For the piecewise linear system* (2.3)*, the following two statements hold.*

(i) If $a_+(t) \geq \eta^2$, $a_-(t) \geq \tau^2$, $t \in [t_0, t_0 + 2\pi]$. Then

$$\rho(a) \geq \frac{2\eta\tau}{\eta + \tau}.$$

(ii) If $a_+(t) \geq \eta^2$, $a_-(t) \geq \tau^2$, $t \in [t_0, t_0 + \chi]$; and $a_+(t)$, $a_-(t)$ take other values for $t \in (t_0 + \chi, t_0 + 2\pi]$. Then

$$\rho(a) \geq [\frac{\eta\tau\chi}{(\eta + \tau)\pi}].$$

Lemma 5.2. *In* (1.13)*, if* $q(t) = -\lambda^2$*,* $v(t) = -\mu^2$*, for* $t \in [\pi, 2\pi]$*, then*

$$-2\max\{\arctan|\lambda|,\ \arctan|\mu|\} < \theta(\pi) - \theta(2\pi) < \pi - \arctan|\lambda| - \arctan|\mu|.$$

Proof of (1.15). First, we prove $\rho(q) > m$. By using Lemma 4.4 in [33], we have $\rho(q) \geq m$. If $\rho(q) = m$, from Lemma 2.1, there is at least one nontrivial 2π-periodic solution $\theta(t)$ of the torus differential equation

$$\theta' = -q(t)\cos^2\theta - \sin^2\theta, \tag{5.6}$$

with $\theta(0) - \theta(2\pi) = 2m\pi$. Using a simple computation, it holds

$$\theta(0) - \theta\Big(\frac{2m\pi}{2m+1}\Big) = 2m\pi.$$

Then $\theta(2\pi) - \theta\Big(\frac{2m\pi}{2m+1}\Big) = 0$. Using Lemma 4.5 in [33], we have $\theta(\pi) - \theta(2\pi) > -2\arctan|\lambda|$, then it follows that

$$\theta\Big(\frac{2m\pi}{2m+1}\Big) - \theta(\pi) < 2\arctan|\lambda|.$$

Since

$$-\theta' = \sin^2\theta + (2m+1)^2\cos^2\theta > 1, \quad \text{for } t \in \Big(\frac{2m\pi}{2m+1}, \pi\Big),$$

then

$$\theta\Big(\frac{2m\pi}{2m+1}\Big) - \theta(\pi) > \pi/(2m+1).$$

Thus $\arctan|\lambda| > \pi/(2(2m+1))$. This contradicts to the definitions of λ in Remark 1.1. Therefore, $\rho(q) > m$.

Second, we prove $\rho(q) < m+1$. Let $\bar\theta(t)$ be a solution of (5.6) with $\bar\theta(0) = 0$. Since $q(t) = (2m+1)^2$ for $t \in [0, \pi]$, it follows that $\bar\theta(\pi) = -(2m+1)\pi$. Furthermore, we can observe that $y' = \mu^2 x$ for $t \in [\pi, 2\pi]$, and it implies that nonzero solutions of (5.6) can never perform clockwise rotations at x-axis when $t \in [\pi, 2\pi]$. Therefore, we have

$$\bar\theta(t) > -(2m+1)\pi, \quad \text{for } t \in [\pi, 2\pi].$$

So it follows that $\bar\theta(2\pi) > -(2m+1)\pi$. Furthermore, by the uniqueness of the solution for Cauchy problem associated to Eq (5.6), we have

$$\bar\theta(2k\pi) > \bar\theta(2(k-1)\pi) - (2m+1)\pi > -(2m+1)k\pi, \quad \text{for } k \in \mathbb{N}.$$

Therefore, by the definition of rotation number in Section 2, we have

$$\rho(q) = \lim_{k\to+\infty} \frac{\bar\theta(0) - \bar\theta(2k\pi)}{2k\pi} \leq m + \frac{1}{2} < m+1.$$

Finally, we prove $\rho(v) > n$. By (ii) of Lemma 5.1 and (1.14), we have $\rho(v) \geq n$. If $\rho(v) = n$, from Lemma 2.1, we can conclude that there is at least one nontrivial 2π-periodic solution $\theta(t)$ of the torus differential equation

$$\theta' = -q(t)((\cos\theta)^+)^2 - v(t)((\cos\theta)^-)^2 - \sin^2\theta,$$

with $\theta(0) - \theta(2\pi) = 2n\pi$. Using a simple computation, it holds

$$\theta(0) - \theta\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) = 2n\pi.$$

Then $\theta(2\pi) - \theta\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) = 0$. Using Lemma 5.2, it follows that

$$\theta(\pi) - \theta(2\pi) > -2\max\{\arctan|\lambda|, \ \arctan|\mu|\},$$

which implies

$$\theta\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) - \theta(\pi) < 2\max\{\arctan|\lambda|, \ \arctan|\mu|\}.$$

Since

$$-\theta' = \sin^2\theta + (2m+1)^2\cos^2\theta > 1, \quad \text{for} \quad t \in \Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}, \pi\Big),$$

then

$$\theta\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) - \theta(\pi) > \pi - \frac{n\pi}{2m+1} - \frac{n\pi}{2\alpha+\varrho}.$$

Thus we have $2\max\{\arctan|\lambda|, \ \arctan|\mu|\} > \pi - n\pi/(2m+1) - n\pi/(2\alpha+\varrho)$. This contradicts to the definitions of λ and μ in Remark 1.1. Therefore, $\rho(v) > n$. □

6. Conclusions

In this paper, we have obtained the existence of multiple periodic solutions for Eq (1.1). It is formulated in an original way, with sufficiently general assumptions. We have overcome the difficulties coming from the indefinite terms and the absence of global existence of solutions for the Cauchy problem associated with the Eq (1.1).

Regarding the difficulty coming from the indefinite terms, the resulting indefinite problems are not easy to deal with. To tackle this issue we use a rotation number approach. Especially, the estimations of rotation numbers, as well as the proof of Lemma 3.3, which is different from that in [1].

Regarding the difficulty coming from the absence of global existence of solutions, using the phase-plane analysis, we have obtained a spiral property and modified the system associated with Eq (4.1) to ensure the global existence of solutions for the modified system. By applying the Poincaré-Birkhoff twist theorem to the modified system, we obtain the multiplicity of periodic solutions. Finally, using the argument property and rotational property of solutions, we obtain the existence of multiple periodic solutions for the original system.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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