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Research article

Closed-form approximate solutions for stop-loss and Russian options with multiscale stochastic volatility

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Abstract: In general, derivation of closed-form analytic formulas for the prices of path-dependent exotic options is a challenging task when the underlying asset price model is chosen to be a stochastic volatility model. Pricing stop-loss and Russian options is studied under a multiscale stochastic volatility model in this paper. Both options are commonly perpetual American-style derivatives with a lookback provision. We derive closed-form formulas explicitly for the approximate prices of these two exotic options by using multiscale asymptotic analysis and partial differential equation method. The formulas can be efficiently computed starting with the Black-Scholes option prices. The accuracy of the analytic approximation is verified via Monte-Carlo simulations and the impacts of the multiscale stochastic volatility on the corresponding Black-Scholes option prices are revealed. Also, the performance of the model is compared with that of other models.

Keywords: option pricing; stop-loss option; Russian option; multiscale; stochastic volatility; asymptotics
Mathematics Subject Classification: 91G20, 91G60

1. Introduction

Following the seminal papers by Black and Scholes [3] and Merton [25] on the pricing formula for European vanilla options, there have been active researches also for path-dependent exotic options based on the geometric Brownian motion framework with constant volatility. The papers by Merton [25] himself and Reiner and Rubinstein [29] for barrier options, Goldman et al. [16] and Conze and Vishwanathan [6] for lookback options are few examples. Refer to Clewlow et al. [5] for a review study of path-dependent options under the Black-Scholes model. Based on the knowledge that the

Black-Scholes model with constant volatility does not account for empirically observed phenomena such as volatility smile effect, there have been extensions to local or stochastic volatility model cases. For instance, Davydov and Linetsky [9] obtained closed-form solutions for barrier and lookback options under the constant elasticity of variance (CEV) model of Cox [7] and Cox and Ross [8]. Park and Kim [28] derived an infinite series form of a pricing formula for a lookback option under a general stochastic volatility model. Kato et al. [20] obtained a semi closed-form approximation formula for the price of a barrier option under a certain type of stochastic volatility model covering the stochastic alpha-beta-rho (SABR) model of Hagan et al. [17]. Aquino and Bernard [1] derived semi-analytical pricing formulas for lookback and barrier options under the Heston model. Kim et al. [21] obtained an analytic approximation formula for the price of an external barrier option under a fast mean-reverting stochastic volatility model.

In this paper, we consider two types of perpetual American options with a path-dependent exotic structure, namely, stop-loss and Russian options. A stop-loss option is a perpetual style option with the structure of both 'knock-in' barrier and lookback options. This option was introduced by Fitt et al. [11] in 1994. If the underlying price reaches a maximum value $S^*_{t^*}$ at time t^* and then never goes up beyond $S^*_{t^*}$ and falls back into a given proportion, say λ, of it at later time t, then the option is knocked in and exercised in such a way that the option holder receives the amount S_t $(= \lambda S^*_t)$ at that time, where $0 < \lambda < 1$ is given as a predetermined value. In this case, the exercise time is a stopping time. On the other hand, a Russian option is a perpetual American option with a free boundary that contains a lookback provision. It was first proposed by Shepp and Shiryaev [31] in 1993. At any time t chosen by the option holder, this option pays out the maximum realized asset price S^*_t up to that time if the holder wants to claim it. We refer readers to Wilmott et al. [32] for more details on these two options together with the corresponding pricing formulas under the geometric Brownian motion with constant volatility.

Obtaining analytic pricing formulas for stop-loss and Russian options under a stochastic volatility model is a challenging task because of the complicated exotic nature of these options. The contribution of this work is to derive closed-form formulas explicitly for the approximate prices of these two options under a multiscale stochastic volatility model. To the best of our knowledge, there was no previous report on the formulas in such a multiscale volatility environment. The accuracy of the analytic formulas are verified via Monte-Carlo simulations. The impacts of the multisale stochastic volatility model on the corresponding Black-Scholes prices of those exotic options are disclosed. The performance of the model is compared with that of other models.

The rest of the paper is organized as follows. In Section 2, we discuss a multiscale stochastic volatility model formulation for the underlying asset prices and the important features of stop-loss and Russian options. Section 3 provides a detailed discussion on how an asymptotic expansion approach can yield partial differential equations (PDEs) for the prices of stop-loss and Russian options and the subsequent ordinary differential equation (ODE) problems. In Section 4, we derive explicitly the closed-form solutions of the PDE problems for the leading-order terms and the first-order corrections. Section 5 verifies that the results given by those analytic formulas match well with those generated by Monte-Carlo simulations and presents the impacts of the multiscale stochastic volatility model on the Black-Scholes option prices and a comparison with other models. Section 6 states a concluding remark. In Appendices A and B, we derive differential equations and their closed-form solutions for the second-order corrections. Appendix C describes the explicit representations of some functions in Appendix A.

2. Model framework and options of interest

For the price S_t of a given underlying asset (stock or market index), we consider

$$\begin{aligned} dS_t &= (r-q)S_t dt + f(Y_t, Z_t)S_t dW_t^s, \\ dY_t &= \left(\frac{1}{\epsilon}\alpha(Y_t) - \frac{1}{\sqrt{\epsilon}}\beta(Y_t)\Lambda(Y_t, Z_t)\right)dt + \frac{1}{\sqrt{\epsilon}}\beta(Y_t)dW_t^y, \\ dZ_t &= \left(\delta c(Z_t) - \sqrt{\delta}g(Z_t)\Gamma(Y_t, Z_t)\right)dt + \sqrt{\delta}g(Z_t)dW_t^z \end{aligned} \tag{2.1}$$

under a martingale probability measure Q, where and r and q are risk-free interest and dividend rates, respectively, the function f is smooth and bounded on $\mathbb{R}^2$, Λ and Γ represent the market prices of volatility risk, the functions α and β are given in such a way that Y_t is an ergodic process that admits a unique invariant distribution, denoted by Φ, and the functions c and g are smooth on $\mathbb{R}$ and at most linearly growing infinitely. W_t^x, W_t^y and W_t^z are standard Brownian motions with a correlation structure given by

$$d\langle W^s, W^y\rangle_t = \rho_{sy}dt, \quad d\langle W^s, W^z\rangle_t = \rho_{sz}dt, \quad d\langle W^y, W^z\rangle_t = \rho_{yz}dt.$$

Moreover, the constants ϵ and δ are such that $0 < \delta \ll \epsilon \ll \sqrt{\delta} \ll 1$. This type of multiscale stochastic volatility model was proposed by Fouque et al. [13] and the extensive study of pricing several types of derivatives under this model can be found in the book of Fouque et al. [14].

In this paper, we study an evaluation problem of stop-loss options and Russian options under the underlying asset price dynamics given by (2.1). Both options contain no expiration date and they have a lookback provision. We recall that the no-arbitrage price, $P(t, s, y, z)$, of a perpetual American option can be expressed as

$$P(t, s, y, z) = \sup_{t\le\tau\le\infty} \mathbb{E}^Q\left[e^{-r(\tau-t)}h(\tau)\,|\,S_t = s, Y_t = y, Z_t = z\right],$$

where τ is a stopping time and $h(\tau)$ denotes the payoff that depends on the 'path' of the underlying price up to time τ. We note that the starting time t does not matter when pricing perpetual options because of an infinite time horizon. Hence, one can write the option price $P(t, s, y, z)$ as $P(s, y, z)$ with the t-dependence. On the other hand, to deal with a lookback type of option, we need to define the maximum value of the underling asset price until arbitrary time t as

$$S_t^* = \sup_{0\le u\le t} S_u, \quad 0 < t < \infty,$$

which becomes another independent variable for the evaluation of options of interest.

The price of a stop-loss option depends on the underlying asset price, the maximum value of it, the levels of the two volatility driving processes and the pre-determined level λ. It is denoted by $P^{s/l}(s, s^*, y, z)$. The payoff h of this option is given by

$$h(\tau) = S_\tau \mathbf{1}_{S_\tau=\lambda S_\tau^*},$$

where '**1**' stands for the indicator function.

The price of a Russian option also depends on the underlying asset price, the maximum value of it, the levels of the two volatility driving processes. It is denoted by $P^R(s, s^*, y, z)$. For a Russian option, the payoff h is given by

$$h(\tau) = S^*_\tau.$$

In this work, we use the combined asymptotic expansion and partial differential equation (PDE) approach to evaluate the prices of these two options based on the Feynman-Kac theorem (see Oksendal [26] for example), a link between parabolic PDEs and stochastic processes. From this theorem, the no-arbitrage price $P(s, s^*, y, z)$ of a perpetual option, which could be a stop-loss option or a Russian option, satisfies

$$\left[\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}'_1 + \mathcal{L}'_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3 + \sqrt{\delta}\mathcal{M}'_1 + \delta\mathcal{M}_2\right]P(s, s^*, y, z) = 0, \tag{2.2}$$

where the operators $\mathcal{L}_0$, $\mathcal{L}'_1$, $\mathcal{L}'_2$, $\mathcal{M}_3$, $\mathcal{M}'_1$ and $\mathcal{M}_2$ are given by

$$\begin{aligned}
\mathcal{L}_0 &= \alpha(y)\frac{\partial}{\partial y} + \frac{1}{2}\beta^2(y)\frac{\partial^2}{\partial y^2},\\
\mathcal{L}'_1 &= \beta(y)\left(\rho_{sy} f(y,z)s\frac{\partial^2}{\partial s\partial y} - \Lambda(y,z)\frac{\partial}{\partial y}\right),\\
\mathcal{L}'_2 &= \frac{1}{2}f^2(y,z)s^2\frac{\partial^2}{\partial s^2} + (r-q)s\frac{\partial}{\partial s} - r,\\
\mathcal{M}_3 &= \rho_{yz}\beta(y)g(z)\frac{\partial^2}{\partial y\partial z},\\
\mathcal{M}'_1 &= g(z)\left(\rho_{sz} f(y,z)s\frac{\partial^2}{\partial s\partial z} - \Gamma(y,z)\frac{\partial}{\partial z}\right),\\
\mathcal{M}_2 &= c(z)\frac{\partial}{\partial z} + \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2},
\end{aligned} \tag{2.3}$$

respectively. We are going to employ the asymptotic analysis of Fouque et al. [14] to derive an approximate solution of this singularly perturbed PDE with appropriate boundary conditions imposed for each of the stop-loss and Russian options.

In the following argument, the following lemma for a Poisson equation is very useful.

Lemma 2.1. *We consider the Poisson equation*

$$\mathcal{L}_0\mathcal{X}(y) + \mathcal{G}(y) = 0.$$

(a) The existence of a solution for the Poisson equation requires the following condition (called the centering condition):

$$\langle\mathcal{G}\rangle := \int_R \mathcal{G}(y)\Phi(y)dy = 0,$$

where Φ is the invariant distribution of the process Y_t.

(b) If the function $\mathcal{G}$ is zero and the solution $\mathcal{X}$ does not move as fast as

$$\frac{\partial \mathcal{X}}{\partial y} \sim e^{-2\int_R \frac{\alpha(y)}{\beta^2(y)}dy}, \quad y \to \infty,$$

then $\mathcal{X}$ is independent of variable y.

Proof. Refer to Ramm [30] (Fredholm alternative theorem) and Fouque et al. [14] for (a) and (b), respectively. □

3. PDE problems for option pricing

In this section, we establish PDE problems for the price $P^{s/l}(s, s^*, y, z)$ of a stop-loss option and the price $P^R(s, s^*, y, z)$ of a Russian option, respectively.

3.1. Stop-loss option

The option price $P^{s/l}(s, s^*, y, z)$ satisfies the PDE (2.2) on the interval $\lambda s^* < s < s^*$ and boundary conditions given by

$$\begin{aligned} &P^{s/l}(s = \lambda s^*, s^*, y, z) = \lambda s^*, \\ &\frac{\partial P^{s/l}}{\partial s^*}(s = s^*, s^*, y, z) = 0. \end{aligned}$$

In addition to these boundary conditions, one might need to use the following linear scaling property:

$$P^{s/l}(\nu s, \nu s^*, y, z) = \nu P^{s/l}(s, s^*, y, z).$$

For dimensionality reduction, we use the change of independent and dependent variables, $s \to x$ and $P^{s/l} \to V^{s/l}$, defined by

$$\begin{aligned} x &= s/s^*, \\ P^{s/l}(s, s^*, y, z) &= s^* V^{s/l}(x, y, z). \end{aligned}$$

Then we obtain a PDE problem for $V^{s/l}$ (instead of $P^{s/l}$) as follows:

$$\begin{aligned} &\mathcal{L}V^{s/l}(x, y, z) = 0, \quad \lambda < x < 1, \\ &V^{s/l}(x = \lambda, y, z) = \lambda, \\ &V^{s/l}(x = 1, y, z) = \frac{\partial V^{s/l}}{\partial x}(1, y, z), \end{aligned} \tag{3.1}$$

where the operator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2, \\ \mathcal{L}_1 &:= \beta(y)\left(\rho_{xy} f(y, z) x \frac{\partial^2}{\partial x \partial y} - \Lambda(y, z)\frac{\partial}{\partial y}\right), \\ \mathcal{L}_2 &:= \frac{1}{2} f^2(y, z) x^2 \frac{\partial^2}{\partial x^2} + (r - q)x\frac{\partial}{\partial x} - r, \\ \mathcal{M}_1 &:= g(z)\left(\rho_{xz} f(y, z) x \frac{\partial^2}{\partial x \partial z} - \Gamma(y, z)\frac{\partial}{\partial z}\right). \end{aligned} \tag{3.2}$$

Note that $\mathcal{L}_0$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are the same as in (2.3).

We are interested in the solution $V^{s/l}(x,y,z)$ of the expansion form

$$V^{s/l}(x,y,z) = \sum_{i,j=0}^{\infty} \epsilon^{i/2}\delta^{j/2} V^{s/l}_{ij}(x,y,z), \tag{3.3}$$

where $V^{s/l}_{ij}$ are assumed to satisfy the growth condition stated in Lemma 2.1 (b) so that the solution can reflect the realistic situation in market. In the rest of this section, we are going to derive PDE problems for the terms $V^{s/l}_{ij}$ with $(ij) = (0,0)$, $(0,1)$ and $(1,0)$. First, rearranging the PDE (3.1) by using the series (3.3), we can get

$$\begin{aligned}
&\frac{1}{\epsilon}\mathcal{L}_0 V^{s/l}_{00} + \frac{\sqrt{\delta}}{\epsilon}\mathcal{L}_0 V^{s/l}_{01} + \frac{\delta}{\epsilon}\mathcal{L}_0 V^{s/l}_{02} \\
&+ \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0 V^{s/l}_{10} + \mathcal{L}_1 V^{s/l}_{00}\right) \\
&\quad + \frac{\sqrt{\delta}}{\sqrt{\epsilon}}\left(\mathcal{L}_0 V^{s/l}_{11} + \mathcal{L}_1 V^{s/l}_{01} + \mathcal{M}_3 V^{s/l}_{00}\right) \\
&\quad + \frac{\delta}{\sqrt{\epsilon}}\left(\mathcal{L}_0 V^{s/l}_{12} + \mathcal{L}_1 V^{s/l}_{02} + \mathcal{M}_3 V^{s/l}_{01}\right) \\
&+ \mathcal{L}_0 V^{s/l}_{20} + \mathcal{L}_1 V^{s/l}_{10} + \mathcal{L}_2 V^{s/l}_{00} \\
&\quad + \sqrt{\delta}\left(\mathcal{L}_0 V^{s/l}_{21} + \mathcal{L}_1 V^{s/l}_{11} + \mathcal{L}_2 V^{s/l}_{01} + \mathcal{M}_1 V^{s/l}_{00} + \mathcal{M}_3 V^{s/l}_{10}\right) \\
&\quad + \delta\left(\mathcal{L}_0 V^{s/l}_{22} + \mathcal{L}_1 V^{s/l}_{12} + \mathcal{L}_2 V^{s/l}_{02} + \mathcal{M}_1 V^{s/l}_{01} + \mathcal{M}_2 V^{s/l}_{00} + \mathcal{M}_3 V^{s/l}_{11}\right) \\
&+ \sqrt{\epsilon}\left(\mathcal{L}_0 V^{s/l}_{30} + \mathcal{L}_1 V^{s/l}_{20} + \mathcal{L}_2 V^{s/l}_{10}\right) \\
&\quad + \sqrt{\epsilon\delta}\left(\mathcal{L}_0 V^{s/l}_{31} + \mathcal{L}_1 V^{s/l}_{21} + \mathcal{L}_2 V^{s/l}_{11} + \mathcal{M}_1 V^{s/l}_{10} + \mathcal{M}_3 V^{s/l}_{20}\right) \\
&\quad + \epsilon\left(\mathcal{L}_0 V^{s/l}_{40} + \mathcal{L}_1 V^{s/l}_{30} + \mathcal{L}_2 V^{s/l}_{20}\right) + \cdots = 0.
\end{aligned} \tag{3.4}$$

The following proposition says that the first few terms of the asymptotic expansion (3.3) are independent of variable y.

Proposition 3.1. *The terms $V^{s/l}_{ij}$ with $i = 0,1$ and $j = 0,1,2$ in the asymptotic series (3.3) for the stop-loss option price $V^{s/l}$ are independent of y; $V^{s/l}_{ij}(x,y,z) = V^{s/l}_{ij}(x,z)$.*

Proof. From the terms of order $\frac{1}{\epsilon}$, $\frac{\sqrt{\delta}}{\epsilon}$ and $\frac{\delta}{\epsilon}$ in (3.4), we have the Poisson equations $\mathcal{L}_0 V^{s/l}_{0j} = 0$ ($j = 0,1,2$). Then, by Lemma 2.1 (a), $V^{s/l}_{0j}$ are independent of y for $j = 0$, 1 and 2. Since $\mathcal{L}_1 V^{s/l}_{0j} = 0$ ($j = 0,1,2$) and $\mathcal{M}_3 V^{s/l}_{0j} = 0$ ($j = 0,1$), we have the Poisson equations $\mathcal{L}_0 V^{s/l}_{1j} = 0$ ($j = 0,1,2$) from the terms of order $\frac{1}{\sqrt{\epsilon}}$, $\frac{\sqrt{\delta}}{\sqrt{\epsilon}}$ and $\frac{\delta}{\sqrt{\epsilon}}$ in (3.4). Thus $V^{s/l}_{1j}$ are independent of y for $j = 0$, 1 and 2. □

In the following argument, we use an operator, $\overline{\mathcal{L}}$, defined as

$$\overline{\mathcal{L}} := \langle \mathcal{L}_2 \rangle = \frac{1}{2}\sigma^2(z)x^2\frac{\partial^2}{\partial x^2} + (r-q)x\frac{\partial}{\partial x} - r, \quad \sigma(z) := \sqrt{\langle f^2(\cdot,z)\rangle}. \tag{3.5}$$

Also, we use the functions ϕ_1, ϕ_2, ψ, ξ_1 and ξ_2 defined by the solutions of

$$\begin{aligned}
\mathcal{L}_0\phi_1(y,z) &= f(y,z) - \langle f(\cdot,z)\rangle\,,\\
\mathcal{L}_0\phi_2(y,z) &= f^2(y,z) - \left\langle f^2(\cdot,z)\right\rangle,\\
\mathcal{L}_0\psi(y,z) &= \Gamma(y,z) - \langle \Gamma(\cdot,z)\rangle\,,\\
\mathcal{L}_0\xi_1(y,z) &= \beta(y)f(y,z)\frac{\partial\phi_2(y,z)}{\partial y} - \left\langle \beta(\cdot)f(\cdot,z)\frac{\partial\phi_2(\cdot,z)}{\partial y}\right\rangle,\\
\mathcal{L}_0\xi_2(y,z) &= \beta(y)\Lambda(y,z)\frac{\partial\phi_2(y,z)}{\partial y} - \left\langle \beta(\cdot)\Lambda(\cdot,z)\frac{\partial\phi_2(\cdot,z)}{\partial y}\right\rangle,
\end{aligned} \tag{3.6}$$

respectively. We note that the function $\phi_2(y,z)$ might need to be assumed to satisfy $\langle\phi_2(\cdot,z)\rangle = 0$ in order to find a solution for the correction $V_{20}^{s/l}$. Refer to Fouque et al. [15] for a detailed discussion on this requirement.

The next proposition provides the required ODE problems that the leading-order term and the first-order corrections ($0 \le i+j \le 1$) have to satisfy. The case for the second-order corrections corresponding to $i+j=2$ is presented in Appendix A.

Proposition 3.2. *The leading-order term and the first-order corrections, $V_{ij}^{s/l}(x,z)$, $0 \le i+j \le 1$, in the asymptotic series (3.3) for the stop-loss option price $V^{s/l}$ satisfy the ODE problems*

$$\begin{cases}
\overline{\mathcal{L}}V_{00}^{s/l}(x,z) = 0, \quad \lambda < x < 1,\\
V_{00}^{s/l}(\lambda,z) = \lambda,\\
V_{00}^{s/l}(1,z) = \frac{\partial}{\partial x}V_{00}^{s/l}(1,z),
\end{cases} \tag{3.7}$$

$$\begin{cases}
\overline{\mathcal{L}}V_{10}^{s/l}(x,z) = \left(U_{00}^{30}(z)x^3\frac{\partial^3}{\partial x^3} + U_{00}^{20}(z)x^2\frac{\partial^2}{\partial x^2}\right)V_{00}^{s/l}(x,z) := B_{10}(x,z), \quad \lambda < x < 1,\\
V_{10}^{s/l}(\lambda,z) = 0,\\
V_{10}^{s/l}(1,z) = \frac{\partial}{\partial x}V_{10}^{s/l}(1,z),
\end{cases} \tag{3.8}$$

and

$$\begin{cases}
\overline{\mathcal{L}}V_{01}^{s/l}(x,z) = \left(U_{00}^{11}(z)x\frac{\partial}{\partial x} + U_{00}^{01}(z)\right)\frac{\partial}{\partial z}V_{00}^{S/L}(x,z) := B_{01}(x,z), \quad \lambda < x < 1,\\
V_{01}^{s/l}(\lambda,z) = 0,\\
V_{01}^{s/l}(1,z) = \frac{\partial}{\partial x}V_{01}^{s/l}(1,z),
\end{cases} \tag{3.9}$$

respectively, where $U_{00}^{kl}(z)$, $(k,l) \in \{(3,0),(2,0),(1,1),(0,1)\}$, are given by

$$\begin{aligned}
U_{00}^{30}(z) &:= \frac{1}{2}\rho_{xy}\left\langle \beta(\cdot)f(\cdot,z)\frac{\partial\phi_2}{\partial y}(\cdot,z)\right\rangle,\\
U_{00}^{20}(z) &:= \rho_{xy}\left\langle \beta(\cdot)f(\cdot,z)\frac{\partial\phi_2}{\partial y}(\cdot,z)\right\rangle - \frac{1}{2}\left\langle \beta(\cdot)\Lambda(\cdot,z)\frac{\partial\phi_2}{\partial y}(\cdot,z)\right\rangle,\\
U_{00}^{11}(z) &:= -g(z)\rho_{xz}\langle f(\cdot,z)\rangle\,,\\
U_{00}^{01}(z) &:= g(z)\langle\Gamma(\cdot,z)\rangle\,,
\end{aligned} \tag{3.10}$$

respectively.

Proof. Firstly, by substituting the asymptotic series (3.3) into the boundary conditions in (3.1), we obtain

$$\begin{aligned}
&\sum_{i,j=0}^{\infty} \epsilon^{i/2}\delta^{j/2} V_{ij}^{s/l}(\lambda, y, z) = \lambda, \\
&\sum_{i,j=0}^{\infty} \epsilon^{i/2}\delta^{j/2} \left(V_{ij}^{s/l}(1, y, z) - \frac{\partial}{\partial x} V_{ij}^{s/l}(1, y, z) \right) = 0
\end{aligned} \tag{3.11}$$

which yields the desired boundary conditions in (3.7)–(3.9) directly.

Next, by applying Proposition 3.1 to the $\mathcal{O}(1)$ terms in Eq (3.4), we obtain the Poisson equation

$$\mathcal{L}_0 V_{20}^{s/l} + \mathcal{L}_2 V_{00}^{s/l} = 0. \tag{3.12}$$

Then by Lemma 2.1 $\overline{\mathcal{L}} V_{00}^{s/l} = 0$ holds and thus the ODE in (3.7) is satisfied.

Similarly, by Proposition 3.1 and Lemma 2.1, the terms of order $\sqrt{\epsilon}$ in Eq (3.4) lead to

$$\overline{\mathcal{L}} V_{10}^{s/l} = -\left\langle \mathcal{L}_1 \right\rangle V_{20}^{s/l}. \tag{3.13}$$

On the other hand, from (3.12) and $\overline{\mathcal{L}} V_{00}^{s/l} = 0$, the term $V_{20}^{s/l}$ satisfies

$$\mathcal{L}_0 V_{20}^{s/l} = -\frac{1}{2}\left(f^2(y,z) - \left\langle f^2(\cdot, z) \right\rangle \right) x^2 \frac{\partial^2}{\partial x^2} V_{00}^{s/l}. \tag{3.14}$$

Then the solution $V_{20}^{s/l}$ is given by

$$V_{20}^{s/l}(x, y, z) = -\frac{1}{2}\phi_2(y, z) x^2 \frac{\partial^2}{\partial x^2} V_{00}^{s/l}(x, z) + F_{20}^{s/l}(x, z) \tag{3.15}$$

for some function $F_{20}^{s/l}(x, z)$ independent of variable y, where ϕ_2 is defined in (3.6). Thus Eq (3.13) becomes

$$\overline{\mathcal{L}} V_{10}^{s/l} = \left(U_{00}^{30} x^3 \frac{\partial^3}{\partial x^3} + U_{00}^{20} x^2 \frac{\partial^2}{\partial x^2} \right) V_{00}^{s/l}$$

and thus the ODE in (3.8) is satisfied.

Again, by Proposition 3.1 and Lemma 2.1, the terms of order $\sqrt{\delta}$ in Eq (3.4) yield $\overline{\mathcal{L}} V_{01}^{s/l} = -\left\langle \mathcal{M}_1 \right\rangle V_{00}^{s/l}$. Since $-\left\langle \mathcal{M}_1 \right\rangle$ is the same as $\left(U_{00}^{11}(z) x \frac{\partial}{\partial x} + U_{00}^{01}(z) \right) \frac{\partial}{\partial z}$, the ODE in (3.9) holds. □

3.2. *Russian option*

To obtain PDE problems for the price $P^R(s, s^*, y, z)$ of a Russian option, we first note that $P^R(s, s^*, y, z)$ satisfies the PDE (2.2) on the interval $s^f < s < s^*$ and the boundary conditions

$$\begin{aligned}
&P^R(s = s^f, s^*, y, z) = s^*, \\
&\frac{\partial P^R}{\partial s^*}(s = s^*, s^*, y, z) = 0,
\end{aligned}$$

where $s^f(y, z)$ stands for a free boundary at which P^R and $\frac{\partial P^R}{\partial s}$ are continuous.

If we use the change of variables $x = s/s^*$ and $P^R(s, s^*, y, z) = s^* V^R(x, y, z)$ again for dimensionality reduction, we obtain a PDE problem given by

$$\begin{aligned} \mathcal{L}V^R(x, x^*, y, z) &= 0, \quad x^f < x < 1, \\ V^R(x^f, y, z) &= 1, \quad \frac{\partial V^R}{\partial x}(x^f, y, z) = 0, \\ V^R(1, y, z) &= \frac{\partial V^R}{\partial x}(1, y, z), \end{aligned} \tag{3.16}$$

where $x^f(y, z)$ is the free boundary corresponding to $s^f(y, z)$.

We are interested in the option price V^R and the free boundary x^f given in the following form:

$$\begin{aligned} V^R(x, y, z) &= \sum_{i,j=0}^{\infty} \epsilon^{i/2} \delta^{j/2} V_{ij}^R(x, y, z), \\ x^f(y, z) &= \sum_{i,j=0}^{\infty} \epsilon^{i/2} \delta^{j/2} x_{ij}^f(y, z). \end{aligned} \tag{3.17}$$

Proposition 3.3. *The terms V_{ij}^R with $i = 0, 1$ and $j = 0, 1, 2$ in the asymptotic series (3.17) for the Russian option price V^R are independent of y; $V_{ij}^R(x, y, z) = V_{ij}^R(x, z)$.*

Proof. The proof of this proposition is similar to the proof of Proposition 3.1 for the stop-loss option case since the proof does not depend on the boundary conditions. So, we omit the proof. □

The PDE form for the the Russian option price is the same as the one for the stop-loss option price. The difference between them lies in the boundary conditions and the existence of a free boundary. Thus, in the following proposition about the leading-order term and the first-order corrections ($0 \le i + j \le 1$), we obtain the same ODEs as in the stop-loss option case but with different boundary conditions and the appearance of a free boundary. The required ODE problems for the second-order corrections corresponding to $i + j = 2$ are given in Appendix B.

Proposition 3.4. *The leading-order term and the first-order corrections, V_{ij}^R, $0 \le i + j \le 1$, in the asymptotic series (3.17) for the Russian option price V^R satisfy the ODE problems*

$$\begin{cases} \overline{\mathcal{L}} V_{00}^R(x, z) = 0, \quad x_{00}^f(z) < x < 1, \\ V_{00}^R(1, z) = \frac{\partial}{\partial x} V_{00}^R(1, z), \\ V_{00}^R(x_{00}^f(z), z) = 1, \quad \frac{\partial}{\partial x} V_{00}^R(x_{00}^f(z), z) = 0, \end{cases} \tag{3.18}$$

$$\begin{cases} \overline{\mathcal{L}} V_{10}^R(x, z) = \left(U_{00}^{30}(z) x^3 \frac{\partial^3}{\partial x^3} + U_{00}^{20}(z) x^2 \frac{\partial^2}{\partial x^2} \right) V_{00}^R(x, z), \quad x_{00}^f(z) < x < 1, \\ V_{10}^R(1, z) = \frac{\partial}{\partial x} V_{10}^R(1, z), \\ V_{10}^R(x_{00}^f(z), z) = 0, \quad x_{10}^f(z) = -\frac{\frac{\partial}{\partial x} V_{10}^R(x_{00}^f(z), z)}{\frac{\partial^2}{\partial x^2} V_{00}^R(x_{00}^f(z), z)}, \end{cases} \tag{3.19}$$

and

$$\begin{cases} \overline{\mathcal{L}} V_{01}^R(x, z) = \left(U_{00}^{11}(z) x \frac{\partial}{\partial x} + U_{00}^{01}(z) \right) \frac{\partial}{\partial z} V_{00}^R(x, z), \quad x_{00}^f(z) < x < 1, \\ V_{01}^R(1, z) = \frac{\partial}{\partial x} V_{01}^R(1, z), \\ V_{01}^R(x_{00}^f(z), z) = 0, \quad x_{01}^f(z) = -\frac{\frac{\partial}{\partial x} V_{01}^R(x_{00}^f(z), z)}{\frac{\partial^2}{\partial x^2} V_{00}^R(x_{00}^f(z), z)}, \end{cases} \tag{3.20}$$

respectively, where $U_{00}^{kl}(z)$, $(k, l) \in \{(3, 0), (2, 0), (1, 1), (0, 1)\}$, are given by (3.10).

Proof. Substituting the series (3.17) into the boundary conditions in (3.16) and applying the Taylor series of the functions $V^R(x,y,z)$ and $\frac{\partial}{\partial x}V^R(x,y,z)$ at $x^f_{00}(y,z)$, we have

$$\begin{aligned}
&\sum_{i,j=0}^{\infty} \epsilon^{i/2}\delta^{j/2}\left(V^R_{ij}(1,y,z) - \frac{\partial}{\partial x}V^R_{ij}(1,y,z)\right) = 0,\\
&\sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{i,j=0}^{\infty} \epsilon^{i/2}\delta^{j/2}\frac{\partial^k}{\partial x^k}V^R_{ij}(x^f_{00}(y,z),y,z)\right)\left(\sum_{i,j=1}^{\infty} \epsilon^{i/2}\delta^{j/2}x^f_{ij}(y,z)\right)^k = 1,\\
&\sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{i,j=0}^{\infty} \epsilon^{i/2}\delta^{j/2}\frac{\partial^{k+1}}{\partial x^{k+1}}V^R_{ij}(x^f_{00}(y,z),y,z)\right)\left(\sum_{i,j=1}^{\infty} \epsilon^{i/2}\delta^{j/2}x^f_{ij}(y,z)\right)^k = 0.
\end{aligned} \tag{3.21}$$

Since V^R_{ij} is independent of variable y for every $(i,j) \in \{(0,0),(1,0),(0,1)\}$, x^f_{ij} is also independent of y for those (i,j). Rearranging the expansion (3.21) in terms of $\epsilon^{i/2}\delta^{j/2}$, we can obtain the desired boundary conditions as stated in (3.18)–(3.20). □

From Propositions 3.2 and 3.4, we find that the leading-order terms $V^{s/l}_{00}$ and V^R_{00} are the Black-Scholes option prices with volatility $\sigma(z)$ for stop-loss and Russian options, respectively. We obtain the solutions for the first-order corrections in the following section.

4. PDE solutions

In this section, we solve the problems for $V^{s/l}_{ij}$ and V^R_{ij}, $0 \le i+j \le 1$, derived in Propositions 3.2 and 3.4, respectively. The solutions for the second-order corrections corresponding to $i+j=2$ are given in Appendix A (stop-loss option) and Appendix B (Russian option).

4.1. General solutions

Both stop-loss and Russian options share the same PDE structure even if the boundary conditions are different. So, there are identical parts of the PDE solutions for both options. For convenience, we eliminate the superscript 's/l' and 'R' of $V^{s/l}$ and V^R and use notation V for the general PDE solution. In this section, we derive the concrete forms of V_{ij} up to $i+j=1$.

First, we have the following proposition.

Proposition 4.1. *The leading-order term and the first-order corrections, $V_{ij}(x,z)$, $0 \le i+j \le 1$, can be expressed as*

$$V_{ij}(x,z) = \sum_{k=1}^{2}\left(\sum_{\zeta=0}^{i+2j} A^{\zeta}_{ij,k}(z)\,(\ln x)^{\zeta}\right)x^{\eta_k(z)}, \quad (i,j) \in \{(0,0),(1,0),(0,1)\}, \tag{4.1}$$

where $A^0_{00,k}(z)$ *and* $\eta_k(z)$ *are defined by*

$$
\begin{aligned}
\textit{stop-loss:}\quad A^0_{00,k}(z) &= \frac{-(1-\eta_l(z))\lambda}{(1-\eta_k(z))\lambda^{\eta_l(z)} - (1-\eta_l(z))\,\lambda^{\eta_k(z)}} \quad (k,l \in \{1,2\}, k \neq l),\\
\textit{Russian:}\quad A^0_{00,k}(z) &= \frac{\eta_l(z)}{(\eta_l(z)-\eta_k(z))\left(x^f_{00}(z)\right)^{\eta_k(z)}} \quad (k,l \in \{1,2\}, k \neq l),\\
x^f_{00}(z) &:= \left(\frac{\eta_1(z)\,(1-\eta_2(z))}{\eta_2(z)\,(1-\eta_1(z))}\right)^{\frac{1}{\eta_2(z)-\eta_1(z)}},\\
\eta_k(z) &:= \frac{1}{2} - \frac{r-q}{\sigma^2(z)} + (-1)^{k-1}\sqrt{\left(\frac{1}{2}-\frac{r-q}{\sigma^2(z)}\right)^2 + \frac{2r}{\sigma^2(z)}} \quad (k \in \{1,2\}),
\end{aligned}
\tag{4.2}
$$

respectively, and $A^{\zeta}_{ij,k}(z)$*,* $\zeta = 0, \cdots, i+2j$ *and* $i+j=1$*, are some functions of* z *to be determined.*

Proof. Since the leading-order term V_{00} is the option price under the Black-Scholes model with volatility $\sigma(z)$, the known result in Wilmott et al. [32] says

$$
V_{00}(x,z) = \sum_{k=1}^{2} A^0_{00,k}(z)x^{\eta_k(z)}, \tag{4.3}
$$

where $A^0_{00,k}(z)$ and $\eta_k(z)$ are given by (4.2). Also, the leading-order term, $x^f_{00}(z)$, of the free boundary for a Russian option is given by the expression in (4.1).

The first-order corrections $V_{ij}(x,z)$, $i+j=1$, depend on the corresponding inhomogeneous terms $B_{ij}(x,z)$ defined in (3.8) and (3.9). Substituting (4.3) into those $B_{ij}(x,z)$, we can obtain

$$
\begin{aligned}
B_{10}(x,z) &= \sum_{k=1}^{2} B^0_{10,k}(z)x^{\eta_k(z)},\\
B^0_{10,k}(z) &:= \left(U^{30}_{00}\prod_{\omega=0}^{2}(\eta_k(z)-\omega) + U^{20}_{00}\prod_{\omega=0}^{1}(\eta_k(z)-\omega)\right)A^0_{00,k}(z),\\
B_{01}(x,z) &= \sum_{k=1}^{2}\left(\sum_{\zeta=0}^{1} B^{\zeta}_{01,k}(z)\,(\ln x)^{\zeta}\right)x^{\eta_k(z)},\\
B^0_{01,k}(z) &:= \left(U^{11}_{00}\left(\frac{\partial \eta_k(z)}{\partial z} + \eta_k(z)\frac{\partial}{\partial z}\right) + U^{01}_{00}\frac{\partial}{\partial z}\right)A^0_{00,k}(z),\\
B^1_{01,k}(z) &:= \left(U^{11}_{00}\eta_k(z) + U^{01}_{00}\right)\frac{\partial \eta_k(z)}{\partial z}A^0_{00,k}(z),
\end{aligned}
\tag{4.4}
$$

where $U^{kl}_{00}(z)$, $(k,l) \in \{(3,0),(2,0),(1,1),(0,1)\}$, are given by (3.10). Since $\overline{\mathcal{L}}$ defined by (3.5) is related to the well-known Cauchy-Euler equation of order 2, one can take the following expressions for $V_{10}(x,z)$ and $V_{01}(x,z)$, hinted at by (4.4), respectively:

$$
\begin{aligned}
V_{10}(x,z) &= \sum_{k=1}^{2}\left(\sum_{\zeta=0}^{1} A^{\zeta}_{10,k}(z)\,(\ln x)^{\zeta}\right)x^{\eta_k(z)},\\
V_{01}(x,z) &= \sum_{k=1}^{2}\left(\sum_{\zeta=0}^{2} A^{\zeta}_{01,k}(z)\,(\ln x)^{\zeta}\right)x^{\eta_k(z)}
\end{aligned}
\tag{4.5}
$$

for some functions $A^{\zeta}_{10,k}(z)$, $\zeta = 0, 1$, and $A^{\zeta}_{01,k}(z)$, $\zeta = 0, 1, 2$. So, Proposition 4.1 is proved. □

In the following argument, we obtain the concrete forms of $A^{\zeta}_{10,k}(z)$, $\zeta = 0, 1$, and $A^{\zeta}_{01,k}(z)$, $\zeta = 0, 1, 2$. Among those terms, $A^1_{10,k}(z)$, $A^1_{01,k}(z)$ and $A^2_{01,k}(z)$ in (4.1) are commonly shared regardless of stop-loss or Russian. They are first solved in the following proposition.

Proposition 4.2. *The terms $A^1_{10,k}(z)$ and $A^{\zeta}_{01,k}(z)$, $\zeta = 1, 2$, in (4.1) are given by*

$$\begin{aligned} A^1_{10,k}(z) &= \frac{B^0_{10,k}(z)}{\frac{1}{2}\sigma^2(z)\,(2\eta_k(z) - 1) + (r - q)}, \\ A^1_{01,k}(z) &= \frac{B^0_{01,k}(z) - \sigma^2(z)A^2_{01,k}(z)}{\frac{1}{2}\sigma^2(z)\,(2\eta_k(z) - 1) + (r - q)}, \\ A^2_{01,k}(z) &= \frac{B^1_{01,k}(z)}{2\left(\frac{1}{2}\sigma^2(z)\,(2\eta_k(z) - 1) + (r - q)\right)}, \end{aligned} \tag{4.6}$$

respectively, where $B^0_{10,k}(z)$, $B^0_{01,k}(z)$ and $B^1_{01,k}(z)$ are defined in (4.4).

Proof. Substituting

$$V_{ij}(x, z) = \sum_{k=1}^{2} \left(\sum_{\zeta=1}^{i+2j} A^{\zeta}_{01,k}(z)\,(\ln x)^{\zeta} \right) x^{\eta_k(z)}, \quad (i, j) \in \{(1, 0), (0, 1)\}$$

into the ODEs in (3.8) and (3.9), we can get

$$\begin{aligned} &\sum_{k=1}^{2} \left(\frac{1}{2}\sigma^2(z)\,(2\eta_k(z) - 1) + (r - q) \right) A^1_{10,k}(z) = \sum_{k=1}^{2} B^0_{10,k}(x, z), \\ &\sum_{k=1}^{2} \left[\sigma^2(z)A^2_{01,k}(z) + \left(\frac{1}{2}\sigma^2(z)\,(2\eta_k(z) - 1) + (r - q) \right) \left(A^1_{01,k}(z) + 2A^2_{01,k}(z) \ln x \right) \right] \\ &\qquad = \sum_{k=1}^{2} \left(\sum_{\zeta=0}^{1} B^{\zeta}_{01,k}(z)\,(\ln x)^{\zeta} \right). \end{aligned}$$

Then, by simple calculation, we can obtain (4.6). □

Next, since the terms $A^0_{ij,k}(z)$, $(i, j) \in \{(1, 0), (0, 1)\}$, in (4.1) depend on the boundary conditions in (3.8) and (3.9) (stop-loss option) or (3.19) and (3.20) (Russian option), we obtain the particular solutions of them as shown below in Propositions 4.3 and 4.4, respectively.

4.2. Particular solutions for stop-loss option

In this section, we find solutions for $A^0_{10,k}(z)$ and $A^0_{01,k}(z)$ which are decided by the boundary conditions in (3.8) and (3.9) corresponding to a stop-loss option.

Proposition 4.3. *The terms $A^0_{10,k}(z)$ and $A^0_{01,k}(z)$, $k \in \{1,2\}$, in (4.1) for a stop-loss option are given by*

$$A^0_{10,k}(z) = \frac{\lambda^{\eta_l(z)} \sum_{\omega=1}^2 A^1_{10,\omega}(z) + (1-\eta_l(z)) \ln\lambda \sum_{\omega=1}^2 \lambda^{\eta_\omega(z)} A^1_{10,\omega}(z)}{(1-\eta_k(z))\,\lambda^{\eta_l(z)} - (1-\eta_l(z))\,\lambda^{\eta_k(z)}},$$
$$A^0_{01,k}(z) = \frac{\lambda^{\eta_l(z)} \sum_{\omega=1}^2 A^1_{01,\omega}(z) + (1-\eta_l(z)) \sum_{\omega=1}^2 \left(\lambda^{\eta_\omega(z)} \sum_{\zeta=1}^2 A^\zeta_{01,\omega}(z)\,(\ln\lambda)^\zeta\right)}{(1-\eta_k(z))\,\lambda^{\eta_l(z)} - (1-\eta_l(z))\,\lambda^{\eta_k(z)}}, \tag{4.7}$$

respectively, where $k, l \in 1,2$ with $k \neq l$ and $A^1_{10,\omega}(z)$ and $A^\zeta_{01,\omega}(z)$ $(\zeta = 1,2)$ are given in Proposition 4.2.

Proof. Substituting

$$V_{ij}(x,z) = \sum_{k=1}^2 \left(\sum_{\zeta=0}^{i+2j} A^\zeta_{01,k}(z)\,(\ln x)^\zeta \right) x^{\eta_k(z)}, \quad (i,j) \in \{(1,0),(0,1)\}$$

into the boundary conditions in (3.8) and (3.9), we obtain the following simultaneous equations:

$$\begin{cases} \sum_{k=1}^2 A^0_{ij,k}(z) = \sum_{k=1}^2 \left(\eta_k(z) A^0_{ij,k}(z) + A^1_{ij,k}(z)\right), \\ \sum_{k=1}^2 \left(\lambda^{\eta_k(z)} \sum_{\zeta=0}^{i+2j} A^\zeta_{ij,k}(z)\,(\ln\lambda)^\zeta\right) = 0, \end{cases} \quad (i,j) \in \{(1,0),(0,1)\}.$$

Then, by simple calculation, we can obtain the solutions (4.7). □

4.3. Particular solutions for Russian option

In this section, we derive solutions for $A^0_{10,k}(z)$ and $A^0_{01,k}(z)$ which are decided by the boundary conditions in (3.19) and (3.20) corresponding to a Russian option.

Proposition 4.4. *The terms $A^0_{10,k}(z)$ and $A^0_{01,k}(z)$, $k \in \{1,2\}$, in (4.1) for a Russian option are given by*

$$A^0_{ij,k}(z) = \frac{\left(x^f_{00}(z)\right)^{\eta_l(z)} \sum_{\omega=1}^2 A^1_{ij,\omega}(z) + (1-\eta_l(z)) \sum_{\omega=1}^2 \left[\left(x^f_{00}(z)\right)^{\eta_\omega(z)} \sum_{\zeta=1}^{i+2j} A^\zeta_{ij,\omega}(z) \left(\ln x^f_{00}(z)\right)^\zeta\right]}{(1-\eta_k(z))\left(x^f_{00}(z)\right)^{\eta_l(z)} - (1-\eta_l(z))\left(x^f_{00}(z)\right)^{\eta_k(z)}}, \tag{4.8}$$

where $k, l \in 1,2$ with $k \neq l$ and the corresponding free boundaries $x^f_{10}(z)$ and $x^f_{01}(z)$ are given by

$$x^f_{ij}(z) = -\frac{\sum_{k=1}^2 \left[\left(x^f_{00}(z)\right)^{\eta_k(z)-1} \left(\sum_{\zeta=1}^{i+2j} \zeta A^\zeta_{ij,k}(z) \left(\ln x^f_{00}(z)\right)^{\zeta-1} + \eta_k(z) \sum_{\zeta=0}^{i+2j} A^\zeta_{ij,k}(z) \left(\ln x^f_{00}(z)\right)^\zeta\right)\right]}{\sum_{k=1}^2 \left[\left(x^f_{00}(z)\right)^{\eta_k(z)-2} \eta_k(z)\,(\eta_k(z)-1)\,A^0_{00,k}(z)\right]}, \tag{4.9}$$

where $(i,j) \in \{(1,0),(0,1)\}$) and $A^\zeta_{ij,k}(z)$, $k = 1,2$, $\zeta = 1, ...i+2j$, are given in Proposition 4.2.

Proof. Substituting

$$V_{ij}(x,z) = \sum_{k=1}^2 \left(\sum_{\zeta=0}^{i+2j} A^\zeta_{01,k}(z)\,(\ln x)^\zeta \right) x^{\eta_k(z)}, \quad (i,j) \in \{(1,0),(0,1)\},$$

into the boundary conditions in (3.19) and (3.20), we obtain the following simultaneous equations:

$$\begin{cases} \sum_{k=1}^{2} A_{ij,k}^{0}(z) = \sum_{k=1}^{2} \left(\eta_k(z) A_{ij,k}^{0}(z) + A_{ij,k}^{1}(z)\right), \\ \sum_{k=1}^{2} \left(\left(x_{00}^{f}(z)\right)^{\eta_k(z)} \sum_{\zeta=0}^{i+2j} A_{ij,k}^{\zeta}(z) \left(\ln x_{00}^{f}(z)\right)^{\zeta}\right) = 0 \end{cases}, \quad (i,j) \in \{(1,0),(0,1)\}.$$

Then we can obtain the solutions (4.8) by calculating this directly.

Next, we compute the free boundary $x_{ij}^{f}(z)$ from the boundary conditions in (3.18)–(3.20). Substituting the solutions $V_{10}(x,z)$ and $V_{01}(x,z)$ given by (4.5) together with (4.6) and (4.8) into the free boundary conditions in (3.18)–(3.20), we can obtain the desired free boundary result (4.9). □

Remark. In the above argument, we have obtained an approximation, $\tilde{V} := V_{00} + \sqrt{\epsilon} V_{10} + \sqrt{\delta} V_{01}$, of the option price V (= $V^{s/l}$ or V^R). Using the same analysis as in Fouque et al. [14], one can obtain a theoretical error of the approximation. Instead of repeating the argument here, however, we demonstrate the accuracy of the approximation numerically in the next section.

5. Numerical experiments

In this section, we perform some numerical experiments for stop-loss and Russian options under the multiscale stochastic volatility model (2.1). Since the real market data of these options are not available, we use the parameter values used in Fouque et al. [15] and Fitt et al. [11].

We approximate the prices, $V^{s/l}$ and V^R, of stop-loss and Russian options by

$$\begin{aligned} V^{s/l} &\approx V_{00}^{s/l} + \tilde{V}_1^{s/l}, \\ V^{R} &\approx V_{00}^{R} + \tilde{V}_1^{R}, \end{aligned} \tag{5.1}$$

respectively, where $V_{00}^{s/l}$ and V_{00}^{R} are the Black-Scholes option prices with volatility $\sigma(z)$ and $\tilde{V}_1^{s/l}$ and $\tilde{V}_1^{R}$ are the first order corrections defined by

$$\begin{aligned} \tilde{V}_1^{s/l} &:= \tilde{V}_{10}^{sl} + \tilde{V}_{01}^{sl} = \sqrt{\epsilon} V_{10}^{sl} + \sqrt{\delta} V_{01}^{sl}, \\ \tilde{V}_1^{R} &:= \tilde{V}_{10}^{R} + \tilde{V}_{01}^{R} = \sqrt{\epsilon} V_{10}^{R} + \sqrt{\delta} V_{01}^{R}, \end{aligned} \tag{5.2}$$

where V_{ij}^{sl} and V_{ij}^{R} are the corresponding terms in the series (3.3) and (3.17), respectively. We define terms $U_{00}^{kl,\epsilon}(z)$, $U_{00}^{kl,\delta}(z)$, $\tilde{B}_{ij}^{\zeta}(z)$ and $\tilde{A}_{ij}^{\zeta}(z)$ by

$$\begin{aligned} U_{00}^{kl,\epsilon}(z) &:= \sqrt{\epsilon} U_{00}^{kl}(z), \quad (k,l) \in \{(3,0),(2,0)\}, \\ U_{00}^{kl,\delta}(z) &:= \sqrt{\delta} \sigma'(z) U_{00}^{kl}(z), \quad (k,l) \in \{(1,1),(1,0)\}, \\ \tilde{B}_{ij,k}^{\zeta}(z) &:= \epsilon^{i/2} \delta^{j/2} B_{ij,k}^{\zeta}(z), \\ \tilde{A}_{ij,k}^{\zeta}(z) &:= \epsilon^{i/2} \delta^{j/2} A_{ij,k}^{\zeta}(z), \end{aligned} \tag{5.3}$$

where the group parameters $U_{00}^{kl}(z)$, $(k,l) \in \{(3,0),(2,0),(1,1),(1,0)\}$, are defined by (3.10) and the functions $B_{ij,k}^{\zeta}(z)$ and $A_{ij,k}^{\zeta}(z)$ are defined by (4.4) and (4.6), respectively. Moreover, for simplicity, we use the unified notations U_{00}^{ϵ} and U_{00}^{δ} defined by $U_{00}^{\delta} := U_{00}^{11,\delta} = U_{00}^{10,\delta}$ and $U_{00}^{\epsilon} := U_{00}^{30,\epsilon} = U_{00}^{20,\epsilon}$ when $U_{00}^{11,\delta} = U_{00}^{10,\delta}$ and $U_{00}^{30,\epsilon} = U_{00}^{20,\epsilon}$.

For a numerical experiment, we apply the change rule $\frac{\partial}{\partial z} = \sigma'(z)\frac{\partial}{\partial \sigma}$ to the functions in (4.4) and use $U_{00}^{kl,\epsilon}$ and $U_{00}^{kl,\delta}$ defined in (5.3). Then the functions in (4.4) become

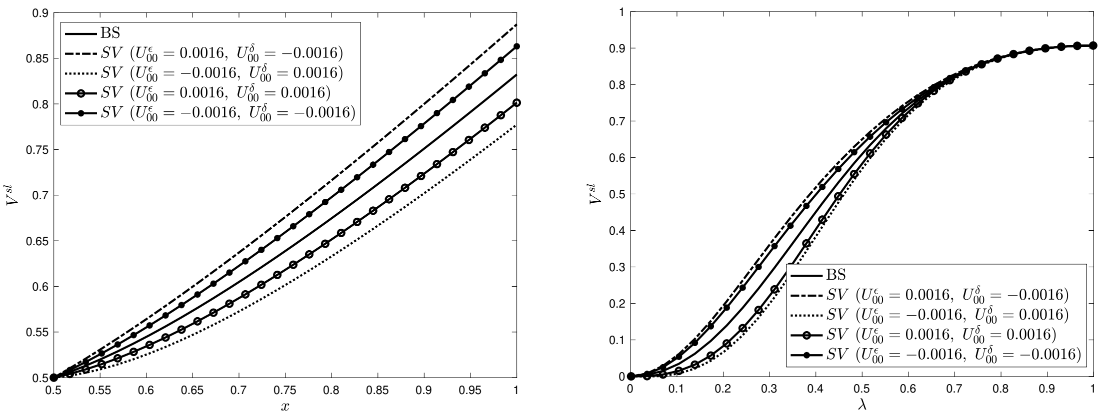
$$\begin{aligned}
\tilde{B}^0_{10,k}(\sigma) &:= \sqrt{\epsilon}B^0_{10,k}(\sigma) = \left(U_{00}^{30,\epsilon}\prod_{\omega=0}^{2}(\eta_k(\sigma)-\omega) + U_{00}^{20,\epsilon}\eta_k(\sigma)(\eta_k(\sigma)-1)\right)A^0_{00,k}(\sigma),\\
\tilde{B}^0_{01,k}(\sigma) &:= \sqrt{\delta}B^0_{01,k}(\sigma) = \left(U_{00}^{11,\delta}\left(\frac{\partial\eta_k}{\partial\sigma}(\sigma) + \eta_k(\sigma)\frac{\partial}{\partial\sigma}\right) + U_{00}^{01,\delta}\frac{\partial}{\partial\sigma}\right)A^0_{00,k}(\sigma),\\
\tilde{B}^1_{01,k}(\sigma) &:= \sqrt{\delta}B^1_{01,k}(\sigma) = \left(U_{00}^{11,\delta}\eta_k(\sigma)\frac{\partial\eta_k}{\partial\sigma}(\sigma) + U_{00}^{01,\delta}\frac{\partial\eta_k}{\partial\sigma}(\sigma)\right)A^0_{00,k}(\sigma),
\end{aligned} \tag{5.4}$$

respectively. On the other hand, from $\tilde{A}^{\zeta}_{ij,k}(\sigma)$ defined in (5.3) and (4.5), we have

$$\begin{aligned}
\tilde{V}_{10} &:= \sqrt{\epsilon}V_{10}(x,\sigma) = \sum_{k=1}^{2}\left(\sum_{\zeta=0}^{1}\tilde{A}^{\zeta}_{10,k}(\sigma)(\ln x)^{\zeta}\right)x^{\eta_k(\sigma)},\\
\tilde{V}_{01} &:= \sqrt{\delta}V_{01}(x,\sigma) = \sum_{k=1}^{2}\left(\sum_{\zeta=0}^{2}\tilde{A}^{\zeta}_{01,k}(\sigma)(\ln x)^{\zeta}\right)x^{\eta_k(\sigma)},
\end{aligned} \tag{5.5}$$

respectively. Therefore, the functions $\tilde{V}^{sl}_{ij}$ and $\tilde{V}^{R}_{ij}$, $(i,j) \in \{(1,0),(0,1)\}$, in (5.5) constitute the correction terms in (5.2) and they are related to $\tilde{A}^{\zeta}_{ij,k}$ which depends on the choice of a stop-loss or Russian option.

Figures 1 and 2 show the effect of the multiscale stochastic volatility on the Black-Scholes option prices. Depending on each parameter set of the values of U^{ϵ}_{00} and U^{δ}_{00}, the option prices affected by the multiscale stochastic volatility (2.1) give a different graphical representation from the Black-Scholes option prices.



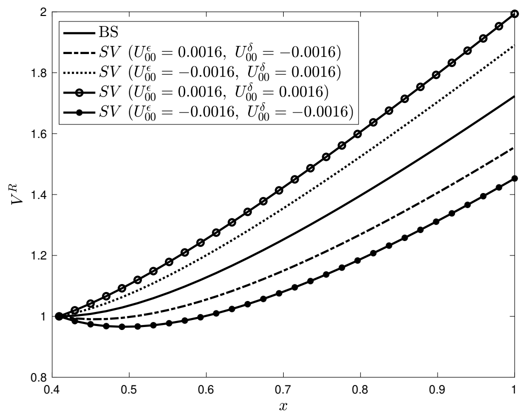
(a) The price $V^{s/l}$ of a stop-loss option as a function of $x (= s/s^*)$; $q = 0.02$, $\lambda = 0.5$.

(b) The price $V^{s/l}$ of a stop-loss option as a function of λ; $q = 0.06$, $x = 0.9$.

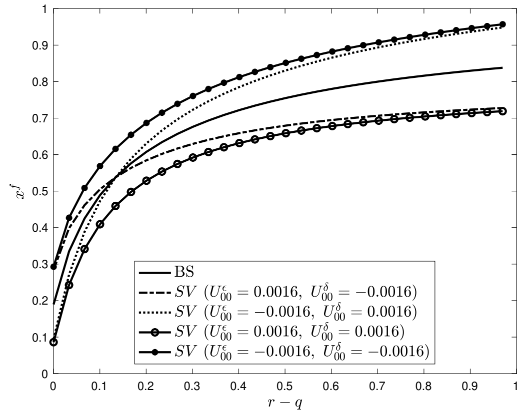
Figure 1. Stop-loss option prices under the Black-Scholes (BS) model and the multiscale stochastic volatility (SV) model for various levels of U^{ϵ}_{00} and U^{δ}_{00}; $r = 0.08$.

Figure 1 demonstrates how flexible the multiscale stochastic volatility model is compared to the Black-Scholes model for a stop-loss option. It shows the dependence of the option price on the variable

$x\,(= s/s^*)$ (the underlying price-the maximum underlying price ratio) and the parameter λ (the rebate level) for different values of the group parameters U^{ϵ}_{00} and U^{δ}_{00}, respectively. Depending on the values of the group parameters, the price of a stop-loss option can be over-priced or under-priced. There is no fixed direction of movement. This suggests that the multiscale stochastic volatility model offers a greater degree of flexibility in the way that it can capture the diverse volatile nature of the stock markets.



(a) The Russian option price V^R against $x\,(= s/s^*)$; $r = 0.08$.



(b) The free boundary x^f against $r - q$

Figure 2. Russian option prices and free boundaries under the Black-Scholes (BS) model and the multiscale stochastic volatility (SV) model for various levels of U^{ϵ}_{00} and U^{δ}_{00}; $q = 0.02$, $\sigma = 0.3$.

In Figure 2, we observe the correction effect on the Black-Scholes price and the free boundary of a Russian option given by the multiscale stochastic volatility model. The multiscale stochastic volatility model offers a greater degree of flexibility in a similar manner to the stop-loss option case. The free boundaries also show a behavior sensitive to the group parameters U^{ϵ}_{00} and U^{δ}_{00}.

Tables 1 and 2 show some comparison results between the Monte-Carlo simulation outcomes $P^{s/l}_{MC}$ and P^{R}_{MC} and our analytic prices $P^{s/l}$ and P^R, respectively, for each different value of the initial asset price s. For the comparison, we calculate the absolute difference and relative difference values, where the absolute difference is defined by the absolute value of the difference between the option price obtained from the Monte-Carlo simulation and the option price from the approximation formula (5.1), and the relative difference is defined by the ratio of the absolute difference over the price obtained from the Monte-Carlo simulation. Here, the repeated sampling number is set to be 100,000 and the discretized time step dt to be $\frac{1}{1500}$ in Monte-Carlo simulation. The common parameters and functions used in both stop-loss and Russian options are as follows: $r = 0.1$, $q = 0.05$, $\sigma = 0.3$, $s^* = 105$, $\epsilon = 0.01$, $\delta = 0.001$, $U^{30,\epsilon}_{00} = 0.007$, $U^{20,\epsilon}_{00} = 0.002$, $U^{11,\delta}_{00} = 0.007$ and $U^{10,\delta}_{00} = 0.002$. We test the multiscale stochastic volatility model by comparing it with Monte-Carlo simulation method because there is no real market data in the case of perpetual American type of options. Here, we brought the values of parameters and the set of functions used in Fouque and Han [12], one of existing research works about the Monte-Carlo simulation for the multiscale stochastic volatility model. They are $f(y,z) = e^{y+z}$, $\alpha(y) = m_1 - y$, $\beta(y) = \nu_1\sqrt{2}$, $c(z) = m_2 - z$, $g(z) = \nu_2\sqrt{2}$, $m_1 = -0.8$, $m_2 = -0.8$, $\nu_1 = 0.5$, $\nu_2 = 0.8$ and $\Lambda = \Gamma = 0$.

Table 1 provides a comparison result between the Monte-Carlo simulation and the approximation (5.1) for the stop-loss option prices. For each different value of the initial asset price s, the absolute difference is within '1' and the relative difference is within '0.01'. They are quite similar to each other. Adding higher order terms to (5.1) would bring more similarity between the two results.

Table 1. Comparison between the approximation formula and the Monte-Carlo simulation for stop-loss option prices against a variety of s; absolute difference = $\left\|P^{sl} - P^{sl}_{MC}\right\|$, relative difference = $\left\|\frac{P^{sl}-P^{sl}_{MC}}{P^{sl}_{MC}}\right\|$, $y = -0.93$, $z = -0.73$ and $\lambda = 0.5$.

s	$P^{s/l}$	$P^{s/l}_{MC}$	absolute difference	relative difference
80	52.6759	52.2988	0.3771	0.0072
85	54.5355	54.2620	0.2735	0.0050
90	56.7345	56.4162	0.3184	0.0056
95	59.2284	58.8681	0.3603	0.0061
100	61.9821	61.5982	0.3839	0.0062
105	64.9677	64.5273	0.4404	0.0068

Table 2 shows a comparison result between the Monte-Carlo simulation and the approximation (5.1) for the Russian option prices. In this case, we used the method suggested by Basso and Pianca [2] for obtaining the Monte-Carlo simulation results. The absolute difference is within '1' and the relative difference is within '0.01' again.

Table 2. Comparison between the approximation formula and the Monte-Carlo simulation for Russian option prices against a variety of s; absolute difference = $\left\|P^{R} - P^{R}_{MC}\right\|$, relative difference = $\left\|\frac{P^{R}-P^{R}_{MC}}{P^{R}_{MC}}\right\|$, $y = -0.745$ and $z = -0.347$.

s	P^{R}	P^{R}_{MC}	absolute difference	relative difference
80	136.9703	137.8832	0.9130	0.0067
85	145.1347	145.4095	0.2748	0.0019
90	153.3948	154.1971	0.8023	0.0052
95	161.7382	162.2437	0.5055	0.0031
100	170.1538	169.8807	0.2731	0.0016
105	178.6319	178.4759	0.1560	0.0009

Tables 3 and 4 provide a comparison result between the Monte-Carlo simulation and the first order approximation (5.1) for each different value of $r-q$. The absolute differences are within '2.5' and '2.8' and the relative differences are within '0.07' and ''0.03', respectively, for the stop-loss and Russian option prices.

Tables 5 and 6 demonstrate a comparison result for the Monte-Carlo simulation and the approximation (5.1) for each different value of s^*. The absolute differences are within '0.5' and '2.8' for the stop-loss and Russian option prices, respectively. Moreover, the relative differences are within '0.009' and '0.02' for the stop-loss and Russian option prices, respectively.

Table 3. Comparison between the approximation formula and the Monte-Carlo simulation for stop-loss option prices against a variety of $r-q$; absolute difference = $\left\|P^{sl} - P^{sl}_{MC}\right\|$, relative difference = $\left\|\frac{P^{sl}-P^{sl}_{MC}}{P^{sl}_{MC}}\right\|$, $y = -0.42$, $z = -0.42$ and $\lambda = 0.5$.

$r-q$	P^{sl}	P^{sl}_{MC}	absolute difference	relative difference
0.1	36.5508	38.9692	2.4184	0.0621
0.12	38.4595	39.6875	1.2281	0.0309
0.14	40.5777	40.4344	0.1432	0.0035
0.16	42.8733	41.8477	1.0256	0.0245
0.18	45.2658	43.3876	1.8782	0.0433
0.2	47.549	45.649	1.9001	0.0416

Table 4. Comparison between the approximation formula and the Monte-Carlo simulation for Russian option prices against a variety of $r-q$; absolute difference = $\left\|P^{R} - P^{R}_{MC}\right\|$, relative difference = $\left\|\frac{P^{R}-P^{R}_{MC}}{P^{R}_{MC}}\right\|$, $y = -0.73$ and $z = -0.33$.

$r-q$	P^{R}	P^{R}_{MC}	absolute difference	relative difference
0.1	111.5176	114.0305	2.5129	0.0220
0.12	113.1129	115.1552	2.0423	0.0177
0.14	115.0676	116.5339	1.4663	0.0126
0.16	117.4950	118.0136	0.5187	0.0044
0.18	120.5682	120.0946	0.4736	0.0039
0.2	124.5698	121.8557	2.7141	0.0223

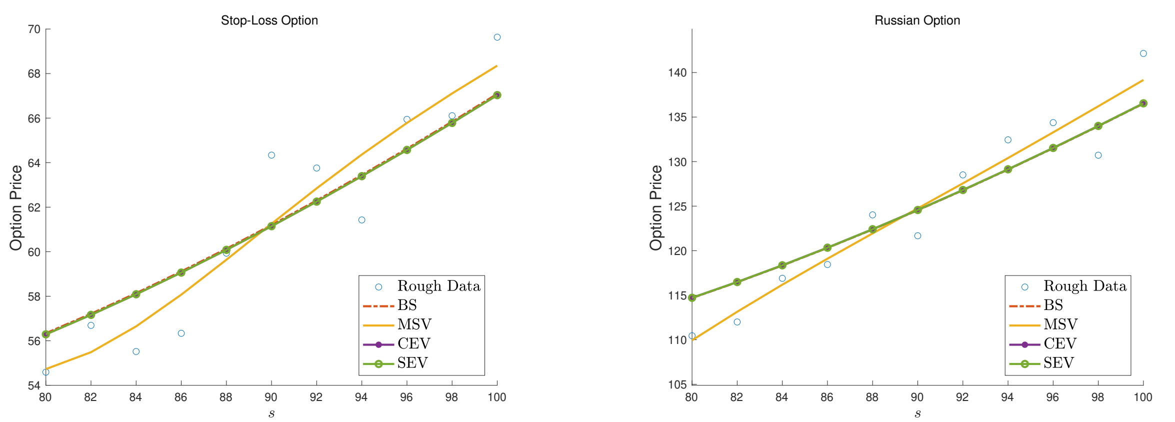
Table 5. Comparison between the approximation formula and the Monte-Carlo simulation for stop-loss option prices against a variety of s^*; absolute difference = $\left\|P^{sl} - P^{sl}_{MC}\right\|$, relative difference = $\left\|\frac{P^{sl}-P^{sl}_{MC}}{P^{sl}_{MC}}\right\|$, $y = -0.93$, $z = -0.73$ and $\lambda = 0.5$.

s^*	P^{sl}	P^{sl}_{MC}	absolute difference	relative difference
85	49.6338	49.4496	0.1842	0.0037
90	50.0286	49.6868	0.3418	0.0069
95	50.6724	50.4782	0.1942	0.0038
100	51.5569	51.1381	0.4188	0.0082
105	52.6759	52.2889	0.3870	0.0074

Table 6. Comparison between the approximation formula and the Monte-Carlo simulation for Russian option prices against a variety of s^*; absolute difference = $\left\|P^R - P^R_{MC}\right\|$, relative difference = $\left\|\frac{P^R - P^R_{MC}}{P^R_{MC}}\right\|$, $y = -0.74$ and $z = -0.34$.

s^*	P^R	P^R_{MC}	absolute difference	relative difference
85	136.1358	135.3692	0.7665	0.0057
90	136.2416	136.887	0.6454	0.0047
95	136.4171	137.5976	1.1805	0.0086
100	136.6607	137.2971	0.6364	0.0046
105	136.9703	139.7656	2.7953	0.02

Figure 3 presents a comparison of the numerical values of stop-loss and Russian options under the multiscale stochastic volatility (MSV) model with those under the Black-Scholes (BS) model, the CEV model and the SEV (stochastic elasticity of variance) model of Kim et al. [22]. We could choose the popular Heston and 3/2 models instead of these models but our model is already a generalization of the (rescaled) Heston and 3/2 models. If $f(y,z) = y^{1/2}$, $\alpha(y) = \theta - y$, $\beta(y) = \xi y^{1/2}$ and $c(z) = g(z) = 0$, then our model is reduced to the Heston model. If $f(y,z) = y^{1/2}$, $\alpha(y) = y(\theta - y)$, $\beta(y) = \xi y^{3/2}$ and $c(z) = g(z) = 0$, then it becomes the 3/2 model. Here, θ and ξ are constants. So, we compare our model with the CEV and SEV models. The real market data of stop-loss and Russian options do not exist since those options are not traded in exchange. So, we generate 'rough data' by Monte-Carlo simulation. Here, for the security of scientific objectivity, we consider the Black-Scholes model for the underlying asset prices on behalf of market data. We use the pricing results in Lee and Kim [24] under the CEV and SEV models. They are obtained by the first-order approximation formulas in [27] and [23] for the CEV and SEV models, respectively. As shown in Figure 3, we find that the MSV model has quite good performance to fit the data in comparison with the CEV and SEV models that tend to produce almost similar performances to each other in the case of the first-order approximation. The numerical values of the fitting errors between the rough data and the option prices under the four models are shown in Table 7, which demonstrates that the MSV model clearly outperforms the other models.



(a) Stop-loss option prices

(b) Russian option prices

Figure 3. Fitting of the BS, MSV, CEV and SEV models to rough data.

Table 7. Numerical error between the BS, MSV, CEV or SEV option prices and rough data; $||P_{data} - P_{model}|| := \sqrt{\sum_{x=80,82,....,100} |P_{data}(x) - P(x)|^2}$.

The stop-loss option

$\|\|P^{sl}_{data} - P^{sl}_{BS}\|\|$	$\|\|P^{sl}_{data} - P^{sl}_{MSV}\|\|$	$\|\|P^{sl}_{data} - P^{sl}_{CEV}\|\|$	$\|\|P^{sl}_{data} - P^{sl}_{SEV}\|\|$
6.5035	5.2410	6.5012	6.5012

The Russian option

$\|\|P^{R}_{data} - P^{R}_{BS}\|\|$	$\|\|P^{R}_{data} - P^{R}_{MSV}\|\|$	$\|\|P^{R}_{data} - P^{R}_{CEV}\|\|$	$\|\|P^{R}_{data} - P^{R}_{SEV}\|\|$
10.9108	7.8487	10.9105	10.9105

6. Conclusions

In this paper, we have studied the pricing of stop-loss and Russian options, two perpetual American-style type of exotic options with a lookback provision, under a multiscale stochastic volatility framework. We obtain a closed-form expression for the approximate price of each option so that the pricing formula is useful for performance optimization since one can then easily compute the first and second order derivatives. Each formula is given as the Black-Scholes option price plus a correction term. The correction terms produced by the multiscale stochastic volatility provide the flexibility of the option prices and the free boundary (in the case of Russian options) so that the model can capture the real market behavior better. The pricing formulas are tested and verified via Monte-Carlo simulations. A possible direction of future research would be an extension of the current work under a stochastic volatility model with multiple time scales to stochastic volatility models with multiple dimensions such as the hybrid stochastic volatility and CEV model of Choi et al. [4], a stochastic volatility model with stochastic liquidity and regime switching of He and Lin [18] or with stochastic interest rates of He and Lin [19], and the hybrid stochastic elasticity of variance and stochastic volatility model of Escobar-Anel and Fan [10].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix

A. ODE problems and solutions for $V_{ij}^{s/l}$, $i+j=2$

In this section, we derive the ODE problems and their solutions for the second-order corrections $V_{ij}^{s/l}$ with $i+j=2$ for a stop-loss option. We first note that the term $V_{20}^{s/l}$ in (3.3) depends on y, whereas the other terms $V_{ij}^{s/l}$, $0 \leq i+j \leq 2$ and $(i,i) \neq (2,0)$, are independent of y. To determine the value of $V_{20}^{s/l}(x,y,z)$, we need to assume that $\left\langle V_{20}^{s/l}(\lambda,\cdot,z)\right\rangle = 0$ and $\left\langle \frac{\partial}{\partial x} V_{20}^{s/l}(1,\cdot,z)\right\rangle = \left\langle V_{20}^{s/l}(1,\cdot,z)\right\rangle$ hold based on the stochastic terminal layer analysis of Fouque et al. [15].

Proposition 6.1. *Under the assumption $\langle \phi_2(\cdot,z)\rangle = 0$, the second-order corrections $V_{ij}^{s/l}$, $i+j=2$, in the asymptotic series (3.3) for a stop-loss option satisfy the following ODE problems, respectively. First, $V_{20}^{s/l}(x,y,z)$ is given by (3.15), i.e.,*

$$V_{20}^{s/l}(x,y,z) = -\frac{1}{2}\phi_2(y,z)x^2\frac{\partial^2}{\partial x^2}V_{00}^{s/l}(x,z) + F_{20}^{s/l}(x,z),$$

where $F_{20}^{s/l}(x,z)$ satisfies

$$\begin{cases} \overline{\mathcal{L}}F_{20}^{s/l}(x,z) = \left(\tilde{U}_{00}^{40}(z)x^4\frac{\partial^4}{\partial x^4} + \tilde{W}_{00}^{30}(z)x^3\frac{\partial^3}{\partial x^3} + \tilde{W}_{00}^{20}(z)x^2\frac{\partial^2}{\partial x^2}\right)V_{00}^{s/l}(x,z) \\ \qquad\qquad + \left(U_{00}^{30}(z)x^3\frac{\partial^3}{\partial x^3} + U_{00}^{20}(z)x^2\frac{\partial^2}{\partial x^2}\right)V_{10}^{s/l}(x,z) := B_{20}(x,z), \quad \lambda < x < 1, \\ F_{20}^{s/l}(\lambda,z) = 0, \\ F_{20}^{s/l}(1,z) = \frac{\partial}{\partial x}F_{20}^{s/l}(1,z). \end{cases} \tag{6.1}$$

Second, $V_{11}^{s/l}(x,z)$ and $V_{02}^{s/l}(x,z)$ satisfy

$$\begin{cases} \overline{\mathcal{L}}V_{11}^{s/l}(x,z) = \left(\tilde{U}_{00}^{21}(z)x^2\frac{\partial^2}{\partial x^2} + W_{00}^{11}(z)x\frac{\partial}{\partial x} + W_{00}^{01}(z)\right)V_{00}^{s/l}(x,z) + \left(U_{00}^{11}(z)x\frac{\partial}{\partial x} + U_{00}^{01}(z)\right)\frac{\partial}{\partial z}V_{10}^{s/l}(x,z) \\ \qquad\qquad + \left(U_{00}^{30}(z)x^3\frac{\partial^3}{\partial x^3} + U_{00}^{20}(z)x^2\frac{\partial^2}{\partial x^2}\right)V_{01}^{s/l}(x,z) := B_{11}(x,z), \quad \lambda < x < 1, \\ V_{11}^{s/l}(\lambda,z) = 0, \\ V_{11}^{s/l}(1,z) = \frac{\partial}{\partial x}V_{11}^{s/l}(1,z), \end{cases} \tag{6.2}$$

and

$$\begin{cases} \overline{\mathcal{L}}V_{02}^{s/l}(x,z) = \left(U_{00}^{11}(z)x\frac{\partial}{\partial x} + U_{00}^{01}(z)\right)\frac{\partial}{\partial z}V_{01}^{s/l}(x,z) \\ \qquad\qquad - \left(c(z)\frac{\partial}{\partial z} + \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2}\right)V_{00}^{s/l}(x,z) := B_{02}(x,z), \lambda < x < 1, \\ V_{02}^{s/l}(\lambda,z) = 0, \\ V_{02}^{s/l}(1,z) = \frac{\partial}{\partial x}V_{02}^{s/l}(1,z), \end{cases} \tag{6.3}$$

respectively, where

$$\begin{aligned}
U_{00}^{40}(z) &:= -\frac{1}{2}\rho_{xy}^2 \left\langle \beta f \frac{\partial \xi_1}{\partial y} \right\rangle, \\
\tilde{U}_{00}^{40}(z) &:= U_{00}^{40}(z) + \frac{1}{4}\left\langle \phi_2 f^2 \right\rangle, \\
W_{00}^{30}(z) &:= -\frac{1}{2}\rho_{xy}\left(5\rho_{xy}\left\langle \beta f \frac{\partial \xi_1}{\partial y} \right\rangle - \left\langle \beta f \frac{\partial \xi_2}{\partial y} \right\rangle - \left\langle \beta \Lambda \frac{\partial \xi_1}{\partial y} \right\rangle\right), \\
\tilde{W}_{00}^{30}(z) &:= W_{00}^{30}(z) + \left\langle \phi_2 f^2 \right\rangle, \\
W_{00}^{20}(z) &:= -\left(2\rho_{xy}^2\left\langle \beta f \frac{\partial \xi_1}{\partial y} \right\rangle - \rho_{xy}\left\langle \beta f \frac{\partial \xi_2}{\partial y} \right\rangle - \rho_{xy}\left\langle \beta \Lambda \frac{\partial \xi_1}{\partial y} \right\rangle + \frac{1}{2}\left\langle \beta \Lambda \frac{\partial \xi_2}{\partial y} \right\rangle\right), \\
\tilde{W}_{00}^{20}(z) &:= W_{00}^{20}(z) + \frac{1}{2}\left\langle \phi_2 f^2 \right\rangle, \\
U_{00}^{21}(z) &:= g(z)\rho_{xy}\rho_{xz}\left\langle \beta f \frac{\partial \phi_1}{\partial y} \right\rangle, \\
\tilde{U}_{00}^{21}(z) &:= U_{00}^{21}(z) + \frac{1}{2}g(z)\rho_{yz}\left\langle \beta \frac{\partial \phi_2}{\partial y} \right\rangle, \\
W_{00}^{11}(z) &:= g(z)\left(\rho_{xy}\rho_{xz}\left\langle \beta f \frac{\partial \phi_1}{\partial y} \right\rangle - \rho_{xz}\left\langle \beta \Lambda \frac{\partial \phi_1}{\partial y} \right\rangle - \rho_{xy}\left\langle \beta f \frac{\partial \psi}{\partial y} \right\rangle\right), \\
W_{00}^{01}(z) &:= g(z)\left\langle \beta \Lambda \frac{\partial \psi}{\partial y} \right\rangle.
\end{aligned}$$

Proof. In the middle of the proof of Proposition 3.2, we have shown that $V_{20}^{s/l}$ is given by (3.15) in which the function $F_{20}^{s/l}$ has not been determined. We now derive an ODE problem for $F_{20}^{s/l}$ as follows. If Lemma 2.1 is applied to the terms of order $\sqrt{\epsilon}$ in (3.4), then we have $\langle \mathcal{L}_1 \rangle V_{20}^{s/l} + \overline{\mathcal{L}} V_{00}^{s/l} = 0$. The terms of order $\sqrt{\epsilon}$ in (3.4) give $\mathcal{L}_0 V_{30}^{s/l} + \mathcal{L}_1 V_{20}^{s/l} + \mathcal{L}_2 V_{10}^{s/l} = 0$. So, we obtain $\mathcal{L}_0 V_{30}^{s/l} = -\left(\mathcal{L}_1 V_{20}^{s/l} - \left\langle \mathcal{L}_1 V_{20}^{s/l} \right\rangle\right) - \left(\mathcal{L}_2 - \overline{\mathcal{L}}\right) V_{10}^{s/l}$ whose general solution is given by

$$\begin{aligned}
V_{30}^{s/l}(x,y,z) = &\left(\frac{1}{2}\rho_{xy}\xi_1(y,z)x^3\frac{\partial^3}{\partial x^3} + \left(\rho_{xy}\xi_1(y,z) - \frac{1}{2}\xi_2(y,z)\right)x^2\frac{\partial^2}{\partial x^2}\right)V_{00}^{s/l}(x,z) \\
&- \frac{1}{2}\phi_2(y,z)x^2\frac{\partial^2}{\partial x^2}V_{10}^{s/l}(x,z) + F_{30}^{s/l}(x,z)
\end{aligned} \tag{6.4}$$

for some function $F_{30}^{s/l}$ independent of y, where the functions ξ_1 and ξ_2 are defined in (3.6). From the terms of order ϵ in (3.4), we have

$$\left\langle \mathcal{L}_2 V_{20}^{s/l} \right\rangle = -\left\langle \mathcal{L}_1 V_{30}^{s/l} \right\rangle \tag{6.5}$$

by Lemma 2.1. With the help of (3.15), the left side of Eq (6.5) becomes

$$\begin{aligned}
\left\langle \mathcal{L}_2 V_{20}^{s/l} \right\rangle &= -\frac{1}{2}\left\langle \phi_2 x^2 \frac{\partial^2}{\partial x^2}\left(\mathcal{L}_2 V_{00}^{s/l} - \overline{\mathcal{L}} V_{00}^{s/l}\right) \right\rangle + \overline{\mathcal{L}} F_{20}^{s/l} \\
&= -\frac{1}{4}\left\langle \phi_2 f^2 \right\rangle \left(x^4\frac{\partial^4}{\partial x^4} + 4x^3\frac{\partial^3}{\partial x^3} + 2x^2\frac{\partial^2}{\partial x^2}\right)V_{00}^{s/l} + \overline{\mathcal{L}} F_{20}^{s/l}.
\end{aligned} \tag{6.6}$$

With the help of (6.4), the right side of Eq (6.5) becomes

$$-\left\langle \mathcal{L}_1 V_{30}^{s/l} \right\rangle = \left(U_{00}^{40} x^4 \frac{\partial^4}{\partial x^4} + W_{00}^{30} x^3 \frac{\partial^4}{\partial x^3} + W_{00}^{20} x^2 \frac{\partial^2}{\partial x^2} \right) V_{00}^{s/l} + \left(U_{00}^{30} x^3 \frac{\partial^3}{\partial x^3} + U_{00}^{20} x^2 \frac{\partial^2}{\partial x^2} \right) V_{10}^{s/l}. \quad (6.7)$$

Putting (6.5)–(6.7) together, we obtain the ODE for $F_{20}^{s/l}$ as in (6.1).

Next, by applying Proposition 3.1 and Lemma 2.1 to the terms of order $\sqrt{\delta}$ in (3.4), we can obtain $\mathcal{L}_0 V_{21}^{s/l} + \mathcal{L}_2 V_{01}^{s/l} + \mathcal{M}_1 V_{00}^{s/l} = 0$ and $\overline{\mathcal{L}} V_{01}^{s/l} + \langle \mathcal{M}_1 \rangle V_{00}^{s/l} = 0$. These two equations yield $\mathcal{L}_0 V_{21}^{s/l} = -\left(\left(\mathcal{L}_2 - \overline{\mathcal{L}} \right) V_{01}^{s/l} + \left(\mathcal{M}_1 - \langle \mathcal{M}_1 \rangle \right) V_{00}^{s/l} \right)$ whose solution $V_{21}^{s/l}$ is given by

$$V_{21}^{s/l}(x,y,z) = -\frac{1}{2}\phi_2(y,z) x^2 \frac{\partial^2}{\partial x^2} V_{01}^{s/l}(x,z) - g(z) \left(\rho_{xz} \phi_1(y,z) x \frac{\partial}{\partial x} - \psi(y,z) \right) \frac{\partial}{\partial z} V_{00}^{s/l}(x,z) + F_{21}^{s/l}(x,z) \quad (6.8)$$

for some function $F_{21}^{s/l}$ independent of variable y. Then, by applying Lemma 2.1 to the terms of order $\sqrt{\epsilon\delta}$ in (3.4) and using the result (6.8), we can obtain the ODE for $V_{11}^{s/l}$ in (6.2).

Moreover, the ODE $\overline{\mathcal{L}} V_{02}^{s/l} = -\left(\langle \mathcal{M}_1 \rangle V_{01}^{s/l} + \mathcal{M}_2 V_{00}^{s/l} \right)$ for $V_{02}^{s/l}$ can be obtained from the terms of order δ in (3.4). Then (6.3) follows from the definitions of $\mathcal{M}_2$ in (2.3) and $\mathcal{M}_1$ in (3.2).

On the other hand, the terms of order $\epsilon^{i/2}\delta^{j/2}$, $(i,j) \in \{(1,1),(0,2)\}$, in (3.11) provide the boundary conditions in (6.2) and (6.3) for $V_{11}^{s/l}$ and $V_{02}^{s/l}$, respectively. For the boundary conditions of $F^{s/l}(x,z)$, if we apply the assumptions $\left\langle V_{20}^{s/l}(\lambda,\cdot,z) \right\rangle = 0$ and $\left\langle \frac{\partial}{\partial x} V_{20}^{s/l}(1,\cdot,z) \right\rangle = \left\langle V_{20}^{s/l}(1,\cdot,z) \right\rangle$ to the terms of order ϵ in (3.11), then the boundary conditions in (6.1) are obtained. □

Proposition 6.2. *The solutions of the ODE problems in Proposition 6.1 are given by*

$$\begin{aligned} V_{20}^{s/l}(x,y,z) &= -\frac{1}{2}\phi_2(y,z) \sum_{k=1}^{2} \eta_k(z)\left(\eta_k(z)-1\right) A_{00,k}^{0}(z) x^{\eta_k(z)} + \sum_{k=1}^{2} \left(\sum_{\zeta=0}^{2} A_{20,k}^{\zeta}(z) (\ln x)^{\zeta} \right) x^{\eta_k(z)}, \\ V_{ij}^{s/l}(x,z) &= \sum_{k=1}^{2} \left(\sum_{\zeta=0}^{i+2j} A_{ij,k}^{\zeta}(z) (\ln x)^{\zeta} \right) x^{\eta_k(z)}, \quad (i,j) \in \{(1,1),(0,2)\}, \end{aligned} \quad (6.9)$$

respectively, where the terms $A_{ij,k}^{\zeta}$, $(i,j) \in \{(1,0),(0,1),(2,0),(1,1),(0,2)\}$ and $\zeta = 1,\cdots,i+2j$, are recursively given by

$$\begin{aligned} A_{ij,k}^{\zeta}(z) &= \frac{B_{ij,k}^{\zeta-1}(z) - \frac{\zeta(\zeta+1)}{2}\sigma^2(z) A_{ij,k}^{\zeta+1}(z)}{\zeta\left(\frac{1}{2}\sigma^2(z)\left(2\eta_k(z)-1\right) + (q-r)\right)}, \quad \zeta = 1,\cdots,i+2j-1, \\ A_{ij,k}^{\zeta}(z) &= \frac{B_{ij,k}^{\zeta-1}(z)}{\zeta\left(\frac{1}{2}\sigma^2(z)\left(2\eta_k(z)-1\right) + (q-r)\right)}, \quad \zeta = i+2j \end{aligned} \quad (6.10)$$

and the terms $A_{ij,k}^{0}$, $(i,j) \in \{(1,0),(0,1),(2,0),(1,1),(0,2)\}$, are given by

$$A_{ij,k}^{0}(z) = \frac{\lambda^{\eta_l(z)} \sum_{\omega=1}^{2} A_{ij,\omega}^{1}(z) + (1-\eta_l(z)) \sum_{\omega=1}^{2} \left(\lambda^{\eta_\omega(z)} \sum_{\zeta=1}^{i+2j} A_{ij,\omega}^{\zeta}(z) (\ln\lambda)^{\zeta} \right)}{(1-\eta_k(z))\lambda^{\eta_l(z)} - (1-\eta_l(z))\lambda^{\eta_k(z)}}, \quad k,l \in \{1,2\},\ k \neq l. \quad (6.11)$$

Proof. Substituting (4.3) and (4.5) into $B_{ij}(x,z)$, $(i,j) \in \{(2,0),(1,1),(0,2)\}$, defined in Proposition 6.1, we first have

$$B_{ij}(x,z) = \sum_{k=1}^{2} \left(\sum_{\zeta=0}^{i+2j-1} B^{\zeta}_{ij,k}(z) (\ln x)^{\zeta} \right) x^{\eta_k(z)}, \tag{6.12}$$

where the explicit (long) representations of $B^{\zeta}_{ij,k}(z)$, $\zeta = 0, 1, \cdots, i+2j-1$, are given below in Appendix C.

To solve the ODE problems (6.1)–(6.3), we consider the functions F_{20}, V_{11} and V_{02} defined by

$$\begin{aligned} F_{20}(x,z) &= \sum_{k=1}^{2} \left(\sum_{\zeta=0}^{2} A^{\zeta}_{20,k}(z) (\ln x)^{\zeta} \right) x^{\eta_k(z)}, \\ V_{ij}(x,z) &= \sum_{k=1}^{2} \left(\sum_{\zeta=0}^{i+2j} A^{\zeta}_{ij,k}(z) (\ln x)^{\zeta} \right) x^{\eta_k(z)}, \quad (i,j) = (1,1),(0,2), \end{aligned} \tag{6.13}$$

where, for legibility, the simplified notation without the superscript 's/l' has been used. Substituting (6.13) into the first equations in (6.1)–(6.3), we obtain $A^{\zeta}_{ij,k}(z)$, $\zeta = 1, \cdots, i+2j$, as follows:

$$\begin{aligned} A^{1}_{ij,k}(z) &= \frac{B^{0}_{ij,k}(z) - \sigma^2(z) A^{2}_{ij,k}(z)}{\frac{1}{2}\sigma^2(z)(2\eta_k(z)-1) + (q-r)}, \quad (i,j) = (2,0),(1,1),(0,2), \\ A^{2}_{20,k}(z) &= \frac{B^{1}_{20,k}(z)}{2\left(\frac{1}{2}\sigma^2(z)(2\eta_k(z)-1) + (q-r)\right)}, \\ A^{2}_{ij,k}(z) &= \frac{B^{1}_{11,k}(z) - 3\sigma^2(z) A^{3}_{ij,k}(z)}{2\left(\frac{1}{2}\sigma^2(z)(2\eta_k(z)-1) + (q-r)\right)}, \quad (i,j) = (1,1),(0,2), \\ A^{3}_{11,k}(z) &= \frac{B^{2}_{11,k}(z)}{3\left(\frac{1}{2}\sigma^2(z)(2\eta_k(z)-1) + (q-r)\right)}, \\ A^{3}_{02,k}(z) &= \frac{B^{2}_{02,k}(z) - 6\sigma^2(z) A^{4}_{02,k}(z)}{3\left(\frac{1}{2}\sigma^2(z)(2\eta_k(z)-1) + (q-r)\right)}, \\ A^{4}_{02,k}(z) &= \frac{B^{3}_{02,k}(z)}{4\left(\frac{1}{2}\sigma^2(z)(2\eta_k(z)-1) + (q-r)\right)}. \end{aligned} \tag{6.14}$$

Then (6.10) follows from (4.6) and (6.14).

Next, substituting (6.13) into the boundary conditions in (6.1)–(6.3), we obtain $A^{0}_{ij,k}$ in (6.13) as follows:

$$A^{0}_{ij,k}(z) = \frac{\lambda^{\eta_l(z)} \sum_{\omega=1}^{2} A^{1}_{ij,\omega}(z) + (1-\eta_l(z)) \sum_{\omega=1}^{2} \left(\lambda^{\eta_\omega(z)} \sum_{\zeta=1}^{i+2j} A^{\zeta}_{ij,\omega}(z) (\ln \lambda)^{\zeta} \right)}{(1-\eta_k(z)) \lambda^{\eta_l(z)} - (1-\eta_l(z)) \lambda^{\eta_k(z)}}, \quad (i,j) = (2,0),(1,1),(0,2). \tag{6.15}$$

Then (6.11) follows from (4.7) and (6.15). □

B. ODE problems and solutions of V_{ij}^R, $i+j=2$

In this section, we derive ODE problems and their solutions for the second-order corrections V_{ij}^R, $i+j=2$ (Russian option). We first note that the terms of order ϵ in the first and third equations in (3.21) given by

$$\begin{cases} V_{20}^R(1,y,z) - \frac{\partial}{\partial x}V_{20}^R(1,y,z), \\ V_{20}^R(x_{00}^f(z),y,z) + \frac{\partial}{\partial x}V_{00}^R\left(x_{00}^f(z),z\right)x_{20}^f(y,z) + \frac{\partial}{\partial x}V_{10}^R\left(x_{00}^f(z),z\right)x_{10}^f(z) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V_{00}^R\left(x_{00}^f(z),z\right)\left(x_{10}^f(z)\right)^2 \end{cases}$$

become

$$\begin{cases} V_{20}^R(1,y,z) - \frac{\partial}{\partial x}V_{20}^R(1,y,z), \\ V_{20}^R(x_{00}^f(z),y,z) - \frac{1}{2}\frac{\partial^2}{\partial x^2}V_{00}^R\left(x_{00}^f(z),z\right)\left(x_{10}^f(z)\right)^2, \end{cases} \tag{6.16}$$

respectively, since $\frac{\partial}{\partial x}V_{00}^R\left(x_{00}^f(z),z\right) = 0$ from (3.18) and $\frac{\partial}{\partial x}V_{10}^R(x_{00}^f(z),z) = -\frac{\partial^2}{\partial x^2}V_{00}^R(x_{00}^f(z),z)x_{10}^f(z)$ from (3.19). In the following proposition, we assume that $\left\langle V_{20}^R(1,\cdot,z)\right\rangle = \left\langle \frac{\partial}{\partial x}V_{20}^R(1,\cdot,z)\right\rangle$ and $\left\langle V_{20}^R\left((x_{00}^f(z),\cdot,z\right)\right\rangle = \frac{1}{2}\frac{\partial^2}{\partial x^2}V_{00}^R\left(x_{00}^f(z),z\right)\left(x_{10}^f(z)\right)^2$ hold.

Proposition 6.3. *Under the assumption $\langle\phi_2(\cdot,z)\rangle = 0$, the second-order corrections, V_{ij}^R, $i+j=2$, in the asymptotic series (3.17) for a Russian option satisfy the following ODE problems. First, V_{20}^R is given by*

$$V_{20}^R(x,y,z) = -\frac{1}{2}\phi_2(y,z)x^2\frac{\partial^2}{\partial x^2}V_{00}^R(x,z) + F_{20}^R(x,z), \tag{6.17}$$

where F_{20}^R solves

$$\begin{cases} \overline{\mathcal{L}}F_{20}^R(x,z) = \left[\tilde{U}_{00}^{40}(z)x^4\frac{\partial^4}{\partial x^4} + \tilde{W}_{00}^{30}(z)x^3\frac{\partial^3}{\partial x^3} + \tilde{W}_{00}^{20}(z)x^2\frac{\partial^2}{\partial x^2}\right]V_{00}^R(x,z) \\ \qquad + \left(U_{00}^{30}(z)x^3\frac{\partial^3}{\partial x^3} + U_{00}^{20}(z)x^2\frac{\partial^2}{\partial x^2}\right)V_{10}^R(x,z), \quad x_{00}^f(z) < x < 1, \\ F_{20}^R(1,z) = \frac{\partial}{\partial x}F_{20}^R(1,z), \\ F_{20}^R(x_{00}^f(z),z) = \frac{1}{2}\frac{\partial^2}{\partial x^2}V_{00}^R(x_{00}^f(z),z)\left(x_{10}^f(z)\right)^2, \\ x_{20}^f(y,z) = -\frac{\frac{\partial}{\partial x}V_{20}^R(x_{00}^f(z),y,z) + \frac{\partial^2}{\partial x^2}V_{10}^R(x_{00}^f(z),z)x_{10}^f(z) + \frac{1}{2}\frac{\partial^3}{\partial x^3}V_{00}^R(x_{00}^f(z),z)(x_{10}^f(z))^2}{\frac{\partial^2}{\partial x^2}V_{00}^R(x_{00}^f(z),z)}. \end{cases} \tag{6.18}$$

Second, V_{11}^R and V_{02}^R satisfy

$$\begin{cases} \overline{\mathcal{L}}V_{11}^R(x,z) = \left(\tilde{U}_{00}^{21}(z)x^2\frac{\partial^2}{\partial x^2} + W_{00}^{11}(z)x\frac{\partial}{\partial x} + W_{00}^{01}(z)\right)V_{00}^R(x,z) + \left(U_{00}^{11}(z)x\frac{\partial}{\partial x} + U_{00}^{01}(z)\right)\frac{\partial}{\partial z}V_{10}^R(x,z) \\ \qquad + \left(U_{00}^{30}(z)x^3\frac{\partial^3}{\partial x^3} + U_{00}^{20}(z)x^2\frac{\partial^2}{\partial x^2}\right)V_{01}^R(x,z), \quad x_{00}^f(z) < x < 1, \\ V_{11}^R(1,z) = \frac{\partial}{\partial x}V_{11}^R(1,z), \\ V_{11}^R(x_{00}^f(z),z) = \frac{\partial^2}{\partial x^2}V_{00}^R(x_{00}^f(z),z)x_{10}^f(z)x_{01}^f(z), \\ x_{11}^f(z) = -\frac{\frac{\partial}{\partial x}V_{11}^R(x_{00}^f(z),z) + \frac{\partial^2}{\partial x^2}V_{10}^R(x_{00}^f(z),z)x_{01}^f(z) + \frac{\partial^2}{\partial x^2}V_{01}^R(x_{00}^f(z),z)x_{10}^f(z) + \frac{\partial^3}{\partial x^3}V_{00}^R(x_{00}^f(z),z)x_{10}^f(z)x_{01}^f(z)}{\frac{\partial^2}{\partial x^2}V_{00}^R(x_{00}^f(z),z)} \end{cases} \tag{6.19}$$

and

$$\begin{cases} \mathcal{L}V_{02}^R(x,z) = \left(U_{00}^{11}(z)x\frac{\partial}{\partial x} + U_{00}^{01}(z)\right)\frac{\partial}{\partial z}V_{01}^R(x,z) \\ \qquad\qquad - \left(c(z)\frac{\partial}{\partial z} + \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2}\right)V_{00}^R(x,z), \quad x_{00}^f(z) < x < 1, \\ V_{02}^R(1,z) = \frac{\partial}{\partial x}V_{02}^R(1,z), \\ V_{02}^R(x_{00}^f(z),z) = \frac{1}{2}\frac{\partial^2}{\partial x^2}V^R(x_{00}^f(z),z)(x_{01}^f(z))^2, \\ x_{02}^f(z) = -\frac{\frac{\partial}{\partial x}V_{02}^R(x_{00}^f(z),z) + \frac{\partial^2}{\partial x^2}V_{01}^R(x_{00}^f(z),z)x_{01}^f(z) + \frac{1}{2}\frac{\partial^3}{\partial x^3}V_{00}^R(x_{00}^f(z),z)(x_{01}^f(z))^2}{\frac{\partial^2}{\partial x^2}V_{00}^R(x_{00}^f(z),z)}, \end{cases} \tag{6.20}$$

respectively.

Proof. In the cases of V_{11}^R and V_{02}^R, one can show the y-independence of x_{11}^f and x_{02}^f and obtain the results (6.19) and (6.20) including the boundary conditions for V_{ij}^R and the free boundary conditions for x_{ij}^f by using the same argument as in the proof of Proposition 4.4. So, we don't repeat the proof here.

Next, in the case of V_{20}^R, as shown in the process of obtaining the solution $V_{20}^{s/l}(x,y,z)$ of (3.15), it can be expressed as (6.17) for some function $F_{20}^R(x,z)$ which is independent of y. As shown in Proposition 6.1 for a stop-loss option, F_{20}^R satisfies the ODE in (6.18).

It remains to find the boundary conditions for F_{20}^R and the free boundary condition for x_{20}^f. Substituting (6.17) into the assumptions $\left\langle V_{20}^R(1,y,z)\right\rangle = \left\langle \frac{\partial}{\partial x}V_{20}^R(1,y,z)\right\rangle$ and $\left\langle V_{20}^R\left((x_{00}^f(z),y,z\right)\right\rangle = \frac{1}{2}\frac{\partial^2}{\partial x^2}V_{00}^R\left(x_{00}^f(z),z\right)\left(x_{10}^f(z)\right)^2$, we obtain the boundary conditions $F_{20}^R(1,z) = \frac{\partial}{\partial x}F_{20}^R(1,z)$ and $F_{20}^R(x_{00}^f(z),z) = \frac{1}{2}\frac{\partial^2}{\partial x^2}V_{00}^R(x_{00}^f(z),z)\left(x_{10}^f(z)\right)^2$. Lastly, from the terms of order ϵ in the third equation of (3.21), one can get directly the free boundary condition for $x_{20}^f(y,z)$ as in (6.18). □

Proposition 6.4. *The solutions of the ODE problems in Proposition 6.3 are given by*

$$V_{20}^R(x,y,z) = -\frac{1}{2}\phi_2(y,z)\sum_{k=1}^{2}\eta_k(z)\left(\eta_k(z)-1\right)A_{00,k}^0(z)x^{\eta_k(z)} + \sum_{k=1}^{2}\left(\sum_{\zeta=0}^{2}A_{20,k}^{\zeta}(z)\left(\ln x\right)^{\zeta}\right)x^{\eta_k(z)},$$

$$V_{ij}^R(x,z) = \sum_{k=1}^{2}\left(\sum_{\zeta=0}^{i+2j}A_{ij,k}^{\zeta}(z)\left(\ln x\right)^{\zeta}\right)x^{\eta_k(z)}, \quad (i,j) \in \{(1,1),(0,2\},$$

respectively, and the corresponding free boundaries x_{ij}^f, $(i,j) \in \{(2,0),(1,1),(0,2)\}$)*, are given by*

$$x_{ij}^f(z) = -\frac{\begin{aligned} &\textstyle\sum_{k=1}^2\left[\begin{array}{l}\sum_{\zeta=1}^{i+2j}\zeta A_{ij,k}^{\zeta}(z)\left(\ln x_{00}^f(z)\right)^{\zeta-1} + \eta_k(z)\sum_{\zeta=0}^{i+2j}A_{ij,k}^{\zeta}(z)\left(\ln x_{00}^f(z)\right)^{\zeta} \\ -\frac{\max\{0,i-1\}}{2}\phi_2(y,z)\eta_k^2(z)\left(\eta_k(z)-1\right)A_{00,k}^0(z)\end{array}\right]\left(x_{00}^f(z)\right)^{\eta_k(z)-1} \\ &+\tfrac{i}{\max\{i,j\}}\left(x_{10}^f(z)\right)^{1_{\{i\neq j\}}}\left(x_{01}^f(z)\right)^{1_{\{i=j\}}} \\ &\quad\times\textstyle\sum_{k=1}^2\left[\begin{array}{l}\left(2\eta_k(z)-1\right)A_{10,k}^1(z) \\ +\eta_k(z)\left(\eta_k(z)-1\right)\sum_{\zeta=0}^{1}A_{10,k}^{\zeta}(z)\left(\ln x_{00}^f(z)\right)^{\zeta}\end{array}\right]\left(x_{00}^f(z)\right)^{\eta_k(z)-2} \\ &+\tfrac{j}{\max\{i,j\}}\left(x_{10}^f(z)\right)^{1_{\{i=j\}}}\left(x_{01}^f(z)\right)^{1_{\{i\neq j\}}} \\ &\quad\times\textstyle\sum_{k=1}^2\left[\begin{array}{l}2A_{01,k}^2(z) + \left(2\eta_k(z)-1\right)\sum_{\zeta=1}^{2}\zeta A_{01,k}^{\zeta}(z)\left(\ln x_{00}^f(z)\right)^{\zeta-1} \\ +\eta_k(z)\left(\eta_k(z)-1\right)\sum_{\zeta=0}^{2}A_{01,k}^{\zeta}(z)\left(\ln x_{00}^f(z)\right)^{\zeta}\end{array}\right]\left(x_{00}^f(z)\right)^{\eta_k(z)-2} \\ &+\tfrac{1}{\max\{i,j\}}\left(x_{10}^f(z)\right)^{i}\left(x_{01}^f(z)\right)^{j}\textstyle\sum_{k=1}^2\prod_{\omega=0}^{2}\left(\eta_k(z)-\omega\right)A_{00,k}^0(z)\left(x_{00}^f(z)\right)^{\eta_k(z)-3}\end{aligned}}{\sum_{k=1}^2\eta_k(z)\left(\eta_k(z)-1\right)A_{00,k}^0(z)\left(x_{00}^f(z)\right)^{\eta_k(z)-2}},$$

respectively, where the (general) solutions $A^{\zeta}_{ij,k}$*,* $(i,j) \in \{(1,0),(0,1),(2,0),(1,1),(0,2)\}$ *and* $\zeta = 1,\cdots,i+2j$*, are the same as (6.10) and the (particular) solutions* $A^{0}_{ij,k}$*,* $(i,j) \in \{(2,0),(1,1),(0,2)\}$ *are given by*

$$A^0_{ij,k}(z) = \frac{\left(x^f_{00}(z)\right)^{\eta_l(z)} \sum_{\omega=1}^{2} A^1_{ij,\omega}(z) + (1-\eta_l(z)) \left[\begin{array}{l} \sum_{\omega=1}^{2} \left(\left(x^f_{00}(z)\right)^{\eta_\omega(z)} \sum_{\zeta=1}^{i+2j} A^{\zeta}_{ij,\omega}(z) \left(\ln x^f_{00}(z)\right)^{\zeta} \right) \\ -\frac{1}{\max(i,j)} \left(x^f_{10}(z)\right)^{i} \left(x^f_{01}(z)\right)^{j} \\ \qquad \times \sum_{\omega=1}^{2} \left(\left(x^f_{00}(z)\right)^{\eta_\omega(z)-2} \eta_\omega(z)\,(\eta_\omega(z)-1)\, A^0_{00,\omega}(z) \right) \end{array}\right]}{(1-\eta_k(z)) \left(x^f_{00}(z)\right)^{\eta_l(z)} - (1-\eta_l(z)) \left(x^f_{00}(z)\right)^{\eta_k(z)}}.$$

Proof. The derivation of the solutions $A^{\zeta}_{ij,k}$, $(i,j) \in \{(1,0),(0,1),(2,0),(1,1),(0,2)\}$ and $\zeta = 1,\cdots,i+2j$, and the solutions $A^{0}_{ij,k}$, $(i,j) \in \{(2,0),(1,1),(0,2)\}$ are similar to the case of a stop-loss option in 6.1. Moreover, from the terms of order $\epsilon^{i/2}\delta^{j/2}$, $(i,j) \in \{(2,0),(1,1),(0,2)\}$, in the third equation of (3.21), one can calculate directly the free boundary x^f_{ij}, $(i,j) \in \{(2,0),(1,1),(0,2)\}$ to get the above result. So, we omit the detailed derivation here. □

C. The explicit representation of $B^{\zeta}_{ij,k}$

In this section, we present the detailed expressions of the functions $B^{\zeta}_{ij,k}(z)$ in (6.12) for $(i,j) = (2,0)$ with $\zeta = 0,1$, $(i,j) = (1,1)$ with $\zeta = 0,1,2$ and $(i,j) = (0,2)$ with $\zeta = 0,1,2,3$.

(1) $B^{\zeta}_{20,k}(z)$ for $\zeta = 0,1$

$$\begin{aligned} B^0_{20,k}(z) = & \left(\tilde{U}^{40}_{00} \prod_{\omega=0}^{3} (\eta_k(z)-\omega) + \tilde{W}^{30}_{00} \prod_{\omega=0}^{2} (\eta_k(z)-\omega) + \tilde{W}^{20}_{00} \eta_k(z)\,(\eta_k(z)-1) \right) A^0_{00,k}(z) \\ & + U^{30}_{00} \left(\prod_{\omega=0}^{2} (\eta_k(z)-\omega)\, A^0_{10,k}(z) + \left(3\eta_k^2(z) - 6\eta_k(z) + 2\right) A^1_{10,k}(z) \right) \\ & + U^{20}_{00} \left(\eta_k(z)\,(\eta_k(z)-1)\, A^0_{10,k}(z) + (2\eta_k(z)-1)\, A^1_{10,k}(z) \right), \\ B^1_{20,k}(z) = & \left(U^{30}_{00} \prod_{\omega=0}^{2} (\eta_k(z)-\omega) + U^{20}_{00} \eta_k(z)\,(\eta_k(z)-1) \right) A^1_{10,k}(z). \end{aligned}$$

(2) $B^{\zeta}_{11,k}(z)$ for $\zeta = 0, 1, 2$

$$
\begin{aligned}
B^{0}_{11,k}(z) = {} & U^{30}_{00}\left(\prod_{\omega=0}^{2}(\eta_k(z)-\omega)\,A^{0}_{01,k}(z) + \left(3\eta_k^2(z) - 6\eta_k(z) + 2\right)A^{1}_{01,k}(z) + 6\,(\eta_k(z)-1)\,A^{2}_{01,k}(z)\right) \\
& + U^{20}_{00}\left(\eta_k(z)\,(\eta_k(z)-1)\,A^{0}_{01,k}(z) + (2\eta_k(z)-1)A^{1}_{01,k}(z) + 2A^{2}_{01,k}(z)\right) \\
& + \tilde{U}^{21}_{00}\left((2\eta_k(z)-1)\frac{\partial\eta_k(z)}{\partial z} + \eta_k(z)\,(\eta_k(z)-1)\,\frac{\partial}{\partial z}\right)A^{0}_{00,k}(z) \\
& + W^{11}_{00}\left(\frac{\partial\eta_k(z)}{\partial z} + \eta_k(z)\frac{\partial}{\partial z}\right)A^{0}_{00,k}(z) + W^{01}_{00}\frac{\partial}{\partial z}A^{0}_{00,k}(z) \\
& + U^{11}_{00}\left(\left(\eta_k(z)\frac{\partial}{\partial z} + \frac{\partial\eta_k(z)}{\partial z}\right)A^{0}_{10,k}(z) + \frac{\partial}{\partial z}A^{1}_{10,k}(z)\right) + U^{01}_{00}\frac{\partial}{\partial z}A^{0}_{10,k}(z), \\
B^{1}_{11,k}(z) = {} & U^{30}_{00}\left(\prod_{\omega=0}^{2}(\eta_k(z)-\omega)\,A^{1}_{01,k}(z) + 2\left(3\eta_k^2(z) - 6\eta_k(z) + 2\right)A^{2}_{01,k}(z)\right) \\
& + U^{20}_{00}\left(\eta_k(z)\,(\eta_k(z)-\omega)\,A^{1}_{01,k}(z) + 2\,(2\eta_k(z)-1)\,A^{2}_{01,k}(z)\right) \\
& + \tilde{U}^{21}_{00}\eta_k(z)\,(\eta_k(z)-1)\,\frac{\partial\eta_k(z)}{\partial z}A^{0}_{00,k}(z) + W^{11}_{00}\eta_k(z)\frac{\partial\eta_k(z)}{\partial z}A^{0}_{00,k}(z) + W^{01}_{00}\frac{\partial\eta_k(z)}{\partial z}A^{0}_{00,k}(z) \\
& + U^{11}_{00}\left(\eta_k(z)\frac{\partial\eta_k(z)}{\partial z}A^{0}_{10,k}(z) + \left(2\frac{\partial\eta_k(z)}{\partial z} + \eta_k(z)\frac{\partial}{\partial z}\right)A^{1}_{10,k}(z)\right) \\
& + U^{01}_{00}\left(\frac{\partial\eta_k(z)}{\partial z} + \frac{\partial}{\partial z}\right)A^{0}_{10,k}(z), \\
B^{2}_{11,k}(z) = {} & \left(U^{30}_{00}\prod_{\omega=0}^{2}(\eta_k(z)-\omega) + U^{20}_{00}\eta_k(z)\,(\eta_k(z)-1)\right)A^{2}_{01,k}(z)) \\
& + \left(U^{11}_{00}\eta_k(z)\frac{\partial\eta_k(z)}{\partial z} + U^{01}_{00}\frac{\partial\eta_k(z)}{\partial z}\right)A^{1}_{10,k}(z).
\end{aligned}
$$

(3) $B^{\zeta}_{02,k}(z)$ for $\zeta = 0, 1, 2, 3$

$$\begin{aligned}
B^0_{02,k}(z) = {} & U^{11}_{00}\left(\left(\frac{\partial \eta_k(z)}{\partial z} + \eta_k(z)\frac{\partial}{\partial z}\right)A^0_{01,k}(z) + \frac{\partial}{\partial z}A^1_{01,k}(z)\right) \\
& + U^{01}_{00}\frac{\partial}{\partial z}A^0_{01,k}(z) - \left(c(z)\frac{\partial}{\partial z} + \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2}\right)A^0_{00,k}(z), \\
B^1_{02,k}(z) = {} & U^{11}_{00}\left(\eta_k(z)\frac{\partial \eta_k(z)}{\partial z}A^0_{01,k}(z) + \left(\eta_k(z)\frac{\partial}{\partial z} + 2\frac{\partial \eta_k(z)}{\partial z}\right)A^1_{01,k}(z) + 2\frac{\partial}{\partial z}A^2_{01,k}(z)\right) \\
& + U^{01}_{00}\left(\frac{\partial \eta_k(z)}{\partial z}A^0_{01,k}(z) + \frac{\partial}{\partial z}A^1_{01,k}(z)\right) \\
& - \left(c(z)\frac{\partial \eta_k(z)}{\partial z} + \frac{1}{2}g^2(z)\left(\frac{\partial^2 \eta_k(z)}{\partial z^2} + 2\frac{\partial \eta_k(z)}{\partial z}\frac{\partial}{\partial z}\right)\right)A^0_{00,k}(z), \\
B^2_{02,k}(z) = {} & U^{11}_{10}\left(\eta_k(z)\frac{\partial \eta_k(z)}{\partial z}A^1_{01,k}(z) + \left(\eta_k(z)\frac{\partial}{\partial z} + 3\frac{\partial \eta_k(z)}{\partial z}\right)A^2_{01,k}(z)\right) \\
& + U^{01}_{10}\left(\frac{\partial \eta_k(z)}{\partial z}A^1_{01,k}(z) + \frac{\partial}{\partial z}A^2_{01,k}(z)\right) - \frac{1}{2}g^2(z)\left(\frac{\partial \eta_k(z)}{\partial z}\right)^2 A^0_{00,k}(z), \\
B^3_{02,k}(z) = {} & \left(U^{11}_{10}\eta_k(z)\frac{\partial \eta_k(z)}{\partial z} + U^{01}_{10}\frac{\partial \eta_k(z)}{\partial z}\right)A^2_{01,k}(z).
\end{aligned}$$