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Research article

Robust optimal reinsurance-investment problem for n competitive and cooperative insurers under ambiguity aversion

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Abstract: We investigate a robust optimal reinsurance-investment problem for n insurers under multiple interactions, which arise from the insurance market, the financial market, the competition mechanism and the cooperation mechanism. Each insurer's surplus process is assumed to follow a diffusion model, which is an approximation of the classical Cramér-Lundberg model. Each insurer is allowed to purchase proportional reinsurance to reduce their claim risk. To reflect the first moment and second moment information on claims, we use the variance premium principle to calculate reinsurance premiums. To increase wealth, each insurer can invest in a financial market, which includes one risk-free asset and n correlated stocks. Each insurer wants to obtain the robust optimal reinsurance and investment strategy under the mean-variance criterion. By applying a stochastic control technique and dynamic programming approach, the extended Hamilton-Jacobi-Bellman (HJB) equation is established. Furthermore, we derive both the robust optimal reinsurance-investment strategy and the corresponding value function by solving the extended HJB equation. Finally, we present numerical experiments, which yield that competition and cooperation have an important influence on the insurer's decision-making.

Keywords: robust optimal strategy; competition; cooperation; stochastic control; ambiguity aversion
Mathematics Subject Classification: 62P05, 91B28, 93E20

1. Introduction

Through reinsurance, the insurer can share part of the claim risk to the reinsurer while paying part of the premium to the reinsurer as the compensation. In addition, the insurer can invest in a finance market for a higher rate of return or to hedge the claim risk. Due to reinsurance and investment (RI) providing effective ways to transfer risk and gain profit, they are garnering interest in the fields of insurance and actuarial science in recent years.

Motivated by recent studies about the optimal RI problem, we provide an integrated framework for studying n insurer-based robust optimal RI problems under the mean-variance (MV) criterion. Considering that different insurers think that the stock with the greatest profit may be different, we assume that each insurer invests in a unique and different stock. In addition, we consider multiple interactions among n insurers, which arise from the insurance market, the financial market, the competition and the cooperation mechanism.

This paper is related to three strands of the literature. The first strand is about the time-consistent MV portfolio selection problem. A tractable framework for time-consistent MV problems was first established by Björk and Murgoci [1] and Basak and Chabakauri [2]. Recently, many scholars considered the RI problem under a time-consistent MV framework. Yang [3] studied the time-consistent RI strategy with common shock dependence between the insurance market and the financial market. Yang et al. [4] studied the time-consistent combining reinsurance problem. Wang et al. [5] considered the time-consistent RI strategy with mispricing. Zhang and Liang [6] considered time-consistent RI strategies for a jump-diffusion financial market without cash. Wang et al. [7] studied the time-consistent RI strategy with a long-range dependent mortality rate. However, these studies only considered a single agent and did not consider the interaction among multiple agents. Therefore, the optimal RI strategy obtained by them may not be available for multiple insurers.

The second strand of the literature is about interaction among multiple insurers. As for this, many scholars quantify the interaction among different insurers based on the relative performance. Bensoussan et al. [8] first quantified the competition between two insurers based on the relative performance. Siu et al. [9] also studied two competitive insurers based on the relative performance, where the two insurers were subject to common claim risks. Afterward, many scholars continued to study the competition between two insurers based on the relative performance. Deng et al. [10] studied the case of default risk, Hu and Wang [11] considered the case of time consistency, Zhu et al. [12] studied the case of the Heston model, Bai et al. [13] studied the case of bounded memory, Dong et al. [14] investigated the case of the Ornstein-Uhlenbeck model. However, these papers only studied the RI for two competitive insurers. Recently, Yang et al. [15] and Guan and Hu [16] studied the competition among n insurers; however, they did not consider the cooperation case and the ambiguity aversion.

The third strand of the literature is about ambiguity aversion. Due to the uncertain investment environment, investors are usually ambiguity-averse. For this reason, Anderson et al. [17] first studied ambiguity aversion under the continuous-time framework, Uppal and Wang [18] first considered the level of ambiguity aversion and Maenhout [19] proposed "homothetic robustness" to study ambiguity aversion. Recently, many scholars have studied the RI problem under the ambiguity aversion framework. Li et al. [20] proposed a class of MV criterion, called the α-maxmin MV criterion, and applied it to the RI problem under ambiguity aversion. Zeng et al. [21] studied the time-consistent RI problem under ambiguity aversion. Li et al. [22] considered a robust optimal excess-of-loss RI problem under ambiguity aversion. Pun [23] established a general analytical framework for an ambiguity-averse agent with time-inconsistent preference. Chen and Yang [24] studied the RI problem under ambiguity aversion with correlated claims. Inside information plays an important role in practice. Peng and Wang [25] and Yang [26] considered RI problems with inside information. Furthermore, Peng et al. [27] extended the results of Peng and Wang [25] and Yang [26] to that under the case of ambiguity aversion. Jiang and Yang [28] considered the robust optimal RI in a liquid financial market under the variance premium principle. However, these papers only derived robust RI strategies under ambiguity aversion for a single insurer. Wang et al. [29] studied the RI problem for two competitive insurers under ambiguity aversion, Yang [30] considered the RI problem for an insurer and a reinsurer with competition under ambiguity aversion however, they did not consider the cooperation case. Therefore, the optimal RI strategy obtained by them may not be available for $n > 2$ competitive and cooperative insurers.

In this paper, we provide an innovative study about the RI problem for n competitive and cooperative insurers under ambiguity aversion. Regarding the first and the third kinds of literature mentioned above, we study the RI problem for n competitive and cooperative insurers. Regarding the second kind of literature mentioned above, we study the RI problem for ambiguity-averse and cooperative insurers. Considering the work of Yang et al. [15] and Wang et al. [29], we propose a competitive and cooperative model for n insurers. To be specific, the surplus process for each insurer follows the Brownian motion with drift; the financial market consists of one risk-free asset and n risky asset, whose price processes are correlated. We assume that the n insurers can be ambiguity-averse and seek a robust optimal RI strategy among a family of alternative probability measures. By the technique of stochastic control theory, the closed-form optimal RI strategy and the corresponding optimal value function (OVF) are obtained.

The main contributions of this paper are as follows:

- We first study a robust RI problem for $n > 2$ insurers under ambiguity aversion and consider multiple interactions, which arise from the insurance market, the financial market, the competition and the cooperation mechanism.
- We first consider n insurers that are competitive and cooperative under an ambiguity aversion framework. Through numerical research, we find that competition and cooperation have an important influence on the insurer's decision-making.
- We obtain many meaningful phenomena and provide important suggestions for RI in reality.

The rest of this paper is organized as follows. The RI problem with ambiguity aversion is presented in Section 2. In Section 3, we introduce the competition and the cooperation mechanism among n insurers and provide the corresponding robust stochastic optimization problem for them. In Section 4, we obtain the main result for n competitive insurers. In Section 5, we derive the main result for n cooperative insurers. In Section 6, we illustrate our theoretical results for two insurers through numerical experiments. The conclusion of this paper is given in Section 7.

2. RI problem with ambiguity aversion

In this section, we propose an RI problem with ambiguity aversion for n insurers. Throughout this paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ is a filtered probability space satisfying the usual conditions. Here $\mathcal{F} := \{\mathcal{F}_t, t \in [0, T]\}$, $\mathcal{F}_t$ is a filtration, which is generated by $2n$ Brownian motions $W_1(t), W_2(t), \cdots, W_n(t)$, $\overline{W}_1(t), \overline{W}_2(t), \cdots, \overline{W}_n(t)$. For a detailed introduction about $\mathcal{F}_t$, one can refer to Karatzas and Shreve [31]. P is a reference measure, T is the termination time of the RI, which is a definite constant. For the case of a random termination time, one can refer to Chen et al. [32] and Xu et al. [33]. In what follows, all stochastic processes are assumed to be adapted to $\mathcal{F}$. We assume that there are no transaction costs or taxes in trading and that the trading occurs continuously.

2.1. RI model

The insurer i's cumulative claims in the time interval $[0,t]$ is denoted by $\sum_{k=1}^{K_i(t)} Y_k^i$. Here $\{Y_k^i, k = 1,2,\cdots\}$ is a sequence of independent and identically distributed random variables, the common random variables of $\{Y_k^i, k = 1,2,\cdots\}$ are denoted by Y^i. Denote the expectation and second-order moment of Y^i by $\mu_{i1} = \mathrm{E}(Y^i) < +\infty$ and $\mu_{i2} = \mathrm{E}[(Y^i)^2] < +\infty$, respectively. $K_i(t) = N(t) + N_i(t)$. $N_i(t)$ and $N(t)$ are two mutually independent homogeneous Poisson processes with intensities $\lambda_i \geq 0$ and $\lambda \geq 0$, respectively. Hence $K_i(t) = N(t) + N_1(t)$ is a Poisson process with an intensity of $\lambda + \lambda_i$. Poisson process $N(t)$ is the first source of the interaction mechanism of n insurers.

To manage and control the risk exposures, we now take into account proportional reinsurance. Let $a_i(t) \in [0,1]$ represent the retention level of reinsurance acquired at time t, which means that the insurer pays $a_i(t)Y^i$ of a claim occurring at time t and the reinsurer pays $(1 - a_i(t))Y^i$. Because the variance premium principle contains the first and second moment information of a claim, while the expected value premium principle only contains the first moment information of a claim, compared with the expected value premium principle, the variance premium principle may better reflect the claim information, thus enabling the insurer and the reinsurer to sign reinsurance contracts as soon as possible. Therefore, we consider the variance premium principle. We assume that $\xi_i > 0$ is the safety loading of the reinsurer. Then, the reinsurance premium is given by

$$\begin{aligned}&\mathrm{E}\left(\sum_{k=1}^{K_i(t)}(Y_k^i - a_i(t)Y_k^i)\right) + \xi_i \mathrm{Var}\left(\sum_{k=1}^{K_i(t)}(Y_k^i - a_i(t)(Y_k^i))\right)\\ =&(1 - a_i(t))(\lambda + \lambda_i)t\mu_{i1} + \xi_i(1 - a_i(t))^2(\lambda + \lambda_i)\mu_{i2}t.\end{aligned} \tag{2.1}$$

To solve the robust stochastic optimization problem in this paper explicitly, similar to Bensoussan et al. [8] and Chen and Yang [24], we consider the diffusion approximation model. Concretely, according to Grandell [34], we have

$$\sum_{k=1}^{K_i(t)} Y_k^i \approx (\lambda + \lambda_i)\mu_{i1}t - \sqrt{(\lambda + \lambda_i)\mu_{i2}}W_i(t), \tag{2.2}$$

where $W_i(t)$ is a standard Brownian motion. For any two Brownian motions $W_i(t)$ and $W_m(t)$, $i \neq m \in \{1,2,\cdots,n\}$, we assume that they are correlated with the correlation coefficient being $\rho_{im} := \frac{\lambda\mu_{i1}\mu_{m1}}{\sqrt{(\lambda+\lambda_i)(\lambda+\lambda_m)\mu_{i2}\mu_{m2}}}$. By using the approximation, the insurer i's surplus process $X_i^{a_i}(t)$ is given by

$$dX_i^{a_i}(t) = \left[\eta_i(\lambda + \lambda_i)\mu_{i1} - \xi_i(1 - a_i(t))^2(\lambda + \lambda_i)\mu_{i2}\right]dt + a_i(t)\sqrt{(\lambda + \lambda_i)\mu_{i2}}W_i(t), \tag{2.3}$$

where $\eta_i > 0$ is the insurer i's safety loading and the premium is calculated by using the expectation value principle. To exclude the insurer's arbitrage behavior, we require that $\xi_i > \eta_i$.

In addition to adopting reinsurance, we assume that the insurers are allowed to invest in a financial market consisting of one risk-free asset and n correlated stocks, referred to as stock 1, 2, $\cdot\cdot\cdot$, n. We assume that $S_0(t)$ and $S_i(t)$ are the price processes for the risk-free asset and the stock i, respectively. $dS_0(t) = rS_0(t)dt$, $r > 0$ is the interest rate; $dS_i(t) = S_i(t)[\mu_i dt + \sigma_i d\overline{W}_i(t)]$,* $\mu_i \geq r$ is the appreciation

*Here, we assume that the price process for a stock satisfies the conditions of geometric Brownian motion. One can also consider the price process for a stock with jumps (see, for example, Yang et al. [4], Chen and Yang [24] and Li et al. [35]), with the mispricing phenomenon (see, for example, Wang et al. [5], Ma et al. [36] and Liu et al. [37]), with stochastic volatility (see, for example, Kang et al. [38] and Wang et al. [39]) and with Markov chain modulation (see, for example Bensoussan et al. [8] and Xu et al. [40]).

rate, $\sigma_i > 0$ is the volatility and $\overline{W}_i(t)$ is a standard Brownian motion. For any $i \neq m \in \{1, 2, \cdots, n\}$, there are two Brownian motions $\overline{W}_i(t)$ and $\overline{W}_m(t)$ with a correlation coefficient of $\bar{\rho}_{im}$, $\bar{\rho}_{im} \in [-1, 1]$. We assume that, for any $i \in \{1, 2, \cdots, n\}$, the Brownian motions $W_i(t)$ and $\overline{W}_i(t)$ are mutually independent. The correlation coefficient $\bar{\rho}_{im}$ is the second source of the interaction mechanism of n insurers.

Each insurer can invest in the financial market to increase their wealth. We assume that each insurer invests in a unique and different stock, i.e., different insurers think that the most profitable stock is different, and each insurer chooses to invest in what they think to be the most profitable stock. It may be assumed that insurer i thinks the most profitable stock is stock i; then, insurer i only invests in stock i, $i = 1, 2, \cdots, n$. Let $\pi_i(t)$ be the amount of money invested in the stock i at time t by the insurer i, and the remaining portion of the money is invested in the risk-free asset. Denote $u_i(t) = (a_i(t), \pi_i(t))$; then, the wealth process $X_i^{u_i}(t)$ of the insurer i with RI becomes

$$\begin{aligned} dX_i^{u_i}(t) &= \Big[\eta_i(\lambda+\lambda_i)\mu_{i1} - \xi_i(1-a_i(t))^2(\lambda+\lambda_i)\mu_{i2} + rX_i^{u_i}(t) + (\mu_i - r)\pi_i(t)\Big]dt \\ &\quad + a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}}dW_i(t) + \pi_i(t)\sigma_i d\overline{W}_i(t). \end{aligned} \tag{2.4}$$

2.2. Ambiguity aversion

As we explained in the introduction, insurers are usually ambiguity-averse. Now, we incorporate ambiguity aversion into our RI problem. To define the wealth process under ambiguity aversion, we first introduce a process $\phi_i(t) = (\theta_i(t), \bar{\theta}_i(t))$ satisfying the following conditions:

(i) For each $t \in [0, T]$, $\phi_i(t)$ is progressively measurable with respect to $\mathcal{F}_t$;

(ii) $\mathrm{E}\left\{\exp\left\{\frac{1}{2}\int_0^T(\theta_i(t))^2dt + \frac{1}{2}\int_0^T(\bar{\theta}_i(t))^2dt\right\}\right\} < +\infty$. Here, the expectation is calculated under probability measure P.

The space of all such processes $\phi_i(t)$ is denoted by $\Phi_i(t)$.

Furthermore, a real-valued process $\{\Lambda^{\phi_i}(t) | t \in [0, T]\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ is defined as

$$\Lambda^{\phi_i}(t) = \exp\left\{-\int_0^t \theta_i(s)dW_i(s) - \frac{1}{2}\int_0^t(\theta_i(s))^2ds - \int_0^t \bar{\theta}_i(s)d\overline{W}_i(s) - \frac{1}{2}\int_0^t(\bar{\theta}_i(s))^2\right\}ds.$$

By the definition of $\phi_i(t)$, we know that $\Lambda^{\phi_i}(t)$ is a P-martingale; then, we have that $\mathrm{E}[\Lambda^{\phi_i}(T)] = 1$. Now, we define a new probability measure Q_i, which is absolutely continuous with respect to P on $\mathcal{F}_T$. Concretely, for each $\phi_i(t) \in \Phi_i(t)$, we define $\frac{d\mathrm{Q}_i}{d\mathrm{P}}\Big|_{\mathcal{F}_T} := \Lambda^{\phi_i}(T)$. According to Girsanov's theorem, under the alternative measure Q_i, the stochastic processes $W_i^{\mathrm{Q}_i}(t)$ and $\overline{W}_i^{\mathrm{Q}_i}(t)$ are standard Brownian motions, where

$$dW_i^{\mathrm{Q}_i}(t) = dW_i(t) + \theta_i(t)dt, \ d\overline{W}_i^{\mathrm{Q}_i}(t) = d\overline{W}_i(t) + \bar{\theta}_i(t)dt. \tag{2.5}$$

Furthermore, under an ambiguity-aversion framework, the wealth process given by (2.4) becomes

$$\begin{aligned} dX_i^{u_i}(t) &= \Big[\eta_i(\lambda+\lambda_i)\mu_{i1} - \xi_i(1-a_i(t))^2(\lambda+\lambda_i)\mu_{i2} + rX_i^{u_i}(t) + (\mu_i - r)\pi_i(t) - \bar{\theta}_i(t)\pi_i(t)\sigma_i \\ &\quad -\theta_i(t)a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}}\Big]dt + a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}}dW_i^{\mathrm{Q}_i}(t) + \pi_i(t)\sigma_i d\overline{W}_i^{\mathrm{Q}_i}(t). \end{aligned} \tag{2.6}$$

Now, we define the following admissible strategy.

Definition 2.1 A strategy $u_i(t)$ ($t \in [0, T]$) is called an admissible strategy if it satisfies the following conditions:

(1) For any $t \in [0, T]$, $u_i(t)$ is measurable with respect to $\mathcal{F}_t$;

(2) For any $t \in [0, T]$, $a_i(t) \in [0, 1]$;

(3) $\mathrm{E}^{\mathrm{Q}_i^*}\left\{\int_0^T \left[\sum_i^n \pi_i^2(t)\right] dt\right\} < +\infty$, where Q_i^* is the probability distribution chosen to describe the worst case;

(4) The stochastic differential equation (2.6) for $u_i(t)$ has a unique strong solution.

The set of all admissible RI strategies of the insurer i is denoted by $\mathcal{U}_i$.

3. Robust stochastic optimization problem for n competitive and cooperative insurers

In this section, we establish two interaction mechanisms among n insurers, i.e., n insurers are competitive and n insurers are cooperative, respectively. Considering these two cases, we define the robust stochastic optimization problem for them, respectively.

3.1. Robust stochastic optimization problem for n competitive insurers

We first establish the competitive mechanism among n insurers.

Many scholars quantified the competition between two insurers based on the relative performance. In this paper, we set up the competition mechanism for n insurers based on the relative performance. The benchmark of each insurer's competition is the average value of the remaining insurer's wealth, which is given by

$$\overline{X}_i^{(u_m)_{m\neq i}}(t) := \frac{1}{n-1}\sum_{m=1,m\neq i}^{n} X_m^{u_m}(t) = \frac{1}{n-1}\sum_{m\neq i} X_m^{u_m}(t),$$

where $(u_m)_{m\neq i} := (u_1(t), u_2(t), \cdots, u_{i-1}(t), u_{i+1}(t), \cdots, u_n(t))$.

The relative wealth process $\hat{X}_i^{u_i,(u_m)_{m\neq i}}(t)$ of the insurer i is defined as

$$\begin{aligned}\hat{X}_i^{u_i,(u_m)_{m\neq i}}(t) &= (1-\tau_i)X_i^{u_i}(t) + \tau_i\left(X_i^{u_i}(t) - \overline{X}_i^{(u_m)_{m\neq i}}(t)\right)\\ &= X_i^{u_i}(t) - \tau_i\overline{X}_i^{(u_m)_{m\neq i}}(t),\end{aligned} \tag{3.1}$$

where the parameter $\tau_i \in [0, 1]$ captures the intensity of insurer i's relative concern and measures their sensitivity to the average performance of their competitors. The larger τ_i means that they give more weight to the relative average performance and care more about increasing their relative wealth, making competition fiercer. When $\tau_i = 0$, we return to the single-insurer case.

Based on the above analysis, the relative wealth process $\hat{X}_i^{u_i,(u_m)_{m\neq i}}(t)$ for the competition case is given by

$$
\begin{aligned}
d\hat{X}_i^{u_i,(u_m)_{m\neq i}}(t) = \Bigg[& \eta_i(\lambda+\lambda_i)\mu_{i1} - \frac{\tau_i}{n-1}\sum_{m\neq i}\eta_m(\lambda+\lambda_m)\mu_{m1} - \xi_i(1-a_i(t))^2(\lambda+\lambda_i)\mu_{i2} + \frac{\tau_i}{n-1} \\
& \cdot \sum_{m\neq i}\xi_m(1-a_m(t))^2(\lambda+\lambda_m)\mu_{m2} + r\hat{X}_i^{u_i,(u_m)_{m\neq i}}(t) + (\mu_i - r)\pi_i(t) - \frac{\tau_i}{n-1}\sum_{m\neq i}(\mu_m - r)\pi_m(t) \\
& - \theta_i(t)a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}} + \frac{\tau_i}{n-1}\sum_{m\neq i}\theta_m(t)a_m(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}} - \bar{\theta}_i(t)\pi_i(t)\sigma_i \\
& + \frac{\tau_i}{n-1}\sum_{m\neq i}\bar{\theta}_m(t)\pi_m(t)\sigma_m\Bigg]dt + a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}}dW_i^{\mathrm{Q}_i}(t) \\
& - \frac{\tau_i}{n-1}\sum_{m\neq i}a_m(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}}dW_m^{\mathrm{Q}_i}(t) + \pi_i(t)\sigma_i d\overline{W}_i^{\mathrm{Q}_i}(t) - \frac{\tau_i}{n-1}\sum_{m\neq i}\pi_m(t)\sigma_m d\overline{W}_m^{\mathrm{Q}_i}(t).
\end{aligned} \tag{3.2}
$$

Now, we construct the robust stochastic optimization problem for n competitive insurers. Each insurer seeks to derive a robust RI strategy during the time interval $[0, T]$ to maximize the expected terminal wealth while minimizing the variance of the terminal wealth. Let

$$
J_i^{u_i,(u_m)_{m\neq i}}(t,\hat{x}_i) = \mathrm{E}_{t,\hat{x}_i}^{\mathrm{Q}_i}[\hat{X}_i^{u_i(u_m)_{m\neq i}}(T)] - \frac{\gamma_i}{2}\mathrm{Var}_{t,\hat{x}_i}^{\mathrm{Q}_i}[\hat{X}_i^{u_i,(u_m)_{m\neq i}}(T)],
$$

where $\mathrm{E}_{t,\hat{x}_i}^{\mathrm{Q}_i}[\cdot] = \mathrm{E}^{\mathrm{Q}_i}[\,\cdot\,|\hat{X}_i^{u_i,(u_m)_{m\neq i}}(t) = \hat{x}_i]$, $\mathrm{Var}_{t,\hat{x}_i}^{\mathrm{Q}_i}[\cdot] = \mathrm{Var}^{\mathrm{Q}_i}[\,\cdot\,|\hat{X}_i^{u_i,(u_m)_{m\neq i}}(t) = \hat{x}_i]$, γ_i is the risk-aversion coefficient for the insurer i. When insurer i is assumed to be ambiguity-neutral, the stochastic optimization problem is given by

$$
\sup_{u_i\in\mathcal{U}_i} J_i^{u_i,(u_m)_{m\neq i}}(t,\hat{x}_i). \tag{3.3}
$$

In this paper, the insurers are assumed to be ambiguity-averse. Then, the robust stochastic optimization problem under the ambiguity aversion framework for the insurer i can be written as

$$
\sup_{u_i\in\mathcal{U}_i}\inf_{\mathrm{Q}_i\in\mathcal{Q}}\left\{J_i^{u_i,(u_m)_{m\neq i}}(t,\hat{x}_i) + D_{t,\hat{x}_i}(\mathrm{Q}_i||\mathrm{P})\right\}, \tag{3.4}
$$

where $D_{t,\hat{x}_i}(\mathrm{Q}_i||\mathrm{P}) \geq 0$ denotes the generalized Kullback-Leibler (KL) divergence between Q_i and P. The introduction of $D_{t,\hat{x}_i}(\mathrm{Q}_i||\mathrm{P})$ allows one to measure the ambiguity aversion of the insurer i and regularize the choices of Q_i. The larger the KL divergence, the less the deviations from the reference model are penalized. When the KL divergence equals 0, the robust stochastic optimization problem (3.4) reduces to the traditional stochastic optimization problem (3.3).

Our aim was to look for the optimal RI strategy for the robust stochastic optimization problem (3.4). From Definition 2.1 given by Espinosa and Touzi [41], we know that a Nash equilibrium for the n insurers is an n-tuple $(u_1^*(t), u_2^*(t), \cdots, u_n^*(t)) \in \mathcal{U}_1\times\mathcal{U}_2\times\cdots\times\mathcal{U}_n$ such that, for each $i = 1, 2, \cdots, n$, given $(u_m^*)_{m\neq i}$, the RI strategy $u_i(t)$ is a solution of the stochastic optimization problem (3.4).

Under the non-zero-sum stochastic differential game framework, our main interest is to seek an equilibrium control strategy $(u_1^*(t), u_2^*(t), \cdots, u_n^*(t)) \in \mathcal{U}_1\times\mathcal{U}_2\times\cdots\times\mathcal{U}_n$ such that

$$
\inf_{\mathrm{Q}_i\in\mathcal{Q}}\left\{J_i^{u_i^*,(u_m^*)_{m\neq i}}(t,\hat{x}_i) + D_{t,\hat{x}_i}(\mathrm{Q}_i||\mathrm{P})\right\} \geq \inf_{\mathrm{Q}_i\in\mathcal{Q}}\left\{J_i^{u_i,(u_m^*)_{m\neq i}}(t,\hat{x}_i) + D_{t,\hat{x}_i}(\mathrm{Q}_i||\mathrm{P})\right\} \tag{3.5}
$$

for each $i = 1, 2, \cdots, n$.

The equilibrium value function is defined as follows.

Definition 3.1. (Equilibrium value function). If (3.5) holds, then we define the equilibrium value function of insurer i as follows:

$$J_i(t, \hat{x}_i) = J_i^{u_i^*,(u_m^*)_{m\neq i}}(t, \hat{x}_i) + D_{t,\hat{x}_i}(\mathrm{Q}_i^*||\mathrm{P}) = \sup_{u_i\in\mathcal{U}_i} \inf_{\mathrm{Q}_i\in\mathcal{Q}} \left\{ J_i^{u_i,(u_m^*)_{m\neq i}}(t, \hat{x}_i) + D_{t,\hat{x}_i}(\mathrm{Q}_i||\mathrm{P}) \right\}. \tag{3.6}$$

Here $u_i^* = u_i^*(t)$ represents the robust RI strategies for insurer i, $i = 1, 2, \cdots, n$. To ensure that this strategy is time-consistent, we define the following strategy.

Definition 3.2. If the control strategy $(u_1^*(t), u_2^*(t), \cdots, u_n^*(t))$ exists, we define the following strategy u_i^ε

$$u_i^\varepsilon(s, \tilde{x}) = \begin{cases} u_i(s, \tilde{x}), & \text{for } (s, \tilde{x}) \in [t, t+\varepsilon) \times R, \\ u_i^*(s, \tilde{x}), & \text{for } (s, \tilde{x}) \in [t+\varepsilon, T] \times R. \end{cases}$$

Here $u_i \in \mathcal{U}_i$, $\varepsilon > 0$ and $(t, \tilde{x}) \in [0, T] \times R$ are arbitrarily choosen. If

$$\liminf_{\varepsilon\to 0} \frac{J_i^{u_i^*,(u_m^*)_{m\neq i}}(t, \hat{x}_i) - J_i^{u_i^\varepsilon,(u_m^*)_{m\neq i}}(t, \hat{x}_i)}{\varepsilon} \geq 0$$

for all $u_i \in \mathcal{U}_i$ and $(t, \tilde{x}) \in [0, T] \times R$, then for insurer i, u_i^* is called an equilibrium strategy and $J_i^{u_i^*,(u_m^*)_{m\neq i}}(t, \hat{x}_i)$ is the equilibrium value function, i.e., the OVF.

In the following Section 4, we will obtain the optimal strategy and the corresponding OVF for the robust stochastic optimization problem (3.4). According to the Definition 3.2 and the proof provided by Björk et al. [42], we can prove that the following optimal strategy is time-consistent.

3.2. Robust stochastic optimization problem for n cooperative insurers

In this part, we define the robust stochastic optimization problem for the cooperation case. First, we present the cooperative mechanism among n insurers.

Today, most major insurance companies exist as insurance groups. They jointly resist risks and make profits from investments. In such a cooperative economy, the common wealth process is controlled by all insurers. We must consider the joint interests of the insurers; the following common wealth process will be studied:

$$X^u(t) = \sum_{i=1}^n \kappa_i X_i^{u_i}(t), \tag{3.7}$$

where κ_i is the weighted coefficient satisfying $\kappa_i \in [0, 1]$, $\sum_{i=1}^n \kappa_i = 1$, and κ_i takes a role of balancing the interests among n insurers. The larger the value of κ_i, the more important the insurer i in the cooperative group. At the termination time T of the RI, the insurers distribute the common wealth according to their weighted coefficient κ_i. Here we can determine κ_i in different ways, such as through the insurer's initial surplus x_i^0, i.e., $\kappa_i = \frac{x_i^0}{\sum_{i=1}^n x_i^0}$. Combining (2.6) and (3.7) yields the following dynamics for n cooperative insurers' common wealth process:

$$\begin{aligned} dX^u(t) = \Bigg[& rX^u(t) + \sum_{i=1}^{n} \kappa_i \eta_i (\lambda + \lambda_i)\mu_{i1} - \sum_{i=1}^{n} \kappa_i \xi_i (1 - a_i(t))^2 (\lambda + \lambda_i)\mu_{i2} \\ & + \sum_{i=1}^{n} \kappa_i (\mu_i - r)\pi_i(t) - \sum_{i=1}^{n} \theta_i(t)\kappa_i a_i(t)\sqrt{(\lambda + \lambda_i)\mu_{i2}} - \sum_{i=1}^{n} \bar{\theta}_i(t)\kappa_i \pi_i(t)\sigma_i \Bigg] dt \\ & + \sum_{i=1}^{n} \kappa_i a_i(t)\sqrt{(\lambda + \lambda_i)\mu_{i2}}\, dW_i^{\mathrm{Q}_i}(t) + \sum_{i=1}^{n} \kappa_i \pi_i(t)\sigma_i d\overline{W}_i^{\mathrm{Q}_i}(t). \end{aligned} \tag{3.8}$$

The robust stochastic optimization problem for n cooperative insurers under a time-consistent MV framework is given as follows:

$$\bar{J}(t,x) = \max_{u \in \mathcal{U}} \inf_{\mathrm{Q} \in \mathcal{Q}} \left\{ \bar{J}^u(t,x) + D_{t,x}(\mathrm{Q}\|\mathrm{P}) \right\} = \max_{u \in \mathcal{U}} \inf_{\mathrm{Q} \in \mathcal{Q}} \left\{ E_{t,x}(X_T^u) - \frac{\gamma}{2} Var_{t,x}(X_T^u) + D_{t,x}(\mathrm{Q}\|\mathrm{P}) \right\}, \tag{3.9}$$

where $\mathrm{Q} = (\mathrm{Q}_1, \mathrm{Q}_2, \cdots, \mathrm{Q}_n)$, $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_1, \cdots, \mathcal{U}_n)$ and $\gamma > 0$ measures the group's risk aversion; $\bar{J}(t,x)$ is OVF.

Similar to Definition 3.2, we can obtain the equilibrium strategy for n cooperative insurers, which is time-consistent. To avoid duplication, we omit it here.

4. Solution to n competitive insurers

In this section, we are devoted to deriving a robust optimal RI strategy and the corresponding OVF for n competitive insurers. Some special cases of our model are also provided, which show that our model and results extend some existing ones in the literature. We first derive the closed-form the robust optimal RI strategy and the corresponding OVF for n insurers; then, we derive the explicit robust optimal RI strategy and the corresponding OVF for two insurers.

To analytically evaluate the robust stochastic optimization problem, we specify the form of $D_{t,\hat{x}_i}(\mathrm{Q}_i\|\mathrm{P})$. Inspired by Maenhout [19], we assume that $D_{t,\hat{x}_i}(\mathrm{Q}_i\|\mathrm{P})$ satisfies the following form

$$D_{t,\hat{x}_i}(\mathrm{Q}_i\|\mathrm{P}) = \mathrm{E}_{t,\hat{x}_i}^{\mathrm{Q}_i} \left\{ \int_t^T \left[\frac{\theta_i^2}{2\psi_i(s, \hat{X}_i^{u_i,(u_m)_{m\neq i}}(s))} + \frac{\bar{\theta}_i^2}{2\bar{\psi}_i(s, \hat{X}_i^{u_i,(u_m)_{m\neq i}}(s))} \right] ds \right\}.$$

Here $\psi_i(t, \hat{X}_i^{u_i,(u_m)_{m\neq i}}(t))$ and $\bar{\psi}_i(t, \hat{X}_i^{u_i,(u_m)_{m\neq i}}(t))$ are nonnegative and capture the insurers' ambiguity aversions. The larger the values of $\psi_i(t, \hat{X}_i^{u_i,(u_m)_{m\neq i}}(t))$ and $\bar{\psi}_i(t, \hat{X}_i^{u_i,(u_m)_{m\neq i}}(t))$, the more ambiguity-averse the insurers. To render the above robust stochastic optimization problem analytically tractable, we assume that $\psi_i(t, \hat{X}_i^{u_i,(u_m)_{m\neq i}}(t)) = \alpha_i$ and $\bar{\psi}_i(t, \hat{X}_i^{u_i,(u_m)_{m\neq i}}(t)) = \bar{\alpha}_i$, where α_i and $\bar{\alpha}_i$ are nonnegative (cf. Maenhout [19]).

Let $C^{1,2}([0,T]\times R)$ denote the space of $\Psi(t,\hat{x}_i)$ such that $\Psi(t,\hat{x}_i)$ and its derivatives $\Psi_t(t,\hat{x}_i)$, $\Psi_{\hat{x}_i}(t,\hat{x}_i)$, $\Psi_{\hat{x}_i\hat{x}_i}(t,\hat{x}_i)$ are continuous on $[0,T]\times R$. For any $\Psi(t,\hat{x}_i) \in C^{1,2}([0,T]\times R)$, and any fixed $u_i(t) \in \mathcal{U}_i$, the infinitesimal generator $\mathcal{A}^{u_i,(u_m^*)_{m\neq i},\phi_i,(\phi_m^*)_{m\neq i}}$ is defined as

$$
\begin{aligned}
&\mathcal{A}^{u_i,(u_m^*)_{m\neq i},\phi_i,(\phi_m^*)_{m\neq i}}\Psi(t,\hat{x}_i)\\
=&\Psi_t(t,\hat{x}_i)+\Bigg[\eta_i(\lambda+\lambda_i)\mu_{i1}-\frac{\tau_i}{n-1}\sum_{m\neq i}\eta_m(\lambda+\lambda_m)\mu_{m1}-\xi_i(1-a_i(t))^2\\
&\cdot(\lambda+\lambda_i)\mu_{i2}+\frac{\tau_i}{n-1}\sum_{m\neq i}\xi_m(1-a_m(t))^2(\lambda+\lambda_m)\mu_{m2}+r\hat{x}+(\mu_i-r)\pi_i(t)\\
&-\frac{\tau_i}{n-1}\sum_{m\neq i}(\mu_m-r)\pi_m(t)-\theta_i(t)a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}}+\frac{\tau_i}{n-1}\sum_{m\neq i}\theta_m(t)a_m(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}}\\
&-\bar{\theta}_i(t)\pi_i(t)\sigma_i+\frac{\tau_i}{n-1}\sum_{m\neq i}\bar{\theta}_m(t)\pi_m(t)\sigma_m\Bigg]\Psi_{\hat{x}_i}(t,\hat{x}_i)+\frac{1}{2}g(u_i,u_m)\Psi_{\hat{x}_i\hat{x}_i}(t,\hat{x}_i),
\end{aligned}
\tag{4.1}
$$

where

$$
g(u_i,u_m)=g_1(a_i,a_m)+g_2(\pi_i,\pi_m)=\mathbf{C}_1^{a_i,(a_m)_{m\neq i}}\mathbf{D_1}(\mathbf{C}_1^{a_i,(a_m)_{m\neq i}})^T+\mathbf{C}_2^{\pi_i,(\pi_m)_{m\neq i}}\mathbf{D_2}(\mathbf{C}_2^{\pi_i,(\pi_m)_{m\neq i}})^T,
\tag{4.2}
$$

$$
\begin{aligned}
\mathbf{C}_1^{a_i,(a_m)_{m\neq i}}=&\Bigg[-\frac{\tau_i a_1(t)\sqrt{(\lambda+\lambda_1)\mu_{12}}}{n-1},\cdots,-\frac{\tau_i a_{i-1}(t)\sqrt{(\lambda+\lambda_{i-1})\mu_{(i-1)2}}}{n-1},a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}},\\
&-\frac{\tau_i a_{i+1}(t)\sqrt{(\lambda+\lambda_{i+1})\mu_{(i+1)2}}}{n-1},\cdots,-\frac{\tau_i a_n(t)\sqrt{(\lambda+\lambda_n)\mu_{n2}}}{n-1}\Bigg],
\end{aligned}
\tag{4.3}
$$

$$
\mathbf{C}_2^{\pi_i,(\pi_m)_{m\neq i}}=\left[-\frac{\tau_i\pi_1(t)\sigma_1}{n-1},\cdots,-\frac{\tau_i\pi_{i-1}(t)\sigma_{i-1}}{n-1},\pi_i(t)\sigma_i,-\frac{\tau_i\pi_{i+1}(t)\sigma_{i+1}}{n-1},\cdots,-\frac{\tau_i\pi_n(t)\sigma_n}{n-1}\right],
\tag{4.4}
$$

$$
\mathbf{D_1}=\begin{pmatrix}1&\rho_{12}&\cdots&\rho_{1(n-1)}&\rho_{1n}\\\rho_{12}&1&\cdots&\rho_{2(n-1)}&\rho_{2n}\\\cdots&\cdots&\cdots&\cdots&\cdots\\\rho_{1n}&\rho_{2n}&\ldots&\rho_{(n-1)n}&1\end{pmatrix},\mathbf{D_2}=\begin{pmatrix}1&\bar{\rho}_{12}&\cdots&\bar{\rho}_{1(n-1)}&\bar{\rho}_{1n}\\\bar{\rho}_{12}&1&\cdots&\bar{\rho}_{2(n-1)}&\bar{\rho}_{2n}\\\cdots&\cdots&\cdots&\cdots&\cdots\\\bar{\rho}_{1n}&\bar{\rho}_{2n}&\ldots&\bar{\rho}_{(n-1)n}&1\end{pmatrix}.
\tag{4.5}
$$

In order to derive the robust optimal RI strategy and the corresponding OVF, we provide the following important theorem.

Theorem 4.1. (Verification theorem). Suppose that there exist two real functions $V_i(t,\hat{x}_i)\in C^{1,2}([0,T]\times R)$, $h_i(t,\hat{x}_i)\in C^{1,2}([0,T]\times R)$ satisfying the following extended Hamilton-Jacobi-Bellman (HJB) equation system:

$$
\begin{aligned}
\sup_{u_i\in\mathcal{U}_i}\inf_{\phi_i\in\Phi_i}\Bigg\{&\mathcal{A}^{u_i,(u_m^*)_{m\neq i},\phi_i,(\phi_m^*)_{m\neq i}}V_i(t,\hat{x}_i)-\mathcal{A}^{u_i,(u_m^*)_{m\neq i},\phi_i,(\phi_m^*)_{m\neq i}}\frac{\gamma_i}{2}h_i^2(t,\hat{x}_i)\\
&+\gamma_i h_i(t,\hat{x}_i)\mathcal{A}^{u_i,(u_m^*)_{m\neq i},\phi_i,(\phi_m^*)_{m\neq i}}h_i(t,\hat{x}_i)+\frac{\theta_i^2(s)}{2\alpha_i}+\frac{\bar{\theta}_i^2(s)}{2\bar{\alpha}_i(s)}\Bigg\}=0,\ V_i(T,\hat{x}_i)=\hat{x}_i,
\end{aligned}
\tag{4.6}
$$

$$
\mathcal{A}^{u_i^*,(u_m^*)_{m\neq i},\phi_i^*,(\phi_m^*)_{m\neq i}}h_i(t,\hat{x}_i)=0,\ h_i(T,\hat{x}_i)=\hat{x}_i;
\tag{4.7}
$$

here,

$$
\begin{aligned}
u_i^*(t) = \arg\sup_{u_i\in\mathcal{U}_i}\inf_{\phi_i\in\Phi_i}\Big\{&\mathcal{A}^{u_i,(u_m^*)_{m\neq i},\phi_i,(\phi_m^*)_{m\neq i}}V_i(t,\hat{x}_i) - \mathcal{A}^{u_i,(u_m^*)_{m\neq i},\phi_i,(\phi_m^*)_{m\neq i}}\frac{\gamma_i}{2}h_i^2(t,\hat{x}_i)\\
&+\gamma_i h_i(t,\hat{x}_i)\mathcal{A}^{u_i,(u_m^*)_{m\neq i},\phi_i,(\phi_m^*)_{m\neq i}}h_i(t,\hat{x}_i) + \frac{\theta_i^2(s)}{2\alpha_i} + \frac{\bar{\theta}_i^2(s)}{2\bar{\alpha}_i(s)}\Big\}.
\end{aligned}
\tag{4.8}
$$

Then $u_i^*(t)$ is the robust optimal RI strategy and $V_i(t,\hat{x}_i)$ is the corresponding OVF, i.e., $V_i(t,\hat{x}_i) = J_i(t,\hat{x}_i)$.

Before giving the main results, we define the following notations:

$$
\begin{cases}
\delta_i = (\lambda+\lambda_i)\mu_{i2}\left[2\xi_i + (\alpha_i+\gamma_i)e^{r(T-t)}\right],\\
\zeta_i = \lambda\mu_{i1}\gamma_i e^{r(T-t)}\frac{\tau_i}{n-1},\ \beta_i = 2\xi_i(\lambda+\lambda_i)\mu_{i2},\\
\bar{\delta}_i = (\gamma_i+\bar{\alpha}_i)\sigma_i^2,\ \bar{\zeta}_i = \frac{\sigma_i\gamma_i\tau_i}{n-1},\ \bar{\beta}_i = (\mu_i - r)e^{-r(T-t)},\\
\hat{\delta}_i = (\lambda+\lambda_i)\mu_{i2}\left[2\xi_i + \gamma_i e^{r(T-t)}\right],\ \hat{\beta}_i = 2\xi_i(\lambda+\lambda_i)\mu_{i2}.
\end{cases}
\tag{4.9}
$$

Now, we give the main results for n competitive insurers.

Theorem 4.2. The solution to the robust stochastic optimization problem (3.4) is as follows. The insurer i's robust optimal reinsurance strategy (ORS) is $a_i^*(t) = (0\vee\bar{a}_i(t))\wedge 1$, $\bar{a}_i(t)$ is the solution of the following system of linear equations

$$
\begin{cases}
\delta_1 a_1(t) - \zeta_1\mu_{21}a_2(t) - \cdots - \zeta_1\mu_{(n-1)1}a_{n-1}(t) - \zeta_1\mu_{n1}a_n(t) = \beta_1,\\
-\zeta_2\mu_{11}a_1(t) + \delta_2 a_2(t) - \cdots - \zeta_2\mu_{(n-1)1}a_{n-1}(t) - \zeta_2\mu_{n1}a_n(t) = \beta_2,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-\zeta_n\mu_{11}a_1(t) - \zeta_n\mu_{21}a_2(t) - \cdots - \zeta_n\mu_{(n-1)1}a_{n-1}(t) + \delta_n a_n(t) = \beta_n;
\end{cases}
\tag{4.10}
$$

the insurers' robust optimal investment strategy (OIS) $(\pi_1^*(t),\pi_2^*(t),\cdots,\pi_n^*(t))$ is the solution of the following system of linear equations

$$
\begin{cases}
\bar{\delta}_1\pi_1(t) - \bar{\zeta}_1\sigma_2\bar{\rho}_{12}\pi_2(t) - \cdots - \bar{\zeta}_1\sigma_{n-1}\bar{\rho}_{1(n-1)}\pi_{n-1}(t) - \bar{\zeta}_1\sigma_n\bar{\rho}_{1n}\pi_n(t) = \bar{\beta}_1,\\
-\bar{\zeta}_2\sigma_1\bar{\rho}_{12}\pi_1(t) + \bar{\delta}_2\pi_2(t) - \cdots - \bar{\zeta}_2\sigma_{n-1}\bar{\rho}_{2(n-1)}\pi_{n-1}(t) - \bar{\zeta}_2\sigma_n\bar{\rho}_{2n}\pi_n(t) = \bar{\beta}_2,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-\bar{\zeta}_n\sigma_1\bar{\rho}_{1n}\pi_1(t) - \bar{\zeta}_n\sigma_2\bar{\rho}_{2n}\pi_2(t) - \cdots - \bar{\zeta}_n\sigma_{n-1}\bar{\rho}_{(n-1)n}\pi_{n-1}(t) + \bar{\delta}_n\pi_n(t) = \bar{\beta}_n;
\end{cases}
\tag{4.11}
$$

the worst-case measures are given by

$$
\theta_i^*(t) = a_i^*(t)\alpha_i\sqrt{(\lambda+\lambda_i)\mu_{i2}}e^{r(T-t)},
\tag{4.12}
$$

$$
\bar{\theta}_i^*(t) = \pi_i^*(t)\sigma_i\bar{\alpha}_i e^{r(T-t)},
\tag{4.13}
$$

and the corresponding OVF for the insurer i is given by

$$
V_i(t,\hat{x}_i) = \hat{x}_i e^{-r(T-t)} + \frac{B_i(t)}{\gamma_i},
\tag{4.14}
$$

where

$$
\begin{aligned}
B_i(t) = \gamma_i \int_t^T \Bigg\{ \Bigg[& \eta_i(\lambda + \lambda_i)\mu_{i1} - \frac{\tau_i}{n-1} \sum_{m \neq i} \eta_m(\lambda + \lambda_m)\mu_{m1} - \xi_i(1 - a_i^*(s))^2(\lambda + \lambda_i)\mu_{i2} \\
& + \frac{\tau_i}{n-1} \sum_{m \neq i} \xi_m(1 - a_m^*(s))^2(\lambda + \lambda_m)\mu_{m2} + (\mu_i - r)\pi_i^*(t) - \frac{\tau_i}{n-1} \sum_{m \neq i} (\mu_m - r)\pi_m^*(s) \\
& + \frac{\tau_i}{n-1} \sum_{m \neq i} \theta_m^*(t) a_m^*(t) \sqrt{(\lambda + \lambda_m)\mu_{m2}} + \frac{\tau_i}{n-1} \sum_{m \neq i} \bar{\theta}_m^*(t)\pi_m^*(t)\sigma_m \Bigg] e^{r(T-s)} \\
& - \frac{\gamma_i}{2} g(u_i^*, u_m^*) e^{2r(T-s)} - \frac{1}{2}\left[a_i^{*2}(s)\alpha_i(\lambda + \lambda_i)\mu_{i2} + \pi_i^{*2}(s)\sigma_i^2\bar{\alpha}_i\right] e^{2r(T-s)} \Bigg\} ds,
\end{aligned}
\tag{4.15}
$$

and $i = 1, 2, \cdots, n$.

Proof. Please see Appendix A. □

From Appendix A, we have the following findings:

$$
\frac{\lambda\gamma_i\mu_{i1}e^{r(T-t)}}{(\lambda + \lambda_i)\mu_{i2}[2\xi_i + (\alpha_i + \gamma_i)e^{r(T-t)}]} \frac{\tau_i}{n-1} \sum_{m \neq i} a_m^*\mu_{m1}
$$

which is a measure of the insurer i's sensitivity to their competitors' reinsurance strategies; also,

$$
\frac{\gamma_i}{(\bar{\alpha}_i + \gamma_i)\sigma_i} \frac{\tau_i}{n-1} \sum_{m \neq i} \pi_m^*(t)\sigma_m\bar{\rho}_{im}
$$

which is a measure of the insurer i's sensitivity to their competitors' investment strategies.

Now, we give some special cases of our general model. We can obtain the corresponding optimal strategies for these cases as we did in Theorem 4.2; the proofs are omitted.

Corollary 4.1. If $\tau_i = 0$, i.e., we do not consider the competition, the insurer i's robust ORS becomes

$$
a_i^*(t) = \frac{2\xi_i}{2\xi_i + (\alpha_i + \gamma_i)e^{r(T-t)}},
\tag{4.16}
$$

and the insurer i's robust OIS becomes

$$
\pi_i^*(t) = \frac{\mu_i - r}{(\bar{\alpha}_i + \gamma_i)\sigma_i^2 e^{r(T-t)}}.
\tag{4.17}
$$

We find that the robust ORS given by (4.16) is similar to that in Theorem 4.1 presented by Yi et al. [43], which considers the robust RI for an insurer under the expected premium principle. Moreover, the robust OIS given by (4.17) is similar to that in Proposition 1 presented by Maenhout [19].

Corollary 4.2. If $\lambda = 0$ and $\bar{\rho}_{im} = 0$, i.e., the interaction arising from the insurance and finance markets is not considered, the insurer i's robust optimal RI strategies are given by (4.16) and (4.17), respectively.

From Corollary 4.2, we find that even though the competition exists, the robust optimal RI strategy is not affected by the competition parameter τ_i. Therefore, we can get such a conclusion that competition plays a role only in interrelated individuals and groups.

Corollary 4.3. If $\alpha_i = 0$ and $\bar{\alpha}_i = 0$, i.e., the ambiguity is not considered, the insurer i's robust ORS is given by

$$a_i^*(t) = \frac{2\xi_i(\lambda + \lambda_i)\mu_{i2} + \frac{\lambda\gamma_i\mu_{i1}\tau_i}{n-1} e^{r(T-t)} \sum_{m\neq i} a_m^*\mu_{m1}}{(\lambda + \lambda_i)\mu_{i2}[2\xi_i + \gamma_i e^{r(T-t)}]} \wedge 1, \tag{4.18}$$

and the insurer i's robust OIS is given by

$$\pi_i^*(t) = \frac{(\mu_i - r) + \frac{\gamma_i\sigma_i\tau_i}{n-1} e^{r(T-t)} \sum_{m\neq i} \pi_m^*(t)\sigma_m\bar{\rho}_{im}}{\gamma_i\sigma_i^2 e^{r(T-t)}}. \tag{4.19}$$

Corollary 4.4. If $\alpha_i = 0$, $\bar{\alpha}_i = 0$ and $\tau_i = 0$ (resp. $\lambda = 0$ and $\bar{\rho}_{im} = 0$), i.e., the ambiguity and relative performance (resp. interaction arising from the insurance and finance markets) are not considered, the insurer i's robust ORS is given by

$$a_i^*(t) = \frac{2\xi_i}{2\xi_i + \gamma_i e^{r(T-t)}}, \tag{4.20}$$

and the insurer i's robust OIS is given by

$$\pi_i^*(t) = \frac{\mu_i - r}{\gamma_i\sigma_i^2 e^{r(T-t)}}. \tag{4.21}$$

We find that the optimal RI strategies given by (4.20) and (4.21) are similar to that in Theorem 2 presented by Zeng and Li [44], which considers RI for an insurer under the expected premium principle. In other words, the model presented by Zeng and Li [44] can be taken as a special case of our model.

In what follows, we solve the stochastic optimization problem (3.4) for the situation with two insurers. In this case, we can derive the explicit solutions.

Theorem 4.3. For the wealth process described by (3.2) with two insurers, the insurer 1's robust ORS is $a_1^*(t) = \check{a}_1(t) \wedge 1$, where $\check{a}_1(t)$ is given by

$$\check{a}_1(t) = \frac{\beta_1\delta_2 + \beta_2\tau_1\gamma_1\lambda\mu_{11}\mu_{21}e^{r(T-t)}}{\delta_1\delta_2 - \tau_1\tau_2\gamma_1\gamma_2 e^{2r(T-t)}(\lambda\mu_{11}\mu_{21})^2}; \tag{4.22}$$

the insurer 1's robust OIS is given by

$$\pi_1^*(t) = \frac{\bar{\beta}_1\bar{\delta}_2 + \bar{\beta}_2\tau_1\gamma_1\sigma_1\sigma_2\bar{\rho}_{12}}{\bar{\delta}_1\bar{\delta}_2 - \tau_1\tau_2\gamma_1\gamma_2(\sigma_1\sigma_2\bar{\rho}_{12})^2}; \tag{4.23}$$

the insurer 2's robust ORS is $a_2^*(t) = \check{a}_1(t) \wedge 1$, where $\check{a}_2(t)$ is given by

$$\check{a}_2(t) = \frac{\beta_2\delta_1 + \beta_1\tau_2\gamma_2\lambda\mu_{11}\mu_{21}e^{r(T-t)}}{\delta_1\delta_2 - \tau_1\tau_2\gamma_1\gamma_2 e^{2r(T-t)}(\lambda\mu_{11}\mu_{21})^2}; \tag{4.24}$$

the insurer 2's robust OIS is given by

$$\pi_2^*(t) = \frac{\bar{\beta}_2\bar{\delta}_1 + \bar{\beta}_1\tau_2\gamma_2\sigma_1\sigma_2\bar{\rho}_{12}}{\bar{\delta}_1\bar{\delta}_2 - \tau_1\tau_2\gamma_1\gamma_2(\sigma_1\sigma_2\bar{\rho}_{12})^2}; \tag{4.25}$$

the worst-case measures are given by

$$\theta_1^*(t) = \frac{(\beta_1\delta_2 + \beta_2\tau_1\gamma_1\lambda\mu_{11}\mu_{21}e^{r(T-t)})\alpha_1\sqrt{(\lambda + \lambda_1)\mu_{12}}e^{r(T-t)}}{\delta_1\delta_2 - \tau_1\tau_2\gamma_1\gamma_2 e^{2r(T-t)}(\lambda\mu_{11}\mu_{21})^2}, \tag{4.26}$$

$$\bar{\theta}_1^*(t) = \frac{(\bar{\beta}_1\bar{\delta}_2 + \bar{\beta}_2\tau_1\gamma_1\sigma_1\sigma_2\bar{\rho}_{12})\pi_1^*\sigma_1\bar{\alpha}_1 e^{r(T-t)}}{\bar{\delta}_1\bar{\delta}_2 - \tau_1\tau_2\gamma_1\gamma_2(\sigma_1\sigma_2\bar{\rho}_{12})^2}, \tag{4.27}$$

$$\theta_2^*(t) = \frac{(\beta_2\delta_1 + \beta_1\tau_2\gamma_2\lambda\mu_{11}\mu_{21}e^{r(T-t)})\alpha_2\sqrt{(\lambda+\lambda_2)\mu_{22}}e^{r(T-t)}}{\delta_1\delta_2 - \tau_1\tau_2\gamma_1\gamma_2 e^{2r(T-t)}(\lambda\mu_{11}\mu_{21})^2}, \tag{4.28}$$

$$\bar{\theta}_2^*(t) = \frac{(\bar{\beta}_2\bar{\delta}_1 + \bar{\beta}_1\tau_2\gamma_2\sigma_1\sigma_2\bar{\rho}_{12})\sigma_2\bar{\alpha}_2 e^{r(T-t)}}{\bar{\delta}_1\bar{\delta}_2 - \tau_1\tau_2\gamma_1\gamma_2(\sigma_1\sigma_2\bar{\rho}_{12})^2}; \tag{4.29}$$

the insurer i's OVF is given by

$$V_i(t, \hat{x}_i) = \hat{x}_i e^{r(T-t)} + \frac{B_i(t)}{\gamma_i}, \tag{4.30}$$

where

$$\begin{aligned} B_i(t) = \gamma_i \int_t^T &\Big\{\Big[\eta_i(\lambda+\lambda_i)\mu_{i1} - \tau_i\eta_m(\lambda+\lambda_m)\mu_{m1} - \xi_i(1-a_i^*(s))^2(\lambda+\lambda_i)\mu_{i2} + \tau_i\xi_m(1-a_m^*(s))^2 \\ &\cdot(\lambda+\lambda_m)\mu_{m2} + (\mu_i - r)\pi_i^*(t) - \tau_i(\mu_m - r)\pi_m^*(t) + \tau_i\theta_m^*(t)a_m^*(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}} + \tau_i\bar{\theta}_m^*(t)\pi_m^*(t)\sigma_m\Big] \\ &\times e^{r(T-s)} - \frac{\gamma_i}{2}\Big[\pi_i^{*2}(t)\sigma_i^2 + \tau_i^2\pi_m^{*2}(t)\sigma_m^2 - 2\bar{\rho}_{im}\tau_i\sigma_i\sigma_m\pi_i^*(t)\pi_m^*(t) + a_i^{*2}(t)(\lambda+\lambda_i)\mu_{i2} + \tau_i^2 a_m^{*2}(t) \\ &\times(\lambda+\lambda_m)\mu_{m2} - 2\lambda\mu_{i1}\mu_{m1}\tau_i a_i^*(t)a_m^*(t)\Big]e^{2r(T-s)} - \frac{1}{2}\Big[a_i^{*2}(s)\alpha_i(\lambda+\lambda_i)\mu_{i2} + \pi_i^{*2}(s)\sigma_i^2\bar{\alpha}_i\Big]e^{2r(T-s)}\Big\}ds, \end{aligned} \tag{4.31}$$

for $i \neq m \in \{1, 2\}$.

Proof. Please see Appendix B. □

Corollary 4.5. If the ambiguity is not considered for two insurers, the insurer 1's ORS is given by

$$a_1^*(t) = \frac{\hat{\beta}_1\hat{\delta}_2 + \hat{\beta}_2\tau_1\gamma_1\lambda\mu_{11}\mu_{21}e^{r(T-t)}}{\hat{\delta}_1\hat{\delta}_2 - \tau_1\tau_2\gamma_1\gamma_2 e^{2r(T-t)}(\lambda\mu_{11}\mu_{21})^2} \wedge 1, \tag{4.32}$$

the insurer 1's OIS is given by

$$\pi_1^*(t) = \frac{e^{-r(T-t)}}{1-\tau_1\tau_2\bar{\rho}_{12}^2}\left[\frac{\mu_1 - r}{\gamma_1\sigma_1^2} + \frac{(\mu_2 - r)\tau_1\bar{\rho}_{12}}{\gamma_2\sigma_1\sigma_2}\right], \tag{4.33}$$

the insurer 2's ORS is given by

$$a_2^*(t) = \frac{\hat{\beta}_2\hat{\delta}_1 + \hat{\beta}_1\tau_2\gamma_2\lambda\mu_{11}\mu_{21}e^{r(T-t)}}{\hat{\delta}_1\hat{\delta}_2 - \tau_1\tau_2\gamma_1\gamma_2 e^{2r(T-t)}(\lambda\mu_{11}\mu_{21})^2} \wedge 1 \tag{4.34}$$

and the insurer 2's OIS is given by

$$\pi_2^*(t) = \frac{e^{-r(T-t)}}{1-\tau_1\tau_2\bar{\rho}_{12}^2}\left[\frac{\mu_2 - r}{\gamma_2\sigma_2^2} + \frac{(\mu_1 - r)\tau_2\bar{\rho}_{12}}{\gamma_1\sigma_1\sigma_2}\right]. \tag{4.35}$$

We find that the ORSs given by (4.32) and (4.34) reduce to that in Theorem 4.3 presented by Deng et al. [10], which considers RI for two insurers under the condition of maximizing the expected exponential utility. Moreover, (4.33) and (4.35) reduce to that in Theorem 1 presented by Hu and Wang [11]. This means that our model extends the model presented by Deng et al. [10] and Hu and Wang [11] to n insurers with ambiguity aversion.

5. Solution to n cooperative insurers

To analytically evaluate the robust stochastic optimization problem (3.9) for the cooperation case, $D_{t,x}(\mathrm{Q}\|\mathrm{P})$ is given as follows:

$$D_{t,x}(\mathrm{Q}\|\mathrm{P}) = E^{\mathrm{Q}}_{t,x}\left\{\int_t^T \sum_{i=1}^{n}\left[\frac{\theta_i^2(s)}{2\alpha_i} + \frac{\bar{\theta}_i^2(s)}{2\bar{\alpha}_i}\right]ds\right\}.$$

Similar to Theorem 4.1, we give the following verification theorem for n cooperative insurers.

Theorem 5.1. Suppose that there exist $\bar{V}(t,x) \in C^{1,2}([0,T]\times R)$ and $\bar{h}(t,x) \in C^{1,2}([0,T]\times R)$ satisfying the following extended HJB equations

(i) For all $(t,x) \in [0,T]\times R$

$$\begin{aligned}\sup_{u\in\mathcal{U}}\inf_{\phi\in\Phi}\Bigg\{&\bar{V}_t(t,x) + \Bigg[rx + \sum_{i=1}^{n}\kappa_i\eta_i(\lambda+\lambda_i)\mu_{i1} - \sum_{i=1}^{n}\kappa_i\xi_i(1-a_i(t))^2(\lambda+\lambda_i)\mu_{i2} + \sum_{i=1}^{n}\kappa_i(\mu_i - r)\pi_i(t)\\ &-\sum_{i=1}^{n}\theta_i(t)\kappa_i a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}} - \sum_{i=1}^{n}\bar{\theta}_i(t)\kappa_i\pi_i(t)\sigma_i\Bigg]\bar{V}_x(t,x) + \frac{1}{2}[\bar{V}_{xx}(t,x) - \gamma\bar{h}_x^2(t,x)]\\ &\times\Bigg[\sum_{i=1}^{n}\kappa_i^2 a_i^2(t)(\lambda+\lambda_i)\mu_{i2} + \sum_{i=1}^{n}\sum_{m\neq i}\kappa_i\kappa_m\lambda\mu_{i1}\mu_{m1}a_i(t)a_m(t) + \sum_{i=1}^{n}\kappa_i^2\sigma_i^2\pi_i^2(t)\\ &+\sum_{i=1}^{n}\sum_{m\neq i}\kappa_i\kappa_m\sigma_i\sigma_m\bar{\rho}_{im}\pi_i(t)\pi_m(t)\Bigg] + \sum_{i=1}^{n}\frac{\theta_i^2(s)}{2\alpha_i} + \sum_{i=1}^{n}\frac{\bar{\theta}_i^2(s)}{2\bar{\alpha}_i}\Bigg\} = 0.\end{aligned} \tag{5.1}$$

(ii) For all $(t,x) \in [0,T]\times R$

$$\begin{aligned}&\bar{h}_t(t,x) + \Bigg[rx + \sum_{i=1}^{n}\kappa_i\eta_i(\lambda+\lambda_i)\mu_{i1} - \sum_{i=1}^{n}\kappa_i\xi_i(1-a_i^*(t))^2(\lambda+\lambda_i)\mu_{i2} + \sum_{i=1}^{n}\kappa_i(\mu_i - r)\pi_i^*(t)\\ &-\sum_{i=1}^{n}\theta_i^*(t)\kappa_i a_i^*(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}} - \sum_{i=1}^{n}\bar{\theta}_i^*(t)\kappa_i\pi_i^*(t)\sigma_i\Bigg]\bar{h}_x(t,x) + \Bigg[\sum_{i=1}^{n}\kappa_i^2 a_i^{*2}(t)(\lambda+\lambda_i)\mu_{i2}\\ &+\sum_{i=1}^{n}\sum_{m\neq i}\kappa_i\kappa_m\lambda\mu_{i1}\mu_{m1}a_i^*(t)a_m^*(t) + \sum_{i=1}^{n}\kappa_i^2\sigma_i^2\pi_i^{*2}(t) + \sum_{i=1}^{n}\sum_{m\neq i}\kappa_i\kappa_m\sigma_i\sigma_m\bar{\rho}_{im}\pi_i^*(t)\pi_m^*(t)\Bigg]\bar{h}_{xx}(t,x)\\ &+\sum_{i=1}^{n}\frac{\theta_i^{*2}(s)}{2\alpha_i} + \sum_{i=1}^{n}\frac{\bar{\theta}_i^{*2}(s)}{2\bar{\alpha}_i} = 0.\end{aligned} \tag{5.2}$$

(iii) For $x \in R$,

$$\bar{V}(T,x) = x, \bar{h}(T,x) = x.$$

Then, $u^*(t)$ is the robust optimal RI strategy and $\bar{V}(T,x)$ is the corresponding OVF, i.e., $\bar{V}(T,x) = \bar{J}(t,x)$.

For notational convenience, we define the following notations:

$$\begin{cases}\breve{\Delta}_i = \left[2\kappa_i\xi_i e^{-r(T-t)} + (\alpha_i+\gamma)\kappa_i^2\right](\lambda+\lambda_i)\mu_{i2},\\ \Upsilon_i = 2\xi_i(\lambda+\lambda_i)\mu_{i2}e^{-r(T-t)}, \bar{\Upsilon}_i = (\mu_i - r)e^{-r(T-t)}.\end{cases} \tag{5.3}$$

By Theorem 5.1, we can obtain the following theorem.

Theorem 5.2. For the wealth process given by (3.8), the solution to the robust stochastic optimization problem (3.9) is as follows. The insurer i's robust ORS is $a_i^*(t) = (0 \vee \acute{a}_i(t)) \wedge 1$, where $\acute{a}_i(t)$ is the solution of the following system of linear equations

$$\begin{cases} \breve{\Delta}_1 a_1(t) + \gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21}a_2(t) + \cdots + \gamma\kappa_1\kappa_n\lambda\mu_{11}\mu_{n1}a_n(t) = \kappa_1\Upsilon_1, \\ \gamma\kappa_2\kappa_1\lambda\mu_{21}\mu_{11}a_1(t) + \breve{\Delta}_2 a_2(t) + \cdots + \gamma\kappa_2\kappa_n\lambda\mu_{21}\mu_{n1}a_n(t) = \kappa_2\Upsilon_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma\kappa_n\kappa_1\lambda\mu_{n1}\mu_{11}a_1(t) + \gamma\kappa_n\kappa_2\lambda\mu_{n1}\mu_{21}a_n(t) + \cdots + \breve{\Delta}_n a_n(t) = \kappa_n\Upsilon_n; \end{cases}, \tag{5.4}$$

the insurers' robust OIS $(\pi_1^*(t), \pi_2^*(t), \cdots, \pi_n^*(t))$ is the solution of the following system of linear equations

$$\begin{cases} \kappa_1^2\sigma_1^2(\bar{\alpha}_1 + \gamma)\pi_1(t) + \gamma\kappa_1\kappa_2\sigma_1\sigma_2\bar{\rho}_{12}\pi_2(t) + \cdots + \gamma\kappa_1\kappa_n\sigma_1\sigma_n\bar{\rho}_{1n}\pi_n(t) = \kappa_1\bar{\Upsilon}_1, \\ \gamma\kappa_2\kappa_1\sigma_2\sigma_1\bar{\rho}_{21}\pi_1(t) + \kappa_2^2\sigma_2^2(\bar{\alpha}_2 + \gamma)\pi_2(t) + \cdots + \gamma\kappa_2\kappa_n\sigma_2\sigma_n\bar{\rho}_{2n}\pi_n(t) = \kappa_2\bar{\Upsilon}_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma\kappa_n\kappa_1\sigma_n\sigma_1\bar{\rho}_{n1}\pi_1(t) + \gamma\kappa_n\kappa_2\sigma_n\sigma_2\bar{\rho}_{n2}\pi_2(t) + \cdots + \kappa_n^2\sigma_n^2(\bar{\alpha}_n + \gamma)\pi_n(t) = \kappa_n\bar{\Upsilon}_n. \end{cases}, \tag{5.5}$$

The market strategies are given by

$$\theta_i(t) = \kappa_i\alpha_i\sqrt{(\lambda+\lambda_i)\mu_{i2}}a_i^*(t)e^{r(T-t)}, \;\; \bar{\theta}_i(t) = \kappa_i\bar{\alpha}_i\sigma_i\pi_i^*(t)e^{r(T-t)}, \; i = 1, 2, \cdots, n. \tag{5.6}$$

The corresponding OVF is given by

$$\bar{V}(t, x) = xe^{r(T-t)} + \frac{\hat{B}(t)}{\gamma}, \tag{5.7}$$

where

$$\begin{aligned} \hat{B}(t) = \gamma\int_t^T &\left\{\left[\sum_{i=1}^n \kappa_i\eta_i(\lambda+\lambda_i)\mu_{i1} - \sum_{i=1}^n \kappa_i\xi_i(1-a_i^*(s))^2(\lambda+\lambda_i)\mu_{i2} + \sum_{i=1}^n \kappa_i(\mu_i - r)\pi_i^*(s)\right]\right. \\ &\times e^{r(T-s)} - \frac{1}{2}\sum_{i=1}^n \kappa_i^2\alpha_i(\lambda+\lambda_i)\mu_{i2}a_i^{*2}(s)e^{2r(T-s)} - \frac{1}{2}\sum_{i=1}^n \kappa_i^2\bar{\alpha}_i\sigma_i^2\pi_i^{*2}(s)e^{2r(T-s)} \\ &- \frac{\gamma}{2}e^{2r(T-s)}\left[\sum_{i=1}^n \kappa_i^2 a_i^{*2}(s)(\lambda+\lambda_i)\mu_{i2} + 2\sum_{m\neq i}\kappa_i\kappa_m\lambda\mu_{i1}\mu_{m1}a_i^*(s)a_m^*(s) + \sum_{i=1}^n \kappa_i^2\sigma_i^2\pi_i^{*2}(s)\right. \\ &\left.\left.+2\sum_{m\neq i}^n \kappa_i\kappa_m\sigma_i\sigma_m\bar{\rho}_{im}\pi_i^*(s)\pi_m^*(s)\right]\right\} ds. \end{aligned} \tag{5.8}$$

Proof. Please see Appendix C. □

In what follows, we solve the robust stochastic optimization problem (3.9) for the two insurers case. We can obtain the following explicit solution.

Theorem 5.3. For the wealth process given by (3.8) with two insurers, the solution to problem (3.9) is as follows. The insurer 1's robust ORS is $a_1^*(t) = (0 \vee \grave{a}_1(t)) \wedge 1$, where $\grave{a}_1(t)$ is given by

$$\grave{a}_1(t) = \frac{\kappa_1\Upsilon_1\breve{\Delta}_2 - \Upsilon_2\gamma\kappa_1\kappa_2^2\lambda\mu_{11}\mu_{21}}{\breve{\Delta}_1\breve{\Delta}_2 - (\gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21})^2}; \tag{5.9}$$

the insurer 1's robust OIS is given by

$$\pi_1^*(t) = \frac{\kappa_1 \bar{\Upsilon}_1 \sigma_2^2(\bar{\alpha}_2 + \gamma) - \bar{\Upsilon}_2 \gamma \kappa_1 \sigma_1 \sigma_2 \bar{\rho}_{12}}{\kappa_1^2 \sigma_1^2 \sigma_2^2 (\bar{\alpha}_1 + \gamma)(\bar{\alpha}_2 + \gamma) - (\gamma \kappa_1 \sigma_1 \sigma_2 \bar{\rho}_{12})^2}; \tag{5.10}$$

the insurer 2's robust ORS is $a_2^*(t) = (0 \vee \grave{a}_2(t)) \wedge 1$, where $\grave{a}_2(t)$ is given by

$$\grave{a}_2(t) = \frac{\kappa_2 \Upsilon_2 \breve{\Delta}_1 - \Upsilon_1 \gamma \kappa_1^2 \kappa_2 \lambda \mu_{11} \mu_{21}}{\breve{\Delta}_1 \breve{\Delta}_2 - (\gamma \kappa_1 \kappa_2 \lambda \mu_{11} \mu_{21})^2}, \tag{5.11}$$

and the insurer 2's robust OIS is given by

$$\pi_2^*(t) = \frac{\kappa_2 \bar{\Upsilon}_2 \sigma_1^2(\bar{\alpha}_1 + \gamma) - \bar{\Upsilon}_1 \gamma \kappa_2 \sigma_1 \sigma_2 \bar{\rho}_{12}}{\kappa_2^2 \sigma_1^2 \sigma_2^2 (\bar{\alpha}_1 + \gamma)(\bar{\alpha}_2 + \gamma) - (\gamma \kappa_2 \sigma_1 \sigma_2 \bar{\rho}_{12})^2}. \tag{5.12}$$

The market strategies are given by

$$\theta_i(t) = \kappa_i \alpha_i \sqrt{(\lambda + \lambda_i)\mu_{i2}}\, a_i^*(t) e^{r(T-t)}, \;\; \bar{\theta}_i(t) = \kappa_i \bar{\alpha}_i \sigma_i \pi_i^*(t) e^{r(T-t)}, \; i = 1, 2. \tag{5.13}$$

The OVF is given by

$$\bar{V}(t, x) = x e^{r(T-t)} + \frac{\hat{B}(t)}{\gamma}, \tag{5.14}$$

where

$$\begin{aligned}
\hat{B}(t) = \gamma \int_t^T \Bigg\{ & \Bigg[\sum_{i=1}^2 \kappa_i \eta_i (\lambda + \lambda_i) \mu_{i1} - \sum_{i=1}^2 \kappa_i \xi_i (1 - a_i^*(s))^2 (\lambda + \lambda_i) \mu_{i2} + \sum_{i=1}^2 \kappa_i (\mu_i - r) \pi_i^*(s) \Bigg] \\
& \times e^{r(T-s)} - \frac{1}{2} \sum_{i=1}^2 \kappa_i^2 \alpha_i (\lambda + \lambda_i) \mu_{i2} a_i^{*2}(s) e^{2r(T-s)} - \frac{1}{2} \sum_{i=1}^2 \kappa_i^2 \bar{\alpha}_i \sigma_i^2 \pi_i^{*2}(s) e^{2r(T-s)} \\
& - \frac{\gamma}{2} e^{2r(T-s)} \Bigg[\sum_{i=1}^2 \kappa_i^2 a_i^{*2}(s)(\lambda + \lambda_i)\mu_{i2} + 2\kappa_1 \kappa_2 \lambda \mu_{11} \mu_{21} a_1^*(s) a_2^*(s) + \sum_{i=1}^2 \kappa_i^2 \sigma_i^2 \pi_i^{*2}(s) \\
& + 2\kappa_1 \kappa_2 \sigma_1 \sigma_2 \bar{\rho}_{12} \pi_1^*(s) \pi_2^*(s) \Bigg] \Bigg\} ds.
\end{aligned} \tag{5.15}$$

Proof. Please see Appendix D. □

In what follows, we discuss two special cases of our model for two cooperative insurers. We can derive the corresponding optimal RI strategies for these cases as we did in Theorem 5.2; the proofs are omitted. The corresponding OVFs can also be similarly obtained; however, the expressions are rather complicated and are thus omitted here.

Corollary 5.1. If $\lambda = 0$, i.e., the interaction arising from the insurance market is not considered, the insurer 1's robust ORS is given by

$$a_1^*(t) = \frac{2\xi_1 e^{-r(T-t)}}{2\xi_1 e^{-r(T-t)} + (\alpha_1 + \gamma)\kappa_1}, \tag{5.16}$$

and the insurer 2's robust ORS is given by

$$a_2^*(t) = \frac{2\xi_2 e^{-r(T-t)}}{2\xi_2 e^{-r(T-t)} + (\alpha_2 + \gamma)\kappa_2}. \tag{5.17}$$

If $\bar{\rho}_{12} = 0$, i.e., the interaction arising from the finance market is not considered, the insurer 1's robust OIS is given by

$$\pi_1^*(t) = \frac{(\mu_1 - r)e^{-r(T-t)}}{\kappa_1\sigma_1^2(\bar{\alpha}_1 + \gamma)}, \tag{5.18}$$

and the insurer 2's robust OIS is given by

$$\pi_2^*(t) = \frac{(\mu_2 - r)e^{-r(T-t)}}{\kappa_2\sigma_2^2(\bar{\alpha}_2 + \gamma)}. \tag{5.19}$$

From (5.16)–(5.19), we can see that insurer 1's (resp. 2's) robust optimal RI strategies decrease with respect to κ_1 (resp. κ_2). In other words, with the increase of the insurer's weight coefficient, insurers are more keen to turn to reinsurers to resist risks and reduce their investment in risky assets. This shows that, as the leader of the group, one should consider the interests of the whole group and take more cautious RI behavior. We find that the OISs given by (5.18) and (5.19) reduce to that in Theorem 3.1 presented by Zhou et al. [45], which considers the joint interests of the insurer and reinsurer. In other words, our model extends the model presented by Zhou et al. [45] to n cooperative insurers.

Corollary 5.2. If $\alpha_i = \bar{\alpha}_i = 0$, i.e., the ambiguity is not considered, the insurer 1's robust ORS is $a_1^*(t) = (0 \vee \tilde{a}_1(t)) \wedge 1$, where $\tilde{a}_1(t)$ is given by

$$\tilde{a}_1(t) = \frac{\kappa_1\Upsilon_1\left(2\kappa_2\xi_2 e^{-r(T-t)} + \gamma\kappa_2^2\right)(\lambda+\lambda_2)\mu_{22} - \kappa_2\Upsilon_2\gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21}}{\left(2\kappa_1\xi_1 e^{-r(T-t)} + \gamma\kappa_1^2\right)\left(2\kappa_2\xi_2 e^{-r(T-t)} + \gamma\kappa_2^2\right)(\lambda+\lambda_1)\mu_{12}(\lambda+\lambda_2)\mu_{22} - (\gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21})^2}; \tag{5.20}$$

the insurer 1's robust OIS is given by

$$\pi_1^*(t) = \frac{e^{-r(T-t)}}{\kappa_1\gamma(1-\bar{\rho}_{12}^2)}\left[\frac{\mu_1 - r}{\sigma_1^2} - \frac{(\mu_2 - r)\bar{\rho}_{12}}{\sigma_1\sigma_2}\right]; \tag{5.21}$$

the insurer 2's robust ORS is $a_2^*(t) = (0 \vee \tilde{a}_2(t)) \wedge 1$, where $\tilde{a}_2(t)$ is given by

$$\tilde{a}_2(t) = \frac{\kappa_2\Upsilon_2\left(2\kappa_1\xi_1 e^{-r(T-t)} + \gamma\kappa_1^2\right)(\lambda+\lambda_1)\mu_{12} - \kappa_1\Upsilon_1\gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21}}{\left(2\kappa_1\xi_1 e^{-r(T-t)} + \gamma\kappa_1^2\right)\left(2\kappa_2\xi_2 e^{-r(T-t)} + \gamma\kappa_2^2\right)(\lambda+\lambda_1)\mu_{12}(\lambda+\lambda_2)\mu_{22} - (\gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21})^2}, \tag{5.22}$$

and the insurer 2's robust OIS is given by

$$\pi_2^*(t) = \frac{e^{-r(T-t)}}{\kappa_2\gamma(1-\bar{\rho}_{12}^2)}\left[\frac{\mu_2 - r}{\sigma_2^2} - \frac{(\mu_1 - r)\bar{\rho}_{12}}{\sigma_1\sigma_2}\right]. \tag{5.23}$$

6. Sensitivity analysis

In this section, we compare the optimal RI strategy and the corresponding OVF for the competition case, the cooperation case and the case without competition and cooperation. Our aim was to find the similarities and differences of RI behaviors for these three cases. Suppose that there are two insurers and two stocks in the insurance and financial markets, respectively. We only report the insurer 1's robust optimal RI strategy and corresponding OVF because the similarity of the analysis and the two insurers are symmetric in terms of properties. The basic parameters are given in Table 1.

Table 1. Values of model parameters in numerical examples.

	r	T	t	λ	$\bar{\rho}_{12}$	γ					
	0.02	10	0	2	0.2	0.7					
Insurer 1	η_1	ξ_1	λ_1	γ_1	α_1	$\bar{\alpha}_1$	τ_1	μ_1	σ_1	$\hat{x}_1$	κ_1
	0.1	0.2	1	1	0.2	0.9	0.7	0.04	0.3	10	0.5
Insurer 2	η_2	ξ_2	λ_2	γ_2	α_2	$\bar{\alpha}_2$	τ_2	μ_2	σ_2	$\hat{x}_2$	κ_2
	0.2	0.25	2	1.2	0.7	0.7	0.5	0.06	0.4	10	0.5

6.1. Numerical results for the robust ORS

In this part, we provide the numerical results for the robust ORS. Specifically, we examine the influences of model parameters on the robust ORS $a_1^*(t)$ for three cases, i.e., the competition case, the cooperation case and the case without competition and cooperation. We assume that the density function of claim size Y^i is e^{-y} for $y > 0$. Therefore, we obtain that $\mu_{11} = \mu_{21} = 1$ and $\mu_{12} = \mu_{22} = 2$.

Figure 1 shows the effect of τ_1 on $a_1^*(t)$ for the competition case. From Figure 1, we find that $a_1^*(t)$ increases with respect to τ_1. A larger τ_1 means that the insurer 1 hopes to surpass the wealth of their competitor, i.e., insurer 2. While taking part in reinsurance can reduce the claim risk, it is nonetheless costly. This is because the insurer 1 needs to pay a reinsurance premium to the reinsurer. In this situation, insurer 1 tends to decrease their reinsurance premium, which increases the competition parameter τ_1, which in turn implies increasing $a_1^*(t)$.

Figure 2 shows the effect of κ_1 on $a_1^*(t)$ for the cooperation case. We find that $a_1^*(t)$ is a decreasing function of κ_1. As κ_1 increases, the status of the insurer 1 in the cooperative group also increases. To keep the cooperation going, the insurer 1 should make a reinsurance decision from the group's perspective. To reduce the risk faced by the entire group, the reinsurance willingness of insurer 1 is enhanced. This causes the retention proportion of reinsurance to decrease.

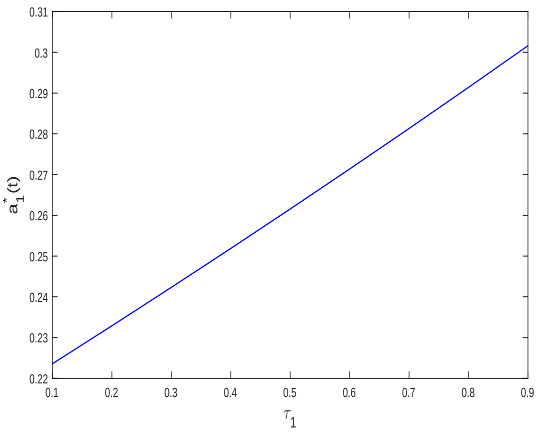


Figure 1. Effect of τ_1 on $a_1^*(t)$.

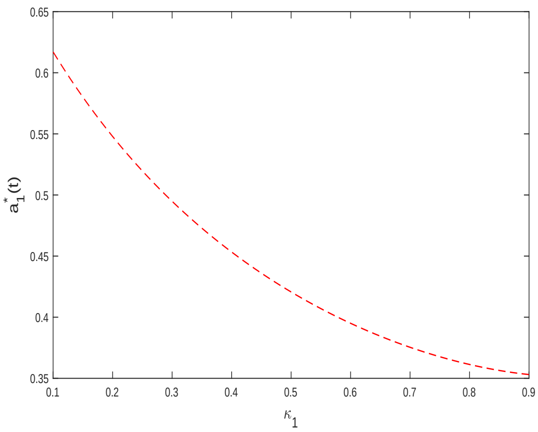


Figure 2. Effect of κ_1 on $a_1^*(t)$.

Figure 3 depicts the effect of ξ_1 on $a_1^*(t)$ for three cases, i.e., the competition case, the cooperation case and the case without competition and cooperation. The numerical result shows that $a_1^*(t)$ is an increasing function of ξ_1. As the parameter ξ_1 increases, the insurer needs to pay more costs to the reinsurer. Therefore, the insurer decreases their demand for reinsurance, i.e., they increase their retention.

Figures 4 and 5 reveal that $a_1^*(t)$ is decreasing about α_1, γ_1 and γ. This is because α_1 represents the ambiguity aversion parameter, and γ_1 and γ represent the risk aversion parameters. With the increase of α_1, γ_1 and γ, the insurer would purchase more reinsurance to transfer claim risk and ambiguity.

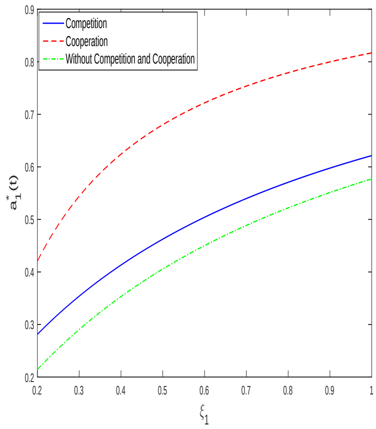


Figure 3. Effect of ξ_1 on $a_1^*(t)$.

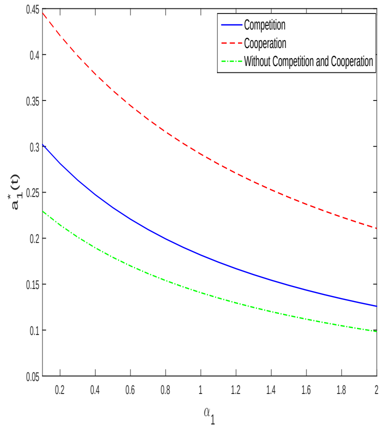


Figure 4. Effect of α_1 on $a_1^*(t)$.

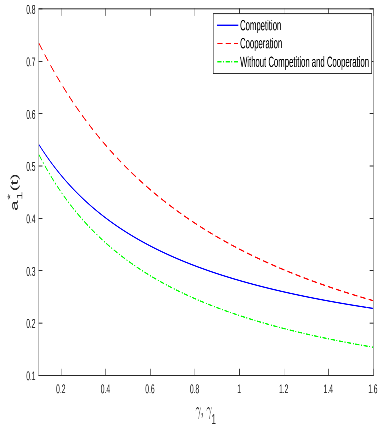


Figure 5. Effects of γ and γ_1 on $a_1^*(t)$.

Further analyzing Figures 3–5, we can see that the insurer 1 keeps more retention in the cooperation case than in the other two cases. This implies that cooperation can improve the insurer's ability to resist claim risk. From Figures 3–5, we also find that the insurer 1 has the weakest ability to resist risk when there is no competition and cooperation. This means that both competition and cooperation can enhance the insurer's ability to resist claim risk.

6.2. Numerical results for the robust OIS

Now, we present the numerical results for the insurer 1's robust OIS $\pi_1^*(t)$ for the competition case, the cooperation case and the case without competition and cooperation.

Figure 6 shows the effect of τ_1 on $\pi_1^*(t)$ for the competition case. From Figure 6, we can see that $\pi_1^*(t)$ increases with respect to τ_1. That is, the more the insurer is concerned about outperforming their competitor, the more money that the insurer would like to invest in the stock. The reason may be because the presence of relative performance, as each insurer desires to perform well relative to their competitor. Investing into the stock has a possibility of rapidly increasing income; hence, the insurer would increase their investments in the stock.

Figure 7 illustrates the effect of κ_1 on $\pi_1^*(t)$ for the cooperation case. It is clear that $\pi_1^*(t)$ is a decreasing function of κ_1. As we explained in Figure 2, as κ_1 increases, the status of the insurer 1 in the cooperative group also increases. Their investment decision should then be made from the group's perspective. To reduce the investment risk faced by the entire group, the insurer 1 invests very little in the stock 1.

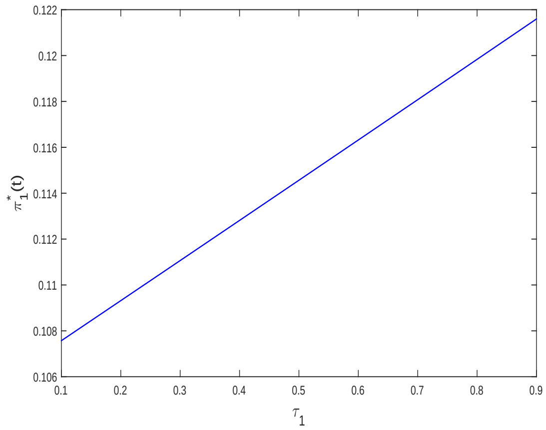


Figure 6. Effect of τ_1 on $\pi_1^*(t)$.

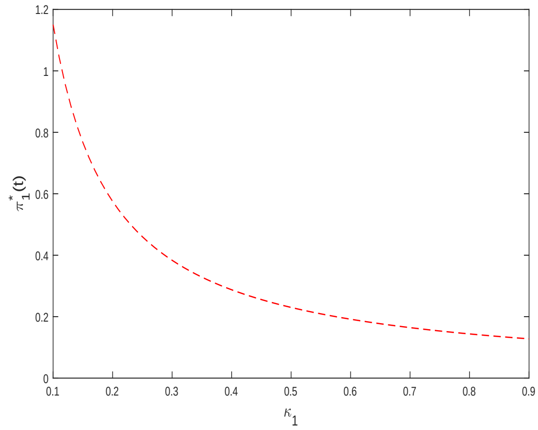


Figure 7. Effect of κ_1 on $\pi_1^*(t)$.

As shown in Figure 8, $\pi_1^*(t)$ is a decreasing function of ambiguity aversion parameter $\bar{\alpha}_1$. Due to the model uncertainty, a larger ambiguity-aversion parameter will force the insurer to reduce their investment in the stock.

Figure 9 describes the effect of r on $\pi_1^*(t)$. As r is the risk-free interest rate, the larger the value of r, the greater the expected income of the risk-free asset, and, hence, the more the insurer will wish to invest in the risk-free asset. Therefore, $\pi_1^*(t)$ decreases with respect to r.

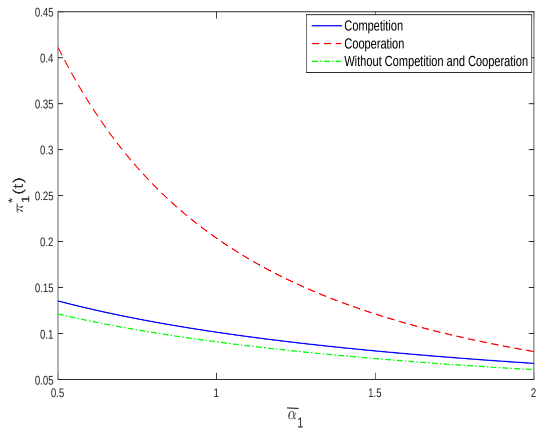


Figure 8. Effect of $\bar{\alpha}_1$ on $\pi_1^*(t)$.

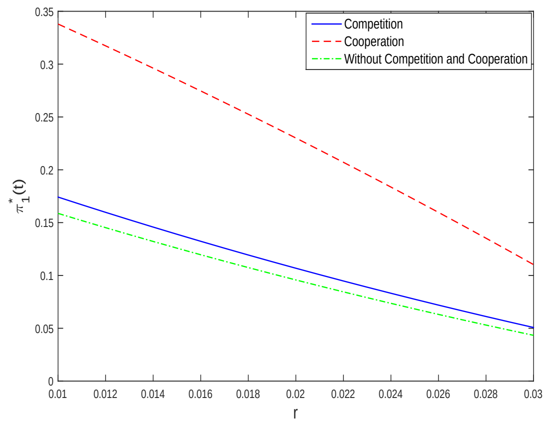


Figure 9. Effect of r on $\pi_1^*(t)$.

Figure 10 provides the effect of μ_1 on $\pi_1^*(t)$. From Figure 10, we can see that $\pi_1^*(t)$ is an increasing function of μ_1. μ_1 stands for the appreciation rate of the stock 1. Therefore, the larger the value of μ_1, the more the insurer will wish to invest in the stock 1.

Figure 11 shows the effect of the volatility σ_1 of the stock 1 on $\pi_1^*(t)$. The larger the value of σ_1, the riskier the stock 1, and, hence, the less the insurer will wish to invest in the stock 1. Therefore, $\pi_1^*(t)$ decreases with respect to σ_1.

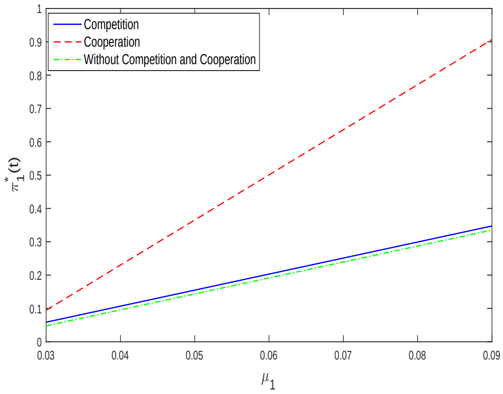


Figure 10. Effect of μ_1 on $\pi_1^*(t)$.

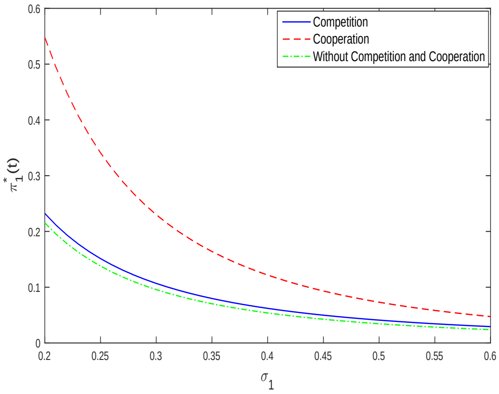


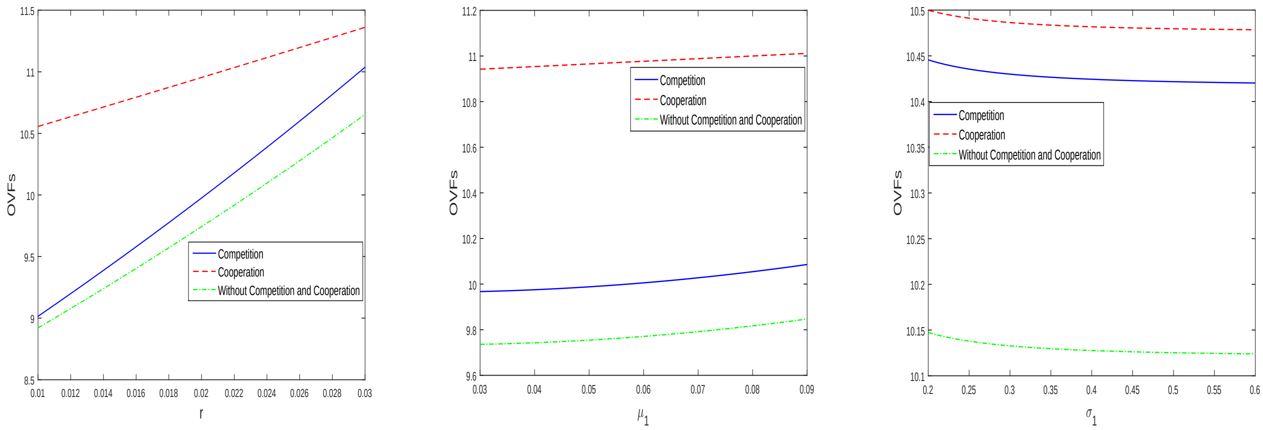
Figure 11. Effect of σ_1 on $\pi_1^*(t)$.

Further analyzing Figures 8–11, we can see that compared with the competition case and the case without competition and cooperation, the insurer tends to invest a higher dollar amount in stock in the cooperation case. This implies that cooperation can improve the insurer's ability to resist investment risk. Compared with the other two cases, the insurer's investment behavior is the most positive in the cooperation case. The insurer has the weakest ability to resist investment risk when there is no competition and cooperation.

6.3. Comparison of the OVFs in three cases

Finally, we compare the optimal value functions among the cooperation case, the competition case and the case without competition and cooperation. We analyze the effects of the financial market model parameters r, μ_1 and σ_1 on the OVFs.

From Figure 12, we can see that the OVF for the cooperation case is larger than that for the competition case and the case without competition and cooperation. This shows that the insurer will get more benefits under the condition of the cooperation case. This is because, in the cooperation case, the RI behaviors of the insurer are more aggressive than what we have derived from the numerical results in Subsections 6.1 and 6.2.



(a) (b) (c)

Figure 12. Effects of r, μ_1 and σ_1 on the OVFs.

7. Conclusions

This article extends the results in the literature from two competitive insurers to $n > 2$ competitive and cooperative insurers. For n insurers, we have considered three aspects of interaction. First, the interaction arises from the claim process, i.e., the correlation coefficient ρ_{im}. Second, the interaction arises from the stock price, i.e., the correlation coefficient $\bar{\rho}_{im}$. Third, the interaction arises from the competition and cooperation, i.e., the competitive weighting coefficient τ_i and the cooperative weighting coefficient κ_i. Furthermore, we have established the new RI models. Under the MV criterion, by applying a stochastic control technique and dynamic programming approach, we derived both the robust optimal RI strategy and the corresponding optimal value function. The numerical experiments demonstrate the effects of model parameters on the robust optimal RI strategy and the OVF for the cooperation case, the competition case and the case without competition and cooperation. Through numerical studies, we have found that the insurers will adopt more active RI decision-making under the condition of cooperation. Both competition and cooperation can improve the insurer's income and the insurer's strength to resist risk; the insurer obtains the most benefits from cooperation. These findings are consistent with our understanding. The results and findings can provide guidance for the insurers making RI decisions.

Use of AI tools declaration

The author declares that he has not used artificial intelligence tools in the creation of this article.

Acknowledgments

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Conflict of interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof of Theorem 4.2.

Proof. First, we rewrite the extended HJB equation (4.6) as

$$\begin{aligned}
&\sup_{u_i\in\mathcal{U}_i}\inf_{\phi_i\in\Phi_i}\Bigg\{\frac{\partial V_i(t,\hat{x}_i)}{\partial t}+\Bigg[r\hat{x}_i+\eta_i(\lambda+\lambda_i)\mu_{i1}-\frac{\tau_i}{n-1}\sum_{m\neq i}\eta_m(\lambda+\lambda_m)\mu_{m1}-\xi_i(1-a_i(t))^2(\lambda+\lambda_i)\mu_{i2}\\
&+\frac{\tau_i}{n-1}\sum_{m\neq i}\xi_m(1-a_m^*(t))^2(\lambda+\lambda_m)\mu_{m2}+(\mu_i-r)\pi_i(t)-\frac{\tau_i}{n-1}\sum_{m\neq i}(\mu_m-r)\pi_m^*(t)-\theta_i(t)a_i(t)\\
&\cdot\sqrt{(\lambda+\lambda_i)\mu_{i2}}+\frac{\tau_i}{n-1}\sum_{m\neq i}\theta_m^*(t)a_m^*(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}}-\bar{\theta}_i(t)\pi_i(t)\sigma_i+\frac{\tau_i}{n-1}\sum_{m\neq i}\bar{\theta}_m^*(t)\pi_m^*(t)\sigma_m\Bigg]\\
&\times\frac{\partial V_i(t,\hat{x}_i)}{\partial \hat{x}_i}+\frac{1}{2}g(u_i,u_m^*)\left[\frac{\partial^2 V_i(t,\hat{x}_i)}{\partial \hat{x}_i^2}-\gamma_i\left(\frac{\partial h_i(t,\hat{x}_i)}{\partial \hat{x}_i}\right)^2\right]+\frac{\theta_i^2(s)}{2\alpha_i}+\frac{\bar{\theta}_i^2(s)}{2\bar{\alpha}_i(s)}\Bigg\}=0.
\end{aligned}\tag{A.1}$$

According to the first-order optimality conditions, the functions $\theta_i(t)$ and $\bar{\theta}_i(t)$, which realize the infimum part of (A.1), are respectively given by

$$\begin{cases}\theta_i^*(t)=a_i(t)\alpha_i\sqrt{(\lambda+\lambda_i)\mu_{i2}}\frac{\partial V_i(t,\hat{x}_i)}{\partial \hat{x}_i},\\ \bar{\theta}_i^*(t)=\pi_i\sigma_i\bar{\alpha}_i\frac{\partial V_i(t,\hat{x}_i)}{\partial \hat{x}_i}.\end{cases}\tag{A.2}$$

Inserting (A.2) into (A.1) yields

$$\begin{aligned}
&\sup_{u_i\in\mathcal{U}_i}\Bigg\{\frac{\partial V_i(t,\hat{x}_i)}{\partial t}+\Bigg[r\hat{x}_i+\eta_i(\lambda+\lambda_i)\mu_{i1}-\frac{\tau_i}{n-1}\sum_{m\neq i}\eta_m(\lambda+\lambda_m)\mu_{m1}-\xi_i(1-a_i(t))^2(\lambda+\lambda_i)\mu_{i2}\\
&+\frac{\tau_i}{n-1}\sum_{m\neq i}\xi_m(1-a_m^*(t))^2(\lambda+\lambda_m)\mu_{m2}+(\mu_i-r)\pi_i(t)-\frac{\tau_i}{n-1}\sum_{m\neq i}(\mu_m-r)\pi_m^*(t)\\
&+\frac{\tau_i}{n-1}\sum_{m\neq i}\theta_m^*(t)a_m^*(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}}+\frac{\tau_i}{n-1}\sum_{m\neq i}\bar{\theta}_m^*(t)\pi_m^*(t)\sigma_m\Bigg]\frac{\partial V_i(t,\hat{x}_i)}{\partial \hat{x}_i}+\frac{1}{2}g(u_i,u_m^*)\\
&\times\left[\frac{\partial^2 V_i(t,\hat{x}_i)}{\partial \hat{x}_i^2}-\gamma_i\left(\frac{\partial h_i(t,\hat{x}_i)}{\partial \hat{x}_i}\right)^2\right]-\frac{1}{2}\left[a_i^2(t)\alpha_i(\lambda+\lambda_i)\mu_{i2}+\pi_i^2(t)\sigma_i^2\bar{\alpha}_i\right]\left(\frac{\partial V_i(t,\hat{x}_i)}{\partial \hat{x}_i}\right)^2\Bigg\}=0.
\end{aligned}\tag{A.3}$$

Similar to Yang et al. [4], we conjecture that the solutions to (A.3) and (4.7) have the following forms:

$$\begin{cases}V_i(t,\hat{x}_i)=A_i(t)\hat{x}_i+\frac{B_i(t)}{\gamma_i},A_i(T)=1,B_i(T)=0,\\ h_i(t,\hat{x}_i)=\bar{A}_i(t)\hat{x}_i+\frac{\bar{B}_i(t)}{\gamma_i},\bar{A}_i(T)=1,\bar{B}_i(T)=0.\end{cases}\tag{A.4}$$

Then the partial derivatives are

$$\begin{cases}\frac{\partial V_i(t,\hat{x}_i)}{\partial t}=A_i'(t)+\frac{B_i'(t)}{\gamma_i},\ \frac{\partial V_i(t,\hat{x}_i)}{\partial \hat{x}_i}=A_i(t),\ \frac{\partial V_i^2(t,\hat{x}_i)}{\partial \hat{x}_i^2}=0,\\ \frac{\partial h_i(t,\hat{x}_i)}{\partial t}=\bar{A}_i'(t)+\frac{\bar{B}_i'(t)}{\gamma_i},\ \frac{\partial h_i(t,\hat{x}_i)}{\partial \hat{x}_i}=\bar{A}_i(t),\ \frac{\partial h_i^2(t,\hat{x}_i)}{\partial \hat{x}_i^2}=0.\end{cases}\tag{A.5}$$

Plugging (A.4) and (A.5) into (A.3), we obtain

$$\begin{aligned}
&\sup_{u_i\in\mathcal{U}_i}\Bigg\{[A_i'(t)+rA_i(t)]\hat{x}_i+\frac{B_i'(t)}{\gamma_i}+A_i(t)\Bigg[\eta_i(\lambda+\lambda_i)\mu_{i1}-\frac{\tau_i}{n-1}\sum_{m\neq i}\eta_m(\lambda+\lambda_m)\mu_{m1}-\xi_i(1-a_i(t))^2\\
&\times(\lambda+\lambda_i)\mu_{i2}+\frac{\tau_i}{n-1}\sum_{m\neq i}\xi_m(1-a_m^*(t))^2(\lambda+\lambda_m)\mu_{m2}+(\mu_i-r)\pi_i(t)-\frac{\tau_i}{n-1}\sum_{m\neq i}(\mu_m-r)\pi_m^*(t)\Bigg]\\
&-\frac{\gamma_i}{2}g(u_i,u_m^*)\bar{A}_i^2(t)-\frac{1}{2}\Big[a_i^2(t)\alpha_i(\lambda+\lambda_i)\mu_{i2}+\pi_i^2(t)\sigma_i^2\bar{\alpha}_i\Big]A_i^2(t)\\
&+\Bigg[\frac{\tau_i}{n-1}\sum_{m\neq i}\theta_m^*(t)a_m^*(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}}+\frac{\tau_i}{n-1}\sum_{m\neq i}\bar{\theta}_m^*(t)\pi_m^*(t)\sigma_m\Bigg]A_i(t)\Bigg\}=0.
\end{aligned}\tag{A.6}$$

Differentiating (A.6) with respect to $a_i(t)$ and $\pi_i(t)$ implies that

$$a_i^*(t)=\frac{2\xi_i(\lambda+\lambda_i)\mu_{i2}A_i(t)+\frac{\lambda\gamma_i\mu_{i1}\tau_i}{n-1}\bar{A}_i^2(t)\sum_{m\neq i}a_m^*\mu_{m1}}{(\lambda+\lambda_i)\mu_{i2}[2\xi_iA_i(t)+\alpha_iA_i^2(t)+\gamma_i\bar{A}_i^2(t)]}\tag{A.7}$$

and

$$\pi_i^*(t)=\frac{(\mu_i-r)A_i(t)+\frac{\gamma_i\sigma_i\tau_i}{n-1}\bar{A}_i^2(t)\sum_{m\neq i}\pi_m^*(t)\sigma_m\bar{\rho}_{im}}{\bar{\alpha}_i\sigma_i^2A_i^2(t)+\gamma_i\sigma_i^2\bar{A}_i^2(t)}.\tag{A.8}$$

Substituting (A.7) and (A.8) into (A.6) and (4.7), respectively, we can derive the following system of ordinary differential equations (ODEs) according to whether it contains $\hat{x}_i$:

$$A_i'(t)+rA_i(t)=0,\ A(T)=0,\tag{A.9}$$

$$\begin{aligned}
&\frac{B_i'(t)}{\gamma_i}+\Bigg[\eta_i(\lambda+\lambda_i)\mu_{i1}-\frac{\tau_i}{n-1}\sum_{m\neq i}\eta_m(\lambda+\lambda_m)\mu_{m1}-\xi_i(1-a_i^*(t))^2(\lambda+\lambda_i)\mu_{i2}\\
&+\frac{\tau_i}{n-1}\sum_{m\neq i}\xi_m(1-a_m^*(t))^2(\lambda+\lambda_m)\mu_{m2}+(\mu_i-r)\pi_i^*(t)-\frac{\tau_i}{n-1}\sum_{m\neq i}(\mu_m-r)\pi_m^*(t)\Bigg]A_i(t)\\
&-\frac{\gamma_i}{2}g(u_i^*,u_m^*)\bar{A}_i^2(t)-\frac{1}{2}[a_i^{*2}(t)\alpha_i(\lambda+\lambda_i)\mu_{i2}+\pi_i^{*2}(t)\sigma_i^2\bar{\alpha}_i]A_i^2(t)+\Bigg[\frac{\tau_i}{n-1}\sum_{m\neq i}\theta_m^*(t)a_m^*(t)\\
&\cdot\sqrt{(\lambda+\lambda_m)\mu_{m2}}+\frac{\tau_i}{n-1}\sum_{m\neq i}\bar{\theta}_m^*(t)\pi_m^*(t)\sigma_m\Bigg]A_i(t)=0,\ B_i(T)=0,
\end{aligned}\tag{A.10}$$

$$\hat{A}_i'(t)+r\hat{A}_i(t)=0,\ \hat{A}(T)=0,\tag{A.11}$$

$$\begin{aligned}
&\frac{\hat{B}_i'(t)}{\gamma_i} + \Bigg[\eta_i(\lambda+\lambda_i)\mu_{i1} - \frac{\tau_i}{n-1}\sum_{m\neq i}\eta_m(\lambda+\lambda_m)\mu_{m1} - \xi_i(1-a_i^*(t))^2(\lambda+\lambda_i)\mu_{i2} \\
&+\frac{\tau_i}{n-1}\sum_{m\neq i}\xi_m(1-a_m^*(t))^2(\lambda+\lambda_m)\mu_{m2} + (\mu_i - r)\pi_i^*(t) - \frac{\tau_i}{n-1}\sum_{m\neq i}(\mu_m - r)\pi_m^*(t)\Bigg]\hat{A}_i(t) \\
&-\frac{1}{2}\left[a_i^{*2}(t)\alpha_i(\lambda+\lambda_i)\mu_{i2} + \pi_i^{*2}(t)\sigma_i^2\bar{\alpha}_i\right]A_i^2(t) \\
&+\left[\frac{\tau_i}{n-1}\sum_{m\neq i}\theta_m^*(t)a_m^*(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}} + \frac{\tau_i}{n-1}\sum_{m\neq i}\bar{\theta}_m^*(t)\pi_m^*(t)\sigma_m\right]\hat{A}_i(t) = 0, \hat{B}_i(T) = 0.
\end{aligned} \tag{A.12}$$

Considering the boundary condition, the solutions to ODEs (A.9)–(A.12) are given by

$$A_i(t) = \hat{A}_i(t) = e^{r(T-t)}, \tag{A.13}$$

$$\begin{aligned}
\hat{B}_i(t) = \gamma_i\int_t^T \Bigg\{&\Bigg[\eta_i(\lambda+\lambda_i)\mu_{i1} - \frac{\tau_i}{n-1}\sum_{m\neq i}\eta_m(\lambda+\lambda_m)\mu_{m1} - \xi_i(1-a_i^*(s))^2(\lambda+\lambda_i)\mu_{i2} \\
&+\frac{\tau_i}{n-1}\sum_{m\neq i}\xi_m(1-a_m^*(s))^2(\lambda+\lambda_m)\mu_{m2} + (\mu_i - r)\pi_i^*(s) - \frac{\tau_i}{n-1}\sum_{m\neq i}(\mu_m - r)\pi_m^*(s) \\
&+\frac{\tau_i}{n-1}\sum_{m\neq i}\theta_m^*(t)a_m^*(t)\sqrt{(\lambda+\lambda_m)\mu_{m2}} + \frac{\tau_i}{n-1}\sum_{m\neq i}\bar{\theta}_{mm}^*(t)\pi_m^*(t)\sigma_m\Bigg]e^{r(T-s)} \\
&-\frac{1}{2}\left[a_i^{*2}(t)\alpha_i(\lambda+\lambda_i)\mu_{i2} + \pi_i^{*2}(s)\sigma_i^2\bar{\alpha}_i\right]e^{2r(T-s)}\Bigg\}ds,
\end{aligned} \tag{A.14}$$

and $B_i(t)$ is given by (4.15).

By substituting (A.13) into (A.7), the insurers' robust ORS is given by

$$a_i^*(t) = \frac{2\xi_i(\lambda+\lambda_i)\mu_{i2} + \frac{\lambda\gamma_i\mu_{i1}\tau_i}{n-1}e^{r(T-t)}\sum_{m\neq i}a_m^*\mu_{m1}}{(\lambda+\lambda_i)\mu_{i2}[2\xi_i + (\alpha_i+\gamma_i)e^{r(T-t)}]}. \tag{A.15}$$

Note that, in (A.15), $i \neq m \in \{1, 2, \cdots, n\}$; (A.15) can be rewritten as the system of linear equations given by (4.10). $\bar{a}_i(t)$ is the solution of (4.10). If $0 < \bar{a}_i(t) < 1$, then the ORS is $a_i^*(t) = \bar{a}_i(t)$. If $\bar{a}_i(t) \leq 0$, the $\sup\{\cdots\}$ in (A.6) with respect to $a_i(t)$ is achieved at the point 0. This is because the function in the interior of $\sup\{\cdots\}$ decreases with respect to $a_i(t)$ in the interval $[0, 1]$. This means that the ORS must be $a_i^*(t) = 0$. Analogously, if $\bar{a}_i(t) \geq 1$, the ORS must be $a_i^*(t) = 1$.

By substituting (A.13) into (A.8), the insurers' robust OIS is given by

$$\pi_i^*(t) = \frac{(\mu_i - r) + \frac{\gamma_i\sigma_i\tau_i}{n-1}e^{r(T-t)}\sum_{m\neq i}\pi_m^*(t)\sigma_m\bar{\rho}_{im}}{(\bar{\alpha}_i+\gamma_i)\sigma_i^2 e^{r(T-t)}}. \tag{A.16}$$

Note that, in (A.16) $i \neq m \in \{1, 2, \cdots, n\}$; (A.16) can be rewritten as the system of linear equations given by (4.11). By substituting (A.13), $a_i^*(t)$ and $\pi_i^*(t)$ into (A.2), the worst-case measures are given by (4.12) and (4.13). This completes the proof. □

Appendix B. Proof of Theorem 4.3.

Proof. For the two-insurers case, the system of ODEs given by (4.10) becomes

$$\begin{cases} \delta_1 a_1^*(t) - \tau_1\gamma_1\lambda\mu_{11}\mu_{21}a_2^*(t) = \beta_1, \\ -\tau_2\gamma_2\lambda\mu_{11}\mu_{21}a_1^*(t) + \delta_2 a_2^*(t) = \beta_2. \end{cases} \tag{B.1}$$

Then, the determinant of the coefficient matrix is given by

$$\begin{aligned}
&\begin{vmatrix} \delta_1 & -\tau_1\gamma_1\lambda\mu_{11}\mu_{21}e^{r(T-t)} \\ -\tau_2\gamma_2\lambda\mu_{11}\mu_{21}e^{r(T-t)} & \delta_2 \end{vmatrix} \\
=&\begin{vmatrix} (\lambda+\lambda_1)\mu_{12}\left[2\xi_1 + (\alpha_1+\gamma_1)e^{r(T-t)}\right] & -\tau_1\gamma_1\lambda\mu_{11}\mu_{21}e^{r(T-t)} \\ -\tau_2\gamma_2\lambda\mu_{11}\mu_{21}e^{r(T-t)} & (\lambda+\lambda_2)\mu_{22}\left[2\xi_2 + (\alpha_2+\gamma_2)e^{r(T-t)}\right] \end{vmatrix} \\
=&(\lambda+\lambda_1)\mu_{12}(\lambda+\lambda_2)\mu_{22}\left[2\xi_1 + (\alpha_1+\gamma_1)e^{r(T-t)}\right]\left[2\xi_2 + (\alpha_2+\gamma_2)e^{r(T-t)}\right] - \gamma_1\gamma_2\tau_1\tau_2 e^{2r(T-t)}(\lambda\mu_{11}\mu_{21})^2 \\
>&\gamma_1\gamma_2\lambda^2 e^{2r(T-t)}[\mu_{12}\mu_{22} - \tau_1\tau_2(\mu_{11})^2(\mu_{21})^2] \geq \gamma_1\gamma_2\lambda^2 e^{2r(T-t)}[\mu_{12}\mu_{22} - (\mu_{11})^2(\mu_{21})^2].
\end{aligned}$$

The last inequality holds because of $\tau_1\tau_2 \in [0,1]$. By the Cauchy-Schwarz inequality, it is clear that $\mu_{12}\mu_{22} - (\mu_{11})^2(\mu_{21})^2 > 0$. This means that the system of linear equations given by (B.1) has a unique root $(\breve{a}_1(t), \breve{a}_2(t))$ which is given by (4.22) and (4.24). From (4.22) and (4.24), it is clear that $\breve{a}_1(t) > 0$ and $\breve{a}_2(t) > 0$; this implies that, if $\breve{a}_1(t) < 1$ and $\breve{a}_2(t) < 1$, then $a_1^*(t) = \breve{a}_1(t)$ and $a_2^*(t) = \breve{a}_2(t)$ are the robust ORSs. Similar to Theorem 4.2, if $\breve{a}_1(t) \geq 1$ and $\breve{a}_2(t) \geq 1$, then $a_1^*(t) = 1$ and $a_2^*(t) = 1$ are the robust ORSs.

In what follows, we obtain the OIS. From the system of linear equations given by (4.11), we have

$$\begin{cases} \bar{\delta}_1\pi_1^*(t) - \tau_1\gamma_1\sigma_1\sigma_2\bar{\rho}_{12}\pi_2^*(t) = \bar{\beta}_1, \\ -\tau_2\gamma_2\sigma_1\sigma_2\bar{\rho}_{12}\pi_1^*(t) + \bar{\delta}_2\pi_2^*(t) = \bar{\beta}_2. \end{cases} \tag{B.2}$$

Then, the determinant of the coefficient matrix is given by

$$\begin{aligned}
&\begin{vmatrix} \bar{\delta}_1 & -\tau_1\gamma_1\sigma_1\sigma_2\bar{\rho}_{12} \\ -\tau_2\gamma_2\sigma_1\sigma_2\bar{\rho}_{12} & \bar{\delta}_2 \end{vmatrix} = \begin{vmatrix} (\gamma_1+\bar{\alpha}_1)\sigma_1^2 & -\tau_1\gamma_1\sigma_1\sigma_2\bar{\rho}_{12} \\ -\tau_2\gamma_2\sigma_1\sigma_2\bar{\rho}_{12} & (\gamma_2+\bar{\alpha}_2)\sigma_2^2 \end{vmatrix} \\
=&(\gamma_1+\bar{\alpha}_1)(\gamma_2+\bar{\alpha}_2)\sigma_1^2\sigma_2^2 - \tau_1\tau_2\gamma_1\gamma_2(\sigma_1\sigma_2\bar{\rho}_{12})^2 \\
>&\gamma_1\gamma_2\sigma_1^2\sigma_2^2[1 - \tau_1\tau_2(\bar{\rho}_{12})^2] \geq \gamma_1\gamma_2\sigma_1^2\sigma_2^2[1 - (\bar{\rho}_{12})^2] \geq 0.
\end{aligned}$$

Therefore, the system of linear equations given by (B.2) has a pair unique roots $(\pi_1^*(t), \pi_2^*(t))$ which is given by (4.23) and (4.25). That is, $\pi_1^*(t)$ and $\pi_2^*(t)$ are the robust OISs.

Inserting (4.22)–(4.25) into (4.12) and (4.13), we have the worst-case measures given by (4.26)–(4.29). Substituting (4.22)–(4.25) into (4.15), we obtain $B_i(t)$ as given by (4.31); substituting (4.22)–(4.25) into (A.14), we obtain $\hat{B}_i(t)$ as given by

$$\begin{aligned}
\hat{B}_i(t) = \gamma_i\int_t^T &\Big\{\Big[\eta_i(\lambda+\lambda_i)\mu_{i1} - \tau_i\eta_m(\lambda+\lambda_m)\mu_{m1} - \xi_i(1-a_i(s))^2(\lambda+\lambda_i)\mu_{i2} \\
&+ \tau_i\xi_m(1-a_m^*(s))^2(\lambda+\lambda_m)\mu_{m2} + (\mu_i - r)\pi_i(s) - \tau_i(\mu_m - r)\pi_m^*(s) + \tau_i\theta_m^*(t)a_m^*(t)\sqrt{(\lambda+\lambda_m)\mu_{i2}} \\
&+\tau_i\bar{\theta}_m^*(t)\pi_m^*(t)\sigma_m\Big]e^{r(T-s)} - \frac{1}{2}\left[a_i^2(t)\alpha_i(\lambda+\lambda_i)\mu_{i2} + \pi_i^2(s)\sigma_i^2\bar{\alpha}_i\right]e^{2r(T-s)}\Big\}ds.
\end{aligned} \tag{B.3}$$

This completes the proof. □

Appendix C. Proof of Theorem 5.2.

Proof. Similar to Theorem 4.2, we assume that the solutions to (5.1) and (5.2) are as follows:

$$\begin{cases} \bar{V}(t,x) = A(t)x + \frac{\hat{B}(t)}{\gamma}, A(T) = 1, B(T) = 0, \\ \bar{h}(t,x) = \bar{A}(t)x + \frac{\tilde{B}(t)}{\gamma}, \bar{A}(T) = 1, \bar{B}(T) = 0. \end{cases} \tag{C.1}$$

Then the partial derivatives are

$$\begin{cases} \bar{V}_t(t,x) = A'(t) + \frac{\hat{B}'(t)}{\gamma}, \bar{V}_x(t,x) = A(t), \bar{V}_{xx}(t,x) = 0, \\ \bar{h}_t(t,x) = \bar{A}'(t) + \frac{\tilde{B}'(t)}{\gamma}, \bar{h}_x(t,x) = \bar{A}_i(t), \bar{h}(t,x)_{xx} = 0. \end{cases} \tag{C.2}$$

Plugging (C.1)–(C.2) into (5.1), we obtain

$$\begin{aligned} \sup_{u\in\mathcal{U}}\inf_{\phi\in\Phi}\Bigg\{&[A'(t)+rA(t)]x + \frac{\hat{B}'(t)}{\gamma} + \Bigg[\sum_{i=1}^{n}\kappa_i\eta_i(\lambda+\lambda_i)\mu_{i1} - \sum_{i=1}^{n}\kappa_i\xi_i(1-a_i(t))^2(\lambda+\lambda_i)\mu_{i2} \\ &+ \sum_{i=1}^{n}\kappa_i(\mu_i - r)\pi_i(t) - \sum_{i=1}^{n}\theta_i(t)\kappa_i a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}} - \sum_{i=1}^{n}\bar{\theta}_i(t)\kappa_i\pi_i(t)\sigma_i\Bigg]A(t) \\ &- \frac{\gamma}{2}\bar{A}^2(t)\Bigg[\sum_{i=1}^{n}\kappa_i^2 a_i^2(t)(\lambda+\lambda_i)\mu_{i2} + \sum_{i=1}^{n}\sum_{m\neq i}\kappa_i\kappa_m\lambda\mu_{i1}\mu_{m1}a_i(t)a_m(t) + \sum_{i=1}^{n}\kappa_i^2\sigma_i^2\pi_i^2(t) \\ &+ \sum_{i=1}^{n}\sum_{m\neq i}^{n}\kappa_i\kappa_m\sigma_i\sigma_m\bar{\rho}_{im}\pi_i(t)\pi_m(t)\Bigg] + \sum_{i=1}^{n}\frac{\theta_i^2(s)}{2\alpha_i} + \sum_{i=1}^{n}\frac{\bar{\theta}_i^2(s)}{2\bar{\alpha}_i}\Bigg\} = 0. \end{aligned} \tag{C.3}$$

Differentiating (C.3) with respect to $\theta_i(t)$ and $\bar{\theta}_i(t)$ implies that

$$\theta_i(t) = \kappa_i\alpha_i a_i(t)\sqrt{(\lambda+\lambda_i)\mu_{i2}}A(t), \quad \bar{\theta}_i(t) = \kappa_i\bar{\alpha}_i\pi_i(t)\sigma_i A(t). \tag{C.4}$$

By substituting (C.4) into (C.3), we obtain

$$\begin{aligned} \sup_{u\in\mathcal{U}}\Bigg\{&[A'(t)+rA(t)]x + \frac{\hat{B}'(t)}{\gamma} + \Bigg[\sum_{i=1}^{n}\kappa_i\eta_i(\lambda+\lambda_i)\mu_{i1} - \sum_{i=1}^{n}\kappa_i\xi_i(1-a_i(t))^2(\lambda+\lambda_i)\mu_{i2} \\ &+ \sum_{i=1}^{n}\kappa_i(\mu_i - r)\pi_i(t)\Bigg]A(t) - \frac{1}{2}\sum_{i=1}^{n}\kappa_i^2\alpha_i(\lambda+\lambda_i)\mu_{i2}A^2(t)a_i^2(t) - \frac{1}{2}\sum_{i=1}^{n}\kappa_i^2\bar{\alpha}_i\sigma_i^2A^2(t)\pi_i^2(t) \\ &- \frac{\gamma}{2}\bar{A}^2(t)\Bigg[\sum_{i=1}^{n}\kappa_i^2 a_i^2(t)(\lambda+\lambda_i)\mu_{i2} + \sum_{i=1}^{n}\sum_{m\neq i}\kappa_i\kappa_m\lambda\mu_{i1}\mu_{m1}a_i(t)a_m(t) + \sum_{i=1}^{n}\kappa_i^2\sigma_i^2\pi_i^2(t) \\ &+ \sum_{i=1}^{n}\sum_{m\neq i}^{n}\kappa_i\kappa_m\sigma_i\sigma_m\bar{\rho}_{im}\pi_i(t)\pi_m(t)\Bigg]\Bigg\} = 0. \end{aligned} \tag{C.5}$$

Differentiating (C.5) with respect to $a_i(t), i = 1, 2, \cdots, n$, we have

$$\begin{aligned} &a_i(t)\left[2\kappa_i\xi_i(\lambda+\lambda_i)\mu_{i2}A(t)+\kappa_i^2\alpha_i(\lambda+\lambda_i)\mu_{i2}A^2(t)+\gamma\kappa_i^2(\lambda+\lambda_i)\mu_{i2}\bar{A}^2(t)\right] \\ &+\gamma\bar{A}^2(t)\sum_{m\neq i}\kappa_i\kappa_m\lambda\mu_{i1}\mu_{m1}a_m(t)=2\kappa_i\xi_i(\lambda+\lambda_i)\mu_{i2}A(t),\ i=1,2,\cdots,n. \end{aligned} \tag{C.6}$$

We assume that the solution of (C.6) is $\acute{a}_i(t), i = 1, 2, \cdots, n$. Similar to what we have explained in Theorem 4.2, the insurer i's ORS is $a_i^*(t) = (0 \vee \acute{a}_i(t)) \wedge 1$.

Differentiating (C.5) with respect to $\pi_i(t), i = 1, 2, \cdots, n$, we obtain the OIS $(\pi_1^*(t), \pi_2^*(t), \cdots \pi_n^*(t))$ as the solution of the system of linear equations given by (5.5).

Substituting $(a_1^*(t), a_2^*(t), \cdots, a_n^*(t))$ and $(\pi_1^*(t), \pi_2^*(t), \cdots, \pi_n^*(t))$ into (C.5), we can obtain the following system of ODEs, according to whether it contains x

$$A'(t) + rA(t) = 0, A(T) = 1, \tag{C.7}$$

$$\begin{aligned} &\frac{\hat{B}'(t)}{\gamma}+\left[\sum_{i=1}^n\kappa_i\eta_i(\lambda+\lambda_i)\mu_{i1}-\sum_{i=1}^n\kappa_i\xi_i(1-a_i^*(t))^2(\lambda+\lambda_i)\mu_{i2}+\sum_{i=1}^n\kappa_i(\mu_i-r)\pi_i^*(t)\right]A(t) \\ &-\frac{1}{2}\sum_{i=1}^n\kappa_i^2\alpha_i(\lambda+\lambda_i)\mu_{i2}A^2(t)a_i^{*2}(t)-\frac{1}{2}\sum_{i=1}^n\kappa_i^2\bar{\alpha}_i\sigma_i^2A^2(t)\pi_i^{*2}(t)-\frac{\gamma}{2}\bar{A}^2(t)\left[\sum_{i=1}^n\kappa_i^2a_i^{*2}(t)(\lambda+\lambda_i)\mu_{i2}\right. \\ &\left.+\sum_{i=1}^n\sum_{m\neq i}\kappa_i\kappa_m\lambda\mu_{i1}\mu_{m1}a_i^*(t)a_m^*(t)+\sum_{i=1}^n\kappa_i^2\sigma_i^2\pi_i^{*2}(t)+\sum_{i=1}^n\sum_{m\neq i}\kappa_i\kappa_m\sigma_i\sigma_m\bar{\rho}_{im}\pi_i^*(t)\pi_m^*(t)\right]=0, \hat{B}(T)=0. \end{aligned} \tag{C.8}$$

Substituting $(a_1^*(t), a_2^*(t), \cdots, a_n^*(t))$ and $(\pi_1^*(t), \pi_2^*(t), \cdots, \pi_n^*(t))$ into (C.4), the optimal market strategies are given by (5.6). Substituting $(a_1^*(t), a_2^*(t), \cdots, a_n^*(t))$ and $(\pi_1^*(t), \pi_2^*(t), \cdots, \pi_n^*(t))$ and $(\theta_1(t), \cdots, \theta_n(t)$ and $(\bar{\theta}_1(t), \cdots, \bar{\theta}_n(t)$ into (5.2), we can derive the following system of ODEs according to whether it contains x

$$\bar{A}'(t) + r\bar{A}(t) = 0, \bar{A}(T) = 1, \tag{C.9}$$

and

$$\begin{aligned} &\frac{\tilde{B}'(t)}{\gamma}+\left[\sum_{i=1}^n\kappa_i\eta_i(\lambda+\lambda_i)\mu_{i1}-\sum_{i=1}^n\kappa_i\xi_i(1-a_i^*(t))^2(\lambda+\lambda_i)\mu_{i2}+\sum_{i=1}^n\kappa_i(\mu_i-r)\pi_i^*(t)\right]\bar{A}(t) \\ &-\frac{1}{2}\sum_{i=1}^n\kappa_i^2\alpha_i(\lambda+\lambda_i)\mu_{i2}A^2(t)a_i^{*2}(t)-\frac{1}{2}\sum_{i=1}^n\kappa_i^2\bar{\alpha}_i\sigma_i^2A^2(t)\pi_i^{*2}(t),\quad \tilde{B}(t)=0. \end{aligned} \tag{C.10}$$

From (C.7) and (C.9), we have

$$A(t) = \bar{A}(t) = e^{r(T-t)}. \tag{C.11}$$

Substituting (C.11) into (C.8), we can obtain $\hat{B}(t)$, which is given by (5.8). Substituting (C.11) into (C.10), $\tilde{B}(t)$ is given by

$$\begin{aligned} \tilde{B}(t)=\gamma\int_t^T&\left\{\left[\sum_{i=1}^n\kappa_i\eta_i(\lambda+\lambda_i)\mu_{i1}-\sum_{i=1}^n\kappa_i\xi_i(1-a_i^*(s))^2(\lambda+\lambda_i)\mu_{i2}+\sum_{i=1}^n\kappa_i(\mu_i-r)\pi_i^*(s)\right]e^{r(T-s)}\right. \\ &\left.-\frac{1}{2}\sum_{i=1}^n\kappa_i^2\alpha_i(\lambda+\lambda_i)\mu_{i2}a_i^{*2}(s)e^{2r(T-s)}-\frac{1}{2}\sum_{i=1}^n\kappa_i^2\bar{\alpha}_i\sigma_i^2\pi_i^{*2}(s)e^{2r(T-s)}\right\}ds. \end{aligned} \tag{C.12}$$

This completes the proof. $\square$

Appendix D. Proof of Theorem 5.3.

Proof. For the two-insurers case, from (5.4), we have

$$\begin{cases} \breve{\Delta}_1 a_1(t) + \gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21}a_2(t) = \kappa_1\Upsilon_1, \\ \gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21}a_1(t) + \breve{\Delta}_2 a_2(t) = \kappa_2\Upsilon_2. \end{cases} \tag{D.1}$$

Then, the determinant of the coefficient matrix is given by

$$\begin{aligned} &\begin{vmatrix} \breve{\Delta}_1 & \gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21} \\ \gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21} & \breve{\Delta}_2 \end{vmatrix} \\ =&\begin{vmatrix} [2\kappa_1\xi_1 e^{-r(T-t)} + (\alpha_1+\gamma)\kappa_1^2](\lambda+\lambda_1)\mu_{12} & \gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21} \\ \gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21} & [2\kappa_2\xi_2 e^{-r(T-t)} + (\alpha_2+\gamma)\kappa_2^2](\lambda+\lambda_2)\mu_{22} \end{vmatrix} \\ =&[2\kappa_1\xi_1 e^{-r(T-t)} + (\alpha_1+\gamma)\kappa_1^2][2\kappa_2\xi_2 e^{-r(T-t)} + (\alpha_2+\gamma)\kappa_2^2](\lambda+\lambda_1)\mu_{12}(\lambda+\lambda_2)\mu_{22} - (\gamma\kappa_1\kappa_2\lambda\mu_{11}\mu_{21})^2 \\ >&(\gamma\kappa_1\kappa_2\lambda)^2(\mu_{12}\mu_{22} - \mu_{11}^2\mu_{21}^2). \end{aligned}$$

By the Cauchy-Schwarz inequality, it is clear that $\mu_{12}\mu_{22} - \mu_{11}^2\mu_{21}^2 > 0$. This means that (D.1) has a unique root $(\grave{a}_1(t), \grave{a}_2(t))$, which is given by (5.9) and (5.11), respectively. Similar to what we have explained in Theorem 4.2, the ORSs of insurer 1 and 2 are respectively given by $a_1^*(t) = (0 \vee \grave{a}_1(t)) \wedge 1$ and $a_2^*(t) = (0 \vee \grave{a}_2(t)) \wedge 1$.

From (5.5) we have

$$\begin{cases} \kappa_1^2\sigma_1^2(\bar{\alpha}_1+\gamma)\pi_1(t) + \gamma\kappa_1\kappa_2\sigma_1\sigma_2\bar{\rho}_{12}\pi_2(t) = \kappa_1(\mu_1 - r)e^{-r(T-t)}, \\ \gamma\kappa_1\kappa_2\sigma_1\sigma_1\bar{\rho}_{12}\pi_1(t) + \kappa_2^2\sigma_2^2(\bar{\alpha}_2+\gamma)\pi_2(t) = \kappa_2(\mu_2 - r)e^{-r(T-t)}. \end{cases} \tag{D.2}$$

Then, the determinant of the coefficient matrix is given by

$$\begin{aligned} &\begin{vmatrix} \kappa_1^2\sigma_1^2(\bar{\alpha}_1+\gamma) & \gamma\kappa_1\kappa_2\sigma_1\sigma_2\bar{\rho}_{12} \\ \gamma\kappa_1\kappa_2\sigma_1\sigma_2\bar{\rho}_{12} & \kappa_2^2\sigma_2^2(\bar{\alpha}_2+\gamma) \end{vmatrix} \\ =&\kappa_1^2\sigma_1^2(\bar{\alpha}_1+\gamma)\kappa_2^2\sigma_2^2(\bar{\alpha}_2+\gamma) - (\gamma\kappa_1\kappa_2\sigma_1\sigma_2\bar{\rho}_{12})^2 \\ >&(\gamma\kappa_1\kappa_2\sigma_1\sigma_2)^2(1-\bar{\rho}_{12}^2) \geq 0. \end{aligned}$$

Therefore, (D.2) has a unique root $(\pi_1^*(t), \pi_2^*(t))$, which is given by (5.10) and (5.12). Substituting (5.9)–(5.12) into (5.6), the optimal market strategies are given by (5.13). By substituting (5.9)–(5.13) into (5.8), we can obtain $\hat{B}(t)$ as given by (5.15). This completes the proof. $\square$

References

1. T. Björk, A. Murgoci, A General Theory of Markovian Time Inconsistent Stochastic Control Problems, Working Paper, Stockholm School of Economics, 2010. https://doi.org/10.2139/ssrn.1694759

2. S. Basak, G. Chabakauri, Dynamic mean-variance asset allocation, *Rev. Financ. Stud.*, **23** (2010), 2970–3016. https://doi.org/10.1093/rfs/hhq028
3. P. Yang, Time-consistent mean-variance reinsurance-investment in a jump-diffusion financial market, *Optimization*, **66** (2017), 737–758. https://doi.org/10.1080/02331934.2017.1296837
4. P. Yang, Z. Chen, L. Wang, Time-consistent reinsurance and investment strategy combining quota-share and excess of loss for meanvariance insurers with jump-diffusion price process, *Commun. Stat. Theor. M.*, **50** (2021), 2546–2568. https://doi.org/10.1080/03610926.2019.1670849
5. Y. Wang, Y. Deng, Y. Huang, J. Zhou, X. Xiang, Optimal reinsurance-investment policies for insurers with mispricing under mean-variance criterion, *Commun. Stat. Theor. M.*, **51** (2022), 5653–5680. https://doi.org/10.1080/03610926.2020.1844239
6. C. Zhang, Z. Liang, Optimal time-consistent reinsurance and investment strategies for a jump-diffusion financial market without cash, *N. Am. J. Econ. Financ.*, **59** (2022), Article ID 101578. https://doi.org/10.1016/j.najef.2021.101578
7. L. Wang, M. C. Chiu, H. Y. Wong, Time-consistent mean-variance reinsurance-investment problem with long-range dependent mortality rate, *Scand. Actuar. J.*, **2023** (2023b), 123–152. https://doi.org/10.1080/03461238.2022.2089050
8. A. Bensoussan, C. C. Siu, S. C. P. Yam, H. L. Yang, A class of non-zero-sum stochastic diffrential investment and reinsurance games, *Automatica*, **50** (2014), 2025–2037. https://doi.org/10.1016/j.automatica.2014.05.033
9. C. C. Siu, S. C. P. Yam, H. Yang, H. Zhao, A class of nonzero-sum investment and reinsurance games subject to systematic risks, *Scand. Actuar. J.*, **2017** (2017), 670–707. https://doi.org/10.1080/03461238.2016.1228542
10. C. Deng, X. Zeng, H. Zhu, Non-zero-sum stochastic differential reinsurance and investment games with default risk, *Eur. J. Oper. Res.*, **264** (2018), 1144–1158. https://doi.org/10.1016/j.ejor.2017.06.065
11. D. Hu, H. Wang, Time-consistent investment and reinsurance under relative performance concerns, *Commun. Stat. Theor. M.*, **47** (2018), 1693–1717. https://doi.org/10.1080/03610926.2017.1324987
12. H. Zhu, M. Cao, C. K. Zhang, Time-consistent investment and reinsurance strategies for mean-variance insurers with relative performance concerns under the Heston model, *Financ. Res. Lett.*, **30** (2019), 280–291. https://doi.org/10.1016/j.frl.2018.10.009
13. Y. Bai, Z. Zhou, H. Xiao, R. Gao, F. Zhong, A hybrid stochastic differential reinsurance and investment game with bounded memory, *Eur. J. Oper. Res.*, **296** (2022), 717–737. https://doi.org/10.1016/j.ejor.2021.04.046
14. X. Dong, X. Rong, H. Zhao, Non-zero-sum reinsurance and investment game with non-trivial curved strategy structure under Ornstein-Uhlenbeck process, *Scand. Actuar. J.*, **2023** (2023), 565–597. https://doi.org/10.1080/03461238.2022.2139631
15. P. Yang, Z. Chen, Y. Xu, Time-consistent equilibrium reinsurance-investment strategy for n competitive insurers under a new interaction mechanism and a general investment framework, *J. Comput. Appl. Math.*, **374** (2020), Article ID 112769. https://doi.org/10.1016/j.cam.2020.112769
16. G. Guan, X. Hu, Time-consistent investment and reinsurance strategies for mean-variance insurers in N-agent and mean-field games, *N. Am. Actuar J.*, **26** (2022), 537–569. https://doi.org/10.1080/10920277.2021.2014891

17. E. W. Anderson, L. P. Hansen, T. J. Sargent, A quartet of semigroups for model specification, robustness, prices of risk, and model detection, *J. Eur. Econ. Assoc.*, **1** (2003), 68–123. https://doi.org/10.1162/154247603322256774
18. R. Uppal, T. Wang, Model misspecification and underdiversification, *J. Finance*, **58** (2003), 2465–2486. https://doi.org/10.1046/j.1540-6261.2003.00612.x
19. P. J. Maenhout, Robust portfolio rules and asset pricing, *Rev. Financ. Stud.*, **17** (2004), 951–983. https://doi.org/10.1093/rfs/hhh003
20. B. Li, D. Li, D. Xiong, Alpha-robust mean-variance reinsurance-investment strategy, *J. Econ. Dyn. Control.*, **70** (2016), 101–123. https://doi.org/10.1016/j.jedc.2016.07.001
21. Y. Zeng, D. Li, A. Gu, Robust equilibrium reinsurance-investment strategy for a mean-variance insurer in a model with jumps, *Insur. Math. Econ.,* **66** (2016), 138–152. https://doi.org/10.1016/j.insmatheco.2015.10.012
22. D. Li, Y. Zeng, H. Yao, Robust optimal excess-of-loss reinsurance and investment strategy for an insurer in a model with jumps, *Scand. Actuar. J.*, **2017** (2017), 1–27. https://doi.org/10.1080/03461238.2017.1309679
23. C. C. Pun, Robust time-inconsistent stochastic control problems, *Automatica*, **94** (2018), 249–257. https://doi.org/10.1016/j.automatica.2018.04.038
24. Z. Chen, P. Yang, Robust optimal reinsurance-investment strategy with price jumps and correlated claims, *Insur. Math. Econ.*, **92** (2020), 27–46. https://doi.org/10.1016/j.insmatheco.2020.03.001
25. X. Peng, W. Wang, Optimal investment and risk control for an insurer under inside information, *Insur. Math. Econ.*, **69** (2016), 104–116. https://doi.org/10.1016/j.insmatheco.2016.04.008
26. P. Yang, Closed-loop equilibrium reinsurance-investment strategy with insider information and default risk, *Math. Probl. Eng.*, **2021** (2021), Article ID 8873473. https://doi.org/10.1155/2021/8873473
27. X. Peng, F. Chen, W. Wang, Robust optimal investment and reinsurance for an insurer with inside information, *Insur. Math. Econ.*, **96** (2021), 15–30. https://doi.org/10.1016/j.insmatheco.2020.10.004
28. W. Jiang, Z. Yang, Optimal robust insurance contracts with investment strategy under variance premium principle, *Math. Control Relat. F.*, (2023). https://doi.org/10.3934/mcrf.2023001
29. N. Wang, N. Zhang, Z. Jin, L. Qian, Robust non-zero-sum investment and reinsurance game with default risk, *Insur. Math. Econ.*, **84** (2019), 115–132. https://doi.org/10.1016/j.insmatheco.2018.09.009
30. P. Yang, Robust optimal reinsurance strategy with correlated claims and competition, *AIMS Math.*, **8** (2023), 15689–15711. https://doi.org/10.3934/math.2023801
31. I. Karatzas, S. E. Shreve, Brownian motion and stochastic calculus, Springer-Verlag, New York, 1988. https://doi.org/10.1007/978-1-4684-0302-2
32. Z. Chen, P. Yang, Y. Gan, Optimal reinsurance and investment with a common shock and a random exit time, *Rairo-Oper. Res.*, **57** (2023), 881–903.https://doi.org/10.1051/ro/2023036
33. L. Xu, L. Wang, X. Liu, H. Wang, Optimal active lifetime investment, *Int. J. Control*, **96** (2023), 48–57. https://doi.org/10.1080/00207179.2021.1979252

34. J. Grandell, Aspects of risk theory, Springer-Verlag, New York, 1991. https://doi.org/10.1007/978-1-4613-9058-9
35. S. Li, W. Yuan, P. Chen, Optimal control on investment and reinsurance strategies with delay and common shock dependence in a jump-diffusion financial market, *J. Ind. Manag. Optim.*, (2022). https://doi.org/10.3934/jimo.2022068
36. J. Ma, H. Zha, X. Rong, Optimal investment strategy for a DC pension plan with mispricing under the Heston model, *Commun. Stat.-Theor. M.*, **49** (2020), 3168–3183. https://doi.org/10.1080/03610926.2019.1586938
37. Z. Liu, Y. Wang, Y. Huang, J. Zhou, Optimal portfolios for the DC pension fund with mispricing under the HARA utility framework, *J. Ind. Manag. Optim.*, **19** (2023), 1262–1281. https://doi.org/10.3934/jimo.2021228
38. J. Kang, M. Wang, N. Huang, Equilibrium strategy for mean-variance-utility portfolio selection under Heston's SV model, *J. Comput. Appl. Math.*, **392** (2021), Article ID 113490. https://doi.org/10.1016/j.cam.2021.113490
39. W. Wang, D. Muravey, Y. Shen, Y. Zeng, Optimal investment and reinsurance strategies under 4/2 stochastic volatility mode, *Scand. Actuar. J.*, **2023** (2023a), 413–449. https://doi.org/10.1080/03461238.2022.2108335
40. L. Xu, S. Xu, D. Yao, Maximizing expected terminal utility of an insurer with high gain tax by investment and reinsurance, *Comput. Math. Appl.*, **79** (2020), 716–734. https://doi.org/10.1016/j.camwa.2019.07.023
41. G. E. Espinosa, N. Touzi, Optimal investment under relative performance concerns, *Math. Finace.*, **25** (2015), 221–257. https://doi.org/10.1111/mafi.12034
42. T. Björk, A. Murgoci, X. Y. Zhou, Mean-variance portfolio optimization with state dependent risk aversion, *Math. Finance*, **24** (2014), 1–24. https://doi.org/10.1111/j.1467-9965.2011.00515.x
43. B. Yi, Z. Li, F. G. Viens, Y. Zeng, Robust optimal control for an insurer with reinsurance and investment under Heston's stochastic volatility model, *Insur. Math. Econ.*, **53** (2013), 601–614. https://doi.org/10.1016/j.insmatheco.2013.08.011
44. Y. Zeng, Z. Li, Optimal time-consistent investment and reinsurance policies for mean-variance insurers, *Insur. Math. Econ.*, **49** (2011), 145–154. https://doi.org/10.1016/j.insmatheco.2011.01.001
45. J. Zhou, X. Yang, Y. Huang, Robust optimal investment and proportional reinsurance toward joint interests of the insurer and the reinsurer, *Commun. Stat. Theor. M.*, **46** (2017), 10733–10757. https://doi.org/10.1080/03610926.2016.1242734

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