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Research article

On concurrent vector fields on Riemannian manifolds

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Abstract: It is shown that the presence of a non-zero concurrent vector field on a Riemannian manifold poses an obstruction to its topology as well as certain aspects of its geometry. It is shown that on a compact Riemannian manifold, there does not exist a non-zero concurrent vector field. Also, it is shown that a Riemannian manifold of non-zero constant scalar curvature does not admit a non-zero concurrent vector field. It is also shown that a non-zero concurrent vector field annihilates de-Rham Laplace operator. Finally, we find a characterization of a Euclidean space using a non-zero concurrent vector field on a complete and connected Riemannian manifold.

Keywords: concurrent vector fields; Euclidean spaces; de-Rham Laplace operator; Ricci curvature
Mathematics Subject Classification: 53C15, 53C40

1. Introduction

Recall that if the holonomy group of a Riemannian manifold (M, g) leaves invariant a point of M, then there exists a vector field ξ on M that satisfies

$$\nabla_T \xi = T, \quad T \in \mathfrak{X}(M), \tag{1.1}$$

where ∇ is the Riemannian connection and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M (cf. [12]). The vector field ξ satisfying (1.1) is called a concurrent vector field on the Riemannian manifold (M, g). The influence of a concurrent vector field on the geometry of a Riemannian manifold has been the subject of interest to many mathematicians (cf. [1, 3, 6–9, 11–14]). In [13], the authors classified immersed submanifolds of the Euclidean space whose tangential component of the position vector field is a concurrent vector field. In [3], the authors studied Ricci solitons whose potential field is a concurrent vector field and have classified such Ricci solitons (cf. Theorem 3.1). Concurrent vector fields are not only important in geometry, they also have significance in physics (cf. [2, 14, 15]).

In this short note, we are interested in two questions. The first is to study the obstruction to the topology of a Riemannian manifold (M, g) possessing a concurrent vector field. In our finding, we discover that a Riemannian manifold that admits a non-zero concurrent vector field cannot be compact (cf. Theorem 3.1). Note that on the Euclidean space $(\mathbf{R}^n, \langle , \rangle)$ the position vector field is a non-zero concurrent vector field. We seek the question: does a Riemannian manifold with non-zero constant scalar curvature admit a non-zero concurrent vector field? We show in Theorem 3.2 that there does not exist a non-zero concurrent vector field on a Riemannian manifold of non-zero constant scalar curvature. An interesting question in geometry is to find different characterizations of the Euclidean sapce. The second question we consider is, can we characterize a Euclidean space using a non-zero concurrent vector field? Finally, we find a characterization of a Euclidean space using a non-zero concurrent vector field on a complete and connected Riemannian manifold (cf. Theorem 4.1).

2. Preliminaries

Let ξ be a concurrent vector field on an n-dimensional Riemannian manifold (M, g). Then, using Eq (1.1), we have the following expression for the curvature tensor vector field

$$R(T_1, T_2)\xi = \nabla_{T_1}\nabla_{T_2}\xi - \nabla_{T_2}\nabla_{T_1}\xi - \nabla_{[T_1,T_2]}\xi = 0, \quad T_1, T_2 \in \mathfrak{X}(M). \tag{2.1}$$

For a local orthonormal frame $\{t_1, ..., t_n\}$, the Ricci tensor of (M, g) is given by

$$Ric\,(T_1, T_2) = \sum_{i=1}^{n} g\,(R\,(t_i, T_1)\,T_2, t_i)\,.$$

Therefore, using Eq (2.1), we have

$$Ric\,(T, \xi) = 0, \quad T \in \mathfrak{X}(M), \tag{2.2}$$

that is,

$$Ric\,(\xi, \xi) = 0. \tag{2.3}$$

The Ricci operator S of the Riemannian manifold (M, g) is a symmetric operator given by

$$Ric\,(T_1, T_2) = g\,(S\,T_1, T_2)\,, \quad T_1, T_2 \in \mathfrak{X}(M)$$

and the scalar curvature τ of (M, g) is given by

$$\tau = \sum_{i=1}^{n} Ric\,(t_i, t_i) = TrS\,.$$

The gradient $\nabla\tau$ of the scalar curvature τ satisfies

$$\frac{1}{2}\nabla\tau = \sum_{i=1}^{n} (\nabla S)\,(t_i, t_i)\,, \tag{2.4}$$

where $(\nabla S)\,(T_1, T_2) = \nabla_{T_1} S\,T_2 - S\,(\nabla_{T_1} T_2), \quad T_1, T_2 \in \mathfrak{X}(M)$.

3. Obstruction to existence of concurrent vector fields

In this section, we find Riemannian manifolds on which concurrent vector fields does not exist. It is interesting to know that the topology of a Riemannian manifold (M, g) obstructs the existence of a concurrent vector field on (M, g).

Theorem 3.1. *There does not exist a non-zero concurrent vector field on an n-dimensional compact Riemannian manifold* (M, g)*,* $n > 1$.

Proof. Let (M, g) be an n-dimensional compact Riemannian manifold, $n > 1$. Suppose (M, g) admits a non-zero concurrent vector field ξ. Then, using Eq (1.1), we see that the Lie derivative of g with respect to ξ is given by

$$\left(\pounds_{\xi} g\right)(T_1, T_2) = 2g\left(T_1, T_2\right), \quad T_1, T_2 \in \mathfrak{X}(M). \tag{3.1}$$

Choosing a local orthonormal frame $\{t_1, ..., t_n\}$, using Eqs (1.1) and (3.1), we get

$$div\xi = \sum_{i=1}^{n} g\left(\nabla_{t_i}\xi, t_i\right) = n, \tag{3.2}$$

$$\|\nabla\xi\|^2 = \sum_{i=1}^{n} g\left(\nabla_{t_i}\xi, \nabla_{t_i}\xi\right) = n \tag{3.3}$$

and

$$\left|\pounds_{\xi} g\right|^2 = \sum_{i,j=1}^{n} \left(\left(\pounds_{\xi} g\right)\left(t_i, t_j\right)\right)^2 = 4n. \tag{3.4}$$

Using the integral formula for compact Riemannian manifold (M, g) (cf. [14])

$$\int_M \left(Ric\left(\xi, \xi\right) + \frac{1}{2}\left|\pounds_{\xi} g\right|^2 - \|\nabla\xi\|^2 - (div\xi)^2\right) = 0$$

and Eqs (2.3) and (3.2)–(3.4), we get

$$\int_M n\left(n-1\right) = 0,$$

a contradiction. Hence, there does not exist a non-zero concurrent vector field on a compact Riemannian manifold. □

Note that the Euclidean space $(\mathbf{R}^n, \langle , \rangle)$ is a space of constant scalar curvature ($\tau = 0$) and admits a non-zero concurrent vector field. A natural question is, whether a Riemannian manifold (M, g) of non-zero constant scalar curvature τ admits a non-zero concurrent vector field. Certainly by virtue of the above theorem, we see that the Riemannian product $\mathbf{S}^m(c_1) \times \mathbf{S}^l(c_2)$ does not admit a non-zero concurrent vector field. However, there are many Riemannian manifolds of non-zero constant scalar curvature such as the Riemannian product $\mathbf{S}^l(c) \times \mathbf{R}^m$ and certain warped product manifolds. In this respect we have the following:

Theorem 3.2. *There does not exist a non-zero concurrent vector field on an n-dimensional Riemannian manifold* (M, g)*, of non-zero constant scalar curvature.*

Proof. Let (M, g) be an n-dimensional Riemannian manifold of constant scalar curvature $\tau \neq 0$. Suppose (M, g) admits a non-zero concurrent vector field ξ. Then, using Eq (2.2), we see that the Ricci operator S satisfies

$$S(\xi) = 0. \tag{3.5}$$

Using a local orthonormal frame $\{t_1, ..., t_n\}$ and symmetry of the Ricci operator S, Eqs (1.1) and (2.4), we have

$$\begin{aligned} div(S\xi) &= \sum_{i=1}^{n} g(\nabla_{t_i} S\xi, t_i) = \sum_{i=1}^{n} (t_i g(S\xi, t_i) - g(S\xi, \nabla_{t_i} t_i)) \\ &= \sum_{i=1}^{n} (t_i g(\xi, S t_i) - g(\xi, S\nabla_{t_i} t_i)) \\ &= \sum_{i=1}^{n} (g(\nabla_{t_i}\xi, S t_i) + g(\xi, \nabla_{t_i} S t_i) - g(\xi, S(\nabla_{t_i} t_i))) \\ &= \tau + g\left(\xi, \frac{1}{2}\nabla\tau\right). \end{aligned}$$

Using Eq (3.5) and the fact that τ is a constant in above equation, we get $\tau = 0$, a contradiction. Hence, there does not exist a non-zero concurrent vector field on a Riemannian manifold of non-zero constant scalar curvature. □

Note an n-dimensional Einstein manifold $n > 2$ has constant scalar curvature. Consequently, by virtue of the above result, we have the following:

Corollary. *On an n-dimensional Einstein manifold $n > 2$ with non-zero Ricci curvature, there does not exist a non-zero concurrent vector field.*

One of the important operators on a Riemannian manifold (M, g) is the de-Rham Laplace operator $\square : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by (cf. [4], p. 83)

$$\square = S + \Delta, \tag{3.6}$$

where $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the rough Laplace operator defined by

$$\Delta T = \sum_{i=1}^{n} \left(\nabla_{t_i}\nabla_{t_i} T - \nabla_{\nabla_{t_i} t_i} T\right), \quad T \in \mathfrak{X}(M). \tag{3.7}$$

This operator is used for obtaining a characterization of a Killing vector field on a Riemannian manifold. In that the Kernal of the operator $Ker\square$ defined by

$$Ker\square = \{T \in \mathfrak{X}(M) : \square T = 0\}$$

is an important subspace of $\mathfrak{X}(M)$ (cf. [5]).

Recall that a subspace V of $\mathfrak{X}(M)$ is said to be a nontrivial space if $\dim V \geq 1$. Therefore, a subspace W of $\mathfrak{X}(M)$ is trivial if $W = \{0\}$. We have the following:

Theorem 3.3. *Let ξ be a non-zero concurrent vector field on an n-dimensional Riemannian manifold (M,g). Then, $Ker\square$ is nontrivial subspace of $\mathfrak{X}(M)$.*

Proof. Let (M,g) be an n-dimensional Riemannian manifold and $\xi \neq 0$ be the concurrent vector field on (M,g). Then, using Eqs (1.1) and (3.7), we have

$$\Delta\xi = \sum_{i=1}^{n}\left(\nabla_{t_i}\nabla_{t_i}\xi - \nabla_{\nabla_{t_i}t_i}\xi\right) = 0.$$

Thus, using Eqs (3.5) and (3.6) and above equation, we have

$$\square\xi = 0,$$

that is, $\xi \in Ker\square$. Since $\xi \neq 0$, it proves that $Ker\square$ is a nontrivial subspace. □

4. Characterizing Euclidean spaces by concurrent vector fields

We know that the Euclidean space $(\mathbf{R}^n, \langle,\rangle)$ possesses a non-zero concurrent vector field $\xi = \Psi$, where

$$\Psi = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i}$$

is the position vector field on $\mathbf{R}^n$. One of interesting questions in differential geometry is to find a characterization of the spaces, the Euclidean sphere $\mathbf{S}^n(c)$ and the Euclidean space $(\mathbf{R}^n, \langle,\rangle)$. In this section, we find a characterization of the Euclidean space using a non-zero concurrent vector field. Indeed we prove the following:

Theorem 4.1. *An n-dimensional complete and connected Riemannian manifold (M,g), $n > 1$ admits a non-zero concurrent vector field, if and only if it is isometric to the Euclidean space $(\mathbf{R}^n, \langle,\rangle)$.*

Proof. Let (M,g) be an n-dimensional complete and connected Riemannian manifold and $\xi \neq 0$ be a concurrent vector field on (M,g). Define the function $f : M \to R$ by

$$f = \frac{1}{2} g(\xi,\xi).$$

Then, using Eq (1.1), we have

$$T(f) = g(T,\xi), \quad T \in \mathfrak{X}(M),$$

which gives the gradient ∇f of f as

$$\nabla f = \xi. \tag{4.1}$$

By Eq (4.1), f is a non-constant function, owing to the fact $\xi \neq 0$. Taking covariant derivative in Eq (4.1) and using Eq (1.1), we conclude

$$\nabla_T \nabla f = T, \quad T \in \mathfrak{X}(M). \tag{4.2}$$

Recall that the Hessian of f is defined by

$$Hess(f)(T_1,T_2) = g(\nabla_{T_1}\nabla f, T_2), \quad T_1,T_2 \in \mathfrak{X}(M),$$

which in view of Eq (4.2), gives

$$Hess(f) = g, \tag{4.3}$$

where f is a non-constant function. Recall that in [10], it is proved that a complete and connected n-dimensional Riemannian manifold (M, g) is isometric to an Euclidean space $(\mathbf{R}^n, \langle , \rangle)$, if and only if there is a non-constant smooth function $f : M \to R$ satisfying $Hess(f) = cg$ for a nonzero constant c. Hence, by Eq (4.3), (M, g) is isometric to the Euclidean space $(\mathbf{R}^n, \langle , \rangle)$ (cf. [10]). The converse is trivial. □

Remark. *In [5], Chen has proved that there does not exist a concurrent vector field on a pseudo-Kaehler manifold of positive (or negative) Ricci curvature (cf. Proposition 3.2). Also, in the same paper, Chen proved tha if an Einstein pseudo-Kaehler manifold admits a concurrent vector field, then it is Ricci-flat (cf. Corollary 3.1). We have seen in the Corollary to Theorem 3.2 that on n-dimensional non-Ricci-flat Einstein manifolds $n > 2$, there does not exist a concurrent vector field. It will be an interesting question to see whether such a result can be proved for quasi-Einstein spaces. We propose to take this question in our future studies.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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