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*Research article*

# Cohomologies of modified $\lambda$-differential Lie triple systems and applications

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**Abstract:** In this paper, we introduce the concept and representation of modified $\lambda$-differential Lie triple systems. Next, we define the cohomology of modified $\lambda$-differential Lie triple systems with coefficients in a suitable representation. As applications of the proposed cohomology theory, we study 1-parameter formal deformations and abelian extensions of modified $\lambda$-differential Lie triple systems.

**Keywords:** Lie triple system; modified $\lambda$-differential operator; cohomology; deformation; extension
**Mathematics Subject Classification:** 17A30, 17A42, 17B10, 17B56

## 1. Introduction

Jacobson [1, 2] introduced the concept of Lie triple systems by quantum mechanics and Jordan theory. In fact, Lie triple systems originated from E. Cartan's research on symmetric spaces and totally geodesic submanifolds. Since then, structure theory, representation theory, cohomology theory, deformation theory and extension theory of Lie triple systems were established in [3–9].

The study of differential algebras began with the algebraic abstraction of Ritt [10] to differential equations. Differentiation is also called derivation. Derivations play an important role in the study of homotopy algebras, deformation formulas, differential Galois theory, differential algebraic geometry, control theory and gauge theories of quantum field theory, with broad applications in mathematics and physics, see [11–20]. Recently, associative algebras with derivations [21], Lie algebras with derivations [22], Leibniz algebras with derivations [23], Leibniz triple systems with derivations [24] and Lie triple systems with derivations [25–27] have been extensively studied. All these results provide a good starting point for our further study.

In recent years, due to the outstanding work of [28–33], more and more scholars have focused on the structure with arbitrary weights. Rota-Baxter Lie algebras of any weights [34], Rota-Baxter 3-Lie algebras of any weights [35, 36] and Rota-Baxter Lie triple systems of any weights [37] appear successively. After that, for $\lambda \in \mathbb{K}$, the cohomology, extension and deformation theory of Lie algebras with differential operators of weight $\lambda$ are introduced by Li and Wang [38]. In addition, the cohomology and deformation theory of modified Rota-Baxter associative algebras and modified Rota-Baxter Leibniz algebras of weight $\lambda$ are given in [39–41]. In [42] the authors have considered cohomology and deformation theory of modified $r$-matrices. The concept of modified $\lambda$-differential Lie algebras is introduced in [43].

Motivated by Peng's [43] terminology of modified $\lambda$-differential Lie algebras and considering the importance of Lie triple systems, representation, cohomology, abelian extension and deformation theories, we mainly study the representation, cohomology, deformation theory and abelian extension of modified $\lambda$-differential Lie triple system in this paper.

The paper is organized as follows. In Section 2, we introduce the concept of a modified $\lambda$-differential Lie triple system, and give its representation. In Section 3, we define a cohomology theory of modified $\lambda$-differential Lie triple systems with coefficients in a representation by using a Yamaguti coboundary operator $\delta$ and a cochain map $\Phi$. In Section 4, we study 1-parameter formal deformations of a modified $\lambda$-differential Lie triple system using the third cohomology group with the coefficient in the adjoint representation. In Section 5, we study abelian extensions of a modified $\lambda$-differential Lie triple system using the third cohomology group with coefficients in a suitable representation.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All the algebras, vector spaces and (multi)linear maps are taken over $\mathbb{K}$.

## 2. Representations of modified $\lambda$-differential Lie triple systems

In this section, we recall concepts of Lie triple systems from [1,4]. Then, we introduce the concept of a modified $\lambda$-differential Lie triple system and its representation.

**Definition 2.1.** (i) [1] A Lie triple system is a pair $(\mathfrak{L}, [\cdot,\cdot,\cdot])$ in which $\mathfrak{L}$ is a vector space together with a ternary operation $[\cdot,\cdot,\cdot]$ on $\mathfrak{L}$ such that

$$[x, y, z] + [y, x, z] = 0, \tag{2.1}$$
$$[x, y, z] + [z, x, y] + [y, z, x] = 0, \tag{2.2}$$
$$[a, b, [x, y, z]] = [[a, b, x], y, z] + [x, [a, b, y], z] + [x, y, [a, b, z]], \tag{2.3}$$

for all $x, y, z, a, b \in \mathfrak{L}$.
(ii) A homomorphism between two Lie triple systems $(\mathfrak{L}_1, [\cdot,\cdot,\cdot]_1)$ and $(\mathfrak{L}_2, [\cdot,\cdot,\cdot]_2)$ is a linear map $\zeta : \mathfrak{L}_1 \to \mathfrak{L}_2$ satisfying $\zeta([x, y, z]_1) = [\zeta(x), \zeta(y), \zeta(z)]_2, \ \forall\, x, y, z \in \mathfrak{L}_1$.

**Definition 2.2.** [4] A representation of a Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot])$ on a vector space $\mathfrak{V}$ is a bilinear map $\theta$: $\mathfrak{L} \times \mathfrak{L} \to \mathrm{End}(\mathfrak{V})$, such that

$$\theta(a, b)\theta(x, y) - \theta(y, b)\theta(x, a) - \theta(x, [y, a, b]) + D(y, a)\theta(x, b) = 0, \tag{2.4}$$
$$\theta(a, b)D(x, y) - D(x, y)\theta(a, b) + \theta([x, y, a], b) + \theta(a, [x, y, b]) = 0, \tag{2.5}$$

for all $x, y, a, b \in \mathfrak{L}$, where $D(x, y) = \theta(y, x) - \theta(x, y)$. In this case, we also call $\mathfrak{V}$ a $\mathfrak{L}$-module.

**Example 2.3.** Any Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot])$ is a representation over itself with

$$\mathcal{R} : \mathfrak{L} \times \mathfrak{L} \to \mathrm{End}(\mathfrak{L}), (a, b) \mapsto (x \mapsto [x, a, b]).$$

It is called the adjoint representation over the Lie triple system.

**Definition 2.4.** (i) Let $\lambda \in \mathbb{K}$ and $(\mathfrak{L}, [\cdot,\cdot,\cdot])$ be a Lie triple system. A modified $\lambda$-differential operator (also called a modified differential operator of weight $\lambda$) on $\mathfrak{L}$ is a linear operator $d\colon \mathfrak{L} \to \mathfrak{L}$, such that

$$d([a,b,c]) = [d(a),b,c] + [a,d(b),c] + [a,b,d(c)] + \lambda[a,b,c]. \tag{2.6}$$

(ii) A modified $\lambda$-differential Lie triple system (also called a modified differential Lie triple system of weight $\lambda$) is a triple $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ consisting of a Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot])$ and a modified $\lambda$-differential operator $d$.
(iii) A homomorphism between two modified $\lambda$-differential Lie triple systems $(\mathfrak{L}_1, [\cdot,\cdot,\cdot]_1, d_1)$ and $(\mathfrak{L}_2, [\cdot,\cdot,\cdot]_2, d_2)$ is a Lie triple system homomorphism $\zeta\colon (\mathfrak{L}_1, [\cdot,\cdot,\cdot]_1) \to (\mathfrak{L}_2, [\cdot,\cdot,\cdot]_2)$ such that $\zeta \circ d_1 = d_2 \circ \zeta$. Furthermore, if $\zeta$ is nondegenerate, then $\zeta$ is called an isomorphism from $\mathfrak{L}_1$ to $\mathfrak{L}_2$.

**Remark 2.5.** Let $d$ be a modified $\lambda$-differential operator on $(\mathfrak{L}, [\cdot,\cdot,\cdot])$. If $\lambda = 0$, then $d$ is a derivation on $\mathfrak{L}$. We denote the set of all derivations on $\mathfrak{L}$ by $\mathrm{Der}(\mathfrak{L})$. See [44] for various derivations of Lie triple systems.

Moreover, there is a close relationship between derivations and modified $\lambda$-differential operators.

**Proposition 2.6.** *Let $(\mathfrak{L}, [\cdot,\cdot,\cdot])$ be a Lie triple system. Then, a linear operator $d\colon \mathfrak{L} \to \mathfrak{L}$ is a modified $\lambda$-differential operator if and only if $d + \frac{\lambda}{2}\mathrm{id}_{\mathfrak{L}}$ is a derivation on $\mathfrak{L}$.*

*Proof.* Equation (2.6) is equivalent to

$$(d + \frac{\lambda}{2}\mathrm{id}_{\mathfrak{L}})([a,b,c]) = [(d + \frac{\lambda}{2}\mathrm{id}_{\mathfrak{L}})(a),b,c] + [a,(d + \frac{\lambda}{2}\mathrm{id}_{\mathfrak{L}})(b),c] + [a,b,(d + \frac{\lambda}{2}\mathrm{id}_{\mathfrak{L}})(c)].$$

The proposition follows. □

**Example 2.7.** Let $(\mathfrak{L}, [\cdot,\cdot], d)$ be a modified $\lambda$-differential Lie algebra (see [43], Definition 2.5). Define trilinear map $[\cdot,\cdot,\cdot]\colon \mathfrak{L}\times\mathfrak{L}\times\mathfrak{L} \to \mathfrak{L}$ by $[a,b,c] = [[a,b],c], \forall a,b,c \in \mathfrak{L}$. Then $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ is a modified $(2\lambda)$-differential Lie triple system.

**Example 2.8.** Let $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ be a modified $\lambda$-differential Lie triple system. Then, for $k \in \mathbb{K}$, $(\mathfrak{L}, [\cdot,\cdot,\cdot], kd)$ is a modified $(k\lambda)$-differential Lie triple system.

**Example 2.9.** Let $(\mathfrak{L}, [\cdot,\cdot,\cdot])$ be a 2-dimensional Lie triple system $\mathfrak{L}$ with the basis $\mathfrak{u}_1$ and $\mathfrak{u}_2$ defined by

$$[\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_2] = \mathfrak{u}_1.$$

Then, the operator $d = \begin{pmatrix} k & k_1 \\ 0 & k_2 \end{pmatrix}$ is a modified $(-2k_2)$-differential operator on $\mathfrak{L}$, for $k, k_1, k_2 \in \mathbb{K}$.

**Example 2.10.** Let $(\mathfrak{L}, [\cdot,\cdot,\cdot])$ be a 4-dimensional Lie triple system with a basis $\mathfrak{u}_1$, $\mathfrak{u}_2$, $\mathfrak{u}_3$ and $\mathfrak{u}_4$ defined by $[\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_1] = \mathfrak{u}_4$. Then, the operator

$$d = \begin{pmatrix} 1 & 0 & k_1 & 0 \\ 0 & 1 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & k_4 & k \end{pmatrix}$$

is a modified $(k-3)$-differential operator on $\mathfrak{L}$, for $k, k_i \in \mathbb{K}, (i = 1,2,3,4)$.

**Definition 2.11.** A representation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ is a triple $(\mathfrak{V};\theta, d_{\mathfrak{V}})$ such that the following conditions are satisfied:

(i) $(\mathfrak{V};\theta)$ is a representation of the Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot])$;

(ii) $d_{\mathfrak{V}}\colon \mathfrak{V}\to\mathfrak{V}$ is a linear map satisfying the following equation

$$d_{\mathfrak{V}}(\theta(x,y)v) = \theta(d(x),y)v + \theta(x,d(y))v + \theta(x,y)d_{\mathfrak{V}}(v) + \lambda\theta(x,y)v, \tag{2.7}$$

for any $x, y \in \mathfrak{L}$ and $v \in \mathfrak{V}$.

By Eq (2.7), we have

$$d_{\mathfrak{V}}(D(x,y)v) = D(d(x),y)v + D(x,d(y))v + D(x,y)d_{\mathfrak{V}}(v) + \lambda D(x,y)v. \tag{2.8}$$

Obviously, $(\mathfrak{L};\mathcal{R}, d)$ is a representation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$.

**Remark 2.12.** Let $(\mathfrak{V};\theta, d_{\mathfrak{V}})$ be a representation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$. If $\lambda = 0$, then $(\mathfrak{V};\theta, d_{\mathfrak{V}})$ is a representation of the Lie triple system with a derivation $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$. One can refer to [25–27] for more information about Lie triple systems with derivations.

Moreover, the following result finds the relation between representations over modified $\lambda$-differential Lie triple systems and over Lie triple systems with derivations.

**Proposition 2.13.** *Let $(\mathfrak{V};\theta)$ be a representation of the Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot])$. Then $(\mathfrak{V};\theta, d_{\mathfrak{V}})$ is a representation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ if and only if $(\mathfrak{V};\theta, d_{\mathfrak{V}}+\frac{\lambda}{2}\mathrm{id}_{\mathfrak{V}})$ is a representation of the Lie triple system with a derivation $(\mathfrak{L}, [\cdot,\cdot,\cdot], d+\frac{\lambda}{2}\mathrm{id}_{\mathfrak{L}})$.*

*Proof.* Equation (2.7) is equivalent to

$$(d_{\mathfrak{V}}+\frac{\lambda}{2}\mathrm{id}_{\mathfrak{V}})(\theta(x,y)v) = \theta((d+\frac{\lambda}{2}\mathrm{id}_{\mathfrak{L}})(x),y)v + \theta(x,(d+\frac{\lambda}{2}\mathrm{id}_{\mathfrak{L}}))(y))v + \theta(x,y)(d_{\mathfrak{V}}+\frac{\lambda}{2}\mathrm{id}_{\mathfrak{V}})(v).$$

The proposition follows. □

**Example 2.14.** Let $(\mathfrak{V};\theta)$ be a representation of the Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot])$. Then, for $k\in\mathbb{K}$, $(\mathfrak{V};\theta,\mathrm{id}_{\mathfrak{V}})$ is a representation of the modified $(-2k)$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], k\mathrm{id}_{\mathfrak{L}})$.

**Example 2.15.** Let $(\mathfrak{V};\theta, d_{\mathfrak{V}})$ be a representation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$. Then, for $k\in\mathbb{K}$, $(\mathfrak{V};\theta, kd_{\mathfrak{V}})$ is a representation of the modified $(k\lambda)$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], kd)$.

Next we construct the semidirect product in the context of modified $\lambda$-differential Lie triple systems.

**Proposition 2.16.** *Let $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ be a modified $\lambda$-differential Lie triple system and $(\mathfrak{V};\theta, d_{\mathfrak{V}})$ be a representation of it. Then $\mathfrak{L}\oplus\mathfrak{V}$ is a modified $\lambda$-differential Lie triple system under the following maps:*

$$\begin{aligned}[x+u, y+v, z+w]_{\ltimes} :=&[x,y,z] + D(x,y)(w) - \theta(x,z)(v) + \theta(y,z)(u),\\ d\oplus d_{\mathfrak{V}}(x+u) :=&d(x) + d_{\mathfrak{V}}(u),\end{aligned}$$

*for all $x, y, z\in\mathfrak{L}$ and $u, v, w\in\mathfrak{V}$.*

*Proof.* First, as we all know, $(\mathfrak{L}\oplus\mathfrak{V},[\cdot,\cdot,\cdot]_{\ltimes})$ is a Lie triple system. Next, for any $x,y,z\in\mathfrak{L}$, $u,v,w\in\mathfrak{V}$, by Eqs (2.6)–(2.8), we have

$$\begin{aligned}
&d\oplus d_{\mathfrak{V}}([x+u,y+v,z+w]_{\ltimes})\\
=&d([x,y,z])+d_{\mathfrak{V}}(D(x,y)(w)-\theta(x,z)(v)+\theta(y,z)(u))\\
=&[d(x),y,z]+[x,d(y),z]+[x,y,d(z)]+\lambda[x,y,z]+D(d(x),y)w+D(x,d(y))w+D(x,y)d_{\mathfrak{V}}(w)\\
&+\lambda D(x,y)w-\theta(d(x),z)v-\theta(x,d(z))v-\theta(x,z)d_{\mathfrak{V}}(v)-\lambda\theta(x,z)v+\theta(d(y),z)u+\theta(y,d(z))u\\
&+\theta(y,z)d_{\mathfrak{V}}(u)+\lambda\theta(y,z)u\\
=&[d(x),y,z]+D(d(x),y)w-\theta(d(x),z)v+\theta(y,z)d_{\mathfrak{V}}(u)\\
&+[x,d(y),z]+D(x,d(y))w-\theta(x,z)d_{\mathfrak{V}}(v)+\theta(d(y),z)u\\
&+[x,y,d(z)]+D(x,y)d_{\mathfrak{V}}(w)-\theta(x,d(z))v+\theta(y,d(z))u\\
&+\lambda([x,y,z]+D(x,y)w-\theta(x,z)v+\theta(y,z)u)\\
=&[d\oplus d_{\mathfrak{V}}(x+u),y+v,z+w]_{\ltimes}+[x+u,d\oplus d_{\mathfrak{V}}(y+v),z+w]_{\ltimes}+[x+u,y+v,d\oplus d_{\mathfrak{V}}(z+w)]_{\ltimes}\\
&+\lambda[x+u,y+v,z+w]_{\ltimes}.
\end{aligned}$$

Therefore, $(\mathfrak{L}\oplus\mathfrak{V},[\cdot,\cdot,\cdot]_{\ltimes},d\oplus d_{\mathfrak{V}})$ is a modified $\lambda$-differential Lie triple system. □

Let $(\mathfrak{V};\theta,d_{\mathfrak{V}})$ be a representation of a modified $\lambda$-differential Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$, and $\mathfrak{V}^*$ be a dual space of $\mathfrak{V}$. We define a bilinear map $\theta^*\colon\mathfrak{L}\times\mathfrak{L}\to\mathrm{End}(\mathfrak{V}^*)$ and a linear map $d_{\mathfrak{V}}^*\colon\mathfrak{V}^*\to\mathfrak{V}^*$ respectively by

$$\langle\theta^*(a,b)u^*,v\rangle=-\langle u^*,\theta(a,b)v\rangle \text{ and } \langle d_{\mathfrak{V}}^*u^*,v\rangle=\langle u^*,d_{\mathfrak{V}}(v)\rangle, \tag{2.9}$$

for any $a,b\in\mathfrak{L}$, $v\in\mathfrak{V}$ and $u^*\in\mathfrak{V}^*$.

Give the switching operator $\tau\colon\mathfrak{L}\otimes\mathfrak{L}\to\mathfrak{L}\otimes\mathfrak{L}$ by $\tau(a\otimes b)=\tau(b\otimes a)$, for any $a,b\in\mathfrak{L}$.

**Proposition 2.17.** *With the above notations,* $(\mathfrak{V}^*;-\theta^*\tau,-d_{\mathfrak{V}}^*)$ *is a representation of modified $\lambda$-differential Lie triple system* $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$*. We call it the dual representation of* $(\mathfrak{V};\theta,d_{\mathfrak{V}})$

*Proof.* Following [8, 45], we can easily get $(\mathfrak{V}^*;-\theta^*\tau)$ is a representation of the Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot])$. Moreover, for any $a,b\in\mathfrak{L}$, $v\in\mathfrak{V}$ and $u^*\in\mathfrak{V}^*$, by Eqs (2.7) and (2.9), we have

$$\begin{aligned}
&\langle-\theta^*\tau(d(a),b)u^*-\theta^*\tau(a,d(b))u^*+\theta^*\tau(a,b)d_{\mathfrak{V}}^*u^*-\lambda\theta^*\tau(a,b)u^*-d_{\mathfrak{V}}^*\theta^*\tau(a,b)u^*,v\rangle\\
=&\langle u^*,\theta(b,d(a))v+\theta(d(b),a)v-d_{\mathfrak{V}}(\theta(b,a)v)+\lambda\theta(b,a)v+\theta(b,a)d_{\mathfrak{V}}(v)\rangle\\
=&0,
\end{aligned}$$

which implies that $(\mathfrak{V}^*;-\theta^*\tau,-d_{\mathfrak{V}}^*)$ is a representation of $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$. □

**Example 2.18.** Let $(\mathfrak{L};\mathcal{R},d)$ be an adjoint representation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$. Then, $(\mathfrak{L}^*;-\mathcal{R}^*\tau,-d^*)$ is a dual adjoint representation of $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$, called the coadjoint representation.

**Example 2.19.** Let $(\mathfrak{V};\theta)$ be a representation of the Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot])$. Then, for $k\in\mathbb{K}$, $(\mathfrak{V}^*;-\theta^*\tau,-\mathrm{id}_{\mathfrak{V}^*})$ is a dual representation of the modified $(-2k)$-differential Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot],k\mathrm{id}_{\mathfrak{L}})$.

**Example 2.20.** Let $(\mathfrak{V};\theta,d_{\mathfrak{V}})$ be a representation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$. Then, for $k\in\mathbb{K}$, $(\mathfrak{V}^*;-\theta^*\tau,-kd_{\mathfrak{V}}^*)$ is a dual representation of the modified $(k\lambda)$-differential Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot],kd)$.

## 3. Cohomology of modified $\lambda$-differential Lie triple systems

In this section, we study the cohomology of a modified $\lambda$-differential Lie triple system with coefficients in its representation.

Let $(\mathfrak{V};\theta)$ be a representation of a Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot])$. Denote the $(2n+1)$-cochains of $\mathfrak{L}$ with coefficients in representation $(\mathfrak{V};\theta)$ by

$$\begin{aligned}C^{2n+1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V}) := \{f\in \mathrm{Hom}(\mathfrak{L}^{\otimes 2n+1},\mathfrak{V}) \mid\ & f(a_1,\cdots,a_{2n-2},a,b,c)+f(a_1,\cdots,a_{2n-2},b,a,c)=0,\\ & \circlearrowleft_{a,b,c} f(a_1,\cdots,a_{2n-2},a,b,c)=0\}.\end{aligned}$$

The Yamaguti coboundary operator $\delta$: $C^{2n-1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V})\to C^{2n+1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V})$, for $a_1,\cdots,a_{2n+1}\in\mathfrak{L}$ and $f\in C^{2n-1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V})$, as

$$\begin{aligned}&\delta f(a_1,\cdots,a_{2n+1})\\ =&\theta(a_{2n},a_{2n+1})f(a_1,\cdots,a_{2n-1})-\theta(a_{2n-1},a_{2n+1})f(a_1,\cdots,a_{2n-2},a_{2n})\\ &+\sum_{i=1}^{n}(-1)^{i+n}D(a_{2i-1},a_{2i})f(a_1,\cdots,a_{2i-2},a_{2i+1},\cdots,a_{2n+1})\\ &+\sum_{i=1}^{n}\sum_{j=2i+1}^{2n+1}(-1)^{i+n+1}f(a_1,\cdots,a_{2i-2},a_{2i+1},\cdots,[a_{2i-1},a_{2i},a_j],\cdots,a_{2n+1}).\end{aligned}$$

Yamaguti proved in [4] that $\delta\circ\delta=0$. One can refer to [1,4,7,9] for more information about Lie triple systems and cohomology theory.

Next, we introduce a cohomology of a modified $\lambda$-differential Lie triple system with coefficients in a representation.

We first give the following lemma.

**Lemma 3.1.** *Let $(\mathfrak{V};\theta,d_{\mathfrak{V}})$ be a representation of a modified $\lambda$-differential Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$. For any $n\geq 1$, we define a linear map $\Phi: C^{2n-1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V})\to C^{2n-1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V})$ by*

$$\begin{aligned}\Phi f(a_1,\cdots,a_{2n-1})=&\sum_{i=1}^{2n-1}f(a_1,\cdots,d(a_i),\cdots,a_{2n-1})\\ &+(n-1)\lambda f(a_1,\cdots,a_{2n-1})-d_{\mathfrak{V}}(f(a_1,\cdots,a_{2n-1})),\end{aligned}$$

*for any $f\in C^{2n-1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V})$ and $a_1,\cdots,a_{2n-1}\in\mathfrak{L}$. Then, $\Phi$ is a cochain map, that is, the following diagram is commutative:*

$$\begin{tikzcd} C^{2n-1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V}) \arrow[r, "\delta"] \arrow[d, "\Phi"] & C^{2n+1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V}) \arrow[d, "\Phi"] \\ C^{2n-1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V}) \arrow[r, "\delta"] & C^{2n+1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V}). \end{tikzcd}$$

*Proof.* The weighted item $(n-1)\lambda f(a_1,\cdots,a_{2n-1})$ is added to the map $\Phi$, and it can be proved by using similar arguments to Appendix A in [26]. Because of space limitations, here we only prove the case of $n=1$. For all $f\in C^{1}_{\mathrm{Lts}}(\mathfrak{L},\mathfrak{V})$ and $a_1,a_2,a_3\in\mathfrak{L}$, by Eqs (2.6)–(2.8), we have

$$
\begin{aligned}
&\Phi \circ \delta f(a_1, a_2, a_3)\\
=&\delta f(d(a_1), a_2, a_3) + \delta f(a_1, d(a_2), a_3) + \delta f(a_1, a_2, d(a_3)) + \lambda\delta f(a_1, a_2, a_3) - d_{\mathfrak{V}}(\delta f(a_1, a_2, a_3))\\
=&\theta(a_2, a_3) f(d(a_1)) - \theta(d(a_1), a_3) f(a_2) + D(d(a_1), a_2) f(a_3) - f([d(a_1), a_2, a_3])\\
&+ \theta(d(a_2), a_3) f(a_1) - \theta(a_1, a_3) f(d(a_2)) + D(a_1, d(a_2)) f(a_3) - f([a_1, d(a_2), a_3])\\
&+ \theta(a_2, d(a_3)) f(a_1) - \theta(a_1, d(a_3)) f(a_2) + D(a_1, a_2) f(d(a_3)) - f([a_1, a_2, d(a_3)])\\
&+ \lambda\theta(a_2, a_3) f(a_1) - \lambda\theta(a_1, a_3) f(a_2) + \lambda D(a_1, a_2) f(a_3) - \lambda f([a_1, a_2, a_3])\\
&- d_{\mathfrak{V}}(\theta(a_2, a_3) f(a_1)) + d_{\mathfrak{V}}(\theta(a_1, a_3) f(a_2)) - d_{\mathfrak{V}}(D(a_1, a_2) f(a_3)) + d_{\mathfrak{V}}(f([a_1, a_2, a_3]))\\
=&\theta(a_2, a_3) f(d(a_1)) - \theta(a_2, a_3) d_{\mathfrak{V}}(f(a_1)) - \theta(a_1, a_3) f(d(a_2)) + \theta(a_1, a_3) d_{\mathfrak{V}}(f(a_2))\\
&+ D(a_1, a_2) f(d(a_3)) - D(a_1, a_2) d_{\mathfrak{V}} f(a_3) - f(d([a_1, a_2, a_3])) + d_{\mathfrak{V}}(f([a_1, a_2, a_3]))\\
=&\theta(a_2, a_3)\Phi f(a_1) - \theta(a_1, a_3)\Phi f(a_2) + D(a_1, a_2)\Phi f(a_3) - \Phi f([a_1, a_2, a_3])\\
=&\delta \circ \Phi f(a_1, a_2, a_3).
\end{aligned}
$$

Therefore, $\Phi \circ \delta = \delta \circ \Phi$. □

We define the set of $(2n-1)$-cochains by

$$
C^{2n-1}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V}) := \begin{cases} C^{2n-1}_{\mathrm{Lts}}(\mathfrak{L}, \mathfrak{V}) \oplus C^{2n-3}_{\mathrm{Lts}}(\mathfrak{L}, \mathfrak{V}), & n \geq 2,\\ C^{1}_{\mathrm{Lts}}(\mathfrak{L}, \mathfrak{V}) = \mathrm{Hom}(\mathfrak{L}, \mathfrak{V}), & n = 1. \end{cases}
$$

When $n \geq 2$, we define the linear map $\partial$: $C^{2n-1}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V}) \to C^{2n+1}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V})$ by

$$
\partial(f, g) = (\delta f, \delta g + (-1)^n \Phi f), \forall (f, g) \in C^{2n-1}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V}).
$$

When $n = 1$, define linear map $\partial$: $C^{1}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V}) \to C^{3}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V})$ by

$$
\partial(f) = (\delta f, -\Phi f), \forall f \in C^{1}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V}).
$$

**Theorem 3.2.** *The pair $(C^{*}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V}), \partial)$ is a cochain complex. So $\partial \circ \partial = 0$.*

*Proof.* For any $f \in C^{1}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V})$, by Lemma 3.1, we have

$$
\partial \circ \partial(f) = \partial(\delta f, -\Phi f) = (\delta(\delta f), \delta(-\Phi f) + \Phi(\delta f)) = 0.
$$

Given any $(f, g) \in C^{2n-1}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V})$ with $n \geq 2$, we have

$$
\partial \circ \partial(f, g) = \partial(\delta f, \delta g + (-1)^n \Phi f) = (\delta(\delta f), \delta(\delta g + (-1)^n \Phi f) + (-1)^{n+1} \Phi(\delta f)) = 0.
$$

Therefore, $(C^{*}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V}), \partial)$ is a cochain complex. □

**Definition 3.3.** The cohomology of the cochain complex $(C^{*}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V}), \partial)$, denoted by $\mathcal{H}^{*}_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V})$, is called the cohomology of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot, \cdot, \cdot], d)$ with coefficients in the representation $(\mathfrak{V}; \theta, d_{\mathfrak{V}})$.

For $n \geq 1$, we denote the set of $(2n+1)$-cocycles by $\mathcal{Z}^{2n+1}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V}) = \{(f, g) \in C^{2n+1}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V}) \mid \partial(f, g) = 0\}$, the set of $(2n+1)$-coboundaries by $\mathcal{B}^{2n+1}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V}) = \{\partial(f, g) \mid (f, g) \in C^{2n-1}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V})\}$ and the $(2n+1)$-th cohomology group of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ with coefficients in the representation $(\mathfrak{V}; \theta, d_{\mathfrak{V}})$ by $\mathcal{H}^{2n+1}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V}) = \mathcal{Z}^{2n+1}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V})/\mathcal{B}^{2n+1}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V})$.

To end this section, we compute 1-cocycles and 3-cocycles of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ with coefficients in the representation $(\mathfrak{V}; \theta, d_{\mathfrak{V}})$.

It is obvious that for all $f \in C^{1}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V})$, $f$ is a 1-cocycle if and only if $\partial(f) = (\delta f, -\Phi f) = 0$, that is,

$$\theta(a_2, a_3)f(a_1) - \theta(a_1, a_3)f(a_2) + D(a_1, a_2)f(a_3) - f([a_1, a_2, a_3]) = 0$$

and

$$d_{\mathfrak{V}}(f(a_1)) - f(d(a_1)) = 0.$$

For all $(f, g) \in C^{3}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{V})$, $(f, g)$ is a 3-cocycle if and only if $\partial(f, g) = (\delta f, \delta g + \Phi f) = 0$, that is,

$$\begin{aligned}&\theta(a_4, a_5)f(a_1, a_2, a_3) - \theta(a_3, a_5)f(a_1, a_2, a_4) - D(a_1, a_2)f(a_3, a_4, a_5) + D(a_3, a_4)f(a_1, a_2, a_5)\\ &+ f([a_1, a_2, a_3], a_4, a_5) + f(a_3, [a_1, a_2, a_4], a_5) + f(a_3, a_4, [a_1, a_2, a_5]) - f(a_1, a_2, [a_3, a_4, a_5]) = 0\end{aligned}$$

and

$$\begin{aligned}&\theta(a_2, a_3)g(a_1) - \theta(a_1, a_3)g(a_2) + D(a_1, a_2)g(a_3) - g([a_1, a_2, a_3])\\ &+ f(d(a_1), a_2, a_3) + f(a_1, d(a_2), a_3) + f(a_1, a_2, d(a_3)) + \lambda f(a_1, a_2, a_3) - d_{\mathfrak{V}}(f(a_1, a_2, a_3)) = 0.\end{aligned}$$

## 4. Deformations of modified $\lambda$-differential Lie triple systems

In this section, we introduce 1-parameter formal deformations of the modified $\lambda$-differential Lie triple system. Furthermore, we show that if the third cohomology group $\mathcal{H}^{3}_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L}, \mathfrak{L}) = 0$, then the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ is rigid. For deformations of Lie triple systems with derivations, see [26, 27].

Let $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ be a modified $\lambda$-differential Lie triple system. Denote by $\nu$ the multiplication of $\mathfrak{L}$, i.e., $\nu = [\cdot,\cdot,\cdot]$. Consider the 1-parameterized family

$$\nu_t = \sum_{i=0}^{\infty} \nu_i t^i, \ \ \nu_i \in C^{3}_{\mathrm{Lts}}(\mathfrak{L}, \mathfrak{L}), \quad d_t = \sum_{i=0}^{\infty} d_i t^i, \ \ d_i \in C^{1}_{\mathrm{Lts}}(\mathfrak{L}, \mathfrak{L}).$$

**Definition 4.1.** A 1-parameter formal deformation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, \nu, d)$ is a pair $(\nu_t, d_t)$ which endows the $\mathbb{K}[[t]]$-module $(\mathfrak{L}[[t]], \nu_t, d_t)$ with the modified $\lambda$-differential Lie triple system over $\mathbb{K}[[t]]$ such that $(\nu_0, d_0) = (\nu, d)$.

Obviously, $(\mathfrak{L}[[t]], \nu_t = \nu, d_t = d)$ is a 1-parameter formal deformation of $(\mathfrak{L}, \nu, d)$.

The pair $(\nu_t, d_t)$ generates a 1-parameter formal deformation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ if and only if for all $a, b, c, x, y \in \mathfrak{L}$, the following equations hold:

$$\nu_t(a, b, c) + \nu_t(b, a, c) = 0, \tag{4.1}$$

$$\nu_t(a, b, c)) + \nu_t(b, c, a) + \nu_t(c, a, b) = 0, \tag{4.2}$$

$$\nu_t(x, y, \nu_t(a, b, c)) = \nu_t(\nu_t(x, y, a), b, c) + \nu_t(a, \nu_t(x, y, b), c) + \nu_t(a, b, \nu_t(x, y, c)), \tag{4.3}$$

$$d_t(\nu_t(a, b, c)) = \nu_t(d_t(a), b, c) + \nu_t(a, d_t(b), c) + \nu_t(a, b, d_t(c)) + \lambda \nu_t(a, b, c). \tag{4.4}$$

Comparing the coefficients of $t^n$ on both sides of the above equations, Eqs (4.1)–(4.4) are equivalent to the following equations:

$$\nu_n(a,b,c) + \nu_n(b,a,c) = 0, \tag{4.5}$$
$$\nu_n(a,b,c)) + \nu_n(b,c,a) + \nu_n(c,a,b) = 0, \tag{4.6}$$
$$\sum_{i+j=n} \nu_i(x,y,\nu_j(a,b,c)) = \sum_{i+j=n} (\nu_i(\nu_j(x,y,a),b,c) + \nu_i(a,\nu_j(x,y,b),c) + \nu_i(a,b,\nu_j(x,y,c))), \tag{4.7}$$
$$\sum_{i+j=n} d_i(\nu_j(a,b,c)) = \sum_{i+j=n} (\nu_i(d_j(a),b,c) + \nu_i(a,d_j(b),c) + \nu_i(a,b,d_j(c))) + \lambda \nu_n(a,b,c). \tag{4.8}$$

**Proposition 4.2.** *Let $(\mathfrak{L}[[t]], \nu_t, d_t)$ be a 1-parameter formal deformation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, \nu, d)$. Then $(\nu_1, d_1)$ is a 3-cocycle of $(\mathfrak{L}, \nu, d)$ with the coefficient in the adjoint representation $(\mathfrak{L}; \mathcal{R}, d)$.*

*Proof.* For $n = 1$, Eq (4.7) is equivalent to

$$\begin{aligned}
&\nu_1(x,y,[a,b,c]) + [x,y,\nu_1(a,b,c)] \\
=&\nu_1([x,y,a],b,c) + [\nu_1(x,y,a),b,c] + \nu_1(a,[x,y,b],c) + [a,\nu_1(x,y,b),c] + \nu_1(a,b,[x,y,c]) \\
&+ [a,b,\nu_1(x,y,c)],
\end{aligned}$$

i.e., $\delta\nu_1 = 0$. In addition, for $n = 1$, Eq (4.8) is equivalent to

$$\begin{aligned}
&d_1([a,b,c]) + d(\nu_1(a,b,c)) \\
=&[d_1(a),b,c] + \nu_1(d(a),b,c) + [a,d_1(b),c] + \nu_1(a,d(b),c) + [a,b,d_1(c)] + \nu_1(a,b,d(c)) \\
&+ \lambda\nu_1(a,b,c),
\end{aligned}$$

that is, $\delta d_1 + \Phi\nu_1 = 0$. In other words, Eqs (4.7) and (4.8) are equivalent to $\partial(\nu_1, d_1) = (\delta\nu_1, \delta d_1 + \Phi\nu_1) = 0$. Therefore, $(\nu_1, d_1)$ is a 3-cocycle of $(\mathfrak{L}, \nu, d)$ with the coefficient in the adjoint representation $(\mathfrak{L}; \mathcal{R}, d)$. □

If $\nu_t = \nu$ in the above 1-parameter formal deformation of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, \nu, d)$, we obtain a 1-parameter formal deformation of the modified $\lambda$-differential operator $d$. So, we have the following result.

**Corollary 4.3.** *Let $d_t$ be a 1-parameter formal deformation of the modified $\lambda$-differential operator $d$. Then $d_1$ is a 1-cocycle of the modified $\lambda$-differential operator $d$ with coefficients in the adjoint representation $(\mathfrak{L}; \mathcal{R}, d)$.*

*Proof.* When $n = 1$, by $\nu_1 = 0$ and Eq (4.8), we have $d_1 \in \mathrm{Der}(\mathfrak{L})$. That is, Eq (4.8) is equivalent to $\delta d_1 = 0$, which implies that $d_1$ is a 1-cocycle of the modified $\lambda$-differential operator $d$ with coefficients in the adjoint representation $(\mathfrak{L}; \mathcal{R}, d)$. □

**Definition 4.4.** The 3-cocycle $(\nu_1, d_1)$ is called the infinitesimal of the 1-parameter formal deformation $(\mathfrak{L}[[t]], \nu_t, d_t)$ of $(\mathfrak{L}, \nu, d)$.

**Definition 4.5.** Let $(\mathfrak{L}[[t]], \nu_t, d_t)$ and $(\mathfrak{L}[[t]], \nu'_t, d'_t)$ be two 1-parameter formal deformations of $(\mathfrak{L}, \nu, d)$. A formal isomorphism from $(\mathfrak{L}[[t]], \nu'_t, d'_t)$ to $(\mathfrak{L}[[t]], \nu_t, d_t)$ is a power series $\varphi_t = \mathrm{id}_{\mathfrak{L}} + \sum_{i=1}^{\infty} \varphi_i t^i$: $(\mathfrak{L}[[t]], \nu'_t, d'_t) \to (\mathfrak{L}[[t]], \nu_t, d_t)$, where $\varphi_i \in \mathrm{End}(\mathfrak{L})$, such that

$$\varphi_t \circ \nu'_t = \nu_t \circ (\varphi_t \times \varphi_t \times \varphi_t), \tag{4.9}$$

$$\varphi_t \circ d'_t = d_t \circ \varphi_t. \tag{4.10}$$

Two 1-parameter formal deformations $(\mathfrak{L}[[t]], \nu_t, d_t)$ and $(\mathfrak{L}[[t]], \nu'_t, d'_t)$ are said to be equivalent if there exists a formal isomorphism $\varphi_t$: $(\mathfrak{L}[[t]], \nu'_t, d'_t) \to (\mathfrak{L}[[t]], \nu_t, d_t)$.

**Theorem 4.6.** *The infinitesimals of two equivalent 1-parameter formal deformations of $(\mathfrak{L}, \nu, d)$ are in the same cohomology class in $\mathcal{H}^3_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{L})$.*

*Proof.* Let $\varphi_t$: $(\mathfrak{L}[[t]], \nu'_t, d'_t) \to (\mathfrak{L}[[t]], \nu_t, d_t)$ be a formal isomorphism. For all $a, b, c \in \mathfrak{L}$, we have

$$\begin{aligned} \varphi_t \circ \nu'_t(a, b, c) &= \nu_t(\varphi_t(a), \varphi_t(b), \varphi_t(c)), \\ \varphi_t \circ d'_t(a) &= d_t \circ \varphi_t(a). \end{aligned}$$

Comparing the coefficients of $t$ on both sides of the above equations, we can get

$$\begin{aligned} \nu'_1(a, b, c) - \nu_1(a, b, c) &= [\varphi_1(a), b, c] + [a, \varphi_1(b), c] + +[a, b, \varphi_1(c)] - \varphi_1([a, b, c]), \\ d'_1(a) - d_1(a) &= d(\varphi_1(a)) - \varphi_1(d(a)). \end{aligned}$$

Therefore, we have

$$(\nu'_1, d'_1) - (\nu_1, d_1) = (\delta\varphi_1, -\Phi\varphi_1) = \partial(\varphi_1) \in C^3_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{L}),$$

which implies that $[(\nu'_1, d'_1)] = [(\nu_1, d_1)]$ in $\mathcal{H}^3_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{L})$. □

**Definition 4.7.** (i) A 1-parameter formal deformation $(\mathfrak{L}[[t]], \nu_t, d_t)$ of $(\mathfrak{L}, \nu, d)$ is said to be trivial if it is equivalent to the deformation $(\mathfrak{L}[[t]], \nu, d)$.
(ii) A modified $\lambda$-differential Lie triple system $(\mathfrak{L}, \nu, d)$ is said to be rigid if every 1-parameter formal deformation is trivial.

**Theorem 4.8.** *If $\mathcal{H}^3_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{L}) = 0$, then the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, \nu, d)$ is rigid.*

*Proof.* Let $(\mathfrak{L}[[t]], \nu_t, d_t)$ be a 1-parameter formal deformation of $(\mathfrak{L}, \nu, d)$. By Proposition 4.2, $(\nu_1, d_1)$ is a 3-cocycle. By $\mathcal{H}^3_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{L}) = 0$, there exists a 1-cochain $\varphi_1 \in C^1_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{L})$ such that

$$(\nu_1, d_1) = \partial(\varphi_1). \tag{4.11}$$

Then, setting $\varphi_t = \mathrm{id}_{\mathfrak{L}} + \varphi_1 t$, we have a deformation $(\mathfrak{L}[[t]], \nu'_t, d'_t)$, where

$$\begin{aligned} \nu'_t &= \varphi_t^{-1} \circ \nu_t \circ (\varphi_t \otimes \varphi_t \otimes \varphi_t), \\ d'_t &= \varphi_t^{-1} \circ d_t \circ \varphi_t. \end{aligned}$$

Thus, $(\mathfrak{L}[[t]], \nu'_t, d'_t)$ is equivalent to $(\mathfrak{L}[[t]], \nu_t, d_t)$. Moreover, we have

$$\begin{aligned} \nu'_t &= (\mathrm{id}_L - \varphi_1 t + \varphi_1^2 t^2 + \cdots + (-1)^i \varphi_1^i t^i + \cdots) \circ \nu_t \circ ((\mathrm{id}_L + \varphi_1 t) \otimes (\mathrm{id}_L + \varphi_1 t) \otimes (\mathrm{id}_L + \varphi_1 t)), \\ d'_t &= (\mathrm{id}_L - \varphi_1 t + \varphi_1^2 t^2 + \cdots + (-1)^i \varphi_1^i t^i + \cdots) \circ d_t \circ (\mathrm{id}_L + \varphi_1 t). \end{aligned}$$

By Eq (4.11), we have

$$\begin{aligned} \nu'_t &= \nu + \nu'_2 t^2 + \cdots, \\ d'_t &= d + d'_2 t^2 + \cdots. \end{aligned}$$

Then by repeating the argument, we can show that $(\mathfrak{L}[[t]], \nu_t, d_t)$ is equivalent to $(\mathfrak{L}[[t]], \nu, d)$. Therefore, $(\mathfrak{L}, \nu, d)$ is rigid. □

## 5. Abelian extensions of modified $\lambda$-differential Lie triple systems

Motivated by the extensions of Lie triple systems with derivations [25–27], in this section, we study abelian extensions of modified $\lambda$-differential Lie triple systems and prove that they are classified by the third cohomology.

**Definition 5.1.** Let $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ and $(\mathfrak{V}, [\cdot,\cdot,\cdot]_{\mathfrak{V}}, d_{\mathfrak{V}})$ be two modified $\lambda$-differential Lie triple systems. An abelian extension of $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ by $(\mathfrak{V}, [\cdot,\cdot,\cdot]_{\mathfrak{V}}, d_{\mathfrak{V}})$ is a short exact sequence of homomorphisms of modified $\lambda$-differential Lie triple systems

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{V} \arrow[r, "i"] \arrow[d, "d_{\mathfrak{V}}"'] & \hat{\mathfrak{L}} \arrow[r, "p"] \arrow[d, "\hat{d}"'] & \mathfrak{L} \arrow[r] \arrow[d, "d"'] & 0 \\ 0 \arrow[r] & \mathfrak{V} \arrow[r, "i"] & \hat{\mathfrak{L}} \arrow[r, "p"] & \mathfrak{L} \arrow[r] & 0 \end{tikzcd}$$

such that $[u, v, \cdot]_{\hat{\mathfrak{L}}} = [u, \cdot, v]_{\hat{\mathfrak{L}}} = [\cdot, u, v]_{\hat{\mathfrak{L}}} = 0$, for all $u, v \in \mathfrak{V}$, i.e., $\mathfrak{V}$ is an abelian ideal of $\hat{\mathfrak{L}}$.

**Definition 5.2.** Let $(\hat{\mathfrak{L}}_1, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_1}, \hat{d}_1)$ and $(\hat{\mathfrak{L}}_2, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_2}, \hat{d}_2)$ be two abelian extensions of $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ by $(\mathfrak{V}, [\cdot,\cdot,\cdot]_{\mathfrak{V}}, d_{\mathfrak{V}})$. They are said to be equivalent if there is an isomorphism of modified $\lambda$-differential Lie triple systems $\zeta : (\hat{\mathfrak{L}}_1, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_1}, \hat{d}_1) \to (\hat{\mathfrak{L}}_2, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_2}, \hat{d}_2)$ such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & (\mathfrak{V}, d_{\mathfrak{V}}) \arrow[r, "i_1"] \arrow[d, equal] & (\hat{\mathfrak{L}}_1, \hat{d}_1) \arrow[r, "p_1"] \arrow[d, "\zeta"'] & (\mathfrak{L}, d) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (\mathfrak{V}, d_{\mathfrak{V}}) \arrow[r, "i_2"] & (\hat{\mathfrak{L}}_2, \hat{d}_2) \arrow[r, "p_2"] & (\mathfrak{L}, d) \arrow[r] & 0. \end{tikzcd}$$

A section of an abelian extension $(\hat{\mathfrak{L}}, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}}, \hat{d})$ of $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ by $(\mathfrak{V}, [\cdot,\cdot,\cdot]_{\mathfrak{V}}, d_{\mathfrak{V}})$ is a linear map $\sigma$: $\mathfrak{L} \to \hat{\mathfrak{L}}$ such that $p \circ \sigma = \mathrm{id}_{\mathfrak{L}}$.

Now, for an abelian extension $(\hat{\mathfrak{L}}, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}}, \hat{d})$ of $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ by $(\mathfrak{V}, [\cdot,\cdot,\cdot]_{\mathfrak{V}}, d_{\mathfrak{V}})$ with a section $\sigma$: $\mathfrak{L} \to \hat{\mathfrak{L}}$, we define linear map $\vartheta : \mathfrak{L} \times \mathfrak{L} \to \mathrm{End}(\mathfrak{V})$ by

$$\vartheta(a, b)u := [u, \sigma(a), \sigma(b)]_{\hat{\mathfrak{L}}}, \quad \forall a, b \in \mathfrak{L}, u \in \mathfrak{V}.$$

In particular, $\mathcal{D}(a, b)u = [\sigma(a), \sigma(b), u]_{\hat{\mathfrak{L}}} = \vartheta(b, a)u - \vartheta(a, b)u$.

**Proposition 5.3.** *With the above notations, $(\mathfrak{V}, \vartheta, d_{\mathfrak{V}})$ is a representation over the modified $\lambda$-differential Lie triple systems $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$.*

*Proof.* First, for any $x, y, a, b \in \mathfrak{L}$ and $u \in \mathfrak{V}$, from $\sigma([y,a,b]) - [\sigma(y), \sigma(a), \sigma(b)]_{\hat{\mathfrak{L}}} \in \mathfrak{V} \cong \ker(p)$, we can get $[u, \sigma(x), \sigma([y,a,b])]_{\hat{\mathfrak{L}}} = [u, \sigma(x), [\sigma(y), \sigma(a), \sigma(b)]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}}$. Furthermore, by Eqs (2.1) and (2.3), we obtain

$$\begin{aligned}
&\vartheta(a,b)\vartheta(x,y)u - \vartheta(y,b)\vartheta(x,a)u - \vartheta(x,[y,a,b])u + \mathcal{D}(y,a)\vartheta(x,b)u\\
=&[[u,\sigma(x),\sigma(y)]_{\hat{\mathfrak{L}}},\sigma(a),\sigma(b)]_{\hat{\mathfrak{L}}} - [[u,\sigma(x),\sigma(a)]_{\hat{\mathfrak{L}}},\sigma(y),\sigma(b)]_{\hat{\mathfrak{L}}} - [u,\sigma(x),\sigma([y,a,b])]_{\hat{\mathfrak{L}}}\\
&+ [\sigma(y),\sigma(a),[u,\sigma(x),\sigma(b)]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}}\\
=&[[u,\sigma(x),\sigma(y)]_{\hat{\mathfrak{L}}},\sigma(a),\sigma(b)]_{\hat{\mathfrak{L}}} + [\sigma(y),[u,\sigma(x),\sigma(a)]_{\hat{\mathfrak{L}}},\sigma(b)]_{\hat{\mathfrak{L}}} - [u,\sigma(x),[\sigma(y),\sigma(a),\sigma(b)]_{\hat{\mathfrak{L}}})]_{\hat{\mathfrak{L}}}\\
&+ [\sigma(y),\sigma(a),[u,\sigma(x),\sigma(b)]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}}\\
=&0,\\
&\vartheta(a,b)\mathcal{D}(x,y)u - \mathcal{D}(x,y)\vartheta(a,b)u + \vartheta([x,y,a],b)u + \vartheta(a,[x,y,b])u\\
=&[[\sigma(x),\sigma(y),u]_{\hat{\mathfrak{L}}},\sigma(a),\sigma(b)]_{\hat{\mathfrak{L}}} - [\sigma(x),\sigma(y),[u,\sigma(a),\sigma(b)]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}} + [u,\sigma([x,y,a]),\sigma(b)]_{\hat{\mathfrak{L}}}\\
&+ [u,\sigma(a),\sigma([x,y,b])]_{\hat{\mathfrak{L}}}\\
=&[[\sigma(x),\sigma(y),u]_{\hat{\mathfrak{L}}},\sigma(a),\sigma(b)]_{\hat{\mathfrak{L}}} - [\sigma(x),\sigma(y),[u,\sigma(a),\sigma(b)]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}} + [u,[\sigma(x),\sigma(y),\sigma(a)]_{\hat{\mathfrak{L}}},\sigma(b)]_{\hat{\mathfrak{L}}}\\
&+ [u,\sigma(a),[\sigma(x),\sigma(y),\sigma(b)]_{\hat{\mathfrak{L}}}]_{\hat{\mathfrak{L}}}\\
=&0.
\end{aligned}$$

In addition, $\hat{d}(\sigma(x)) - \sigma(d(x)) \in \mathfrak{V} \cong \ker(p)$, means that $[u, \sigma(d(x)), \sigma(y)]_{\hat{\mathfrak{L}}} = [u, \hat{d}(\sigma(x)), \sigma(y)]_{\hat{\mathfrak{L}}}$. Thus, we have

$$\begin{aligned}
&d_{\mathfrak{V}}(\vartheta(x,y)u) = d_{\mathfrak{V}}([u,\sigma(x),\sigma(y)]_{\hat{\mathfrak{L}}})\\
=&[d_{\mathfrak{V}}(u),\sigma(x),\sigma(y)]_{\hat{\mathfrak{L}}} + [u,\hat{d}(\sigma(x)),\sigma(y)]_{\hat{\mathfrak{L}}} + [u,\sigma(x),\hat{d}(\sigma(y))]_{\hat{\mathfrak{L}}} + \lambda[u,\sigma(x),\sigma(y)]_{\hat{\mathfrak{L}}}\\
=&[d_{\mathfrak{V}}(u),\sigma(x),\sigma(y)]_{\hat{\mathfrak{L}}} + [u,\sigma(d(x)),\sigma(y)]_{\hat{\mathfrak{L}}} + [u,\sigma(x),\sigma(d(y)))]_{\hat{\mathfrak{L}}} + \lambda[u,\sigma(x),\sigma(y)]_{\hat{\mathfrak{L}}}\\
=&\vartheta(x,y)d_{\mathfrak{V}}(u) + \vartheta(d(x),y)u + \vartheta(x,d(y))u + \lambda\vartheta(x,y)u.
\end{aligned}$$

Hence, $(\mathfrak{V}, \vartheta, d_{\mathfrak{V}})$ is a representation over $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$. □

We further define linear maps $\varsigma : \mathfrak{L}\times\mathfrak{L}\times\mathfrak{L} \to \mathfrak{V}$ and $\varpi : \mathfrak{L} \to \mathfrak{V}$ respectively by

$$\begin{aligned}
\varsigma(a,b,c) &= [\sigma(a),\sigma(b),\sigma(c)]_{\hat{\mathfrak{L}}} - \sigma([a,b,c]),\\
\varpi(a) &= \hat{d}(\sigma(a)) - \sigma(d(a)), \quad \forall a,b,c \in \mathfrak{L}.
\end{aligned}$$

We transfer the modified $\lambda$-differential Lie triple system structure on $\hat{\mathfrak{L}}$ to $\mathfrak{L}\oplus\mathfrak{V}$ by endowing $\mathfrak{L}\oplus\mathfrak{V}$ with a multiplication $[\cdot,\cdot,\cdot]_\varsigma$ and the modified $\lambda$-differential operator $d_\varpi$ defined by

$$[a+u, b+v, c+w]_\varsigma = [a,b,c] + \vartheta(b,c)u - \vartheta(a,c)v + \mathcal{D}(a,b)w + \varsigma(a,b,c), \tag{5.1}$$

$$d_\varpi(a+u) = d(a) + \varpi(a) + d_{\mathfrak{V}}(u),\ \forall a,b,c \in \mathfrak{L},\ u,v,w \in \mathfrak{V}. \tag{5.2}$$

**Proposition 5.4.** *The triple $(\mathfrak{L}\oplus\mathfrak{V}, [\cdot,\cdot,\cdot]_\varsigma, d_\varpi)$ is a modified $\lambda$-differential Lie triple system if and only if $(\varsigma,\varpi)$ is a 3-cocycle of the modified $\lambda$-differential Lie triple system $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ with the coefficient in $(\mathfrak{V};\theta, d_{\mathfrak{V}})$.*

*Proof.* The triple $(\mathfrak{L}\oplus\mathfrak{V},[\cdot,\cdot,\cdot]_{\varsigma},d_{\varpi})$ is a modified $\lambda$-differential Lie triple system if and only if

$$
\begin{aligned}
&\varsigma(a,b,c)+\varsigma(b,a,c)=0,\\
&\varsigma(a,b,c)+\varsigma(c,a,b)+\varsigma(b,c,a)=0,\\
&\varsigma(a,b,[x,y,z])+\mathcal{D}(a,b)\varsigma(x,y,z)-\varsigma([a,b,x],y,z)-\varsigma(x,[a,b,y],z)-\varsigma(x,y,[a,b,z])\\
&-\vartheta(y,z)\varsigma(a,b,x)+\vartheta(x,z)\varsigma(a,b,y)-\mathcal{D}(x,y)\varsigma(a,b,z)=0, \qquad (5.3)\\
&\vartheta(b,c)\varpi(a)+\varsigma(d(a),b,c)-\vartheta(a,c)\varpi(b)+\varsigma(a,d(b),c)+\mathcal{D}(a,b)\varpi(c)+\varsigma(a,b,d(c))\\
&+\lambda\varsigma(a,b,c)-\varpi([a,b,c])-d_{\mathfrak{V}}(\varsigma(a,b,c))=0, \qquad (5.4)
\end{aligned}
$$

for any $a,b,c,x,y,z\in\mathfrak{L}$. Using Eqs (5.3) and (5.4), we get $\delta\varsigma=0$ and $\delta\varpi+\Phi\varsigma=0$, respectively. Therefore, $\partial(\varsigma,\varpi)=(\delta\varsigma,\delta\varpi+\Phi\varsigma)=0$, that is, $(\varsigma,\varpi)$ is a 3-cocycle.

Conversely, if $(\varsigma,\varpi)$ satisfying Eqs (5.3) and (5.4), one can easily check that $(\mathfrak{L}\oplus\mathfrak{V},[\cdot,\cdot,\cdot]_{\varsigma},d_{\varpi})$ is a modified $\lambda$-differential Lie triple system. □

Next, we are ready to classify abelian extensions of a modified $\lambda$-differential Lie triple system.

**Theorem 5.5.** *Abelian extensions of a modified $\lambda$-differential Lie triple system $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$ by $(\mathfrak{V},[\cdot,\cdot,\cdot]_{\mathfrak{V}},d_{\mathfrak{V}})$ are classified by the third cohomology group $\mathcal{H}^3_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L},\mathfrak{V})$ of $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$ with coefficients in the representation $(\mathfrak{V};\vartheta,d_{\mathfrak{V}})$.*

*Proof.* Let $(\hat{\mathfrak{L}},[\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}},d_{\hat{\mathfrak{L}}})$ be an abelian extension of $(\mathfrak{L},[\cdot,\cdot,\cdot],d)$ by $(\mathfrak{V},[\cdot,\cdot,\cdot]_{\mathfrak{V}},d_{\mathfrak{V}})$. We choose a section $\sigma\colon\ \mathfrak{L}\to\hat{\mathfrak{L}}$ to obtain a 3-cocycle $(\varsigma,\varpi)$ by Proposition 5.4. First, we show that the cohomological class of $(\varsigma,\varpi)$ is independent of the choice of $\sigma$. Let $\sigma_1,\sigma_2\colon\ \mathfrak{L}\to\hat{\mathfrak{L}}$ be two distinct sections providing 3-cocycles $(\varsigma_1,\varpi_1)$ and $(\varsigma_2,\varpi_2)$ respectively. Define linear map $\xi\colon\ \mathfrak{L}\to\mathfrak{V}$ by $\xi(a)=\sigma_1(a)-\sigma_2(a)$. Then

$$
\begin{aligned}
&\varsigma_1(a,b,c)\\
&=[\sigma_1(a),\sigma_1(b),\sigma_1(c)]_{\hat{\mathfrak{L}}_1}-\sigma_1([a,b,c])\\
&=[\sigma_2(a)+\xi(a),\sigma_2(b)+\xi(b),\sigma_2(c)+\xi(c)]_{\hat{\mathfrak{L}}_1}-(\sigma_2([a,b,c])+\xi([a,b,c]))\\
&=[\sigma_2(a),\sigma_2(b),\sigma_2(c)]_{\hat{\mathfrak{L}}_2}+\theta(b,c)\xi(a)-\theta(a,c)\xi(b)+D(a,b)\xi(c)-\sigma_2([a,b,c])-\xi([a,b,c])\\
&=([\sigma_2(a),\sigma_2(b),\sigma_2(c)]_{\hat{\mathfrak{L}}_2}-\sigma_2([a,b,c]))+\theta(b,c)\xi(a)-\theta(a,c)\xi(b)+D(a,b)\xi(c)-\xi([a,b,c])\\
&=\varsigma_2(a,b,c)+\delta\xi(a,b,c)
\end{aligned}
$$

and

$$
\begin{aligned}
\varpi_1(x)&=\hat{d}(\sigma_1(a))-\sigma_1(d(a))\\
&=\hat{d}(\sigma_2(a)+\xi(a))-(\sigma_2(d(a))+\xi(d(a)))\\
&=(\hat{d}(\sigma_2(a))-\sigma_2(d(a)))+\hat{d}(\xi(a))-\xi(d(a))\\
&=\varpi_2(a)+d_{\mathfrak{V}}(\xi(a))-\xi(d(a))\\
&=\xi_2(x)-\Phi\xi(a).
\end{aligned}
$$

i.e., $(\varsigma_1,\varpi_1)-(\varsigma_2,\varpi_2)=(\delta\xi,-\Phi\xi)=\partial(\xi)\in C^3_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L},\mathfrak{V})$. So $(\varsigma_1,\varpi_1)$ and $(\varsigma_2,\varpi_2)$ are in the same cohomological class in $\mathcal{H}^3_{\mathrm{mDLts}^{\lambda}}(\mathfrak{L},\mathfrak{V})$.

Next, assume that $(\hat{\mathfrak{L}}_1, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_1}, \hat{d}_1)$ and $(\hat{\mathfrak{L}}_2, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_2}, \hat{d}_2)$ are two equivalent abelian extensions of $(\mathfrak{L}, [\cdot,\cdot,\cdot], d)$ by $(\mathfrak{V}, [\cdot,\cdot,\cdot]_{\mathfrak{V}}, d_{\mathfrak{V}})$ with the associated isomorphism $\zeta$: $(\hat{\mathfrak{L}}_1, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_1}, \hat{d}_1) \to (\hat{\mathfrak{L}}_2, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_2}, \hat{d}_2)$. Let $\sigma_1$ be a section of $(\hat{\mathfrak{L}}_1, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_1}, \hat{d}_1)$. As $p_2 \circ \zeta = p_1$, we get

$$p_2 \circ (\zeta \circ \sigma_1) = p_1 \circ \sigma_1 = \mathrm{id}_{\mathfrak{L}}.$$

That is, $\zeta \circ \sigma_1$ is a section of $(\hat{\mathfrak{L}}_2, [\cdot,\cdot,\cdot]_{\hat{\mathfrak{L}}_2}, \hat{d}_2)$. Denote $\sigma_2 := \zeta \circ \sigma_1$. Since $\zeta$ is an isomorphism of modified $\lambda$-differential Lie triple systems such that $\zeta|_{\mathfrak{V}} = \mathrm{id}_{\mathfrak{V}}$, we have

$$\begin{aligned}
\varsigma_2(a,b,c) &= [\sigma_2(a), \sigma_2(b), \sigma_2(c)]_{\hat{\mathfrak{L}}_2} - \sigma_2([a,b,c]) \\
&= [\zeta(\sigma_1(a)), \zeta(\sigma_1(b)), \zeta(\sigma_1(c))]_{\hat{\mathfrak{L}}_2} - \zeta(\sigma_1([a,b,c])) \\
&= \zeta([\sigma_1(a), \sigma_1(b), \sigma_1(c)]_{\hat{\mathfrak{L}}_1} - \sigma_1([a,b,c])) \\
&= \zeta(\varsigma_1(a,b,c)) \\
&= \varsigma_1(a,b,c)
\end{aligned}$$

and

$$\begin{aligned}
\varpi_2(a) &= \hat{d}_2(\sigma_2(a)) - \sigma_2(d(a)) = \hat{d}_2(\zeta(\sigma_1(a))) - \zeta(\sigma_1(d(a))) \\
&= \zeta(\hat{d}_1(\sigma_1(a)) - \sigma_1(d(x))) \\
&= \zeta(\varpi_1(a)) \\
&= \varpi_1(a).
\end{aligned}$$

Hence, all equivalent abelian extensions give rise to the same element in $\mathcal{H}^3_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V})$.

Conversely, given two cohomologous 3-cocycles $(\varsigma_1, \varpi_1)$ and $(\varsigma_2, \varpi_2)$ in $\mathcal{H}^3_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V})$, we can construct two abelian extensions $(\mathfrak{L} \oplus \mathfrak{V}, [\cdot,\cdot,\cdot]_{\varsigma_1}, d_{\varpi_1})$ and $(\mathfrak{L} \oplus \mathfrak{V}, [\cdot,\cdot,\cdot]_{\varsigma_2}, d_{\varpi_2})$ via Eqs (5.1) and (5.2). Then, there is a linear map $\xi$: $\mathfrak{L} \to \mathfrak{V}$ such that

$$(\varsigma_1, \varpi_1) - (\varsigma_2, \varpi_2) = \partial(\xi) = (\delta\xi, -\Phi\xi).$$

Define linear map $\zeta_\xi : \mathfrak{L} \oplus \mathfrak{V} \to \mathfrak{L} \oplus \mathfrak{V}$ by $\zeta_\xi(a+u) := a + \xi(a) + u,\ a \in \mathfrak{L}, u \in \mathfrak{V}$. It is obvious that $\zeta_\xi$ is an automorphism on vector space $\mathfrak{L} \oplus \mathfrak{V}$. Moreover, for all $a, b, c \in \mathfrak{L}$ and $u, v, w \in \mathfrak{V}$, by Eqs (5.1) and (5.2), we have

$$\begin{aligned}
&[\zeta_\xi(a+u), \zeta_\xi(b+v), \zeta_\xi(c+w)]_{\varsigma_2} - \zeta_\xi([a+u, b+v, c+w]_{\varsigma_1}) \\
=&[a + \xi(a) + u, b + \xi(b) + v, c + \xi(c) + w]_{\varsigma_2} \\
&- \zeta_\xi([a,b,c] + \vartheta(b,c)u - \vartheta(a,c)v + \mathcal{D}(a,b)w + \varsigma_1(a,b,c)) \\
=&[a,b,c] + \vartheta(b,c)(\xi(a) + u) - \vartheta(a,c)(\xi(b) + v) + \mathcal{D}(a,b)(\xi(c) + w) + \varsigma_2(a,b,c) \\
&- ([a,b,c] + \xi([a,b,c]) + \vartheta(b,c)u - \vartheta(a,c)v + \mathcal{D}(a,b)w + \varsigma_1(a,b,c)) \\
=&\vartheta(b,c)\xi(a) - \vartheta(a,c)\xi(b) + \mathcal{D}(a,b)\xi(c) - \xi([a,b,c]) + \varsigma_2(a,b,c) - \varsigma_1(a,b,c) \\
=&\vartheta(b,c)\xi(a) - \vartheta(a,c)\xi(b) + \mathcal{D}(a,b)\xi(c) - \xi([a,b,c]) - \delta\xi(a,b,c) \\
=&0, \\
&d_{\varpi_2}(\zeta_\xi(a+u)) - \zeta_\xi(d_{\varpi_1}(a+u))
\end{aligned}$$

$$
\begin{aligned}
&= d_{\varpi_2}(a + \xi(a) + u) - \zeta_\xi(d(a) + \varpi_1(a) + d_{\mathfrak{V}}(u)) \\
&= d(a) + \varpi_2(a) + d_{\mathfrak{V}}(\xi(a)) + d_{\mathfrak{V}}(u) - (d(a) + \xi(d(a)) + \varpi_1(a) + d_{\mathfrak{V}}(u)) \\
&= \varpi_2(a) - \varpi_1(a) + d_{\mathfrak{V}}(\xi(a)) - \xi(d(a)) \\
&= \Phi\xi(a) + d_{\mathfrak{V}}(\xi(a)) - \xi(d(a)) \\
&= 0,
\end{aligned}
$$

that is, $\zeta_\xi([a+u, b+v, c+w]_{\varsigma_1}) = [\zeta_\xi(a+u), \zeta_\xi(b+v), \zeta_\xi(c+w)]_{\varsigma_2}$ and $d_{\varpi_2} \circ \zeta_\xi = \zeta_\xi \circ d_{\varpi_1}$. Therefore, $\zeta_\xi$ is an isomorphism of these two abelian extensions. □

## 6. Conclusions

In the current study, we develop a cohomology theory of modified $\lambda$-differential Lie triple systems that controls the 1-parameter formal deformations and abelian extensions of modified $\lambda$-differential Lie triple systems. More precisely, we first give the concept and representation of modified $\lambda$-differential Lie triple systems, and present some examples. So, we establish the cohomology theory of modified $\lambda$-differential Lie triple systems with coefficients in a representation. As applications of the proposed cohomology theory, if the third cohomology group is trivial, then the modified $\lambda$-differential Lie triple system is rigid. Infinitesimals of 1-parameter formal deformations and abelian extensions are classified by the third cohomology group $\mathcal{H}^3_{\mathrm{mDLts}^\lambda}(\mathfrak{L}, \mathfrak{V})$. All the results in this paper can be regarded as generalizations of Lie triple systems with derivations [25–27].

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest in this paper.

## References

1. N. Jacobson, Lie and Jordan triple systems, *J. Am. Math. Soc.*, **71** (1949), 49–170. https://doi.org/10.2307/2372102

2. N. Jacobson, General representation theory of Jordan algebras, *T. Am. Math. Soc.*, **70** (1951), 509–530. https://doi.org/10.2307/1990612
3. W. Lister, A structure theory of Lie triple systems, *T. Am. Math. Soc.*, **72** (1952), 217–242. https://doi.org/10.2307/1990753
4. K. Yamaguti, On the cohomology space of Lie triple system, *Kumamoto J. Sci. Ser. A*, **5** (1960), 44–52. https://doi.org/10.1007/s40840-016-0334-2
5. T. Hodge, B. Parshall, On the representation theory of Lie triple systems, *T. Am. Math. Soc.*, **354** (2002), 4359–4391. https://doi.org/10.2307/3072903
6. B. Harris, Cohomology of Lie triple systems and Lie algebras with involution, *T. Am. Math. Soc.*, **98** (1961), 148–162. https://doi.org/10.2307/1993515
7. F. Kubo, Y. Taniguchi, A controlling cohomology of the deformation theory of Lie triple systems, *J. Algebra*, **278** (2004), 242–250. https://doi.org/10.1016/j.jalgebra.2004.01.005
8. J. Lin, Y. Wang, S. Deng, $T^*$-extension of Lie triple systems, *Linear Algebra Appl.*, **431** (2009), 2071–2083. https://doi.org/10.1016/j.laa.2009.07.001
9. T. Zhang, Notes on cohomologies of Lie triple systems, *J. Lie Theory*, **24** (2014), 909–929.
10. J. F. Ritt, *Differential algebra*, AMS Colloquium Publications, 1950. https://doi.org/10.1007/978-94-010-0854-9-5
11. T. Voronov, Higher derived brackets and homotopy algebras, *J. Pure Appl. Algebra*, **202** (2005), 133–153. https://doi.org/10.1016/j.jpaa.2005.01.010
12. V. Coll, M. Gerstenhaber, A. Giaquinto, An explicit deformation formula with noncommuting derivations, *Ring Theory*, 1989, 396–403.
13. A. R. Magid, *Lectures on differential Galois theory*, University Lecture Series, American Mathematical Society, **7** (1994).
14. V. Ayala, E. Kizil, I. A. Tribuzy, On an algorithm for finding derivations of Lie algebras, *Proyecciones*, **31** (2012), 81–90. https://doi.org/10.4067/S0716-09172012000100008
15. V. Ayala, J. Tirao, *Linear control systems on Lie groups and controllability*, Proc. Sympos. Pure Math., **64**, (1999). https://doi.org/10.1090/pspum/064/1654529
16. I. Batalin, G. Vilkovisky, Gauge algebra and quantization, *Phys. Lett. B*, **102** (1981), 27–31. https://doi.org/10.1016/0370-2693(81)90205-7
17. S. Y. Chou, C. Attanayake, C. Thapa, A homotopy perturbation method for a class of truly nonlinear oscillators, *Ann. Math. Sci. Appl.*, **6** (2021), 3–23. https://doi.org/10.4310/AMSA.2021.v6.n1.a1
18. Y. Lin, Y. Wei, Q. Ye, A homotopy method for multikernel-based approximation, *J. Nonlinear Var. Anal.*, **6** (2022), 139–154.
19. E. Kolchin, *Differential algebra and algebraic groups*, Academic Press, 1973.
20. M. Singer, M. V. Put, *Galois theory of linear differential equations*, Springer, 2003. https://doi.org/10.1017/CBO9780511721564.002
21. L. Loday, On the operad of associative algebras with derivation, *Georgian Math. J.*, **17** (2010), 347–372. https://doi.org/10.1515/GMJ.2010.010

22. R. Tang, Y. Frégier, Y. Sheng, Cohomologies of a Lie algebra with a derivation and applications, *J. Algebra*, **534** (2019), 65–99. https://doi.org/10.1016/j.jalgebra.2019.06.007
23. A. Das, Leibniz algebras with derivations, *J. Homotopy Relat. Str.*, **16** (2021), 245–274. https://doi.org/10.1007/s40062-021-00280-w
24. X. Wu, Y. Ma, B. Sun, L. Chen, Cohomology of Leibniz triple systems with derivations, *J. Geom. Phys.*, **179** (2022), 104594. https://doi.org/10.1016/j.geomphys.2022.104594
25. X. Wu, Y. Ma, B. Sun, L. Chen, Abelian extensions of Lie triple systems with derivations, *Electron. Res. Arch.*, **30** (2022), 1087–1103. https://doi.org/10.3934/era.2022058
26. Q. Sun, S. Chen, Cohomologies and deformations of Lie triple systems with derivations, *J. Algebra Appl.*, **2024** (2024), 2450053. https://doi.org/10.1142/S0219498824500531
27. S. Guo, Central extensions and deformations of Lie triple systems with a derivation, *J. Math. Res. Appl.*, **42** (2022), 189–198. https://doi.org/:10.3770/j.issn:2095-2651.2022.02.009
28. R. Bai, L. Guo, J. Li, Y. Wu, Rota-Baxter 3-Lie algebras, *J. Math. Phys.*, **54** (2013), 063504. https://doi.org/10.1063/1.4808053
29. L. Guo, W. Keigher, On differential Rota-Baxter algebras, *J. Pure Appl. Algebra*, **212** (2008), 522–540. https://doi.org/10.1016/j.jpaa.2007.06.008
30. L. Guo, G. Regensburger, M. Rosenkranz, On integro-differential algebras, *J. Pure Appl. Algebra*, **218** (2014), 456–471. https://doi.org/10.1016/j.jpaa.2013.06.015
31. L. Guo, Y. Li, Y. Sheng, G. Zhou, Cohomology, extensions and deformations of differential algebras with any weights, *Theor. Appl. Categ.*, **38** (2020), 1409–1433. https://doi.org/10.48550/arXiv.2003.03899
32. A. Das, Cohomology and deformations of weighted Rota-Baxter operators, *J. Math. Phys.*, **63** (2022), 091703. https://doi.org/10.1063/5.0093066
33. K. Wang, G. Zhou, Deformations and homotopy theory of Rota-Baxter algebras of any weight, *ArXiv Preprint*, 2021. https://doi.org/10.48550/arXiv.2108.06744
34. A. Das, Cohomology of weighted Rota-Baxter Lie algebras and Rota-Baxter paired operators, *ArXiv Preprint*, 2021. https://doi.org/10.48550/arXiv.2109.01972
35. S. Hou, Y. Sheng, Y. Zhou, 3-post-Lie algebras and relative Rota-Baxter operators of nonzero weight on 3-Lie algebras, *J. Algebra*, **615** (2023), 103–129. https://doi.org/10.1016/j.jalgebra.2022.10.016
36. S. Guo, Y. Qin, K. Wang, G. Zhou, Deformations and cohomology theory of Rota-Baxter 3-Lie algebras of arbitrary weights, *J. Geom. Phys.*, **183** (2023), 104704. https://doi.org/10.1016/j.geomphys.2022.104704
37. S. Chen, Q. Lou, Q. Sun, Cohomologies of Rota-Baxter Lie triple systems and applications, *Commun. Algebra*, **51** (2023), 1–17. https://doi.org/10.1080/00927872.2023.2205938
38. Y. Li, D. Wang, Lie algebras with differential operators of any weights, *Electron. Res. Arch.*, **31** (2022), 1195–1211. https://doi.org/10.3934/era.2023061
39. A. Das, A cohomological study of modified Rota-Baxter algebras, *ArXiv Preprint*, 2022. https://doi.org/10.48550/arXiv.2207.02273

40. Y. Li, D. Wang, Cohomology and Deformation theory of Modified Rota-Baxter Leibniz algebras, *ArXiv Preprint*, 2022. https://doi.org/10.48550/arXiv.2211.09991

41. B. Mondal, R. Saha, Cohomology of modified Rota-Baxter Leibniz algebra of weight $\kappa$, *ArXiv Preprint*, 2022. https://doi.org/10.48550/arXiv.2211.07944

42. J. Jiang, Y. Sheng, Cohomologies and deformations of modified $r$-matrices, *ArXiv Preprint*, 2022. https://doi.org/10.48550/arXiv.2206.00411

43. X. Peng, Y. Zhang, X. Gao, Y. Luo, Universal enveloping of (modified) $\lambda$-differential Lie algebras, *Linear Multilinear A.*, **70** (2022), 1102–1127. https://doi.org/10.1080/03081087.2020.1753641

44. J. Zhou, L. Chen, Y. Ma, Generalized derivations of Lie triple systems, *B. Malays. Math. Sci. So.*, **41** (2018), 637–656. https://doi.org/10.1007/s40840-016-0334-2

45. Y. Sheng, J. Zhao, Relative Rota-Baxter operators and symplectic structures on Lie-Yamaguti algebras, *Commun. Algebra*, **50** (2022), 4056–4073. https://doi.org/10.1080/00927872.2022.2057517

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