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Research article

Dynamic behavior of a stochastic SIR model with nonlinear incidence and recovery rates

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Abstract: The spread of infectious diseases are inevitably affected by natural and social factors, and their evolution presents oscillations and other uncertainties. Therefore, it is of practical significance to consider stochastic noise interference in the studies of infectious disease models. In this paper, a stochastic SIR model with nonlinear incidence and recovery rate is studied. First, a unique global positive solution for any initial value of the system is proved. Second, we provide the sufficient conditions for disease extinction or persistence, and the influence of threshold $\tilde{R}_0$ of the stochastic SIR model on disease state transition is analyzed. Additionally, we prove that the system has a stationary distribution under some given parameter conditions by building an appropriate stochastic Lyapunov function as well as using the equivalent condition of the Hasminskii theorem. Finally, the correctness of these theoretical results are validated by numerical simulations.

Keywords: stochastic SIR model; extinction; persistence in the mean; stochastic Lyapunov function; stationary distribution
Mathematics Subject Classification: 37H05, 37H30, 60H10, 62M15

1. Introduction

The most crucial mathematical model in epidemiology and the prevention of disease is the SIR warehouse model built by Kermack and Mckendrick [1] in 1927. They used the model to examine the transmission laws of the outbreaks of plague in 1906 and the Black Death in 1665 to 1666. The specific model is as follows:

$$\begin{cases} \frac{dS(t)}{dt} = -\beta S(t)I(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t), \\ \frac{dR(t)}{dt} = \gamma I(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$ and $R(t)$ refer to susceptible population, infected population and removed population at time t respectively. β is the coefficient of infectivity and γ is the coefficient of removal rate. Since then, the model had been studied in depth and generalized to various forms by many scientists (see [2–5]).

The incidence of a disease plays an important role in the study of the epidemic model. $\beta S(t)I(t)$ was widely used earlier as a standard incidence to describe the infectivity of the disease (see [6–8]). However, when the population is large or subject to some random factors, the standard incidence rate is not reasonable. A number of nonlinear incidence functions were proposed by many scholars. For example, in 1978, Anderson and May introduced a saturated incidence rate function $\frac{\beta S(t)I(t)}{1+aI(t)}$, where a is the saturation parameter, and $\frac{1}{1+aI(t)}$ describes the saturation caused by interventions when the number of infected people increases [9]. Beddington [10] and DeAngelis et al. [11] independently introduced nonlinear incidence $\frac{\beta S(t)I(t)}{1+a_1S(t)+a_2I(t)}$, a_1 is the saturation coefficient of susceptible individuals, and a_2 is the saturation coefficient of infected individuals. Li and Zhang [12] modified this incidence rate to $\frac{\beta S(t)I(t)}{1+a_1S(t)+a_2I^2(t)}$. In [13], Alshammari and Khan extend the SIR model by introducing nonlinear recovery rate and considering the nonlinear Monod equation for morbidity [14].

Given that reality is full of uncertainty and randomness, the spread of disease will inevitably be affected by various forms of random factors. Spencer [15] points out that since human contact is unpredictable, the growth and spread of infectious diseases are essentially random. Taking random factors into account in epidemic models is conducive to understand the transmission laws of infectious diseases more scientifically and deeply in the real world. There are three methods to introduce stochastic factors into deterministic models in existing literatures: parameter perturbation [16–18], equilibrium disturbance [19–21] and linear disturbance of the system [22–24]. Based on the three methods, a series of generalized epidemic models were studied, and many valuable results were obtained. For example, in [25–27] the SIQS model with isolation and random fluctuations was studied. Stochastic infectious disease models with a standard incidence rate was described in [28, 29]. In [30, 31] the stochastic infectious disease model with delay was introduced. Chen and Kang [32] considered a stochastic vaccination model with backward bifurcation and investigated its asymptotic behavior. In [33], a stochastic heroin epidemic model was investigated and the sufficient conditions for the extinction of the drug users and the existence of ergodic stationary distribution to the model were established respectively. In recent years, the stochastic epidemic model had some new studies. For example, in [34], Bekiros and Kouloumpou provided a new stochastic epidemic model to describe the spread of COVID-19. In 2021, Tocino and Del Rey [35] given a new method that how sufficient conditions for the local stochastic asymptotic stability of a nonlinear system can be derived from the stability analysis of an ordinary linear system.

Inspired by the predecessors' research, we generalized a class of SIR models with nonlinear incidence and recovery rates (2.1) to a stochastic model (2.3) by referring to parametric perturbation methods and analyzed its dynamic characteristics for exploring the influence of stochastic disturbance.

The organization of this paper is as follows: In Section 2, a stochastic epidemic model with nonlinear incidence and recovery rates is formulated. In Section 3, it is proved that system (2.3) has a unique global positive solution. The sufficient conditions for disease extinction and persistence are deduced in Section 4. In Section 5, we prove that the system (2.3) has an ergodic stationary distribution under certain conditions. Corresponding numerical simulations are given to validate the

theoretical results in Section 6. Finally, the paper ends with a brief conclusion.

2. Model formulation

According to the transmission mechanism of SIR infectious diseases, the total population $N(t)$ at t is divided into susceptible individuals $S(t)$, infected individuals $I(t)$ and recovered individuals $R(t)$. That is, $N(t) = S(t) + I(t) + R(t)$.

Based on the above description, Fehaid Salem Alshammari et al. establish the following nonlinear dynamical system composed of nonlinear differential equations in [13] as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \frac{\beta I(t)S(t)}{m+I(t)} - \mu S(t), \\ \frac{dI(t)}{dt} = \frac{\beta I(t)S(t)}{m+I(t)} - \left(\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b+I(t)}\right) I(t) - (\gamma + \mu)I(t), \\ \frac{dR(t)}{dt} = \left(\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b+I(t)}\right) I(t) - \mu R(t), \\ N(t) = S(t) + I(t) + R(t), \end{cases} \tag{2.1}$$

where $N(t)$ is a positive constant and the total population at time t, Λ represents the birth rate, m is the intervention level, μ denotes the natural death rate at each compartment, γ is the disease death rate, β is the disease transmission rate. α_0 and α_1 represent minimum and maximum recovery rates per capita, respectively. Here, b is an indicator of the available resources of hospitals, i.e., the importance of the number of beds in controlling the spread of infectious diseases. $\frac{\beta I(t)S(t)}{m+I(t)}$ refers to transmission rate of individuals moving from $S(t)$ chamber to $I(t)$ chamber. Once an individual is infected, the individual either recovers at a recovery rate of $(\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b+I(t)})$ or dies at a rate of γ.

Since $R(t)$ has no effect on the propagation of disease, only the following reduced system is discussed in [13].

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \frac{\beta I(t)S(t)}{m+I(t)} - \mu S(t), \\ \frac{dI(t)}{dt} = \frac{\beta I(t)S(t)}{m+I(t)} - \left(\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b+I(t)}\right) I(t) - (\gamma + \mu)I(t). \end{cases} \tag{2.2}$$

The basic reputation number of the system (2.2) is $R_0 = \frac{\beta\Lambda}{m\mu(\alpha_1+\gamma+\mu)}$ [13]. According to [13], when $R_0 \leq 1$, the system (2.2) has a globally asymptotically stable disease-free equilibrium point. When $R_0 > 1$, the system (2.2) has the only globally asymptotically stable endemic disease equilibrium point.

Environmental fluctuations have a large impact on all aspects of real life. The influence of random perturbations of white noise on the propagation rate is considered in this paper, that is, disease transmission rate βdt is replaced with $\beta dt + \sigma B(t)$, where $B(t)$ is standard Brownian movement which is defined on the complete probability, and σ denotes the intensity of white noise. Then, the stochastic model is as follows:

$$\begin{cases} dS(t) = \left(\Lambda - \frac{\beta I(t)S(t)}{m+I(t)} - \mu S(t)\right) dt - \frac{\sigma I(t)S(t)}{m+I(t)} dB(t), \\ dI(t) = \left[\frac{\beta I(t)S(t)}{m+I(t)} - \left(\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b+I(t)}\right) I(t) - (\gamma + \mu)I(t)\right] dt + \frac{\sigma I(t)S(t)}{m+I(t)} dB(t). \end{cases} \tag{2.3}$$

In this paper, we assume that model (2.3) will be in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

3. Existence and uniqueness of global positive solutions

In order to study the dynamic behavior of the system (2.3), it is necessary to analyze whether the system has a global positive solution. Some preliminaries need to be introduced for preparation.

In general, we denote

$$\mathbf{R}^n_+ := \{x \in \mathbf{R}^n : x_i > 0\}$$

for all $i = 1, 2, \cdots, n$ and consider an n-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t)$$

with the initial value $x(0) = x_0 \in \mathbf{R}^n$, then the following conclusions are valid [36].
(I) For function $V(x, t) \in C^{2,1}(\mathbf{R}^n \times \mathbf{R}_+; \mathbf{R}_+)$, according to Itô′s formula,

$$dV(x(t), t) = LV(x(t), t)dt + V_x(x(t), t)g(x(t), t)dB(t), \tag{3.1}$$

where L is an operator, and

$$LV(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}trace[g^T(x, t)V_{xx}(x, t)g(x, t)], \tag{3.2}$$

in which

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}), \quad V_{xx} = (\frac{\partial^2 V}{\partial x_i x_j})_{n\times n}.$$

(II)When f, g satisfy
(i) for any $x, y \in \mathbf{R}^n, t \in [t_0, T], H_1 > 0$,

$$|f(x, t) - f(y, t)|^2 \vee |g(x, t) - g(y, t)|^2 \leq H_1 |x - y|, \tag{3.3}$$

(ii) for any $x, y \in \mathbf{R}^n, t \in [t_0, T], H_2 > 0$,

$$|f(x, t)|^2 \vee |g(x, t)|^2 \leq H_2(1 + |x|)^2, \tag{3.4}$$

then for any initial value $x(0) = x_0$, there is a unique global solution to equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t)$$

on the interval $[0, +\infty)$.

Next, we prove that the system (2.3) has a global positive solution by the following theorem.

Theorem 1. *For any given initial value $(S(0), I(0)) \in \mathbf{R}^2_+$, the system (2.3) has a unique positive solution $(S(t), I(t))$, and this solution stays in $\mathbf{R}^2_+$ with probability one. In other words $(S(t), I(t)) \in \mathbf{R}^2_+$ for all $t \geq 0$ almost surely (a.s.).*

Proof. Since the coefficients of the system (2.3) satisfy the local Lipschitz condition, for any given initial value $(S(0), I(0)) \in \mathbf{R}^2_+$ on $t \in (0, \tau_e)$, the system (2.3) has a unique local solution $(S(t), I(t))$, where τ_e is the explosion time. To prove that the solution is global, we only need to prove that $\tau_e = \infty$. Let $k_0 > 0$ is sufficient to make $S(0), I(0)$ falling in the interval $\left[\frac{1}{k_0}, k_0\right]$, and define the stopping time for each integer $k \geq k_0$ as

$$\tau_k = \inf\left\{t \in [0, \tau_e) : S(t) \notin \left(\frac{1}{k}, k\right) or\ I(t) \notin \left(\frac{1}{k}, k\right)\right\}. \tag{3.5}$$

Obviously, τ_k is increasing when $k \to \infty$. Set $\tau_\infty = \lim\limits_{k\to\infty} \tau_k$ and $\tau_\infty \leq \tau_e$. If $\tau_\infty = \infty$ can be proved, $\tau_e = \infty$ a.s., the theorem should be proved. Next we need to prove $\tau_\infty = \infty$ a.s. Suppose this assertion is incorrect, i.e., $\tau_\infty \neq \infty$, there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon. \tag{3.6}$$

Therefore, there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} > \epsilon,\ \forall\ k \geq k_1. \tag{3.7}$$

Let $N_1(t) = S(t) + I(t)$, then

$$\frac{dN_1(t)}{dt} = \Lambda - \mu N_1(t) - \left(\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b + I(t)}\right) I(t) - \gamma I(t). \tag{3.8}$$

Thus

$$N_1(t) \leq N_1(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}), \tag{3.9}$$

that is, $\limsup\limits_{t\to\infty} N_1(t) \leq \frac{\Lambda}{\mu}$.

Define a binary Lyapunov function V: $\mathbf{R}^2_+ \to \mathbf{R}_+$

$$V(S(t), I(t)) = (S(t) - 1 - \ln S(t)) + (I(t) - 1 - \ln I(t)). \tag{3.10}$$

It's non-negative because we can see from $u \geq 0$, $u - 1 - \ln u \geq 0$. Using Itô′s formula to V, we can calculate that

$$dV(S(t), I(t)) = LV(S(t), I(t))dt + \frac{\sigma}{m + I(t)}(I(t) - S(t))dB(t). \tag{3.11}$$

According to the definition of the operator L, we can gain

$$\begin{aligned} LV(S(t), I(t)) &= \left(1 - \frac{1}{S(t)}\right)\left(\Lambda - \frac{\beta I(t)S(t)}{m + I(t)} - \mu S(t)\right) \\ &\quad + \left(1 - \frac{1}{I(t)}\right)\left(\frac{\beta I(t)S(t)}{m(t) + I(t)} - \left(\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b + I(t)}\right) I(t) - (\gamma + \mu)I(t)\right) \\ &\quad + \frac{\sigma^2}{2}\left(\frac{I(t)}{m + I(t)}\right)^2 + \frac{\sigma^2}{2}\left(\frac{S(t)}{m + I(t)}\right)^2 \\ &\leq -\mu S(t) - (\gamma + \mu + \alpha_0)I(t) + \Lambda + 2\mu + \gamma + \alpha_1 + \beta N_1(t) + (\sigma N_1(t))^2. \end{aligned} \tag{3.12}$$

By using of inequality $x - a - a\ln\frac{x}{a} \geq 0$ for any positive constant a, we derive that

$$\begin{aligned}
d(e^{-t}V(S(t),I(t))) &= e^{-t}(-V(S(t),I(t)) + dV(S(t),I(t))) \\
&\leq e^{-t}\left[-S(t) + 1 + \ln S(t) - I(t) + 1 + \ln I(t) - \mu S(t) - (\gamma+\mu+\alpha_0)I(t)\right. \\
&\quad \left. +\Lambda + 2\mu + \gamma + \alpha_1 + \beta N_1(t) + (\sigma N_1(t))^2\right]dt + \frac{\sigma}{m+I(t)}(I(t)-S(t))dB(t) \\
&= e^{-t}\left[-(1+\mu)\left(S(t) - \frac{1}{1+\mu}\ln S(t)\right) - (1+\gamma+\mu+\alpha_0)\,(I(t)\right. \\
&\quad \left. -\frac{1}{1+\gamma+\mu+\alpha_0}\ln I(t)) + 2 + \Lambda + 2\mu + \gamma + \alpha_1 + \beta N_1(t) + (\sigma N_1(t))^2\right]dt \\
&\quad + \frac{\sigma}{m+I(t)}(I(t)-S(t))dB(t) \\
&\leq e^{-t}\left[\ln\frac{1}{1+\mu} + \ln\frac{1}{1+\gamma+\mu+\alpha_0} + 2 + \Lambda + 2\mu + \gamma + \alpha_1 + \beta N_1(t)\right. \\
&\quad \left. +(\sigma N_1(t))^2\right]dt + \frac{\sigma}{m+I(t)}(I(t)-S(t))dB(t) \\
&= e^{-t}Kdt + \frac{\sigma}{m+I(t)}(I(t)-S(t))dB(t),
\end{aligned} \tag{3.13}$$

where

$$K = \left|\ln\frac{1}{1+\mu}\right| + \left|\ln\frac{1}{1+\gamma+\mu+\alpha_0}\right| + 2 + \Lambda + 2\mu + \gamma + \alpha_1 + \beta N_1(t) + (\sigma N_1(t))^2.$$

We set

$$\tilde{V}(S(t),I(t)) = K + V(S(t),I(t)), \tag{3.14}$$

and obtain that

$$d\tilde{V}(S(t),I(t)) \leq \tilde{V}(S(t),I(t))dt - \frac{\sigma}{m+I(t)}(S(t)-I(t))dB(t). \tag{3.15}$$

Integrate Eq (3.15) from 0 to $t \wedge \tau_k$ and take expectation

$$E\tilde{V}(S(t\wedge\tau_k),I(t\wedge\tau_k)) \leq \tilde{V}(S(0),I(0)) + \int_0^{\tau} E\tilde{V}(S(t\wedge\tau_k),I(t\wedge\tau_k))dr, \tag{3.16}$$

the following inequality can be obtained by Gronwall inequality

$$E\tilde{V}(S(t\wedge\tau_k),I(t\wedge\tau_k)) \leq \tilde{V}(S(0),I(0))e^T. \tag{3.17}$$

When $k > k_1$, set $\Omega_k = (\tau_k \leq T)$, so $P(\Omega_k) \geq \epsilon$. For every $\omega \in \Omega_k$, at least one of $S(\tau_k\wedge\omega), I(\tau_k\wedge\omega)$ is equal to k or $\frac{1}{k}$, that is $k-1-\ln k$ or $\frac{1}{k}-1-\ln\frac{1}{k} = (\frac{1}{k}-1+\ln k)$, so

$$\tilde{V}(S(\tau_k\wedge\omega),I(\tau_k\wedge\omega)) \geq (k-1-\ln k)\wedge(\frac{1}{k}-1+\ln k). \tag{3.18}$$

Consequently, we have

$$\tilde{V}(S(0),I(0))e^T \geq E(I\Omega_k\tilde{V}(S(\tau_k\wedge\omega),I(\tau_k\wedge\omega)) \geq \epsilon(k-1-\ln k)\wedge(\frac{1}{k}-1+\ln k), \tag{3.19}$$

where $I\Omega_k$ is Ω_k indicator function. Letting $k\to\infty$, one have

$$\infty > \tilde{V}(S(0),I(0))e^T = \infty. \tag{3.20}$$

This is a contradiction, therefore $\tau_\infty = \infty$. The theorem is proved. □

4. Extinction and persistence of disease of system (2.3)

Before the main results of this section, we introduce the relevant stochastic theory knowledge used in this section.

(I) Burkholder-Davis-Gundy inequality [36]. Assumption $g \in \ell^2(\mathbb{R}_+, \mathbb{R}^{n\times m})$. For $t \geq 0$, define

$$x(t) = \int_0^t g(s)dB(s), \quad A(t) = \int_0^t |g(s)|^2 \, ds,$$

then for any $p > 0$, there are constants c_p and C_p that are only dependent on p, such that

$$c_p E\,|A(t)|^{\frac{p}{2}} \leq E(\sup_{0\leq s\leq t} |x(s)|^p) \leq C_p E\,|A(t)|^{\frac{p}{2}}, \quad t \geq 0.$$

(II) Doob's martingale inequality theorem [36]. Let $\{M_t\}_{t\geq 0}$ be the martingale on R_d and $[a, b]$ be the bounded interval on R_+.

(i) If $p \geq 1$, $M_t \in L^p(\Omega; R^d)$, for $c > 0$, then

$$\mathbb{P}\left\{\omega : \sup_{a\leq t\leq b} |M_t(\omega)| \geq c\right\} \leq \frac{E\,|M_b|^p}{c^p}.$$

(ii) If $p > 1$, $M_t \in L^p(\Omega; R^d)$, then

$$E(\sup_{a\leq t\leq b} |M_t|^p) \leq (\frac{p}{p-1})^p E\,|M_b|^p\,.$$

(III) Borel-Cantelli theorem [37]. If $B_k \in \mathcal{F}(k = 1, 2, \cdots)$, and $\sum_{k=1}^{\infty} \mathbb{P}(B_k) < \infty$, then

$$\mathbb{P}(\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} B_k) = 0.$$

(IV) Strong number theorem [38]. If $M = \{M_t\}_{t\geq 0}$ is real-valued continuous and there is a local martingale with $M(0) = 0$, then it satisfies

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \quad a.s. \Rightarrow \lim_{t\to\infty} \frac{M}{\langle M, M\rangle_t} = 0 \quad a.s.,$$

and has

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{M_t} \leq \infty \quad a.s. \Rightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s.$$

4.1. Disease extinction

Sufficient conditions for the almost inevitable extinction of disease of system (2.3) is deduced in this subsection.

Lemma 1. *Let* $(S(t), I(t))$ *be the solution of system (2.3),* $(S(0), I(0)) \in \mathbf{R}^2_+$*, then*

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \frac{\sigma S(t)}{m + I(t)} dB(t) = 0. \tag{4.1}$$

Proof. Set

$$F(t) = \int_0^t \frac{\sigma S(t)}{m + I(t)} dB(t)$$

for any $\theta > 2$. According to Burkholder-Davis-Gundy inequality [36] and Theorem 1, we have

$$E\left[\sup_{0\le\tau\le t} |F(\tau)|^\theta\right] \le C_\theta E\left[\int_0^t \frac{\sigma^2 S^2(\tau)}{(m + I(\tau))^2} d\tau\right]^{\frac{\theta}{2}} \le C_\theta t^{\frac{\theta}{2}} E\left[\sup_{0\le\tau\le t} \frac{\sigma^\theta S^\theta(\tau)}{(m + I(\tau))^\theta}\right] \le Z_\theta C_\theta t^{\frac{\theta}{2}}, \tag{4.2}$$

where C_θ is a constant related to θ, $Z_\theta = \left(\frac{\sigma\Lambda}{\mu}\right)^\theta$. For any $0 < \varepsilon < \frac{\theta}{2} - 1$, by the Doob's martingale inequality [36], one have

$$\mathbb{P}\left\{\omega : \sup_{d\delta\le t\le(d+1)\delta} |F(t)|^\theta > (d\delta)^{1+\varepsilon+\frac{\theta}{2}}\right\} \le \frac{E(|F((d+1)\delta)|^\theta)}{(d\delta)^{1+\varepsilon+\frac{\theta}{2}}} \le \frac{Z_\theta C_\theta [(d+1)\delta]^{\frac{\theta}{2}}}{(d\delta)^{1+\varepsilon+\frac{\theta}{2}}} \le \frac{2^{\frac{\theta}{2}} Z_\theta C_\theta}{(d\delta)^{1+\varepsilon}}, \tag{4.3}$$

where δ is an arbitrary real number and d is a finite constant. For almost all $\omega \in \Omega$, the following inequality holds for all but finitely many k by the Borel-Cantelli theorem [37]:

$$\sup_{d\delta\le t\le(d+1)\delta} |F(t)|^\theta \le (d\delta)^{1+\varepsilon+\frac{\theta}{2}}. \tag{4.4}$$

Therefore, for almost all $\omega \in \Omega$, there is a positive number $d_0(\omega)$, when $d \ge d_0(\omega)$ the above inequality holds. Moreover, if $d \ge d_0(\omega)$ and $d\delta \le t \le (d+1)\delta$ hold, for almost all $\omega \in \Omega$, we have

$$\frac{\ln |F(t)|^\theta}{\ln t} \le \frac{(1 + \varepsilon + \frac{\theta}{2}) \ln(d\delta)}{\ln(d\delta)} = 1 + \varepsilon + \frac{\theta}{2}, \tag{4.5}$$

so we have

$$\lim_{t\to+\infty} sup \frac{\ln |F(t)|}{\ln t} \le \frac{1 + \varepsilon + \frac{\theta}{2}}{\theta}. \tag{4.6}$$

Let $\varepsilon \to 0$, it can be obtained that

$$\lim_{t\to+\infty} sup \frac{\ln |F(t)|}{\ln t} \le \frac{1}{2} + \frac{1}{\theta}. \tag{4.7}$$

Then, for the above small positive $\varepsilon(\varepsilon < \frac{1}{2} - \frac{1}{\theta})$, there exists a constant $T(\omega)$ and a set Ω_ε such that $P(\Omega_\varepsilon) \ge 1 - \varepsilon$, $t \ge T(\omega)$, $\omega \in \Omega_\varepsilon$,

$$\ln \|F(t)\| \le (\frac{1}{2} + \frac{1}{\theta} + \varepsilon) \ln t. \tag{4.8}$$

Therefore

$$\lim_{t\to+\infty} \sup \frac{|F(t)|}{t} \le \lim_{t\to\infty} \frac{t^{\frac{1}{2}+\frac{1}{\theta}+\varepsilon}}{t} = 0. \tag{4.9}$$

Notice that

$$0 \le \lim_{t\to+\infty} \inf \frac{|F(t)|}{t},$$

then we have

$$\lim_{t\to+\infty} \frac{|F(t)|}{t} = 0, a.s., \quad \lim_{t\to+\infty} \frac{F(t)}{t} = \lim_{t\to+\infty} \frac{1}{t} \int_0^t \frac{\sigma S(t)}{m + I(t)} dB(t) = 0, \tag{4.10}$$

the proof completes. □

Theorem 2. *Let $(S(t), I(t))$ be the solution of the system (2.3) with any initial value $(S(0), I(0)) \in \mathbf{R}^2_+$.*
(i) If $\sigma^2 > \max\left\{\frac{\beta m\mu}{\Lambda}, \frac{\beta^2}{2(\mu+\alpha_1+\gamma)}\right\}$, then $\lim_{t\to+\infty} I(t) = 0$ a.s.
(ii) If $\sigma^2 < \frac{\beta m\mu}{\Lambda}$ and $\tilde{R}_0 < 1$ then $\lim_{t\to+\infty} I(t) = 0$ a.s., where

$$\tilde{R}_0 = R_0 - \frac{\sigma^2\Lambda^2}{2m^2\mu^2(\alpha_1+\gamma+\mu)},$$

and R_0 is the basic regeneration number noted in Section 2.

Proof. Since $(S(t), I(t))$ is the solution of system (2.3) and satisfies the initial condition $(S(0), I(0)) \in \mathbf{R}^2$, we can obtain by applying Itô's formula

$$d\ln I(t) = \left[\frac{\beta S(t)}{m+I(t)} - \left(\alpha_0 + (\alpha_1-\alpha_0)\frac{b}{b+I(t)}\right) - (\gamma+\mu) - \frac{\sigma^2 S^2(t)}{2(m+I(t))^2}\right]dt + \frac{\sigma S(t)}{m+I(t)}dB(t). \quad (4.11)$$

Integrating both sides of Eq (4.11) from 0 to t, we get

$$\begin{aligned}\ln I(t) = &\int_0^t \left[\frac{\beta S(u)}{m+I(u)} - (\alpha_1-\alpha_0)\frac{b}{b+I(u)} - \frac{\sigma^2 S^2(u)}{2(m+I(u))^2}\right]du \\ &- (\alpha_0+\gamma+\mu)t + M(t) + \ln I(0),\end{aligned} \quad (4.12)$$

where $M(t) = \int_0^t \frac{\sigma S(u)}{m+I(u)}dB(u)$ is a locally continuous martingale and $M(0) = 0$. Let's move on to the following two cases:
(i) If $\sigma^2 > \frac{\beta m\mu}{\Lambda}$, by Eq (4.12), we have

$$\ln I(t) \le \frac{\beta^2}{2\sigma^2}t - (\mu+\gamma+\alpha_1)t + M(t) + \ln I(0). \quad (4.13)$$

Divide both sides of Eq (4.13) by t:

$$\frac{\ln I(t)}{t} \le \frac{\beta^2}{2\sigma^2} - (\mu+\gamma+\alpha_1) + \frac{M(t)}{t} + \frac{\ln I(0)}{t}. \quad (4.14)$$

It can be seen from Lemma 1 that

$$\lim_{t\to\infty}\frac{M(t)}{t} = 0. \quad (4.15)$$

When $\sigma^2 > \frac{\beta^2}{2(\mu+\alpha_1+\gamma)}$, taking the limit of both sides of Eq (4.13), we get

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \le -\left(\mu+\gamma+\alpha_1-\frac{\beta^2}{2\sigma^2}\right) < 0. \quad (4.16)$$

So when $\sigma^2 > \max\left\{\frac{\beta m\mu}{\Lambda}, \frac{\beta^2}{2(\mu+\alpha_1+\gamma)}\right\}$, we have

$$\lim_{t\to\infty} I(t) = 0.$$

(ii) If $\sigma^2 < \frac{\beta m\mu}{\Lambda}$ is satisfied, note that

$$\ln I(t) \le \frac{\Lambda}{m\mu}\left(\beta - \frac{\sigma^2}{2}\left(\frac{\Lambda}{m\mu}\right)\right)t - (\mu+\alpha_1+\gamma)t + M(t) + \ln I(0), \quad (4.17)$$

thus

$$\frac{\ln I(t)}{t} \leq (\mu + \alpha_1 + \gamma)\left[\frac{\beta\Lambda}{m\mu(\mu + \alpha_1 + \gamma)} - \frac{\sigma^2\Lambda^2}{2m^2\mu^2(\mu + \alpha_1 + \gamma)} - 1\right] + \frac{M(t)}{t} + \frac{\ln I(0)}{t}. \tag{4.18}$$

Take the limit of both sides of inequality (4.18)

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq (\mu + \alpha_1 + \gamma)(\tilde{R_0} - 1). \tag{4.19}$$

Because of $\tilde{R_0} < 1$, we get

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} < 0, \tag{4.20}$$

that is

$$\lim_{t\to\infty} I(t) = 0.$$

□

Theorem 2 shows that when $\sigma^2 > \max\left\{\frac{\beta m\mu}{\Lambda}, \frac{\beta^2}{2(\mu+\alpha_1+\gamma)}\right\}$, the infectious disease of model (2.3) must be extinct, that is, the random disturbance with strong white noise is beneficial to control the infectious disease.

4.2. Disease persistence in the mean

We focus on the behavior of diseases over time, and analyze the persistence of diseases in this subsection. A necessary definition is given first.

Definition 1. *([39]) System (2.3) is considered to be the average value of persistence if* $\lim_{t\to\infty} inf \frac{1}{t}\int_0^t I(u)du > 0$ *a.s.*

We have the following results:

Theorem 3. *If*

$$\tilde{R_0} = \frac{\beta\Lambda}{m\mu(\alpha_1 + \gamma + \mu)} - \frac{\sigma^2\Lambda^2}{2m^2\mu^2(\alpha_1 + \gamma + \mu)} = R_0 - \frac{\sigma^2\Lambda^2}{2m^2\mu^2(\alpha_1 + \gamma + \mu)} > 1,$$

for any initial value $(S(0), I(0)) \in \mathbf{R}^2_+$*, the solution* $(S(t), I(t))$ *of the system (2.3) has the following properties:*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(u)du \geq \frac{m\mu(\tilde{R_0} - 1)}{\beta + \mu} \quad a.s.$$

Proof. Integrating system (2.3) yields

$$\begin{aligned}\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} &= \Lambda - \frac{\mu}{t}\int_0^t S(u)du - \frac{\alpha_0 + \gamma + \mu}{t}\int_0^t I(u)du \\ &\quad - \frac{(\alpha_1 - \alpha_0)b}{t}\int_0^t \frac{I(u)}{b + I(u)}du \\ &\leq \Lambda - \frac{\mu}{t}\int_0^t S(u)du - \frac{\alpha_0 + \gamma + \mu}{t}\int_0^t I(u)du,\end{aligned} \tag{4.21}$$

so

$$\frac{1}{t}\int_0^t S(u)du \le \frac{\Lambda}{\mu} - \frac{\alpha_0+\gamma+\mu}{\mu t}\int_0^t I(u)du + \Phi(t), \tag{4.22}$$

where

$$\Phi(t) = -\frac{1}{\mu t}\left[S(t) - S(0) + I(t) - I(0)\right].$$

By Itô′*s* formula, one can get that

$$\begin{aligned} d(\ln(mI(t)) + I(t)) &= \left[\beta S(t) - (m+I(t))\left(\alpha_0 + (\alpha_1-\alpha_0)\frac{b}{b+I(t)} + (\gamma+\mu)\right)\right. \\ &\quad \left. -\frac{\sigma^2 S(t)^2 m}{2(m+I(t))^2}\right]dt + \sigma S(t)dB(t) \\ &\ge \left[\beta S(t) - (m+I(t))(\alpha_1+\gamma+\mu) - \frac{\sigma^2 S(t)^2}{2m}\right]dt + \sigma S(t)dB(t). \end{aligned} \tag{4.23}$$

Integrating this inequality from 0 to t and dividing by t on both sides, it leads to

$$\begin{aligned} & m\frac{\ln I(t) - \ln I(0)}{t} + \frac{I(t)-I(0)}{t} \ge \frac{\beta}{t}\int_0^t S(u)du - m(\alpha_1+\gamma+\mu) - \frac{\sigma^2\Lambda^2}{2m\mu^2} \\ &\quad - \frac{(\alpha_1+\gamma+\mu)}{t}\int_0^t I(u)du + \frac{\sigma}{t}\int_0^t S(u)dB(u) \\ &\ge \frac{\beta\Lambda}{\mu} - m(\alpha_1+\gamma+\mu) - \frac{\sigma^2\Lambda^2}{2m\mu^2} - \left[\frac{\beta(\alpha_1+\gamma+\mu)}{\mu} + (\alpha_1+\gamma+\mu)\right]\frac{1}{t}\int_0^t I(u)du \\ &\quad + \beta\Phi(t) + \frac{Z(t)}{t} \\ &= m(\tilde{R}_0 - 1)(\alpha_1+\gamma+\mu) - \left[\frac{\beta(\alpha_1+\gamma+\mu)}{\mu} + (\alpha_1+\gamma+\mu)\right]\frac{1}{t}\int_0^t I(u)du + \beta\Phi(t) + \frac{Z(t)}{t}. \end{aligned} \tag{4.24}$$

It can be obtained from the above inequality

$$\frac{1}{t}\int_0^t I(u)du \ge \frac{1}{W}\left[m(\tilde{R}_0-1)(\alpha_1+\gamma+\mu) + \beta\Phi(t) + \frac{Z(t)}{t} - m\frac{\ln I(t) - \ln I(0)}{t} - \frac{I(t)-I(0)}{t}\right], \tag{4.25}$$

where

$$W = \frac{\beta(\alpha_1+\gamma+\mu)}{\mu} + (\alpha_1+\gamma+\mu),$$

and

$$Z(t) = \sigma\int_0^t S(u)du$$

is a local continuous martingale and $Z(0) = 0$. Furthermore,

$$\limsup_{t\to+\infty}\frac{\langle Z(t), Z(t)\rangle}{t} \le \frac{\sigma^2\Lambda^2}{\mu^2} < +\infty$$

a.s. It follows from the large number theorem for martingales (see [38]) that $\lim\limits_{t\to+\infty}\frac{Z(t)}{t}=0$ a.s. By Theorem 1, one can obtain that$-\infty<\ln I(t)<\ln\frac{\Lambda}{\mu}$ and $\lim\limits_{t\to+\infty}\Phi(t)=0$ a.s. Taking the limit inferior on both sides of (4.26) yields

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t I(u)du\geq\frac{m\mu(\tilde{R}_0-1)}{\beta+\mu}\quad a.s. \tag{4.26}$$

This completes the proof of Theorem 3. □

5. Existence of the stationary distribution of the system (2.3)

We are interested in how long the disease persists in the population. In deterministic models, this problem can be solved by proving the global asymptotical stability of the endemic equilibrium point of the model. However, for the stochastic system (2.3), the above-mentioned method is not applicable. In this section, the equivalent condition of the Hasminskii theorem [40] is used to obtain the traversal smooth distribution of the system (2.3), which indicates that the disease will continue.

Let $X(t)$ be the Markov process defined in the state space $\mathbf{R}_+^d$, and satisfy

$$dX(t)=b(X(t))dt+\sum_{r=1}^{d}\sigma_r(X(t))dB_r(t). \tag{5.1}$$

The corresponding diffusion matrix is

$$A(X)=(a_{ij}(x)),\quad a_{ij}(x)=\sum_{r=1}^{d}\sigma_r^i(x)\sigma_r^j(x), \tag{5.2}$$

where $x\in D$, D is a bounded open subset of $\mathbf{R}^d$ with a regular boundary, then it is necessary to introduce the following lemma.

Lemma 2. *[6] If the bounded region of the regular boundary D, $D\subset\mathbf{R}^d$, and its closure $\bar{D}\subset\mathbf{R}^d$, satisfy the following conditions:*

(i) For any $x\in D$, there are some $i=1,2,3\cdots,n$ and the positive constant $q>0$, such that $a_{ii}(x)\geq q$.

(ii) For any $x\in\mathbf{R}^d\setminus D$, there is a nonnegative C^2-function V that makes LV negative. Then the Markov processes X(t) has a unique ergodic stationary distribution. Let $f(\cdot)$ be integrable with respect to the measure π, then there is

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt=\int_{R^d}f(x)\pi(dx)\right\}=1$$

for all $x\in\mathbf{R}^d$.

Remark 1. *To verify condition (ii), it is sufficient to prove that there is a nonnegative C^2-function and a neighborhood U, so for some $\kappa>0$, $L\phi(x)<-\kappa$, where $x\in\mathbf{R}^d\setminus D$ (see [41]) and L represents the difference operator defined in (3.2).*

It is easy to obtain that the deterministic systems (2.2) have two equilibrium points: a disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0)$ and a disease endemic equilibrium $E_1 = (S_*, I_*)$. We prove the main result of this section.

Theorem 4. *Consider the stochastic systems (2.3) with initial conditions* $(S(0), I(0)) \in \mathbf{R}^2_+$, *and suppose that* $R_0 > 1$, $\tilde{R_0} > 1$ *and*

$$0 < \chi < \min\left\{h_1 S_*^2, h_2 I_*^2\right\}, \tag{5.3}$$

where

$$\chi = \frac{\sigma^2 I_*(m + I_*)(2\mu + \gamma + \alpha_0)}{m^2\beta} S_*^2, \;\; h_1 = \mu - \frac{\sigma^2 I_*(m + I_*)(2\mu + \gamma + \alpha_0)}{m^2\beta}, \;\; h_2 = \alpha_0 + \gamma + \mu.$$

Then there is a stable distribution π, *and the solution* $(S(t), I(t))$ *of system (2.3) is ergodic. In particular, we get*

$$\lim_{t\to+\infty} \sup \frac{1}{t} E \int_0^t [h_1(S(u) - S_*) + h_2(I(u) - I_*)]du \le \chi. \tag{5.4}$$

Proof. Since $R_0 > 1$, the system (2.2) has a positive balance $E_* = (S_*, I_*)$, and

$$\Lambda = \frac{\beta I_* S_*}{m + I_*} + \mu S_*, \;\; \frac{\beta I_* S_*}{m + I_*} = \left[\alpha_0 + \gamma + \mu + (\alpha_1 - \alpha_0)\frac{b}{b + I_*}\right] I_*. \tag{5.5}$$

Define the following function

$$V(S(t), I(t)) = V_1(S(t), I(t)) + \frac{(2\mu + \gamma + \alpha_0)(m + I_*)}{\beta} V_2(I(t)), \tag{5.6}$$

where

$$V_1(S(t), I(t)) = \frac{1}{2}(S(t) + I(t) - S_* - I_*)^2, \; V_2(I(t)) = I(t) - I_* - I_* \ln \frac{I(t)}{I_*}. \tag{5.7}$$

Using the Itô′*s* formula, yields

$$dV(S(t), I(t)) = dV_1(S(t), I(t)) + \frac{(2\mu + \gamma + \alpha_0)(m + I_*)}{\beta} dV_2(I(t)), \tag{5.8}$$

in detail

$$\begin{aligned} dV_1(S(t), I(t)) &= (S(t) + I(t) - S_* - I_*)(dS(t) + dI(t)) + \frac{1}{2}(dS(t) + dI(t))^2 \\ &= LV_1(S(t), I(t))dt, \\ dV_2(I(t)) &= (1 - \frac{I_*}{I(t)})dI(t) + \frac{I_*}{2I^2(t)}(dI(t))^2 \\ &= LV_2(I(t))dt + (I(t) - I*)\frac{\sigma S(t)}{m + I(t)} dB(t), \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} LV_1(S(t), I(t)) = (S(t) + I(t) - S_* - I_*)[&\Lambda - \mu S(t) \\ &- (\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b + I(t)})I(t) - (\gamma + \mu)I(t)], \end{aligned} \tag{5.10}$$

$$LV_2(I(t)) = (I(t) - I_*)[\frac{\beta S(t)}{m + I(t)} - (\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b + I(t)}) - (\gamma + \mu)] + \frac{I_*}{2}\frac{\sigma^2 S^2(t)}{(m + I(t)^2}. \tag{5.11}$$

Substitute Eq (5.5) into Eq (5.10)

$$\begin{aligned} LV_1(S(t), I(t)) &= (S(t) + I(t) - S_* - I_*)[-\mu(S(t) - S_*) + (\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b + I_*})I_* + (r + \mu)I_* \\ &\quad - (\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b + I(t)})I(t) - (\gamma + \mu)I(t)] \\ &\leq (S(t) + I(t) - S_* - I_*)[-\mu(S(t) - S_*) - (\alpha_0 + \gamma + \mu)(I(t) - I_*)] \\ &= -\mu(S(t) - S_*)^2 - (\alpha_0 + \gamma + \mu)(I(t) - I_*)^2 \\ &\quad - (2\mu + \alpha_0 + \gamma)(S(t) - S_*)(I(t) - I_*). \end{aligned} \tag{5.12}$$

Using (5.5), we can get

$$\begin{aligned} LV_2(I(t)) &= (I(t) - I_*)\left[(\frac{\beta S(t)}{m + I(t)} - \frac{\beta S_*}{m + I_*})\right] + \frac{I_*}{2}\frac{\sigma^2 S^2(t)}{(m + I(t))^2} \\ &= (I(t) - I_*)\left[\beta S(t)(\frac{1}{m + I(t)} - \frac{1}{m + I_*}) + \frac{\beta}{m + I_*}(S(t) - S_*)\right] + \frac{I_*}{2}\frac{\sigma^2 S^2(t)}{(m + I(t))^2}. \end{aligned} \tag{5.13}$$

From $m + I(t) \geq m$ and using the inequality $(a + b)^2 \leq 2a^2 + 2b^2$ one can see that

$$\begin{aligned} LV_2(I(t)) &\leq \frac{\beta}{m + I_*}(S(t) - S_*)(I(t) - I_*) + \frac{\sigma^2 I_*}{2m^2}[(S(t) - S_*) + S_*]^2 \\ &\leq \frac{\sigma^2 I_*}{m^2}(S(t) - S_*)^2 + \frac{\beta}{m + I_*}(S(t) - S_*)(I(t) - I_*) + \frac{\sigma^2 I_*}{m^2}S_*^2. \end{aligned} \tag{5.14}$$

Through (5.8), we can deduce

$$LV(S(t), I(t)) = LV_1(S(t), I(t)) + \frac{(2\mu + \gamma + \alpha_0)(m + I_*)}{\beta}LV_2(I(t)). \tag{5.15}$$

Substituting (5.12) and (5.14) into (5.15):

$$\begin{aligned} LV(S(t), I(t)) &\leq -\left[\mu - \frac{\sigma^2 I_*(m + I_*)(2\mu + \gamma + \alpha_0)}{m^2\beta}\right](S(t) - S_*)^2 - (\alpha_0 + \gamma + \mu)(I(t) - I_*)^2 \\ &\quad + \frac{\sigma^2 I_*(m + I_*)(2\mu + \gamma + \alpha_0)}{m^2\beta}S_*^2 \\ &= -h_1(S(t) - S_*)^2 - h_2(I(t) - I_*)^2 + \chi, \end{aligned} \tag{5.16}$$

where h_1, h_3 and χ are defined in theorem 4 respectively, we have

$$dV(S(t), I(t)) \leq -h_1(S(t) - S_*)^2 - h_2(I(t) - I_*)^2 + \chi + (I(t) - I_*)\frac{\sigma S(t)}{m + I(t)}dB(t). \tag{5.17}$$

Integrating the above equation from 0 to t, the following inequality can be obtained:

$$V(S(t), I(t)) - V(S(0), I(0)) \leq \int_0^t [-h_1(S(u) - S_*)^2 - h_2(I(u) - I_*)^2]du + \chi t + M(t), \tag{5.18}$$

where $M(t)$ is a local martingale noted as

$$M(t) = \int_0^t (I(u) - I_*)\frac{\sigma S(u)}{m + I(u)}dB(u).$$

Taking the expectation from both sides of Eq (5.18)

$$EV(S(t), I(t)) - EV(S(0), I(0)) \leq E\int_0^t [-h_1(S(u) - S_*)^2 - h_2(I(u) - I_*)^2]du + \chi t, \tag{5.19}$$

we have

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t [h_1(S(u) - S_*)^2 + h_2(I(u) - I_*)^2]du \leq \chi. \tag{5.20}$$

That is to say, we get Eq (5.4).

Noting that if $0 < \chi < min\left\{h_1 S_*^2, h_2 I_*^2\right\}$, then the ellipsoid

$$-h_1(S(t) - S_*)^2 - h_2(I(t) - I_*)^2 + \chi = 0 \tag{5.21}$$

lies entirely in $\mathbf{R}_+^2$. We can take D as any neighborhood of the ellipsoid such that $\bar{D} \subset \mathbf{R}_+^2$, So for $(S(t), I(t)) \in \mathbf{R}_+^2 \setminus D$, $LV(S(t), I(t)) < 0$, which means that the second condition in Lemma 2 is satisfied.

On the other hand, we can write the system (2.3) as follows

$$d\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} = \begin{pmatrix} \Lambda - \frac{\beta I(t)S(t)}{m+I(t)} - \mu S(t) \\ \frac{\beta I(t)S(t)}{m+I(t)} - (\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b+I(t)})I(t) - (\gamma + \mu)I(t) \end{pmatrix} dt + \begin{pmatrix} -\frac{\sigma\beta I(t)S(t)}{m+I(t)} \\ \frac{\sigma\beta I(t)S(t)}{m+I(t)} \end{pmatrix} dB(t). \tag{5.22}$$

The diffusion matrix corresponding to system (2.3) is

$$A(S(t), I(t)) = \frac{\sigma^2 I^2(t)S^2(t)}{(m + I(t))^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{5.23}$$

Since $\bar{D} \subset \mathbf{R}_+^2$, then

$$a_{11}(S(t), I(t)) = a_{22}(S(t), I(t)) = \frac{\sigma^2 S^2(t)I^2(t)}{(m + I(t))^2} \geq \min_{(S(t),I(t))\in\bar{D}} \frac{\sigma^2 S^2(t)I^2(t)}{(m + I(t))^2} = q. \tag{5.24}$$

Therefore, we verify the condition of Lemma 2, and the proof of Theorem 4 has been completed. □

Remark 2. *From Theorem 4, we have,*

$$\lim_{\sigma^2\to\infty} \chi = 0, \quad \lim_{\sigma^2\to\infty} h_1 = \mu > 0, \quad \lim_{\sigma^2\to\infty} h_2 = \alpha_0 + \gamma + \mu > 0,$$

namely, the solution of system (2.3) fluctuates around the endemic equilibrium $E_*(S_*, I_*)$ of deterministic system (2.2).

6. Numerical simulation

In this section, numerical simulations are used to validate the theoretical results obtained in Sections 4 and 5. To better understand the results, the parameters of models (2.2) and (2.3) are referenced from [13].

First, we use the Milstein method mentioned in Higham [42] to validate the effect of white noise on disease extinction and persistence. According to the Milstein method, model (2.3) can be rewritten as the following discretization equation:

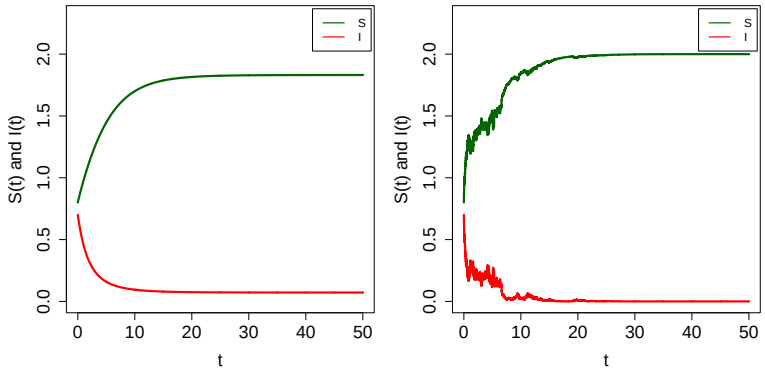
$$\begin{cases} S_{k+1} = S_k + \left(\Lambda - \frac{\beta I_k S_k}{m+I_k} - \mu S_k\right) dt - \frac{\sigma I_k S_k}{m+I_k}\xi_k \sqrt{\Delta t} + \frac{\sigma^2 I_k S_k}{2(m+I_k)^2}(\xi_k^2 - 1)\Delta t, \\ I_{k+1} = I_k + \left(\frac{\beta I_k S_k}{m+I_k} - (\alpha_0 + (\alpha_1 - \alpha_0)\frac{b}{b+I_k})I_k - (\gamma + \mu)I_k\right) dt + \frac{\sigma I_k S_k}{m+I_k}\xi_k \sqrt{\Delta t} + \frac{\sigma^2 I_k S_k}{2(m+I_k)^2}(\xi_k^2 - 1)\Delta t, \end{cases} \tag{6.1}$$

where Δt is time increment and $\xi_k (k = 1, 2, \cdots, n)$ is the Gaussian random variable which follows $N(0, 1)$.

According to [38], the parameters are chosen as

$$\Lambda = 0.6,\ \beta = 0.8,\ m = 2,\ \mu = 0.3,\ \alpha_0 = 0.2,\ \alpha_1 = 0.21,\ b = 0.2,\ \gamma = 0.2,$$

an initial value is selected as $(S(0), I(0)) = (0.8, 0.7)$ for simulation. It is easy to calculate that model (2.2) has a unique stable disease equilibrium point $E_* = (1.8316, 0.0715)$. For comparison, the evolutions of susceptible population and infected population of the deterministic model (2.2) are shown in Figure 1(a). Obviously, the disease is persistent. However, when the model is exposed to strong white noise interference, the disease would go extinct after a while (see Figure 1(b)).



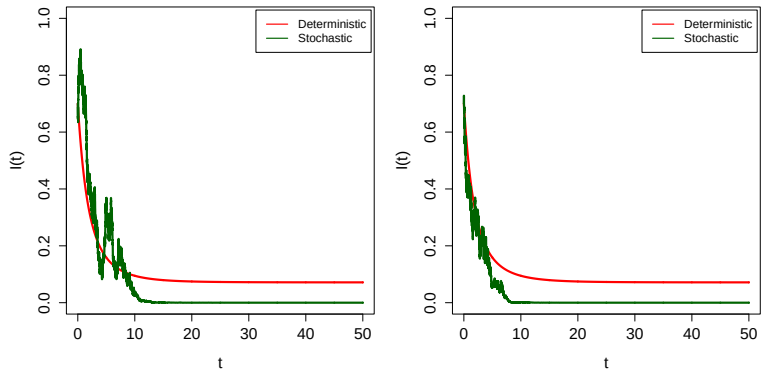
(a) (b)

Figure 1. Time series diagram of $S(t), I(t)$, (a) is the deterministic model and (b) is the stochastic model with $\sigma = 0.9$, where the initial condition and parameter are $S(0) = 0.8, I(0) = 0.7, \Lambda = 0.6, \beta = 0.8, m = 2, \mu = 0.3, \alpha_0 = 0.2, \alpha_1 = 0.21, b = 0.2, \gamma = 0.2, R_0 = 1.1268 > 1$.

The effects of different intensities of white noise on model evolution are shown on Figures 2–4. It can be calculated that

$$\frac{\beta m \mu}{\lambda} = 0.8, \quad \frac{\beta^2}{2(\mu + \alpha_1 + \gamma)} = 0.4507.$$

When $\sigma = 0.95$ or $\sigma = 0.9$, according to (i) of Theorem 2, the disease eliminates and the result is shown in Figure 2.



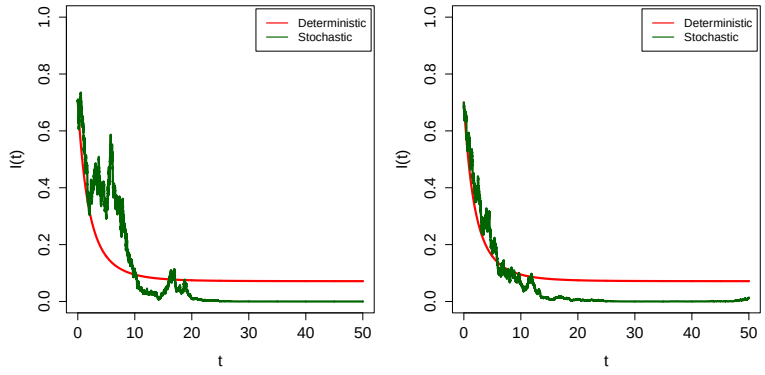
(a) (b)

Figure 2. Comparison of $I(t)$ time series between stochastic and deterministic systems under different white noise intensities, (a): $\sigma = 0.95$ and (b): $\sigma = 0.9$, where the initial condition and parameter are $S(0) = 0.8, I(0) = 0.7, \Lambda = 0.6, \beta = 0.8, m = 2, \mu = 0.3, \alpha_0 = 0.2, \alpha_1 = 0.21, b = 0.2, \gamma = 0.2, R_0 = 1.1268 > 1$.

When

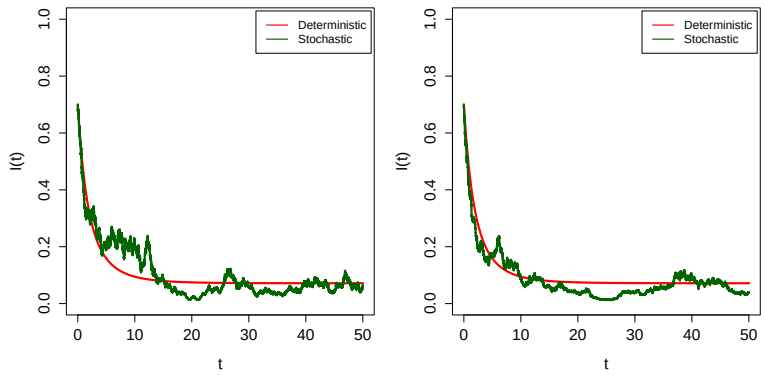
$$\sigma = 0.7,\ \ \tilde{R}_0 = 0.7817 < 1 \ \text{ or } \ \sigma = 0.6,\ \ \tilde{R}_0 = 0.8733 < 1,\ \ \sigma^2 < \frac{\beta m \mu}{\lambda},$$

the disease still eliminates as time goes by. The numerical simulation shown in Figure 3 is consistent with (ii) of Theorem 2. However, when the white noise σ decreases to 0.4 *or* 0.3, it can be seen that disease will continue to exist (see Figure 4). In fact, in this case, $\tilde{R}_0 = 1.0141 > 1$ *or* $\tilde{R}_0 = 1.0634 > 1$.



(a) (b)

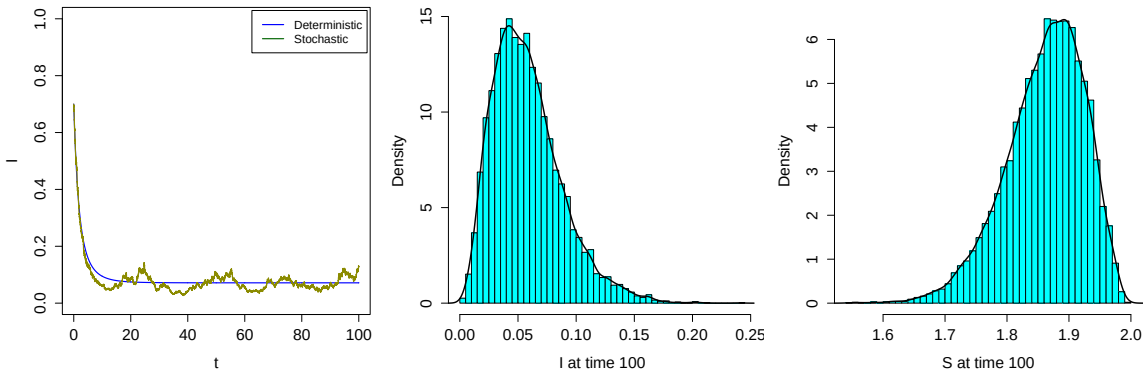
Figure 3. Comparison of $I(t)$ time series between stochastic and deterministic systems under different white noise intensities, (a): $\sigma = 0.7$, $\tilde{R}_0 = 0.7817 < 1$ and (b): $\sigma = 0.6$, $\tilde{R}_0 = 0.8733 < 1$, where the initial condition and parameter are $S(0) = 0.8, I(0) = 0.7, \Lambda = 0.6, \beta = 0.8, m = 2, \mu = 0.3, \alpha_0 = 0.2, \alpha_1 = 0.21, b = 0.2, \gamma = 0.2, R_0 = 1.1268 > 1$.



(a) (b)

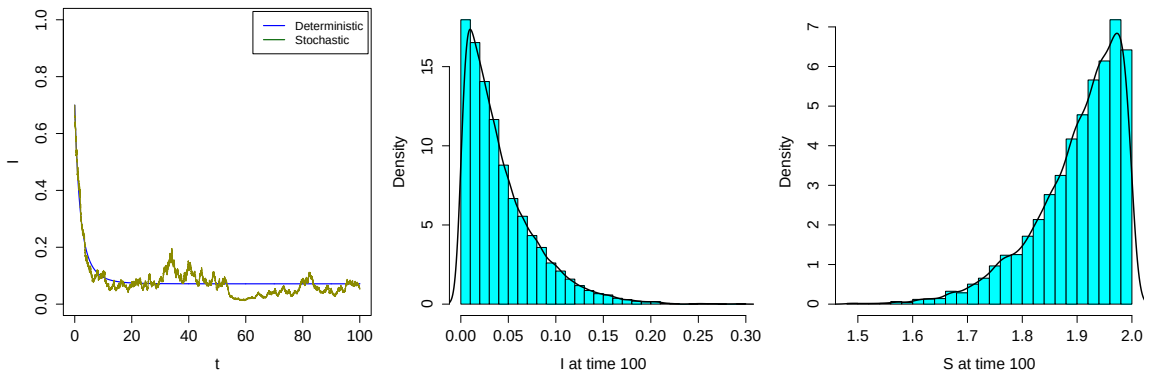
Figure 4. Comparison of $I(t)$ time series between stochastic and deterministic systems under different white noise intensities, (a): $\sigma = 0.4$, $\tilde{R}_0 = 1.0141 > 1$ and (b): $\sigma = 0.3$, $\tilde{R}_0 = 1.0634 > 1$, where the initial condition and parameter are $S(0) = 0.8, I(0) = 0.7, \Lambda = 0.6, \beta = 0.8, m = 2, \mu = 0.3, \alpha_0 = 0.2, \alpha_1 = 0.21, b = 0.2, \gamma = 0.2, R_0 = 1.1268 > 1$.

Next, we give the corresponding numerical simulation for the theoretical analysis of stationary distribution. The histogram and probability density function diagram of $S(t)$ and $I(t)$ distributions of system (2.3) are shown in Figures 5–7 corresponding to different intensities white noise. The selected white noise intensity satisfies the condition of the Theorem 4. In order to better illustrate the results, the longer evolution processes of infected population $I(t)$ in deterministic and stochastic systems are simulated.



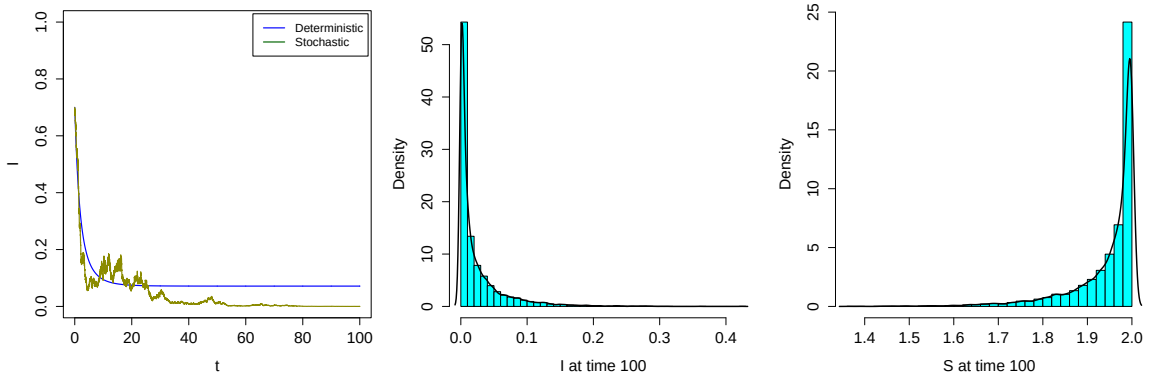
(a) (b) (c)

Figure 5. (a): Trajectory plot of $I(t)$ of the stochastic model (2.3) and its corresponding deterministic model (2.2). $(b), (c)$: when $\sigma = 0.2$, $\tilde{R}_0 = 1.0986 > 1$, the histogram and probability density function plot of $I(t)$ and $S(t)$ of stochastic model (2.3) at $t = 100$.



(a) (b) (c)

Figure 6. (a): Trajectory plot of $I(t)$ of the stochastic model (2.3) and its corresponding deterministic model (2.2). $(b), (c)$: when $\sigma = 0.3$, $\tilde{R}_0 = 1.0634 > 1$, the histogram and probability density function plot of $I(t)$ and $S(t)$ of stochastic model (2.3) at $t = 100$.



(a) (b) (c)

Figure 7. (a): Trajectory plot of $I(t)$ of the stochastic model (2.3) and its corresponding deterministic model (2.2). $(b), (c)$: when $\sigma = 0.4$, $\tilde{R}_0 = 1.0141 > 1$, the histogram and probability density function plot of $I(t)$ and $S(t)$ of stochastic model (2.3) at $t = 100$.

It can be seen from Figures 5–7: when the noise intensity is high, the fluctuation amplitude of $I(t)$ is relatively large before the extinction, and the distribution of infected population and susceptible population is a partial normal distribution (see $(b), (c)$ of Figures 6 and 7). However, when the noise intensity is low, the fluctuation amplitude of $I(t)$ is relatively small, and the distributions of infected population and susceptible population are close to normal distribution. Obviously, when $\tilde{R}_0 > 1$, there exists a unique stationary distribution in the system (2.3).

7. Conclusions and discussion

A lot of studies have proved that introducing random interference into deterministic systems can better reflect the evolution of the real world. In this paper, we study a stochastic SIR model (2.3) with nonlinear incidence rate and recovery rate by replacing the disease transmission rate βdt with $\beta dt + \sigma dB(t)$. First, we prove that the system (2.3) has a unique global positive solution. Then, a sufficient condition based on the relationship of white noise and model parameters is deduced for the extinction of diseases of system (2.3), and the condition of disease persistence is obtained in the sense of the mean. Especially, a threshold value expression

$$\tilde{R_0} = \frac{\beta\Lambda}{m\mu(\alpha_1 + \gamma + \mu)} - \frac{\sigma^2\Lambda^2}{2m^2\mu^2(\alpha_1 + \gamma + \mu)} = R_0 - \frac{\sigma^2\Lambda^2}{2m^2\mu^2(\alpha_1 + \gamma + \mu)}$$

is found by transforming the stochastic model (2.3) into the stratonovich stochastic differential equation model and calculating the basic regeneration number of the corresponding averaging system. The threshold value obtained by the above-mentioned method is available for determining extinction or persistent of the disease and is noted as a basic regeneration number of stochastic epidemic model in some references. In fact, theoretical analysis shows that when $\tilde{R_0} < 1$, the disease becomes extinct under a small disturbance. If the disturbance is large enough, the disease is likely to die out regardless of $\tilde{R_0}$. The results show that random noise has a great influence on the spread of infectious diseases. Besides, under the restriction of some parameters conditions, if threshold of the stochastic system (2.3) satisfies $\tilde{R_0} > 1$, the corresponding stochastic system (2.3) has a ergodic stationary distribution. Furthermore, with the help of mathematical software, some necessary numerical simulations verify the correctness of the theoretical analysis.

Based on theoretical analysis and numerical simulation, we find that the random noise has an impact on the evolution of epidemic model (2.2). We consider that the contact rate is interfered by environmental noise, i.e., $\beta dt \to \beta dt + \sigma dB(t)$, and it is multiplicative noise. It$\hat{o}'s$ formula is recognized as an effective mathematical tool to deal with this kind of model. It is adopted in this paper to obtain some valuable results of the stochastic SIR model (2.3). Due to the inclusion of nonlinear incidence rates, nonlinear recovery rates and random disturbance terms with parameter interference in the model, the theoretical analysis and proof process posed certain difficulties. We have overcome these challenges and achieved favorable results. However, there are some other dynamic properties of the proposed model that have not been completely solved and we will carry out further research in our future work. Otherwise, there are many interfering factors that affect the evolution of the infectious disease model. In future studies, adding random terms from different perspectives to generalize the model and analyze its dynamic characteristics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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