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Research article

Insider trading with dynamic asset under market makers' partial observations

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Abstract: This paper studies an extended continuous-time insider trading model of Calentey and Stacchetti (2010, Econometrica), which allows market makers to observe some partial information about a dynamic risky asset. For each of the two cases with trading until either a fixed time or a random time, we establish the existence and uniqueness of linear Bayesian equilibrium, consisting of insider trading intensity, price pressure on market orders and price pressure on asset observations. It shows that at each of the two equilibria, all information on the risky asset is incorporated in the market price and when the volatility of observation noise keeps constant, the more information observed by market makers, the smaller price pressure on market orders but the greater price pressure on asset observations such that the insider earns less profit and vice versa. It suggests that the partial observation of market makers weakens the information advantage of the insider, which prevents the insider from monopolizing the market to make excessive profit, then reduces the losses of noise traders, thus improving the fairness and effectiveness in the insider trading market.

Keywords: Bayesian equilibrium; continuous-time insider trading; dynamic programming; filtering; partial observation
Mathematics Subject Classification: 60G35, 60H10, 91G80, 93E11

1. Introduction

In recent years, the studying of financial micro-structures and characteristics for risky asset markets has become a hot topic. In a setting of semi-strong effective pricing rule, Kyle [1] first proposed an insider trading model of multi-stage auction with asymmetric information on a static risky asset and proved the existence and uniqueness of its linear sequential equilibrium, consisting of insider trading intensity and market liquidity. It showed that as the time step approaching zero, the equilibrium converges to a continuous-time limit version in which market liquidity is a time-independent constant and all information is incorporated in the market price. Later, Back [2] extended Kyle's model [1] to continuous-time version and also obtained a unique market equilibrium when the risky asset value follows more general distributions, where market liquidity is also independent of time. Collins-Dufresne and Fos [3] generalized Back's model by assuming that the liquidity trading volatility follows a general stochastic process and proved that market depth, market liquidity and price dynamics are characterized by a martingale, submartingale and stochastic Brown Bridge process, respectively. Yang, He and Huang [4] pointed out that even if noise traders have their own trading memories, the properties of equilibrium are similar to those of equilibrium in [3]. Real financial phenomena remind us that the value of risky asset often varies with time. Caldentey and Stacchetti [5] studied an insider trading model, where the risky asset value follows an arithmetic Brownian motion and the trade ends at a random time with life-time distributed. It indicates that in equilibrium both the market liquidity and the insider's value function are exponential functions with respect to time. In the market for defaultable claims, Campi, Çetin and Danilova [6] found that from the market's perspective, the information released by the insider while trading optimally makes the default time predictable in equilibrium. Furthermore, Ma, Sun and Zhou [7] considered the setting that the value of a risky is driven by a conditional mean-field Ornstein-Uhlenback-type dynamic and obtained a closed form of optimal trading intensity. There is much literature on continuous-time insider trading, see [8–16] and so on.

Note that all the above work on insider trading focuses on an insider with perfect information. However, insiders often acquire only partial information on the underlying risky asset. Back, Cao and Willard [17] took the lead in establishing a continuous-time insider trading model of imperfect competition among informed traders. Then, Back, Crotty and Li [18] investigated the case that an insider acquires partial information about a risky asset in a high or low probability way. Under the framework of Collins-Dufresne and Fos [3], Banerjee and Breon-Drish [19,20] assumed that the insider may need to pay a certain cost to acquire the dynamic information flow of asset and demonstrated that the market depth is a semimartingale. Recently, Han, Li, Ma and Kennedy [21] continued to explore insider trading behaviors that noise traders have their own memories of historical trading and found that in a transparent market, to prevent the private information from rapid information leakage, the insider should adopt a mixed insider trading strategy. Qiu and Zhou [22] solved the problem of insider trading that an insider possesses some memory about the on-going observation of the underlying asset. Moreover, Crane, Crotty and Umar [23] illustrated that hedge funds which acquire public information earn higher annualized abnormal returns than nonacquirers. In the study of a new economic benefit of common institutional ownership, Chen, Ma, Wu and Zhang [24] pointed out that there is a significant negative relation between common ownership and insider trading profitability. For a fuzzy model to consider the robust portfolio selection problem of an agent with limited attention, Ma and Li [25] established an explicit solution of robust optimal strategy, which shows that more attention leads to smaller variance in estimated return.

In fact, in real financial markets, market makers can also observe some information on the underlying risky asset. Nishide [26] added the correlation of liquidity trading and the public signal to the Back's model [2] and revealed that market liquidity in equilibrium is not necessarily monotonous with time and the public signal does not necessarily make the market more efficient. Zhou [27] showed that when market makers observe some partial observations on a risky asset at the very beginning, the market liquidity in equilibrium remains a constant and the partial observation makes more information on the risky asset to be incorporated into the market price, thus improving the informativeness of the market. Xiao and Zhou [28] verified this economic intuition further.

In this paper, we study an extension of Caldentey and Stacchetti's [5] continuous-time insider trading model, in which market makers can observe some on-going partial information on a dynamic risky asset. Since market makers observe not only the total market orders, but also some partial information about the dynamic asset, then when pricing they must consider impacts of the two kinds of information. Therefore, we introduce a linear Bayesian equilibrium, which consists of insider trading intensity, price pressure on market orders (market liquidity) and price pressure on asset observations. Then, we establish the existence and uniqueness of equilibrium for insider trading until a fixed time T or a random time τ respectively. There are several findings in our study. First, when trading until at the fixed time T price pressure on market orders is constant as in literature [1, 2, 8, 15, 27], while it is an exponential function of time when trading until the random τ as in [5]. Second, at the end of the transaction in both cases, the insider's private information is completely released. Third, the more information observed by market makers, the smaller the weight that market makers give to price pressure on market orders, while the greater the weight to price pressure on asset observations and vice versa. Finally, the more information the market makers observe, the weaker the information advantage of the insider and the lower the expected profit earned by the insider as in [27]. These results show that the partial observations of market makers can prevent effectively the insider from profiteering by monopolizing information.

It is mentioned that our research is based on pure models, that is, only considering that all agents will process and respond to the information they have in the same way. However, empirically and theoretically it is not the case. For example, in a statistical method, Dang, Foerster, Li and Tang [29] found that financial analysts with high ability can produce firmer specific information through more accurate forecasts, which can effectively reduce the critical information asymmetry between insiders and external investors. In the future research, we will check some famous insider trading events in history and link those to information and market conditions, especially when agents process their information in different ways.

This article consists of six sections. A model of continuous-time insider trading with partial observations is described in Section 2. In Section 3, some necessary conditions on market efficiency are given. The existence and uniqueness of linear Bayesian equilibria for insider trading until a fixed time or a random time is established in Section 4. Some comparative static analyses on properties of equilibria in some special markets are given in Section 5. Conclusions are drawn in Section 6.

2. The model

We here consider an extension model of continuous-time insider trading in [5], which allows market makers to observe some partial information about a dynamic risky asset trading until either a deterministic time T or a random time τ. Since there could be many equilibria in a continuous-time insider trading market [2, 15], then for simplification, only linear strategies for the insider and market makers are considered in our model. All of random variables or processes are assumed to be defined on a complete and filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual condition [30].

In a financial market, there is a risky asset traded in continuous-time whose value v_t evolves as

$$v_t = v_0 + \int_0^t \sigma_{vs} dW_s^v, \tag{2.1}$$

where v_0 is normally distributed as $N(0, \sigma_{v0}^2)$, σ_{vt} is a deterministic, differentiable and positive function, and W_t^v is a standard Brownian motion [5, 16]. There are three types of agents in the market:

(i) *liquidity traders:* who can not observe any information about the value of the risky asset and submit their trading volume z_t randomly [8, 10], which satisfies

$$z_t = \int_0^t \sigma_{zs} dW_s^z, \tag{2.2}$$

where σ_{zt} is a deterministic, differentiable and positive function and W_t^z is a standard Brownian motion

(ii) *an insider:* who is risk-neutral and knows the realization of the risky asset value v_t and the current price p_t of the risky asset, then submits her/his trading volume x_t, evolving as in [1, 3, 5, 8, 15, 20]

$$dx_t = \beta_t(v_t - p_t)dt, \tag{2.3}$$

where β is a trading strategy of the insider, called *insider trading intensity*, which is a deterministic, differentiable and positive function

(iii) *market makers:* who observe the aggregate trading volume

$$y_t = x_t + z_t, \tag{2.4}$$

(but can not observe x_t and z_t separately, otherwise the market maker can infer the perfect information of risky asset through the trading volume submitted by the insider and there is no insider trading) and some partial information of the risky asset as

$$u_t = v_t + \varepsilon_t, \tag{2.5}$$

where $\varepsilon_t = \varepsilon_0 + \int_0^t \sigma_{\varepsilon s} dW_s^\varepsilon$ with ε_0 normally distributed as $N(0, \sigma_{\varepsilon 0}^2)$, $\sigma_{\varepsilon t}$ is a deterministic, differentiable and positive function, W_t^ε is a standard Brownian motion and set for the risky asset a market price p_t, whose dynamic is

$$dp_t = \lambda_{1t} dy_t + \lambda_{2t} du_t, \tag{2.6}$$

where $p_0 = 0$, λ_1 and λ_2 are two deterministic, differentiable and positive functions, called *price pressure on market orders* and *price pressure on asset observations* respectively.

Here, v_0, ε_0, $\{W_t^z\}$, $\{W_t^v\}$ and $\{W_t^\varepsilon\}$ are supposed mutually independent. Denote the insider's information and market makers' information by $\mathcal{F}_t^I = \sigma\{p_s;\ 0 \le s \le t\} \vee \sigma\{v_s;\ 0 \le s \le t\}$ and $\mathcal{F}_t^M = \sigma\{y_s;\ 0 \le s \le t\} \vee \sigma\{u_s;\ 0 \le s \le t\}$, respectively. Then,

Case I: If the trading continues to a fixed time T, the insider's profit of self-financing [2] is

$$E\{\int_0^T \beta_s(v_s - p_s)^2 ds | \mathcal{F}_t^I\}. \tag{2.7}$$

Case II: If the trading continues until a random time τ, which is exponentially distributed with rate $\eta > 0$ and is independent of the history of transactions and prices, the insider's profit of self-financing [5] is

$$E\{\int_0^\tau \beta_s(v_s - p_s)^2 ds | \mathcal{F}_t^I\} = E\{\int_0^\infty e^{-\eta s} \beta_s(v_s - p_s)^2 ds | \mathcal{F}_t^I\}. \tag{2.8}$$

For well-postness in each of the two cases, given any insider trading intensity β of the insider, any price pressure on market orders λ_1 and any price pressure on asset observations λ_2, there must be $E\int_0^T|\beta_s|(v_s-p_s)^2ds<\infty$ in Case I and $E\int_0^\infty e^{-\eta s}|\beta_s|(v_s-p_s)^2ds<\infty$ in Case II. $(\mathcal{S},\mathcal{P})$ denotes the choice space where $\mathcal{S}$ is the set of insider's strategies β and $\mathcal{P}$ is the set of pricing rules (λ_1,λ_2).

Assume that market makers are all risk-neutral and have a Bertrand competition. Then, similar to those in [1,7,10], for any market trade volume y_t, the total profit of market makers should be 0, that is,

$$E[(y_t(v_t-p_t)|\mathcal{F}_t^M]=0$$

or

$$p_t=E[v_t|\mathcal{F}_t^M], \tag{2.9}$$

which implies that the price p_t satisfies *semi-strong market efficiency*.

Now a concept of linear Bayesian equilibrium in our model is given below.

Definition 2.1. *A linear Bayesian equilibrium in the market is a triple* $(\beta,(\lambda_1,\lambda_2))\in(\mathcal{S},\mathcal{P})$ *such that for any time t,*

(i) (maximization of profit) for the given (λ_1,λ_2), function β in Case I (Case II) maximizes

$$E[\int_0^T\beta'_s(v_s-p_s)^2ds|\mathcal{F}_t^I],\ \ (E[\int_0^\infty e^{-\eta s}\beta'_s(v_s-p_s)^2ds|\mathcal{F}_t^I]),\ \ for\,\beta'\in\mathcal{S} \tag{2.10}$$

(ii) (market efficiency) for the given β, (λ_1,λ_2) such that pricing p satisfies

$$p_t=\int_0^t\lambda_{1s}dy_s+\lambda_{2s}du_s=E[v_t|\mathcal{F}_t^M]. \tag{2.11}$$

3. Necessary conditions of market efficiency

Before to establish the existence of linear equilibrium, we first discuss some necessary conditions of market efficiency.

Proposition 3.1. *Let a strategy profile* $(\beta,(\lambda_1,\lambda_2))\in(\mathcal{S},\mathcal{P})$. *If* (λ_1,λ_2) *such that the corresponding market price* p_t *satisfies the market efficiency (2.11). Then,*

$$\lambda_{1t}=\frac{\Sigma_t\beta_t}{\sigma_{zt}^2},\quad \lambda_{2t}=\frac{\sigma_{vt}^2}{\sigma_{vt}^2+\sigma_{\varepsilon t}^2}, \tag{3.1}$$

where the residual information $\Sigma_t=E(v_t-p_t)^2$ satisfies the following dynamic

$$\frac{d\Sigma_t}{dt}=[(1-\lambda_{2t})\sigma_{vt}^2-\lambda_{1t}^2\sigma_{zt}^2] \tag{3.2}$$

with $\Sigma_0=\frac{\sigma_{v0}^2\sigma_{\varepsilon 0}^2}{\sigma_{v0}^2+\sigma_{\varepsilon 0}^2}$.

Proof. According to Eqs (2.1)–(2.6), for market makers there is a signal-observation system of (v_t,ξ_t) satisfying

$$\begin{cases} dv_t=\sigma_{vt}dW_t^v,\\ d\xi_t=(A_0+A_1v_t)dt+B_1dW_{1t}+B_2dW_{2t}, \end{cases} \tag{3.3}$$

where

$v_0 \sim N(0, \sigma_{v0}^2)$, $\xi_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix}$, $\xi_0 = \begin{pmatrix} 0 \\ v_0 + \varepsilon_0 \end{pmatrix}$, $b_1 = \begin{pmatrix} \sigma_v \end{pmatrix}$, $A_0 = \begin{pmatrix} -p_t\beta_t \\ 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} \beta_t \\ 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 \\ \sigma_{vt} \end{pmatrix}$, $B_2 = \begin{pmatrix} \sigma_{zt} & 0 \\ 0 & \sigma_{\varepsilon t} \end{pmatrix}$, $W_{1t} = \begin{pmatrix} W_t^v \end{pmatrix}$, and $W_{2t} = \begin{pmatrix} W_t^z \\ W_t^\varepsilon \end{pmatrix}$.

Denote $B \circ B = B_1 B_1^* + B_2 B_2^*$, $b \circ B = b_1 B_1^* + b_2 B_2^*$ and $b \circ b = b_1 b_1^* + b_2 b_2^*$. Then,

$$B \circ B = \begin{pmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_v^2 + \sigma_\varepsilon^2 \end{pmatrix},\ b \circ B = (0, \sigma_v^2),\ b \circ b = \sigma_v^2.$$

Since p_t satisfies the market efficiency (2.9), that is,

$$p_t = E[v_t | \mathcal{F}_t^M] = E[v_t | \mathcal{F}_t^\xi].$$

Then, Theorem 12.7 in [31] (or see Lemma 3.3 in [27]) tells us that

$$dp_t = \left(\frac{\Sigma_t \beta_t}{\sigma_{zt}^2}, \frac{\sigma_{vt}^2}{\sigma_{vt}^2 + \sigma_{\varepsilon t}^2}\right) d\xi_t = \frac{\Sigma_t \beta_t}{\sigma_{zt}^2} dy_t + \frac{\sigma_{vt}^2}{\sigma_{vt}^2 + \sigma_{\varepsilon t}^2} du_t, \tag{3.4}$$

where the residual information Σ_t satisfies

$$\frac{d\Sigma_t}{dt} = \left(\frac{\sigma_{vt}^2 \sigma_{\varepsilon t}^2}{\sigma_{vt}^2 + \sigma_{\varepsilon t}^2} - \frac{\Sigma_t^2 \beta_t^2}{\sigma_{zt}^2}\right) \tag{3.5}$$

with $\Sigma_0 = \frac{\sigma_{v0}^2 \sigma_{\varepsilon 0}^2}{\sigma_{v0}^2 + \sigma_{\varepsilon 0}^2}$ by Theorem 13.1 in [31].

Since (λ_1, λ_2) satisfies the market efficiency (2.11), that is,

$$p_t = \int_0^t \lambda_{1t} dy_t + \lambda_{2t} du_t = E[v_t | \mathcal{F}_t^M],$$

the results follow from (3.4) and (3.5), and the proof is complete.

4. Existence and uniqueness of linear Bayesian equilibrium

4.1. Case I: When the asset value is released at the fixed time T

Let the pricing profile $(\lambda_1, \lambda_2) \in \mathcal{P}$ be given. Then, for any $(t, m) \in [0, T) \times R$, for any $\beta' \in \mathcal{S}[t, T) = \{\beta' : \beta'(r) = \beta(r), t \le r < T, \beta \in \mathcal{S}\}$, there is a gap process

$$m_s = v_s - p_s,\ s \in [t, T).$$

which by Eqs (2.1)–(2.6), satisfies the stochastic differential equation

$$dm_s = -\lambda_{1s} \beta_s' m_s dt + [(1 - \lambda_{2s})\sigma_{vs}, -\lambda_{2s}\sigma_{\varepsilon s}, -\lambda_{1s}\sigma_{zs}] \begin{bmatrix} dW_s^v \\ dW_s^\varepsilon \\ dW_s^z \end{bmatrix} \tag{4.1}$$

with $m_t = m$. Then, we have the conditional value function

$$J_1(t,m) = \sup_{\beta' \in \mathcal{S}[t,T)} E[\int_t^T \beta'_s m_s^2(\beta') ds | \mathcal{F}_t^I]. \tag{4.2}$$

Clearly this is a classical stochastic control problem, and by employing Proposition 3.5 of dynamic programming principle in [32], we can easily get Hamilton-Jacobi-Bellman equation below with its proof omitted.

Proposition 4.1. *The Hamilton-Jacobi-Bellman equation of insider's value function (4.2) (if it exists in $C^{1,2}([0,T) \times R)$) is driven by*

$$\sup_{\theta \in R}\{\frac{\partial J_1}{\partial t} + \frac{1}{2}[(1-\lambda_{2t})^2\sigma_{vt}^2 + \lambda_{2t}^2\sigma_{\varepsilon t}^2 + \lambda_{1t}^2\sigma_{zt}^2]\frac{\partial^2 J_1}{\partial m^2} + [-\lambda_{1t} m \frac{\partial J_1}{\partial m} + m^2]\theta\} = 0. \tag{4.3}$$

Now the existence and uniqueness of linear Bayesian equilibrium can be given below.

Theorem 4.2. *Let for any time $t \in [0,T)$*

$$\Sigma_0 = \frac{\sigma_{v0}^2\sigma_{\varepsilon 0}^2}{\sigma_{v0}^2 + \sigma_{\varepsilon 0}^2},\ \Gamma_{zt} = \int_t^T \sigma_{zs}^2 ds,\ \Gamma_{v\varepsilon t} = \int_t^T \frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2 + \sigma_{\varepsilon s}^2} ds.$$

Then, if the following inequality holds for any time $t > 0$,

$$\Gamma_{zt}\Sigma_0 + \Gamma_{zt}\Gamma_{v\varepsilon 0} - \Gamma_{z0}\Gamma_{v\varepsilon t} > 0, \tag{4.4}$$

there is a unique linear Bayesian equilibrium $(\beta, (\lambda_1, \lambda_2)) \in (\mathcal{P}, \mathcal{S})$ satisfying

$$\beta_t = \frac{\lambda_{1t}\sigma_{zt}^2}{\Sigma_t},\ \lambda_{1t} \equiv \lambda^*,\ \lambda_{2t} = \frac{\sigma_{vt}^2}{\sigma_{vt}^2 + \sigma_{\varepsilon t}^2} \tag{4.5}$$

with

$$\lambda^* = \sqrt{\frac{\Sigma_0 + \Gamma_{v\varepsilon 0}}{\Gamma_{z0}}},\ \Sigma_t = \frac{\Gamma_{zt}\Sigma_0 + \Gamma_{zt}\Gamma_{v\varepsilon 0} - \Gamma_{z0}\Gamma_{v\varepsilon t}}{\Gamma_{z0}}. \tag{4.6}$$

Also, the residual information Σ_t satisfies

$$\lim_{t \to T} \Sigma_t = 0,$$

the value function is

$$J_1(t, m_t) = \frac{m_t^2}{2\lambda^*} + \frac{\Gamma_{v\varepsilon t}}{2\lambda^*} + \frac{\lambda^*\Gamma_{zt}}{2}, \tag{4.7}$$

and the expected total profit of insider is

$$E(J_1(0, m_0)) = \sqrt{\Gamma_{z0}(\Sigma_0 + \Gamma_{v\varepsilon 0})}.$$

Proof. The proof is broken down into three steps:

Step I: Find a solution J_1 to the Hamilton-Jacobi-Bellman equation (4.3).

According to the Hamilton-Jacobi-Bellman equation (4.3), the following system follows

$$\begin{cases} -\lambda_{1t}\dfrac{\partial J_1}{\partial m}+m=0, \\ \dfrac{\partial J_1}{\partial t}+\dfrac{1}{2}[(1-\lambda_{2t})^2\sigma_{vt}^2+\lambda_{2t}^2\sigma_{\varepsilon t}^2+\lambda_{1t}^2\sigma_{zt}^2]\dfrac{\partial^2 J_1}{\partial m^2}=0. \end{cases} \tag{4.8}$$

By the first equation in the above system, we have

$$\frac{\partial J_1}{\partial m}=\frac{m}{\lambda_{1t}},\quad \frac{\partial^2 J_1}{\partial m^2}=\frac{1}{\lambda_{1t}},\quad \frac{\partial^2 J_1}{\partial t\partial m}=m\frac{d}{dt}(\frac{1}{\lambda_{1t}}). \tag{4.9}$$

So, the second equation in (4.8) implies that

$$\frac{\partial J_1}{\partial t}+\frac{1}{2\lambda_{1t}}[(1-\lambda_{2t})^2\sigma_{vt}^2+\lambda_{2t}^2\sigma_{\varepsilon t}^2+\lambda_{1t}^2\sigma_{zt}^2]=0. \tag{4.10}$$

Then, by differentiating with respect to m,

$$\frac{\partial^2 J_1}{\partial t\partial m}=0.$$

So,

$$\frac{d}{dt}(\frac{1}{\lambda_{1t}})=0,$$

which reveals that λ_{1t} is a constant, denoted by λ^*, that is, $\lambda_{1t}\equiv\lambda^*$.

Equation (4.9) shows that

$$J_1(t,m)=\frac{m^2}{2\lambda_{1t}}+g(t), \tag{4.11}$$

for some determinate, continuous, differentiable function $g(t)$. Now plugging Eq (4.11) into Eq (4.10), we have

$$\frac{m^2}{2}\frac{d(\frac{1}{\lambda_{1t}})}{dt}+\frac{dg(t)}{dt}+\frac{1}{2\lambda_{1t}}[(1-\lambda_{2t})^2\sigma_{vt}^2+\lambda_{2t}^2\sigma_{\varepsilon t}^2+\lambda_{1t}^2\sigma_{zt}^2]=0. \tag{4.12}$$

Since $\frac{d}{dt}(\frac{1}{\lambda_{1t}})=0$, then (4.12) degenerates to

$$\frac{dg(t)}{dt}+\frac{1}{2\lambda_{1t}}[(1-\lambda_{2t})^2\sigma_{vt}^2+\lambda_{2t}^2\sigma_{\varepsilon t}^2+\lambda_{1t}^2\sigma_{zt}^2]=0. \tag{4.13}$$

Thus,

$$g(t)=\frac{\int_t^T\frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds}{2\lambda^*}+\frac{\lambda^*\int_t^T\sigma_{zs}^2ds}{2}.$$

So,

$$J_1(t,m)=\frac{m^2}{2\lambda^*}+\frac{\int_t^T\frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds}{2\lambda^*}+\frac{\lambda^*\int_t^T\sigma_{zs}^2ds}{2}, \tag{4.14}$$

which can be expressed as (4.7).

Step II: Find a necessary condition for the optimal insider trading intensity:

$$\lim_{t\to T}\Sigma_t=0,$$

which means that

$$\lambda^* = \sqrt{\frac{\Sigma_0 + \int_0^T \frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds}{\int_0^T \sigma_{zs}^2 ds}},$$

and

$$E[J_1(0,m_0)] = \sqrt{\Gamma_{z0}(\Sigma_0 + \Gamma_{v\varepsilon 0})}.$$

In fact, for any $\beta' \in \mathcal{S}$, since stochastic process m_t follows (4.1), then using Itô formula to $J_1(t,m_t)$, we observe that

$$\begin{aligned} J_1(t,m_t(\beta')) = J_1(0,m_0) &+ \int_0^t \left\{\frac{\partial J_1}{\partial t} + \frac{1}{2}[(1-\lambda_{2s})^2\sigma_{vs}^2 + \lambda_{2s}^2\sigma_{\varepsilon s}^2 + \lambda_{1s}^2\sigma_{zs}^2]\frac{\partial^2 J_1}{\partial m^2}\right\} ds \\ &- \int_0^t \lambda_{1s}\beta_s' m_s \frac{\partial J_1}{\partial m} ds + \int_0^t \frac{\partial J_1}{\partial m}(1-\lambda_{2s})\sigma_{vs} dW_s^v - \int_0^t \frac{\partial J_1}{\partial m}\lambda_{2s}\sigma_{\varepsilon s} dW_s^\varepsilon \\ &- \int_0^t \frac{\partial J_1}{\partial m}\lambda_{1s}\sigma_{zs} dW_s^z. \end{aligned}$$

According to the properties of the value function J_1 in Step I,

$$\frac{\partial J_1}{\partial t} + \frac{1}{2}[(1-\lambda_{2s})^2\sigma_{vs}^2 + \lambda_{2s}^2\sigma_{\varepsilon s}^2 + \lambda_{1s}^2\sigma_{zs}^2]\frac{\partial^2 J_1}{\partial m^2} = 0,$$

and

$$\frac{\partial J_1}{\partial m} = \frac{m_t}{\lambda_{1t}}.$$

Then, taking conditional expectation on both sides of the above equation, we see that

$$E[J_1(T,m_T(\beta'))] = E[J_1(0,m_0)] - E[\int_0^T \beta_s' m_s^2 ds], \tag{4.15}$$

that is,

$$E[J_1(0,m_0)] = E[J_1(T,m_T(\beta'))] + E[\int_0^T \beta_s' m_s^2 ds]. \tag{4.16}$$

Note that $J_1(T,m_T) \geq 0$, which implies that

$$E[J_1(0,m_0)] \geq E[\int_0^T \beta_s' m_s^2 ds], \tag{4.17}$$

and the equality in (4.17) holds if and only if $E[J_1(T,m_T)] = 0$.
Indeed,

$$E[J_1(t,m_t)] = \frac{\Sigma_t}{2\lambda^*} + \frac{\int_t^T \frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds}{2\lambda^*} + \frac{\lambda^*\int_t^T \sigma_{zs}^2 ds}{2}, \tag{4.18}$$

which tells that $\lim_{t\to T} E[J(t,m_t)] = 0$ if and only if $\lim_{t\to T}\Sigma_t = 0$.

On the other hand, by market efficiency (3.2) and the inequality condition (4.4), we get

$$\Sigma_t = \Sigma_0 + \int_0^t \frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds - (\lambda^*)^2\int_0^t \sigma_{zs}^2 ds. \tag{4.19}$$

So, using the requirement $\lim_{t\to T}\Sigma_t = 0$, we observe that

$$\lambda^* = \sqrt{\frac{\Sigma_0 + \int_0^T \frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds}{\int_0^T \sigma_{zs}^2 ds}}. \tag{4.20}$$

From this it follows that (4.19) satisfies the second equation in (4.6). Plugging (4.20) into (4.18) yields

$$E[J_1(0, m_0)] = \sqrt{\Gamma_{z0}(\Sigma_0 + \Gamma_{v\varepsilon 0})}.$$

Step III: By Proposition 3.1, taking $(\beta, (\lambda_1, \lambda_2))$ as the forms in (4.5), respectively. Thus, it is a market equilibrium:

(i) Given the pricing rule with (λ_1, λ_2), by computing directly, the insider trading intensity β is such that

$$E\int_0^T \beta_s(v_s - p_s)^2 ds = E\int_0^T \beta_s(v_s - p_s)^2 ds = \int_0^T \beta_s\Sigma_s ds = \sqrt{\Gamma_{z0}(\Sigma_0 + \Gamma_{v\varepsilon 0})} = E[J_1(0, m_0)] < \infty,$$

that is, $\beta \in \mathcal{S}$ is optimal.

(ii) Given the insider trading intensity β, the local linear pricing rule p_t with dynamics $dp_t = \lambda_1(t)dy_t + \lambda_2(t)du_t$ must satisfy the market efficiency $p_t = E[v_t|\mathcal{F}_t^M]$. In fact, there exists one signal-observation system in terms of (v_t, ξ_t) following

$$\begin{cases} dv_t = \sigma_{vt}dW_t^v, \\ d\xi_t = (A_0 + A_1 v_t)dt + B_1 dW_{1t} + B_2 dW_{2t}, \end{cases} \tag{4.21}$$

where
$v_0 \sim N(0, \sigma_{v0}^2)$, $\xi_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix}$, $\xi_0 = \begin{pmatrix} 0 \\ v_0 + \varepsilon_0 \end{pmatrix}$, $b_1 = \begin{pmatrix} \sigma_v \end{pmatrix}$, $A_0 = \begin{pmatrix} -p_t\beta_t \\ 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} \beta_t \\ 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 \\ \sigma_{vt} \end{pmatrix}$, $B_2 = \begin{pmatrix} \sigma_{zt} & 0 \\ 0 & \sigma_{\varepsilon t} \end{pmatrix}$, $W_{1t} = \begin{pmatrix} W_t^v \end{pmatrix}$ and $W_{2t} = \begin{pmatrix} W_t^z \\ W_t^\varepsilon \end{pmatrix}$.

Let

$$\widetilde{p_t} = E[v_t|\mathcal{F}_t^M] = E[v_t|\mathcal{F}_t^\xi],$$

with $\widetilde{p_0} = 0$. Then, applying Theorem 12.7 in [31] (or see Lemma 3.3 in [27]) again, we know that

$$d\widetilde{p_t} = (\frac{\Sigma_t\beta_t}{\sigma_{zt}^2}, \frac{\sigma_{vt}^2}{\sigma_{vt}^2 + \sigma_{\varepsilon t}^2})\left[d\xi_t - \begin{pmatrix} (\widetilde{p_t} - p_t)\beta_t \\ 0 \end{pmatrix}\right]. \tag{4.22}$$

Now taking Eq (4.22) minus Eq (3.4) gives

$$d(\widetilde{p_t} - p_t) = \lambda_{1t}\beta_t(\widetilde{p_t} - p_t)dt,$$

with $\widetilde{p_0} - p_0 = 0$, which leads to $\widetilde{p_t} - p_t = 0$ a.s., that is $\widetilde{p_t} = p_t$ a.s. The proof is complete.

4.2. Case II: When the asset value is released at a random time τ

In this subsection, we further consider the case when the risky asset is traded until a random time τ, life-time distributed with parameter $\eta > 0$.

Let the pricing profile $(\lambda_1, \lambda_2) \in \mathcal{P}$ be given. Then, for any $(t, m) \in [0, \infty) \times R$, for any $\beta' \in \mathcal{S}[t, \infty) = \{\beta' : \beta'(r) = \beta(r), t \leq r < \infty, \beta \in \mathcal{S}\}$, the gap process $m_s = v_s - p_s,\ s \in [t, \infty)$ satisfies (4.1). Then, it follows from (2.8) that the conditional value function is

$$J_2(t, m) = \sup_{\beta' \in \mathcal{S}[t,\infty)} E[\int_t^\infty e^{-\eta(s-t)} \beta'_s m_s^2(\beta') ds | \mathcal{F}_t^I]. \tag{4.23}$$

Again we can easily obtain the corresponding Hamilton-Jacobi-Bellman equation below.

Proposition 4.3. *The Hamilton-Jacobi-Bellman equation of insider's value function (4.23) (if it exists in $C^{1,2}([0, T) \times R)$) is driven by*

$$\sup_{\theta \in R} \{\frac{\partial J_2}{\partial t} + \frac{1}{2}[(1 - \lambda_{2t})^2 \sigma_{vt}^2 + \lambda_{2t}^2 \sigma_{\varepsilon t}^2 + \lambda_{1t}^2 \sigma_{zt}^2] \frac{\partial^2 J_2}{\partial m^2} - \eta J_2 + [-\lambda_{1t} m_t \frac{\partial J_2}{\partial m} + m_t^2] \theta\} = 0. \tag{4.24}$$

Proof. By the value function (4.23), it can be written as

$$e^{-\eta t} J_2(t, m) = \sup_{\beta_t \in \mathcal{S}} E[\int_t^\infty e^{-\eta s} \beta_s m_s^2 ds | \mathcal{F}_t^I].$$

Let $\widetilde{J}(t, m) = e^{-\eta t} J_2(t, m)$. Then,

$$\widetilde{J}(t, m) = \sup_{\beta_t \in \mathcal{S}} E[\int_t^\infty e^{-\eta s} \beta_s m_s^2 ds | \mathcal{F}_t^I].$$

Hence, by employing Proposition 3.5 of dynamic programming principle in [32], $\widetilde{J}(t, m)$ satisfies the following Hamilton-Jacobi-Bellman equation

$$\sup_{\theta \in R} \{\frac{\partial \widetilde{J}}{\partial t} + \frac{1}{2}[(1 - \lambda_{2t})^2 \sigma_{vt}^2 + \lambda_{2t}^2 \sigma_{\varepsilon t}^2 + \lambda_{1t}^2 \sigma_{zt}^2] \frac{\partial^2 \widetilde{J}}{\partial m^2} + [-\lambda_{1t} m_t \frac{\partial \widetilde{J}}{\partial m} + m_t^2 e^{-\eta t}] \theta\} = 0.$$

It follows that

$$\sup_{\theta \in R} \{(\frac{\partial J_2}{\partial t} + \frac{1}{2}[(1 - \lambda_{2t})^2 \sigma_{vt}^2 + \lambda_{2t}^2 \sigma_{\varepsilon t}^2 + \lambda_{1t}^2 \sigma_{zt}^2] \frac{\partial^2 J_2}{\partial m^2} - \eta J_2 + [-\lambda_{1t} m_t \frac{\partial J_2}{\partial m} + m_t^2] \theta) e^{-\eta t}\} = 0,$$

which directly leads to the Hamilton-Jacobi-Bellman equation (4.24). This proof is complete.

As in the previous subsection, the existence and uniqueness of linear Bayesian equilibrium can be given below.

Theorem 4.4. *Let for any time $t \geq 0$,*

$$\Sigma_0 = \frac{\sigma_{v0}^2 \sigma_{\varepsilon 0}^2}{\sigma_{v0}^2 + \sigma_{\varepsilon 0}^2}, \quad \Upsilon_{zt} = \int_t^\infty \sigma_{zs}^2 e^{-2\eta s} ds, \quad \Upsilon_{v\varepsilon t} = \int_t^\infty \frac{\sigma_{vs}^2 \sigma_{\varepsilon s}^2}{\sigma_{vs}^2 + \sigma_{\varepsilon s}^2} ds.$$

Then, if the following inequalities hold

$$\Upsilon_{z0} < \infty, \quad \Upsilon_{v\varepsilon 0} < \infty, \quad \Upsilon_{zt} \Sigma_0 + \Upsilon_{zt} \Upsilon_{v\varepsilon 0} - \Upsilon_{z0} \Upsilon_{v\varepsilon t} > 0,$$

there is a unique linear Bayesian equilibrium $(\beta, (\lambda_1, \lambda_2)) \in (\mathcal{P}, \mathcal{S})$ satisfying

$$\beta_t = \frac{\lambda_{1t}^* \sigma_{zt}^2}{\Sigma_t},\ \lambda_{1t} = \lambda_1^* e^{-\eta t},\ \lambda_{2t} = \frac{\sigma_{vt}^2}{\sigma_{vt}^2 + \sigma_{\varepsilon t}^2} \tag{4.25}$$

with

$$\lambda_1^* = \sqrt{\frac{\Sigma_0 + \Upsilon_{v\varepsilon 0}}{\Upsilon_{z0}}}$$

and

$$\Sigma_t = \frac{\Upsilon_{zt}\Sigma_0 + \Upsilon_{zt}\Upsilon_{v\varepsilon 0} - \Upsilon_{z0}\Upsilon_{v\varepsilon t}}{\Upsilon_{z0}}. \tag{4.26}$$

The residual information Σ_t satisfies

$$\lim_{t\to\infty} \Sigma_t = 0,$$

the value function is

$$J_2(t, m_t) = \frac{m_t^2 e^{\eta t}}{2\lambda_1^*} + \frac{\Upsilon_{v\varepsilon t} e^{\eta t}}{2\lambda_1^*} + \frac{\lambda_1^* \Upsilon_{zt} e^{\eta t}}{2} \tag{4.27}$$

and the expected total profit of insider is

$$E(J_2(0, m_0)) = \sqrt{\Upsilon_{z0}(\Sigma_0 + \Upsilon_{v\varepsilon 0})}. \tag{4.28}$$

Proof. Similar to the proof of Theorem 4.2. The proof is divided into three steps:
Step I: Find a solution J_2 to the Hamilton-Jacobi-Bellman equation (4.24).

The Hamilton-Jacobi-Bellman equation (4.24) states that

$$\begin{cases} -\lambda_{1t}\dfrac{\partial J_2}{\partial m} + m = 0, \\ \dfrac{\partial J_2}{\partial t} + \dfrac{1}{2}[(1-\lambda_{2t})^2\sigma_{vt}^2 + \lambda_{2t}^2\sigma_{\varepsilon t}^2 + \lambda_{1t}^2\sigma_{\varepsilon t}^2]\dfrac{\partial^2 J_2}{\partial m^2} - \eta J_2 = 0. \end{cases} \tag{4.29}$$

By the first equation in (4.29), we have

$$J_2(t, m) = \alpha_t m^2 + \delta_t, \tag{4.30}$$

where α_t and δ_t are two determinate, continuous, differentiable and positive functions with $\alpha_t = \frac{1}{2\lambda_{1t}}$.

Accordingly, it follows from the second equation in (4.29) that

$$\begin{cases} \dfrac{d\alpha_t}{dt} - \eta\alpha_t = 0, \\ \dfrac{d\delta_t}{dt} - \eta\delta_t + \dfrac{1}{2}[(1-\lambda_{2t})^2\sigma_{vt}^2 + \lambda_{2t}^2\sigma_{\varepsilon t}^2 + \lambda_{1t}^2\sigma_{zt}^2]\dfrac{1}{\lambda_{1t}} = 0. \end{cases} \tag{4.31}$$

So, similar to the proof as in [5], we can prove that

$$\lambda_{1t} = \lambda_1^* e^{-\eta t},$$

for some constant $\lambda_1^* > 0$ and that

$$\alpha_t = \frac{1}{2\lambda_1^*}e^{\eta t},\ \delta_t = \alpha_t \int_t^\infty \frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2 + \sigma_{\varepsilon s}^2}ds + \frac{\lambda_1^*}{2}\int_t^\infty \sigma_{zs}^2 e^{-2\eta s}ds, \tag{4.32}$$

that is,

$$J_2(t,m) = \frac{m^2 e^{\eta t}}{2\lambda_1^*} + \frac{e^{\eta t}\int_t^\infty \frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds}{2\lambda_1^*} + \frac{\lambda_1^* e^{\eta t}\int_t^\infty \sigma_{zs}^2 e^{-2\eta s}ds}{2}.$$

Then, the value function $J_2(t, m_t)$ can be written as (4.27).
Step II: Find a necessary condition for the optimal trading intensity:

$$\lim_{t\to\infty} \Sigma_t = 0,$$

which implies that

$$\lambda_1^* = \sqrt{\frac{(\frac{\sigma_v^2\sigma_\varepsilon^2}{\sigma_v^2+\sigma_\varepsilon^2} + \int_0^{+\infty}\frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds)}{(\int_0^{+\infty}\sigma_{zs}^2 e^{-2\eta s}ds)}},$$

and

$$E(J_2(0,m_0)) = \sqrt{\Upsilon_{z0}(\Sigma_0 + \Upsilon_{v\varepsilon 0})}.$$

Let $\widetilde{J}(t,m) = e^{-\eta t}J_2(t,m)$. Then, from the value function (4.23), we see that

$$\widetilde{J}(t,m) = \sup_{\beta_t\in\mathcal{S}} E[\int_t^\infty e^{-\eta s}\beta_s m_s^2 ds|\mathcal{F}_t^I].$$

Indeed, for any $\beta' \in \mathcal{S}$, by the Eq (4.1), applying Itô formula to $\widetilde{J}(t,m_t)$ yields

$$\begin{aligned} d\widetilde{J}(s,m_s(\beta')) = &\left\{\frac{\partial\widetilde{J}}{\partial t} + \frac{1}{2}[(1-\lambda_{2s})^2\sigma_{vs}^2 + \lambda_{2s}^2\sigma_{\varepsilon s}^2 + \lambda_{1s}^2\sigma_{zs}^2]\frac{\partial^2\widetilde{J}}{\partial m^2}\right\}ds \\ &- \lambda_{1s}\beta_s' m_s\frac{\partial\widetilde{J}}{\partial m}ds + \frac{\partial\widetilde{J}}{\partial m}(1-\lambda_{2s})\sigma_{vs}dW_s^v - \frac{\partial\widetilde{J}}{\partial m}\lambda_{2s}\sigma_{\varepsilon s}dW_s^\varepsilon \\ &- \frac{\partial\widetilde{J}}{\partial m}\lambda_{1s}\sigma_{zs}dW_s^z, \end{aligned}$$

that is,

$$\begin{aligned} dJ_2(s,m_s(\beta')) = &\left\{\frac{\partial J_2}{\partial t} + \frac{1}{2}[(1-\lambda_{2s})^2\sigma_{vs}^2 + \lambda_{2s}^2\sigma_{\varepsilon s}^2 + \lambda_{1s}^2\sigma_{\varepsilon s}^2]\frac{\partial^2 J_2}{\partial m^2} - \eta J_2\right\}ds \\ &- \lambda_{1s}\beta_s' m_s\frac{\partial J_2}{\partial m}ds + \frac{\partial J_2}{\partial m}(1-\lambda_{2s})\sigma_{vs}dW_s^v - \frac{\partial J_2}{\partial m}\lambda_{2s}\sigma_{\varepsilon s}dW_s^\varepsilon \\ &- \frac{\partial J_2}{\partial m}\lambda_{1s}\sigma_{zs}dW_s^z + \eta J_2(s,m_s(\beta'))ds. \end{aligned}$$

Note that,

$$\frac{\partial J_2}{\partial t} + \frac{1}{2}[(1-\lambda_{2t})^2\sigma_{vt}^2 + \lambda_{2t}^2\sigma_{\varepsilon t}^2 + \lambda_{1t}^2\sigma_{\varepsilon t}^2]\frac{\partial^2 J_2}{\partial m^2} - \eta J_2 = 0,\quad \frac{\partial J_2}{\partial m} = \frac{m}{\lambda_{1t}}.$$

So,

$$dJ_2(s,m_s(\beta')) = \eta J_2(s,m_s(\beta'))ds - \beta'_s m_s^2 ds + \frac{m_s}{\lambda_{1s}}(1-\lambda_{2s})\sigma_{vs}dW_s^v$$
$$- \frac{m_s}{\lambda_{1s}}\lambda_{2s}\sigma_{\varepsilon s}dW_s^{\varepsilon} - \frac{m_s}{\lambda_{1s}}\lambda_{1s}\sigma_{zs}dW_s^z,$$

which follows that

$$J_2(t,m_t(\beta')) = J_2(0,m_0)e^{\eta t} - e^{\eta t}\int_0^t e^{-\eta s}\beta'_s m_s^2 ds + e^{\eta t}\int_0^t e^{-\eta s}\frac{m_s}{\lambda_{1s}}(1-\lambda_{2s})\sigma_{vs}dW_s^v$$
$$- e^{\eta t}\int_0^t e^{-\eta s}\frac{m_s}{\lambda_{1s}}\lambda_{2s}\sigma_{\varepsilon s}dW_s^{\varepsilon} - e^{\eta t}\int_0^t e^{-\eta s}\frac{m_s}{\lambda_{1s}}\lambda_{1s}\sigma_{zs}dW_s^z.$$

Multiplying both sides of the above equation by $e^{-\eta t}$ gives

$$e^{-\eta t}J_2(t,m_t(\beta')) = J_2(0,m_0) - \int_0^t e^{-\eta s}\beta'_s m_s^2 ds + \int_0^t e^{-\eta s}\frac{m_s}{\lambda_{1s}}(1-\lambda_{2s})\sigma_{vs}dW_s^v$$
$$- \int_0^t e^{-\eta s}\frac{m_s}{\lambda_{1s}}\lambda_{2s}\sigma_{\varepsilon s}dW_s^{\varepsilon} - \int_0^t e^{-\eta s}\frac{m_s}{\lambda_{1s}}\lambda_{1s}\sigma_{zs}dW_s^z.$$

Hence,

$$E[J_2(0,m_0)] = e^{-\eta t}E[J_2(t,m_t(\beta'))] + E[\int_0^t e^{-\eta s}\beta'_s m_s^2 ds],$$

for any $\beta' \in \mathcal{S}$. Since $\lim_{t\to\infty} e^{-\eta t}J_2(t,m_t(\beta')) \geq 0$, for any $t \in [0,\infty)$, then

$$E[J_2(0,m_0)] \geq E[\int_0^{\infty} e^{-\eta s}\beta'_s m_s^2 ds]$$

and the above equality hold if and only if $\lim_{t\to\infty} e^{-\eta t}J_2(t,m_t(\beta')) = 0$.
Since

$$E[J_2(t,m_t)] = \frac{e^{\eta t}\Sigma_t}{2\lambda_1^*} + \frac{e^{\eta t}\int_t^{\infty}\frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds}{2\lambda_1^*} + \frac{\lambda_1^* e^{\eta t}\int_t^{\infty}\sigma_{zs}^2 e^{-2\eta s}ds}{2}, \tag{4.33}$$

then $\lim_{t\to\infty} e^{-\eta t}E[J_2(t,m_t(\beta))] = 0$ if and only if $\lim_{t\to\infty}\Sigma_t = 0$.

According to market efficiency (3.2), it shows that

$$\Sigma_t = \Sigma_0 + \int_0^t \frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds - (\lambda_1^*)^2\int_0^t \sigma_{zs}^2 e^{-2\eta s}ds. \tag{4.34}$$

Together with $\lim_{t\to\infty}\Sigma_t = 0$, we get

$$\lambda_1^* = \sqrt{\frac{\left(\frac{\sigma_v^2\sigma_\varepsilon^2}{\sigma_v^2+\sigma_\varepsilon^2} + \int_0^{+\infty}\frac{\sigma_{vs}^2\sigma_{\varepsilon s}^2}{\sigma_{vs}^2+\sigma_{\varepsilon s}^2}ds\right)}{\left(\int_0^{+\infty}\sigma_{zs}^2 e^{-2\eta s}ds\right)}}.$$

Thus, (4.2) holds. Taking λ_1^* back into (4.33) deduces $E(J_2(0,m_0)) = \sqrt{\Upsilon_{z0}(\Sigma_0 + \Upsilon_{v\varepsilon 0})}$.
Step III: From Proposition 3.1, taking $(\beta,(\lambda_1,\lambda_2)$ as the forms in (4.25) respectively, we now verify that it is a market equilibrium:

(i) For the given pricing rule with (λ_1, λ_2), the insider trading intensity β satisfies

$$E\int_0^{\infty} e^{-\eta s}|\beta_s|(v_s - p_s)^2 ds = \int_0^{\infty} e^{-\eta s}\beta_s\Sigma_s ds = \sqrt{\Upsilon_{z0}(\Sigma_0 + \Upsilon_{v\varepsilon 0})} < \infty,$$

which means that β in $\mathcal{S}$ is optimal.

(ii) For the given insider trading intensity β, the local linear pricing p_t with (λ_1, λ_2) must satisfy the market efficiency by repeating the procedure of (ii) in Step 3 of the proof for Theorem 4.2. This proof is complete.

5. Comparative statics

In this section we will investigate some influences of market makers' partial observation on insider trading in our model. Note that from Theorems 4.2 or 4.5, the corresponding price pressures and the insider's expected total profit depend on the volatility function $\sigma_{\varepsilon t}^2$ of market makers' observation intricately. To explain the economic implication better, we only consider some special cases, especially when the volatility function $\sigma_{\varepsilon t}^2$ keeps constant.

5.1. In Case I.

Now recalling from Theorem 4.2, we can obtain the following proposition.

Proposition 5.1. *Let $(\beta, (\lambda_1, \lambda_2)) \in (\mathcal{S}, \mathcal{P})$ be the linear Bayesian equilibrium in Case I*

$$\sigma_{vt} \equiv \sigma_v > 0,\ \sigma_{zt} \equiv \sigma_z > 0,\ \sigma_{\varepsilon t} \equiv \sigma_\varepsilon > 0.$$

Then

$$\frac{\partial \lambda_{1t}}{\partial \sigma_\varepsilon^2} > 0,\ \frac{\partial \lambda_{2t}}{\partial \sigma_\varepsilon^2} < 0,\ \frac{\partial E(J_1(\beta))}{\partial \sigma_\varepsilon^2} > 0. \tag{5.1}$$

In particular,

(i) if $\sigma_{\varepsilon 0}^2 \to \infty$, $\sigma_\varepsilon^2 \to \infty$, then

$$\lambda_{1t} \to \sqrt{\frac{\sigma_{v0}^2 + T\sigma_v^2}{T\sigma_z^2}},\quad \lambda_{2t} \to 0,\quad \beta_t \to \sqrt{\frac{T}{(T-t)^2}\frac{\sigma_z^2(\sigma_{v0}^2 + T\sigma_v^2)}{\sigma_{v0}^4}}, \tag{5.2}$$

$$E(J_1(\beta)) \to \sqrt{T\sigma_z^2(\sigma_{v0}^2 + T\sigma_v^2)}; \tag{5.3}$$

(ii) if $\sigma_{\varepsilon 0}^2 \to 0$, $\sigma_\varepsilon^2 \to 0$, then

$$\lambda_{2t} \to 1,\ \ \lambda_{1t} \to 0,\ \ \beta_t \to \infty,\quad E(J_1(\beta)) \to 0; \tag{5.4}$$

(iii) $E(J_1(\sigma_{\varepsilon 0}^2, \sigma_\varepsilon^2)) < E(J_1(\infty, \infty))$.

Proof. Let σ_{vt}, σ_{zt} and $\sigma_{\varepsilon t}$ be the corresponding constants in the assumption. Then, by Theorem 4.2, it follows that

$$\lambda_{1t} = \frac{1}{\sqrt{T\sigma_z^2}}\sqrt{\frac{\sigma_{v0}^2\sigma_{\varepsilon 0}^2}{\sigma_{v0}^2 + \sigma_{\varepsilon 0}^2} + \frac{T\sigma_v^2\sigma_\varepsilon^2}{\sigma_v^2 + \sigma_\varepsilon^2}},\ \lambda_{2t} = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\varepsilon^2},$$

$$\beta_t = \sqrt{\frac{T\sigma_z^2}{(T-t)^2}(\frac{\sigma_{v0}^2+\sigma_{\varepsilon 0}^2}{\sigma_{v0}^2\sigma_{\varepsilon 0}^2} + \frac{T\sigma_v^2\sigma_\varepsilon^2}{\sigma_v^2+\sigma_\varepsilon^2}\frac{(\sigma_{v0}^2+\sigma_{\varepsilon 0}^2)^2}{(\sigma_{v0}^2\sigma_{\varepsilon 0}^2)^2})}, \tag{5.5}$$

where the residual information Σ_t and the expected total profit $E(J_1(\beta^*))$ satisfy respectively

$$\Sigma_t = \frac{T-t}{T}\cdot\frac{\sigma_{v0}^2\sigma_{\varepsilon 0}^2}{\sigma_{v0}^2+\sigma_{\varepsilon 0}^2},\ E(J_1(\beta)) = \sqrt{T\sigma_z^2(\frac{\sigma_{v0}^2\sigma_{\varepsilon 0}^2}{\sigma_{v0}^2+\sigma_{\varepsilon 0}^2} + \frac{T\sigma_v^2\sigma_\varepsilon^2}{\sigma_v^2+\sigma_\varepsilon^2})}. \tag{5.6}$$

Finally, it is easy to obtain our results above from Eqs (5.5) and (5.6). This proof is complete.

Beyond that, some propositions for several special cases when the value of risky asset is static, that is, $\sigma_{vt}\equiv 0$, will be listed below one by one:

1) In the case when market makers only observe the total market order $y_t = x_t + z_t$, which means $\sigma_{\varepsilon 0}^2\to\infty$, it follows from Theorem 4.2 that

$$\lambda_{1t}\to\frac{\sigma_{v0}}{\sqrt{T}\sigma_z},\ \beta_t\to\frac{\sqrt{T}\sigma_z}{(T-t)\sigma_{v0}},\ \Sigma_0\to\sigma_{v0}^2,\ \Sigma_t\to\frac{(T-t)\sigma_{v0}^2}{T},\ E[J_1(\beta)]\to\sqrt{\sigma_{v0}^2\sigma_z^2 T},$$

which are the same as those results in [1, 2, 8, 15].

2) In the case when market makers can observe two signals $y_t = x_t + z_t$ and $u_t = v_0 + \varepsilon_0 + \int_0^t \sigma_{\varepsilon t}dW_t^\varepsilon$ in the market, it is easy to see from Theorem 4.2 that

$$\lambda_{1t} = \sqrt{\frac{\Sigma_0}{\Gamma_{z0}}},\ \beta_t = \frac{\sigma_{zt}^2\sqrt{\Sigma_0\Gamma_{z0}}}{\Sigma_0\Gamma_{zt}},\ \Sigma_0 = \frac{\sigma_{v0}^2\sigma_{\varepsilon 0}^2}{\sigma_{v0}^2+\sigma_{\varepsilon 0}^2},\ \Sigma_t = \frac{\Sigma_0\Gamma_{zt}}{\Gamma_{z0}},\ E[J_1(\beta)] = \sqrt{\Sigma_0\Gamma_{z0}},$$

which are consistent with those results in [27].

5.2. In Case II

Similarly, some properties about linear Bayesian equilibrium for special markets in Case II will be presented below with their proofs omitted.

Proposition 5.2. *Let $(\beta,(\lambda_1,\lambda_2))\in(\mathcal{S},\mathcal{P})$ be the linear Bayesian equilibrium in Case II when*

$$\sigma_{zt}\equiv\sigma_z>0,\ \sigma_{\varepsilon t}\equiv\sigma_\varepsilon>0.$$

Then,

$$\frac{\partial\lambda_{1t}}{\partial\sigma_\varepsilon^2}>0,\ \frac{\partial\lambda_{2t}}{\partial\sigma_\varepsilon^2}<0,\ \frac{\partial E(J_2(\beta))}{\partial\sigma_\varepsilon^2}>0. \tag{5.7}$$

In particular,

(i) if $\sigma_{\varepsilon 0}^2\to\infty$, $\sigma_\varepsilon^2\to\infty$, then

$$\lambda_{1t}\to\sqrt{\frac{2\eta(\Sigma_0+\Upsilon_{v0})}{\sigma_z^2}}e^{-\eta t},\ \lambda_{2t}\to 0,\ \beta_t\to\frac{\sqrt{2\eta(\Sigma_0+\Upsilon_{v0})}\sigma_z e^{-\eta t}}{(\Sigma_0+\Upsilon_{v0})e^{-2\eta t}-\Upsilon_{vt}}, \tag{5.8}$$

$$E(J_2(\beta))\to\sqrt{\frac{\sigma_z^2(\Sigma_0+\Upsilon_{v0})}{2\eta}} \tag{5.9}$$

and

$$\Sigma_0 \to \sigma_{v0}^2,\ \Sigma_t \to (\Sigma_0 + \Upsilon_{v0})e^{-2\eta t} - \Upsilon_{vt},\ \Upsilon_{v\varepsilon t} \to \Upsilon_{vt} = \int_t^{+\infty} \sigma_{vs}^2 ds; \tag{5.10}$$

(ii) if $\sigma_{\varepsilon 0}^2 \to 0$, $\sigma_\varepsilon^2 \to 0$, then

$$\lambda_{2t} \to 1,\ \lambda_{1t} \to 0,\ \beta_t \to \infty,\ E(J_2(\beta)) \to 0; \tag{5.11}$$

(iii) $E(J_2(\sigma_{\varepsilon 0}^2, \sigma_\varepsilon^2)) < E(J_2(\infty, \infty))$.

We remark that our results (5.8), (5.9) and (5.10) in Proposition 5.2 show that the model in our setting degenerates into the case of Caldentey and Stacchetti [5].

As in the previous subsection, from Theorem 4.5 some propositions for two more special cases if the asset value is static, that is, $\sigma_{vt} \equiv 0$, are stated as follows:

1) When $\sigma_{\varepsilon t} = 0$ and $\sigma_{zt} = \sigma_z > 0$,

$$\lambda_{1t} = \sqrt{\frac{2\eta\Sigma_0}{\sigma_z^2}} e^{-\eta t},\ \beta_t = \sqrt{\frac{2\eta}{\Sigma_0}} \sigma_z e^{\eta t},\ \Sigma_0 = \frac{\sigma_{v0}^2 \sigma_{\varepsilon 0}^2}{\sigma_{v0}^2 + \sigma_{\varepsilon 0}^2},\ \Sigma_t = \Sigma_0 e^{-2\eta t},\ E(J_2(\beta)) = \sqrt{\frac{\sigma_z^2 \Sigma_0}{2\eta}}.$$

2) In the above case when market makers only observe the total market order $y_t = x_t + z_t$, which means $\sigma_{\varepsilon 0}^2 \to \infty$,

$$\lambda_{1t} \to \frac{\sigma_{v0}\sqrt{2\eta}}{\sigma_z} e^{-\eta t},\ \beta_t \to \frac{\sigma_z \sqrt{2\eta} e^{\eta t}}{\sigma_{v0}},\ \Sigma_0 \to \sigma_{v0}^2,\ \Sigma_t \to \sigma_{v0}^2 e^{-2\eta t},\ E(J_2(\beta)) \to \frac{\sigma_z \sigma_{v0}}{\sqrt{2\eta}}.$$

6. Summary

Based on the Caldentey-Stacchetti's model [5], this article investigates a new insider trading model, in which all information about a dynamic risky asset is known to an insider, while some partial information are observed by market makers. By applying filtering theory and dynamic programming principle, we establish the existence and uniqueness of linear Bayesian equilibrium trading either until a fixed maturity time T or a random time τ, respectively, which consists of insider trading intensity, price pressure on market orders and price pressure on asset observations. It shows that in equilibrium, $\lim_{t\to T} \Sigma_t = 0$ which means that all information on the risky asset is incorporated in the market price (see Theorems 4.2 and 4.5). Our results cover some classical results in literature [1, 2, 5, 8, 15, 27].

To explain the economic implication better, we further study our insider trading model for some special settings especially when the volatility function $\sigma_{\varepsilon t}^2$ keeps constant. According to Proposition 5.1 for Case I or Proposition 5.2 for Case II, it shows that the larger the noise σ_ε^2, the greater the weight that market makers give to price pressure on market orders, but the smaller the weight to price pressure on asset observations such that the insider earns more profit. Particularly, when $\sigma_{\varepsilon 0}^2 \to \infty$ and $\sigma_\varepsilon^2 \to \infty$ (which means market makers observe fewer information on the asset), the price pressure on asset observations tends to 0, which reveals that the weight that market makers give to asset observations in pricing tends to 0. In addition, the expected aggregate profit of the insider reaches the maximum. When $\sigma_{\varepsilon 0}^2 \to 0$ and $\sigma_\varepsilon^2 \to 0$ (which means market makers observe almost of information on the asset), the price pressure on market orders tends to 0 and the price pressure on asset observations tends to 1, which reveals that the weight that market makers give to market orders in pricing tends to 0, the weight to asset observations tends to 1 and the insider can make no money. These results suggest that market makers can acquire information about the risky asset through a variety of channels to prevent the insider from monopolizing the market to seek excessive profit.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest that could affect the publication of this paper.

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