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Research article

Infinity norm upper bounds for the inverse of SDD_k matrices

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Abstract: In this paper, we introduce a new subclass of H-matrices called SDD_k matrices, which contains SDD matrices and SDD_1 matrices as special cases, and present some properties of SDD_k matrices. Based on these properties, we also provide new infinity norm bounds for the inverse of SDD matrices and SDD_k matrices. It is proved that these new bounds are better than some existing results in some cases. Numerical examples demonstrate the effectiveness of the obtained results.

Keywords: SDD_k matrices; SDD matrices; SDD_1 matrices; infinity norm; upper bound
Mathematics Subject Classification: 15A18, 15A69, 65G50, 90C33

1. Introduction

Let n be an integer number, $N = \{1, 2, \ldots, n\}$, and let $C^{n\times n}$ be the set of all complex matrices of order n. A matrix $M = (m_{ij}) \in C^{n\times n} (n \geq 2)$ is called a strictly diagonally dominant (SDD) matrix [1] if

$$|m_{ii}| > r_i(M) = \sum_{j=1, j\neq i}^{n} |m_{ij}|, \qquad \forall i \in N.$$

It was shown that an SDD matrix is an H-matrix [1], where matrix $M = (m_{ij}) \in C^{n\times n} (n \geq 2)$ is called an H-matrix [1–3] if and only if there exists a positive diagonal matrix X such that MX is an SDD matrix [1, 2, 4]. H-matrices are widely applied in many fields, such as computational mathematics, economics, mathematical physics and dynamical system theory, see [1,4–6]. Meanwhile, upper bounds for the infinity norm of the inverse matrices of H-matrices can be used in the convergence analysis of matrix splitting and matrix multi-splitting iterative methods for solving the large sparse of linear equations [7], as well as linear complementarity problems. Moreover, upper bounds of the infinity norm of the inverse for different classes of matrices have been widely studied, such as CKV-type matrices [8], S-$SDDS$ matrices [9], DZ and DZ-type matrices [10, 11], *Nekrasov* matrices [12–15], S-*Nekrasov* matrices [16], Q-*Nekrasov* matrices [17], $GSDD_1$ matrices [18] and so on.

In 2011, Peña [19] proposed a new subclass of H-matrices called SDD_1 matrices, whose definition is listed below. A matrix $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ is called an SDD_1 matrix if

$$|m_{ii}| > p_i(M), \quad \forall i \in N_1(M),$$

where

$$\begin{aligned} p_i(M) &= \sum_{j\in N_1(M)\setminus\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\setminus\{i\}} \frac{r_j(M)}{|m_{jj}|}|m_{ij}|, \\ N_1(M) &= \{i \| m_{ii}| \leq r_i(M)\}, \quad N_2(M) = \{i \| m_{ii}| > r_i(M)\}. \end{aligned}$$

In 2023, Dai et al. [18] gave a new subclass of H-matrices named generalized SDD_1 ($GSDD_1$) matrices, which extends the class of SDD_1 matrices. Here, a matrix $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ is said a $GSDD_1$ matrix if

$$r_i(M) - p_i^{N_2(M)}(M) > 0, \quad \forall i \in N_2(M),$$

and

$$\left(r_i(M) - p_i^{N_2(M)}(M)\right)\left(|a_{jj}| - p_j^{N_1(M)}(M)\right) > p_i^{N_1(M)}(M)p_j^{N_2(M)}(M), \quad \forall i \in N_2(M),\ \forall j \in N_1(M),$$

where

$$p_i^{N_2(M)}(M) = \sum_{j\in N_2(M)\setminus\{i\}} \frac{r_j(M)}{|m_{jj}|}|m_{ij}|, \quad p_i^{N_1(M)}(M) = \sum_{j\in N_1(M)\setminus\{i\}} |m_{ij}|, \quad i \in N.$$

Subsequently, some upper bounds for the infinite norm of the inverse matrices of SDD matrices, SDD_1 matrices and $GSDD_1$ matrices are presented, see [18,20,21]. For example, the following results that will be used later are listed.

Theorem 1. (Varah bound) [21] *Let matrix $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an SDD matrix. Then*

$$\|M^{-1}\|_\infty \leq \frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}| - r_i(M))}.$$

Theorem 2. [20] *Let matrix $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an SDD matrix. Then*

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i(M)}{|m_{ii}|} + \varepsilon}{\min\limits_{i\in N} Z_i}, \quad 0 < \varepsilon < \min_{i\in N} \frac{|m_{ii}| - p_i(M)}{r_i(M)},$$

where

$$Z_i = \varepsilon(|m_{ii}| - r_i(M)) + \sum_{j\in N\setminus\{i\}} \left(\frac{r_j(M) - p_j(M)}{|m_{jj}|}\right)|m_{ij}|.$$

Theorem 3. [20] *Let matrix $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an SDD matrix. If $r_i(M) > 0(\forall i \in N)$, then*

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i(M)}{|m_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\setminus\{i\}} \frac{r_j(M)-p_j(M)}{|m_{jj}|}|m_{ij}|}.$$

Theorem 4. [18] *Let $M = (m_{ij}) \in C^{n\times n}$ be a $GSDD_1$ matrix. Then*

$$\|M^{-1}\|_\infty \le \frac{\max\left\{\varepsilon, \max\limits_{i\in N_2(M)} \frac{r_i(M)}{|m_{ii}|}\right\}}{\min\left\{\min\limits_{i\in N_2(M)} \phi_i, \min\limits_{i\in N_1(M)} \psi_i\right\}},$$

where

$$\begin{aligned} \phi_i &= r_i(M) - \sum_{j\in N_2(M)\backslash\{i\}} \frac{r_j(M)}{|m_{jj}|}|m_{ij}| - \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}|\varepsilon, \\ \psi_i &= |m_{ii}|\varepsilon - \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}|\varepsilon + \sum_{j\in N_2(M)\backslash\{i\}} \frac{r_j(M)}{|m_{jj}|}|m_{ij}|, \end{aligned}$$

and

$$\max_{i\in N_1(M)} \frac{p_i^{N_2(M)}(M)}{|m_{ii}| - p_i^{N_1(M)}(M)} < \varepsilon < \min_{j\in N_2(M)} \frac{r_j(M) - p_j^{N_2(M)}(M)}{p_j^{N_1(M)}(M)}.$$

On the basis of the above articles, we continue to study special structured matrices and introduce a new subclass of H-matrices called SDD_k matrices, and provide some new upper bounds for the infinite norm of the inverse matrices for SDD matrices and SDD_k matrices, which improve the previous results. The remainder of this paper is organized as follows: In Section 2, we propose a new subclass of H-matrices called SDD_k matrices, which include SDD matrices and SDD_1 matrices, and derive some properties of SDD_k matrices. In Section 3, we present some upper bounds for the infinity norm of the inverse matrices for SDD matrices and SDD_k matrices, and provide some comparisons with the well-known Varah bound. Moreover, some numerical examples are given to illustrate the corresponding results.

2. SDD_k matrices

In this section, we propose a new subclass of H-matrices called SDD_k matrices, which include SDD matrices and SDD_1 matrices, and derive some new properties.

Definition 1. *A matrix $M = (m_{ij}) \in C^{n\times n}(n \ge 2)$ is called an SDD_k matrix if there exists $k \in N$ such that*

$$|m_{ii}| > p_i^{(k-1)}(M), \quad \forall i \in N_1(M),$$

where

$$\begin{aligned} p_i^{(k)}(M) &= \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|, \\ p_i^{(0)}(M) &= \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{r_j(M)}{|m_{jj}|}|m_{ij}|. \end{aligned}$$

Immediately, we know that SDD_k matrices contain SDD matrices and SDD_1 matrices, so

$$\{SDD\} \subseteq \{SDD_1\} \subseteq \{SDD_2\} \subseteq \cdots \subseteq \{SDD_k\}.$$

Lemma 1. *A matrix $M = (m_{ij}) \in C^{n\times n} (n \geq 2)$ is an $SDD_k (k \geq 2)$ matrix if and only if for $\forall i \in N$, $|m_{ii}| > p_i^{(k-1)}(M)$.*

Proof. For $\forall i \in N_1(M)$, from Definition 1, it holds that $|m_{ii}| > p_i^{(k-1)}(M)$.

For $\forall i \in N_2(M)$, we have that $|m_{ii}| > r_i(M)$. When $k = 2$, it follows that

$$\begin{aligned} |m_{ii}| > r_i(M) &\geq \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{r_j(M)}{|m_{jj}|}|m_{ij}| = p_i^{(0)}(M) \\ &\geq \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(0)}(M)}{|m_{jj}|}|m_{ij}| = p_i^{(1)}(M). \end{aligned}$$

Suppose that $|m_{ii}| > p_i^{(k-1)}(M) (k \leq l, l > 2)$ holds for $\forall i \in N_2(M)$. When $k = l + 1$, we have

$$\begin{aligned} |m_{ii}| > r_i(M) &\geq \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(l-2)}(M)}{|m_{jj}|}|m_{ij}| = p_i^{(l-1)}(M) \\ &\geq \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(l-1)}(M)}{|m_{jj}|}|m_{ij}| = p_i^{(l)}(M). \end{aligned}$$

By induction, we obtain that for $\forall i \in N_2(M)$, $|m_{ii}| > p_i^{(k-1)}(M)$. Consequently, it holds that matrix M is an SDD_k matrix if and only if $|m_{ii}| > p_i^{(k-1)}(M)$ for $\forall i \in N$. The proof is completed. □

Lemma 2. *If $M = (m_{ij}) \in C^{n\times n} (n \geq 2)$ is an $SDD_k (k \geq 2)$ matrix, then M is an H-matrix.*

Proof. Let X be the diagonal matrix $diag\{x_1, x_2, \cdots, x_n\}$, where

$$(0 <)\ x_j = \begin{cases} 1, & j \in N_1(M), \\ \frac{p_j^{(k-1)}(M)}{|m_{jj}|} + \varepsilon, & j \in N_2(M), \end{cases}$$

and

$$0 < \varepsilon < \min_{i\in N} \frac{|m_{ii}| - p_i^{(k-1)}(M)}{\sum\limits_{j\in N_2(M)\backslash\{i\}} |m_{ij}|}.$$

If $\sum\limits_{j\in N_2(M)\backslash\{i\}} |m_{ij}| = 0$, then the corresponding fraction is defined to be ∞. Next we consider the following two cases.

Case 1: For each $i \in N_1(M)$, it is not difficult to see that $|(MX)_{ii}| = |m_{ii}|$, and

$$\begin{aligned}
r_i(MX) &= \sum_{j=1, j\neq i} |m_{ij}| x_j \\
&= \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-1)}(M)}{|m_{jj}|} + \varepsilon \right) |m_{ij}| \\
&\leq \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-2)}(M)}{|m_{jj}|} + \varepsilon \right) |m_{ij}| \\
&= \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(k-2)}(M)}{|m_{jj}|} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \varepsilon |m_{ij}| \\
&= p_i^{(k-1)}(M) + \varepsilon \sum_{j\in N_2(M)\backslash\{i\}} |m_{ij}| \\
&< p_i^{(k-1)}(M) + |m_{ii}| - p_i^{(k-1)}(M) \\
&= |m_{ii}| = |(MX)_{ii}|.
\end{aligned}$$

Case 2: For each $i \in N_2(M)$, we can obtain that

$$|(MX)_{ii}| = |m_{ii}| \left(\frac{p_i^{k-1}(M)}{|m_{ii}|} + \varepsilon \right) = p_i^{(k-1)}(M) + \varepsilon |m_{ii}|,$$

and

$$\begin{aligned}
r_i(MX) &= \sum_{j=1, j\neq i} |m_{ij}| x_j \\
&= \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-1)}(M)}{|m_{jj}|} + \varepsilon \right) |m_{ij}| \\
&\leq \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-2)}(M)}{|m_{jj}|} + \varepsilon \right) |m_{ij}| \\
&= \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(k-2)}(M)}{|m_{jj}|} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \varepsilon |m_{ij}| \\
&= p_i^{(k-1)}(M) + \varepsilon \sum_{j\in N_2(A)\backslash\{i\}} |m_{ij}| \\
&< p_i^{(k-1)}(M) + \varepsilon |m_{ii}| \\
&= |(MX)_{ii}|.
\end{aligned}$$

Based on Cases 1 and 2, we have that MX is an SDD matrix, and consequently, M is an H-matrix. The proof is completed. □

According to the definition of SDD_k matrix and Lemma 1, we obtain some properties of SDD_k matrices as follows:

Theorem 5. *If $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ is an SDD_k matrix and $N_1(M) \neq \emptyset$, then for $\forall i \in N_1(M)$, $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| > 0$.*

Proof. Suppose that for $\forall i \in N_1(M)$, $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| = 0$. By Definition 1, we have that $p_i^{(k-1)}(M) = r_i(M)$, $\forall i \in N_1(M)$. Thus, it is easy to verify that $|m_{ii}| > p_i^{(k-1)}(M) = r_i(M) \geq |m_{ii}|$, which is a contradiction. Thus for $\forall i \in N_1(M)$, $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| > 0$. The proof is completed. □

Theorem 6. *Let $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an $SDD_k(k \geq 2)$ matrix. It holds that $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| > 0$, $\forall i \in N_2(M)$. Then*

$$|m_{ii}| > p_i^{(k-2)}(M) > p_i^{(k-1)}(M) > 0, \quad \forall i \in N_2(M),$$

and

$$|m_{ii}| > p_i^{(k-1)}(M) > 0, \quad \forall i \in N.$$

Proof. By Lemma 1 and the known conditions that for $\forall i \in N_2(M)$, $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| > 0$, it holds that

$$|m_{ii}| > p_i^{(k-2)}(M) > p_i^{(k-1)}(M) > 0, \quad \forall i \in N_2(M),$$

and

$$|m_{ii}| > p_i^{(k-1)}(M), \quad \forall i \in N.$$

We now prove that $|m_{ii}| > p_i^{(k-1)}(M) > 0(\forall i \in N)$ and consider the following two cases.

Case 1: If $N_1(M) = \emptyset$, then M is an SDD matrix. It is easy to verify that $|m_{ii}| > p_i^{(k-1)}(M) > 0$, $\forall i \in N_2(M) = N$.

Case 2: If $N_1(M) \neq \emptyset$, by Theorem 5 and the known condition that for $\forall i \in N_2(M)$, $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| > 0$, then it is easy to obtain that $|m_{ii}| > p_i^{(k-1)}(M) > 0(\forall i \in N)$.

From Cases 1 and 2, we have that $|m_{ii}| > p_i^{(k-1)}(M) > 0(\forall i \in N)$. The proof is completed. □

Theorem 7. *Let $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an $SDD_k(k \geq 2)$ matrix and for $\forall i \in N_2(M)$, $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| > 0$. Then there exists a diagonal matrix $X = diag\{x_1, x_2, \cdots, x_n\}$ such that MX is an SDD matrix. Elements $x_1, x_2, \ldots, x_n$ are determined by*

$$x_i = \frac{p_i^{(k-1)}(M)}{|m_{ii}|}, \quad \forall i \in N.$$

Proof. We need to prove that matrix MX satisfies the following inequalities:

$$|(MX)_{ii}| > r_i(MX), \quad \forall i \in N.$$

From Theorem 6 and the known condition that for $\forall i \in N_2(M)$, $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| > 0$, it is easy to verify that

$$0 < \frac{p_i^{(k-1)}(M)}{|m_{ii}|} < \frac{p_i^{(k-2)}(M)}{|m_{ii}|} < 1, \quad \forall i \in N_2(M).$$

For each $i \in N$, we have that $|(MX)_{ii}| = p_i^{(k-1)}(M)$ and

$$\begin{aligned} r_i(MX) &= \sum_{j=1, j\neq i} |m_{ij}| x_j \\ &= \sum_{j\in N_1(M)\backslash\{i\}} \frac{p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}| \\ &< \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(k-2)}(M)}{|m_{jj}|}|m_{ij}| \\ &= p_i^{(k-1)}(M) = |(MX)_{ii}|, \end{aligned}$$

that is,

$$|(MX)_{ii}| > r_i(MX).$$

Therefore, MX is an SDD matrix. The proof is completed. □

3. Infinity norm upper bounds for the inverse of SDD and SDD_k matrices

In this section, by Lemma 2 and Theorem 7, we provide some new upper bounds of the infinity norm of the inverse matrices for SDD_k matrices and SDD matrices, respectively. We also present some comparisons with the Varah bound. Some numerical examples are presented to illustrate the corresponding results. Specially, when the involved matrices are SDD_1 matrices as subclass of SDD_k matrices, these new bounds are in line with that provided by Chen et al. [20].

Theorem 8. *Let $M = (m_{ij}) \in C^{n\times n} (n \geq 2)$ be an $SDD_k (k \geq 2)$ matrix. Then*

$$||M^{-1}||_\infty \leq \frac{\max\left\{1, \max\limits_{i\in N_2(M)} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} + \varepsilon\right\}}{\min\left\{\min\limits_{i\in N_1(M)} U_i, \min\limits_{i\in N_2(M)} V_i\right\}},$$

where

$$\begin{aligned} U_i &= |m_{ii}| - \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| - \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-1)}(M)}{|m_{jj}|} + \varepsilon\right)|m_{ij}|, \\ V_i &= \varepsilon(|m_{ii}| - \sum_{j\in N_2(M)\backslash\{i\}} |m_{ij}|) + \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}\right)|m_{ij}|, \end{aligned}$$

and

$$0 < \varepsilon < \min_{i\in N} \frac{|m_{ii}| - p_i^{(k-1)}(M)}{\sum\limits_{j\in N_2(M)\backslash\{i\}} |m_{ij}|}.$$

Proof. By Lemma 2, we have that there exists a positive diagonal matrix X such that MX is an SDD matrix, where X is defined as Lemma 2. Thus,

$$\|M^{-1}\|_\infty = \|X(X^{-1}M^{-1})\|_\infty = \|X(MX)^{-1}\|_\infty \leq \|X\|_\infty \|(MX)^{-1}\|_\infty,$$

and

$$\|X\|_\infty = \max_{1\leq i\leq n} x_i = \max\left\{1, \max_{i\in N_2(M)} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} + \varepsilon\right\}.$$

Notice that MX is an SDD matrix. Hence, by Theorem 1, we have

$$\|(MX)^{-1}\|_\infty \quad \leq \quad \frac{1}{\min\limits_{1\leq i\leq n}(|(MX)_{ii}| - r_i(MX))}.$$

Thus, for any $i \in N_1(M)$, it holds that

$$|(MX)_{ii}| - r_i(MX) = |m_{ii}| - \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| - \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-1)}(M)}{|m_{jj}|} + \varepsilon\right)|m_{ij}| = U_i.$$

For any $i \in N_2(M)$, it holds that

$$\begin{aligned}
|(MX)_{ii}| - r_i(MX) &= p_i^{(k-1)}(M) + \varepsilon|m_{ii}| - \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| - \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-1)}(M)}{|m_{jj}|} + \varepsilon\right)|m_{ij}| \\
&= \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| + \sum_{j\in N_2(M)\backslash\{i\}} \frac{p_j^{(k-2)}(M)}{|m_{jj}|}|m_{ij}| + \varepsilon|m_{ii}| \\
&\quad - \sum_{j\in N_1(M)\backslash\{i\}} |m_{ij}| - \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-1)}(M)}{|m_{jj}|} + \varepsilon\right)|m_{ij}| \\
&= \varepsilon(|m_{ii}| - \sum_{j\in N_2(M)\backslash\{i\}} |m_{ij}|) + \sum_{j\in N_2(M)\backslash\{i\}} \left(\frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}\right)|m_{ij}| \\
&= V_i.
\end{aligned}$$

Therefore, we get

$$\|M^{-1}\|_\infty \leq \frac{\max\left\{1, \max\limits_{i\in N_2(M)} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} + \varepsilon\right\}}{\min\{\min\limits_{i\in N_1(M)} X_i, \min\limits_{i\in N_2(M)} Y_i\}}.$$

The proof is completed. □

From Theorem 8, it is easy to obtain the following result.

Corollary 1. *Let $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an SDD matrix. Then*

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} + \varepsilon}{\min\limits_{i\in N} Z_i},$$

where $k \geq 2$,

$$Z_i = \varepsilon(|m_{ii}| - r_i(M)) + \sum_{j\in N\setminus\{i\}} \left(\frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|} \right) |m_{ij}|,$$

and

$$0 < \varepsilon < \min_{i\in N} \frac{|m_{ii}| - p_i^{(k-1)}(M)}{r_i(M)}.$$

Example 1. Consider the $n \times n$ matrix:

$$M = \begin{pmatrix} 4 & 2 & 1.5 & & & & & & \\ 1.5 & 4 & 2 & & & & & & \\ & 4 & 8 & 2 & & & & & \\ & & 4 & 8 & 2 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 4 & 8 & 2 & & \\ & & & & & 4 & 8 & 2 & \\ & & & & & & 4 & 8 & 2 \\ & & & & & & & 3.5 & 4 \end{pmatrix}.$$

Take that $n = 20$. It is easy to verify that M is an SDD matrix.

By calculations, we have that for $k = 2$,

$$\max_{i\in N} \frac{p_i^{(1)}(M)}{|m_{ii}|} + \varepsilon_1 = 0.5859 + \varepsilon_1, \quad \min_{i\in N} Z_i = 0.4414 + 0.5\varepsilon_1, \quad 0 < \varepsilon_1 < 0.4732.$$

For $k = 4$,

$$\max_{i\in N} \frac{p_i^{(3)}(M)}{|a_{ii}|} + \varepsilon_2 = 0.3845 + \varepsilon_2, \quad \min_{i\in N} Z_i = 0.2959 + 0.5\varepsilon_2, \quad 0 < \varepsilon_2 < 0.7034.$$

For $k = 6$,

$$\max_{i\in N} \frac{p_i^{(5)}(M)}{|m_{ii}|} + \varepsilon_3 = 0.2504 + \varepsilon_3, \quad \min_{i\in N} Z_i = 0.1733 + 0.5\varepsilon_3, \quad 0 < \varepsilon_3 < 0.8567.$$

For $k = 8$,

$$\max_{i\in N} \frac{p_i^{(7)}(M)}{|m_{ii}|} + \varepsilon_4 = 0.1624 + \varepsilon_4, \quad \min_{i\in N} Z_i = 0.0990 + 0.5\varepsilon_4, \quad 0 < \varepsilon_4 < 0.9572.$$

So, when $k = 2, 4, 6, 8$, by Corollary 1 and Theorem 1, we can get the upper bounds for $\|M^{-1}\|_\infty$, see Table 1. Thus,

$$\|M^{-1}\|_\infty \leq \frac{0.5859 + \varepsilon_1}{0.4414 + 0.5\varepsilon_1} < 2, \quad \|M^{-1}\|_\infty \leq \frac{0.3845 + \varepsilon_2}{0.2959 + 0.5\varepsilon_2} < 2,$$

and

$$\|M^{-1}\|_\infty \leq \frac{0.2504 + \varepsilon_3}{0.1733 + 0.5\varepsilon_3} < 2, \quad \|M^{-1}\|_\infty \leq \frac{0.1624 + \varepsilon_4}{0.0990 + 0.5\varepsilon_4} < 2.$$

Moreover, when $k = 1$, by Theorem 2, we have

$$\|M^{-1}\|_\infty \leq \frac{0.7188 + \varepsilon_5}{0.4844 + 0.5\varepsilon_5}, \quad 0 < \varepsilon_5 < 0.3214.$$

Table 1. The bounds in Corollary 1 and Theorem 1.

k	2	4	6	8
Cor 1	$\frac{0.5859+\varepsilon_1}{0.4414+0.5\varepsilon_1}$	$\frac{0.3845+\varepsilon_2}{0.2959+0.5\varepsilon_2}$	$\frac{0.2504+\varepsilon_3}{0.1733+0.5\varepsilon_3}$	$\frac{0.1624+\varepsilon_4}{0.0990+0.5\varepsilon_4}$
Th 1	2	2	2	2

The following Theorem 9 shows that the bound in Corollary 1 is better than that in Theorem 1 of [20] in some cases.

Theorem 9. *Let matrix $M = (m_{ij}) \in C^{n\times n} (n \geq 2)$ be an SDD matrix. If there exists $k \geq 2$ such that*

$$\max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} \min_{i\in N}(|m_{ii}| - r_i(M)) \leq \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|,$$

then

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} + \varepsilon}{\min\limits_{i\in N} Z_i} \leq \frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}| - r_i(M))},$$

where Z_i and ε are defined as in Corollary 1, respectively.

Proof. From the given condition, we have that there exists $k \geq 2$ such that

$$\max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} \min_{i\in N}(|m_{ii}| - r_i(M)) \leq \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|,$$

then

$$\begin{aligned} &\max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} \min_{i\in N}(|m_{ii}| - r_i(M)) + \varepsilon \min_{i\in N}(|m_{ii}| - r_i(M)) \\ \leq\ &\min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}| + \varepsilon \min_{i\in N}(|m_{ii}| - r_i(M)). \end{aligned}$$

Thus, we get

$$\begin{aligned}
&\left(\max_{i\in N}\frac{p_i^{(k-1)}(M)}{|m_{ii}|}+\varepsilon\right)\min_{i\in N}(|m_{ii}|-r_i(M))\\
\leq\ & \min_{i\in N}\sum_{j\in N\setminus\{i\}}\frac{p_j^{(k-2)}(M)-p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|+\varepsilon\min_{i\in N}(|m_{ii}|-r_i(M))\\
=\ & \min_{i\in N}\sum_{j\in N\setminus\{i\}}\frac{p_j^{(k-2)}(M)-p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|+\min_{i\in N}(\varepsilon(|m_{ii}|-r_i(M)))\\
\leq\ & \min_{i\in N}\left(\varepsilon(|m_{ii}|-r_i(M))+\sum_{j\in N\setminus\{i\}}\frac{p_j^{(k-2)}(M)-p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|\right)\\
=\ & \min_{i\in N}Z_i.
\end{aligned}$$

Since M is an SDD matrix, then

$$|m_{ii}|>r_i(M),\quad Z_i>0,\quad \forall i\in N.$$

It's easy to verify that

$$\frac{\max\limits_{i\in N}\frac{p_i^{(k-1)}(M)}{|m_{ii}|}+\varepsilon}{\min\limits_{i\in N}Z_i}\leq\frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}|-r_i(M))}.$$

Thus, by Corollary 1, it holds that

$$||M^{-1}||_\infty\leq\frac{\max\limits_{i\in N}\frac{p_i^{(k-1)}(M)}{|m_{ii}|}+\varepsilon}{\min\limits_{i\in N}Z_i}\leq\frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}|-r_i(M))}.$$

The proof is completed. □

We illustrate Theorem 9 by the following Example 2.

Example 2. This is the previous Example 1. For $k=4$, we have

$$\max_{i\in N}\frac{p_i^{(3)}(M)}{|m_{ii}|}\min_{i\in N}(|m_{ii}|-r_i(M))=0.1923<0.2959=\min_{i\in N}\sum_{j\in N\setminus\{i\}}\frac{p_j^{(2)}(M)-p_j^{(3)}(M)}{|m_{jj}|}|m_{ij}|.$$

Thus, by Theorem 8, we obtain that for each $0<\varepsilon_2<0.7034$,

$$||M^{-1}||_\infty\leq\frac{0.3845+\varepsilon_2}{0.2959+0.5\varepsilon_2}<2=\frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}|-r_i(M))}.$$

However, we find that the upper bounds in Theorems 8 and 9 contain the parameter ε. Next, based on Theorem 7, we will provide new upper bounds for the infinity norm of the inverse matrices of SDD_k matrices, which only depend on the elements of the given matrices.

Theorem 10. *Let $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an $SDD_k(k \geq 2)$ matrix and for each $i \in N_2(M)$, $\sum\limits_{j\neq i, j\in N_2(M)} |m_{ij}| > 0$. Then*

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min\limits_{i\in N}\left(p_i^{(k-1)}(M) - \sum\limits_{j\in N\setminus\{i\}} \frac{p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|\right)}.$$

Proof. By Theorems 7 and 8, we have that there exists a positive diagonal matrix X such that MX is an SDD matrix, where X is defined as in Theorem 7. Thus, it holds that

$$\|M^{-1}\|_\infty = \|X(X^{-1}M^{-1})\|_\infty = \|X(MX)^{-1}\|_\infty \leq \|X\|_\infty \|(MX)^{-1}\|_\infty,$$

and

$$\|X\|_\infty = \max_{1\leq i\leq n} x_i = \max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}.$$

Notice that MX is an SDD matrix. Thus, by Theorem 1, we get

$$\begin{aligned}\|(MX)^{-1}\|_\infty &\leq \frac{1}{\min\limits_{1\leq i\leq n}(|(MX)_{ii}| - r_i(MX))} \\ &= \frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}x_i| - r_i(MX))} \\ &= \frac{1}{\min\limits_{i\in N}\left(p_i^{(k-1)}(M) - \sum\limits_{j\in N\setminus\{i\}} \frac{p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|\right)}.\end{aligned}$$

Therefore, we have that

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min\limits_{i\in N}\left(p_i^{(k-1)}(M) - \sum\limits_{j\in N\setminus\{i\}} \frac{p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|\right)}.$$

The proof is completed. □

Since SDD matrices are a subclass of SDD_k matrices, by Theorem 10, we can obtain the following result.

Corollary 2. *Let $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an SDD matrix. If $r_i(M) > 0 (\forall i \in N)$, then there exists $k \geq 2$ such that*

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|}.$$

Two examples are given to show the advantage of the bound in Theorem 10.

Example 3. Consider the following matrix:

$$M = \begin{pmatrix} 40 & -1 & -2 & -1 & -2 \\ 0 & 10 & -4.1 & -4 & -6 \\ -20 & -2 & 33 & -4 & -8 \\ 0 & -4 & -6 & 20 & -2 \\ -30 & -4 & -2 & 0 & 40 \end{pmatrix}.$$

It is easy to verify that M is not an SDD matrix, an SDD_1 matrix, a $GSDD_1$ matrix, an S-SDD matrix, nor a CKV-type matrix. Therefore, we cannot use the error bounds in [1, 8, 9, 18, 20] to estimate $\|M^{-1}\|_\infty$. But, M is an SDD_2 matrix. So by the bound in Theorem 10, we have that $\|M^{-1}\|_\infty \leq 0.5820$.

Example 4. Consider the tri-diagonal matrix $M \in R^{n\times n}$ arising from the finite difference method for free boundary problems [18], where

$$M = \begin{pmatrix} b+\alpha sin\left(\frac{1}{n}\right) & c & 0 & \cdots & 0 \\ a & b+\alpha sin\left(\frac{2}{n}\right) & c & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & a & b+\alpha sin\left(\frac{n-1}{n}\right) & c \\ 0 & \cdots & 0 & a & b+\alpha sin\,(1) \end{pmatrix}.$$

Take that $n = 4$, $a = 1$, $b = 0$, $c = 3.7$ and $\alpha = 10$. It is easy to verify that M is neither an SDD matrix nor an SDD_1 matrix. However, we can get that M is a $GSDD_1$ matrix and an SDD_3 matrix. By the bound in Theorem 10, we have

$$\|M^{-1}\|_\infty \leq 8.2630,$$

while by the bound in Theorem 4, it holds that

$$\|M^{-1}\|_\infty \leq \frac{\varepsilon}{\min\{2.1488-\varepsilon, 0.3105, 2.474\varepsilon - 3.6272\}}, \quad \varepsilon \in (1.4661, 2.1488).$$

The following two theorems show that the bound in Corollary 2 is better than that in Theorem 1 in some cases.

Theorem 11. *Let $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$ be an SDD matrix. If $r_i(M) > 0(\forall i \in N)$ and there exists $k \geq 2$ such that*

$$\min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}| \geq \min_{i\in N}(|m_{ii}| - r_i(M)),$$

then

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M)-p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|} < \frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}| - r_i(M))}.$$

Proof. Since M is an SDD matrix, then $N_1(M) = \emptyset$ and M is an SDD_k matrix. By the given condition that $r_i(M) > 0(\forall i \in N)$, it holds that

$$|m_{ii}| \quad > \quad r_i(M) > \sum_{j\in N\setminus\{i\}} \frac{r_j(M)}{|m_{jj}|}|m_{ij}| = p_i^{(0)}(M) > 0, \quad \forall i \in N,$$

$$p_i^{(0)}(M) \;=\; \sum_{j\in N\backslash\{i\}} \frac{r_j(M)}{|m_{jj}|}|m_{ij}| > \sum_{j\in N\backslash\{i\}} \frac{p_j^{(0)}(M)}{|m_{jj}|}|m_{ij}| = p_i^{(1)}(M) > 0, \quad \forall i \in N.$$

Similarly, we can obtain that

$$|m_{ii}| > r_i(M) > p_i^{(0)}(M) > \cdots > p_i^{(k-1)}(M) > 0, \quad \forall i \in N,$$

that is,

$$\max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} < 1.$$

Since there exists $k \geq 2$ such that

$$\min_{i\in N} \sum_{j\in N\backslash\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}| \geq \min_{i\in N}(|m_{ii}| - r_i(M)),$$

then we have

$$\frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\backslash\{i\}} \frac{p_j^{(k-2)}(M)-p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|} < \frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}| - r_i(M))}.$$

Thus, from Corollary 2, we get

$$\|M^{-1}\|_\infty \leq \frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\backslash\{i\}} \frac{p_j^{(k-2)}(M)-p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|} < \frac{1}{\min\limits_{1\leq i\leq n}(|m_{ii}| - r_i(M))}.$$

The proof is completed. □

We illustrate the Theorem 11 by following Example 5.

Example 5. Consider the matrix $M = (m_{ij}) \in C^{n\times n}(n \geq 2)$, where

$$M = \begin{pmatrix} 4 & 3 & 0.9 & & & & & & \\ 1 & 6 & 2 & & & & & & \\ & 2 & 5 & 2 & & & & & \\ & & 2 & 5 & 2 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 2 & 5 & 2 & & \\ & & & & & 2 & 5 & 2 & \\ & & & & & & 1 & 6 & 2 \\ & & & & & & 0.9 & 3 & 4 \end{pmatrix}.$$

Take that $n = 20$. It is easy to check that M is an SDD matrix. Let

$$l_k = \min_{i\in N} \sum_{j\in N\backslash\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|, \quad m = \min_{i\in N}(|m_{ii}| - r_i(M)).$$

By calculations, we have

$$\begin{aligned} l_2 &= 0.2692 > 0.1 = m, \quad l_3 = 0.2567 > 0.1 = m, \quad l_4 = 0.1788 > 0.1 = m, \\ l_5 &= 0.1513 > 0.1 = m, \quad l_6 = 0.1037 > 0.1 = m. \end{aligned}$$

Thus, when $k = 2, 3, 4, 5, 6$, the matrix M satisfies the conditions of Theorem 11. By Theorems 1 and 11, we can derive the upper bounds for $\|M^{-1}\|_\infty$, see Table 2. Meanwhile, when $k = 1$, by Theorem 3, we get that $\|M^{-1}\|_\infty \leq 1.6976$.

Table 2. The bounds in Theorem 11 and Theorem 1.

k	2	3	4	5	6
Th 11	1.9022	1.5959	1.8332	1.7324	2.0214
Th 1	10	10	10	10	10

From Table 2, we can see that the bounds in Theorem 11 are better than that in Theorems 1 and 3 in some cases.

Theorem 12. *Let $M = (m_{ij}) \in C^{n\times n} (n \geq 2)$ be an SDD matrix. If $r_i(M) > 0 (\forall i \in N)$ and there exists $k \geq 2$ such that*

$$\begin{aligned} \max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} \min_{i\in N}(|m_{ii}| - r_i(M)) &\leq \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}| \\ &< \min_{i\in N}(|m_{ii}| - r_i(M)), \end{aligned}$$

then

$$\|M^{-1}\|_\infty \leq \frac{\max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|} \leq \frac{1}{\min_{1\leq i\leq n}(|m_{ii}| - r_i(M))}.$$

Proof. By Theorem 7 and the given condition that $r_i(M) > 0 (\forall i \in N)$, it is easy to get that

$$\sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}| > 0, \quad \forall i \in N.$$

From the condition that there exists $k \geq 2$ such that

$$\max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|} \min_{i\in N}(|m_{ii}| - r_i(M)) \leq \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|,$$

we have

$$\frac{\max_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M) - p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|} \leq \frac{1}{\min_{1\leq i\leq n}(|m_{ii}| - r_i(M))}.$$

Thus, from Corollary 2, it holds that

$$\|M^{-1}\|_\infty \le \frac{\max\limits_{i\in N} \frac{p_i^{(k-1)}(M)}{|m_{ii}|}}{\min\limits_{i\in N} \sum\limits_{j\in N\setminus\{i\}} \frac{p_j^{(k-2)}(M)-p_j^{(k-1)}(M)}{|m_{jj}|}|m_{ij}|} \le \frac{1}{\min\limits_{1\le i\le n}(|m_{ii}|-r_i(M))}.$$

The proof is completed. □

Next, we illustrate Theorem 12 by the following Example 6.

Example 6. Consider the tri-diagonal matrix $M = (m_{ij}) \in C^{n\times n}(n \ge 2)$, where

$$M = \begin{pmatrix} 3 & -2.5 & & & & & & & \\ -1.2 & 4 & -2 & & & & & & \\ & -2.8 & 5 & -1 & & & & & \\ & & -2.8 & 5 & -1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & -2.8 & 5 & -1 & & \\ & & & & & -2.8 & 5 & -1 & \\ & & & & & & -1.2 & 4 & -2 \\ & & & & & & & -2.5 & 3 \end{pmatrix}.$$

Take that $n = 20$. It is easy to verify that M is an SDD matrix.

By calculations, we have that for $k = 2$,

$$\begin{aligned} \max_{i\in N} \frac{p_i^{(1)}(M)}{|m_{ii}|} \min_{i\in N}(|m_{ii}| - r_i(M)) &= 0.2686 < \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(0)}(M) - p_j^{(1)}(M)}{|m_{jj}|}|m_{ij}| = 0.3250 \\ &< 0.5 = \min_{i\in N}(|m_{ii}| - r_i(M)). \end{aligned}$$

For $k = 5$, we get

$$\begin{aligned} \max_{i\in N} \frac{p_i^{(4)}(M)}{|m_{ii}|} \min_{i\in N}(|m_{ii}| - r_i(M)) &= 0.1319 < \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(3)}(M) - p_j^{(4)}(M)}{|m_{jj}|}|m_{ij}| = 0.1685 \\ &< 0.5 = \min_{i\in N}(|m_{ii}| - r_i(M)). \end{aligned}$$

For $k = 10$, it holds that

$$\begin{aligned} \max_{i\in N} \frac{p_i^{(9)}(M)}{|m_{ii}|} \min_{i\in N}(|m_{ii}| - r_i(M)) &= 0.0386 < \min_{i\in N} \sum_{j\in N\setminus\{i\}} \frac{p_j^{(8)}(M) - p_j^{(9)}(M)}{|m_{jj}|}|m_{ij}| = 0.0485 \\ &< 0.5 = \min_{i\in N}(|m_{ii}| - r_i(M)). \end{aligned}$$

Thus, for $k = 2, 5, 10$, the matrix M satisfies the conditions of Theorem 12. Thus, from Theorems 12 and 1, we get the upper bounds for $\|M^{-1}\|_\infty$, see Table 3. Meanwhile, when $k = 1$, by Theorem 3, we have that $\|M^{-1}\|_\infty \le 1.7170$.

From Table 3, we can see that the bound in Theorem 12 is sharper than that in Theorems 1 and 3 in some cases.

Table 3. The bounds in Theorem 12 and Theorem 1.

k	2	5	10
Th 12	1.6530	1.5656	1.5925
Th 1	2	2	2

4. Conclusions

SDD_k matrices as a new subclass of H-matrices are proposed, which include SDD matrices and SDD_1 matrices, and some properties of SDD_k matrices are obtained. Meanwhile, some new upper bounds of the infinity norm of the inverse matrices for SDD matrices and SDD_k matrices are presented. Furthermore, we prove that the new bounds are better than some existing bounds in some cases. Some numerical examples are also provided to show the validity of new results. In the future, based on the proposed infinity norm bound, we will explore the computable error bounds of the linear complementarity problems for SDD_k matrices.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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