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Research article

The weighted Lindley exponential distribution and its related properties

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Abstract: Both the exponential and Lindley distributions can be used to model the lifetime of a system or process, as well as the distribution of waiting times. In this study, we introduce the $WLE(\theta, \lambda, \alpha)$ notation for the weighted Lindley exponential distribution. Using two distinct asymmetrical distributions, the skewness mechanism of Azzalini was implemented in this distribution. In other words, we multiplied the density function of the Lindley distribution by the distribution function of the exponential distribution after adding the skewness parameter $\alpha > 0$. This WLE distribution contains the Lindley [1], the two parameters weighed Lindley [2] and the new weighted Lindley [3] distributions as special cases. We investigated the proposed model's mathematical properties. In addition to studying the central moments, we also investigate maximum likelihood estimators. To demonstrate the superiority of our model, we employ the MLE method to fit the weighted Lindley exponential model to the actual data set.

Keywords: extended exponential; extended Lindley; skew distribution; weighted Lindley; weighted exponential
Mathematics Subject Classification: 60E05, 62F10

1. Introduction

As an alternate to the normal distribution, reference [4] introduced the skew normal distribution to deal with asymmetry. This skewed normal distribution is generated by introducing a skewness parameter based on a weighted function into the normal distribution. Azzalini's skewness mechanism has been utilized with different symmetric distributions to generate skew symmetric distributions, such as the skew-t distribution, skew uniform, skew Laplace, skew Cauchy and skew logistic, to name a few.

The exponential distribution is a well-known probability distribution utilized in a variety of disciplines, including statistics, physics and engineering. It is frequently used to describe the waiting time between events that occur at a consistent rate and independently. This distribution is characterized by a single parameter, $\lambda > 0$, which measures the rate of events occurrence. The exponential distribution defined by the probability density function pdf

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0, \quad \lambda > 0 \tag{1.1}$$

where x is the time between events. The expression for the associated cumulative distribution function (cdf) is:

$$F(x) = 1 - e^{-\lambda x} \qquad x > 0, \quad \lambda > 0. \tag{1.2}$$

Due to the widespread use of the exponential distribution, its properties have been explored in a variety of contexts. Reference [5] introduced a three parameter distribution called the generalized exponential distribution which provides a generalization of the exponential distribution that allows for greater flexibility in modeling the behavior of the data. Using Azzalini's skewness mechanism, reference [6] presented weighted exponential distributions.

References [1, 7] it introduced the Lindley distribution as a mixture of exponential and gamma distributions.Is similar in form to the exponential distribution and is given by the pdf

$$f(x) = \frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x}, \qquad x > 0, \quad \theta > 0. \tag{1.3}$$

The cumulative distribution function corespondent to (1.3) is defined by

$$F(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{\theta x}, \qquad x > 0, \quad \theta > 0. \tag{1.4}$$

The shape of the Lindley distribution is determined by a single parameter that governs the skewness of the distribution. This distribution has been widely used in various fields, including economics, engineering and finance, due to its simplicity and flexibility. The Lindley distribution can be utilized to characterize the lifetime of a process or equipment with varying hazard rate forms. It has applications in numerous sectors, including biology, engineering and medicine among other sectors.

Several scientists investigated the characteristics and applications of the Lindley distribution in further depth, as seen by [8], whereas other academics concentrated on generalizing the Lindley distribution by combining it with other well-known distributions.

Reference [9] presented the T-Lindley{Y} class of distribution, which is formed by combining the quantile functions of the uniform, exponential, weibull, log-logistic, logistic and Cauchy distributions. Using [4]'s approach, reference [3] combined the Lindley density function with its distribution function to create the new weighted Lindley distribution. The authors concluded in this research that this generalization yields superior fits than the Lindley distribution and all of its known generalizations.

In statistical modeling, both the exponential distribution and the Lindley distribution are widely applied. In a Poisson process, the time between two consecutive events is typically modeled using an exponential distribution, while the time until a system failure occurs is usually modeled using a Lindley distribution. A single parameter controls the shape of each distribution for both distributions. In the exponential distribution, this parameter is the rate parameter, whereas in the Lindley distribution, it determines the skewness of the distribution as a shape parameter.

Since the exponential distribution and the Lindley distribution are so widely used, we are interested in developing a new class of distribution that combines these two distributions while offering a flexible statistical model. In this study, we develop this new model by combining the two distributions using [4]'s technique. As far as we are aware, this method has never been used as a continuous distribution. The rest of the paper is structured as follows. In Section 2, we introduced the the Weighted Lindley Exponential distribution, WLE, and we presented a wide variety of shapes of its density and cumulative functions. In Section 3, we studied some theoretical properties of WLE distribution such as limiting distribution as one of the three parameters approaches its boundaries. Some graphical representation of the WLE distribution are presented in Section 4. Explicit expressions for the moments of WLE random variable are provided in Section 5. In Section 6, the maximum likelihood estimators are constructed for the distribution. Application and data analysis of real data is given in Section 7 as well as generating a random sample using Inverse Transform Sampling method. Finally, the paper is concluded in Section 8.

2. Distribution and density functions

We define the three parameters weighted Lindley exponential distribution denoted by WLE in this section. A graphical representation of the pdf shapes is useful for determining whether or not a data set can be described using the WLE distribution.

Definition 2.1. A random variable X follows the WLE distribution with parameters θ and $\lambda \in \Re^+$ if it has the pdf

$$f(x) = B(1+x)e^{-x\theta}(1-e^{-x\alpha\lambda}), \qquad x \geq 0 \tag{2.1}$$

where

$$B = \frac{\theta^2(\theta+\alpha\lambda)^2}{\alpha\lambda(\theta(\theta+2)+\alpha\lambda(1+\theta))} \quad ,$$

and $\alpha \in \Re^+$. We say $X \sim LE(\theta, \lambda, \alpha)$.

Definition 2.2. The associated cdf of (2.1) is defined by

$$F(x) = B\{\frac{1-e^{-x\theta}}{\theta} + \frac{e^{-x\theta}(-1+e^{x\theta}-x\theta)}{\theta^2} + \frac{e^{-x(\theta+\alpha\lambda)}-1}{\theta+\alpha\lambda} + \frac{e^{-x(\theta+\alpha\lambda)}(1-e^{x(\theta+\alpha\lambda)}+x(\theta+\alpha\lambda))}{(\theta+\alpha\lambda)^2}\}. \tag{2.2}$$

Figures 1 and 2 represent the pdf and cdf curves of the new distribution defined in (2.1) and (2.2), respectively. We can see that the proposed WLE class of distribution is very flexible in the sense that it contains decreasing, increasing and upside-down bathtub class of distribution, which makes it a flexible class of distributions for modeling various lifetime data.

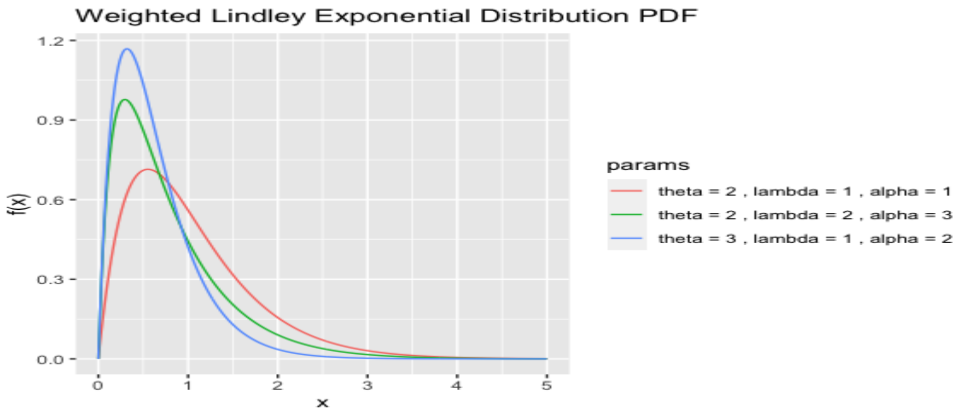


Figure 1. $WLE(\theta, \lambda, \alpha)$ density as the parameter θ, λ, α varying.

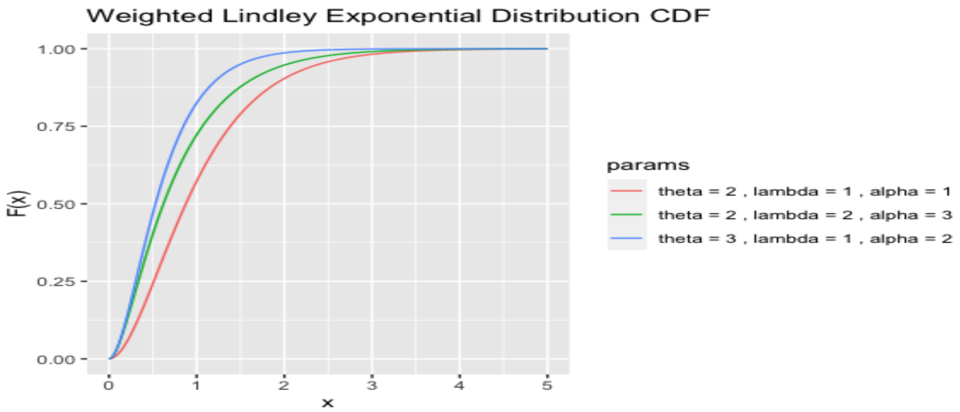


Figure 2. $WLE(\theta, \lambda, \alpha)$ cumulative function as the parameter θ, λ, α varying.

The survival function and failure rate (hazard rate) function for a continuous distribution are defined as follows.

$$S(x) = 1 - F(x).$$
$$H(x) = \frac{f(x)}{1 - F(x)}.$$

Assuming $f(x)$ and $F(x)$ are defined as in (2.1) and (2.2), respectively, we can define the survival function and failure rate (hazard rate) function for the Weighted Lindely Exponential distribution.

Figures 3 and 4 show different shapes for the hazard and survival rate functions of the WLE distribution, considering distinct values of θ, λ and α. It can be observed that the hazard rate function is rising monotonically and the survival rate function has a bathtub shape for all parameters value ≥ 1.

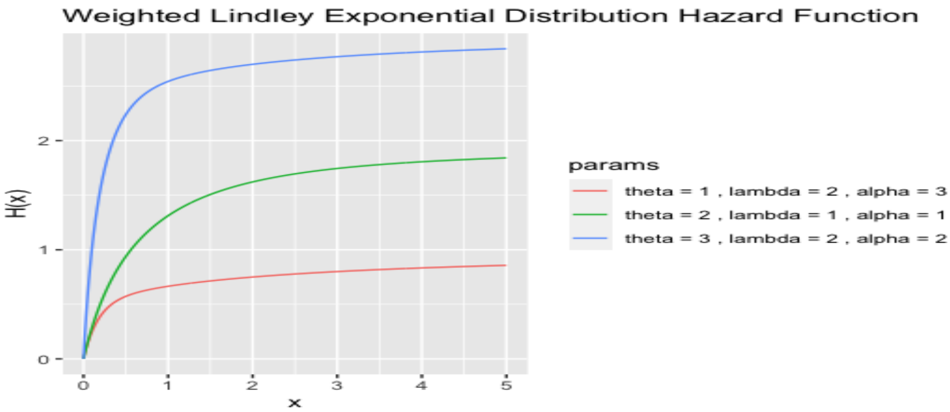


Figure 3. Plot of the hazard rate function of the WLE.

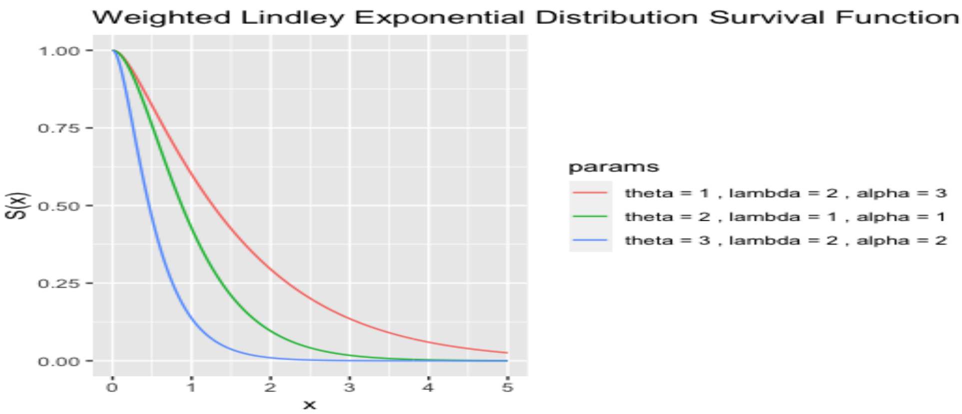


Figure 4. Plot of the survival rate function of the WLE.

3. Properties

In this section, we study some theoretical properties of the weighted Lindley exponential distribution.

In an effort to better comprehend the weighted Lindley exponential model, we examined the behavior of its probability density function when one of its distribution parameters approaches one of its boundaries. We applied this investigation to each parameter individually.

Proposition 3.1. *Let $X \sim WLE(\theta, \lambda, \alpha)$ be a random variable with pdf $f(x)$ defined in (2.1). Then,*

a. $\lim_{\alpha \to 0} f(x) = 0$

b. $\lim_{\lambda \to 0} f(x) = \frac{e^{-x\theta} x(1+x)\theta^3}{2+\theta}$

c. $\lim_{\alpha \to 0} f(x) = \frac{e^{-x\theta} x(1+x)\theta^3}{2+\theta}$.

Proposition 3.2. *Let $X \sim WLE(\theta, \lambda, \alpha)$ be a random variable with pdf $f(x)$ defined in (2.1). Then,*

a. $\lim_{\alpha \to \infty} f(x) = 0$

b. $\lim_{\lambda \to \infty} f(x) = \frac{\theta^2}{1+\theta} e^{-x\theta}(1 + x)$

c. $\lim_{\alpha\to\infty} f(x) = \frac{\theta^2}{1+\theta} e^{-x\theta}(1 + x)$.

Note that when $\lambda \to 0$ or $\alpha \to 0$ the pdf of WLE converges to the weighted Lindley distribution with parameters 2 and θ proposed by [2] as seen in 3.1. On the other hand, when $\lambda \to \infty$ or $\alpha \to \infty$ the pdf of WLE converges to the Lindley(θ) pdf as seen in 3.2. Thus, we can conclude that Lindley(θ) and WL(2, θ) are a sub-model of the proposed weighted Lindley exponential distribution.

Proposition 3.3. *Let $X \sim WLE(\theta, \lambda, \alpha)$ be a random variable with pdf $f(x)$ defined in (2.1). Then, $LE(\theta, 1, 1) = NWL(\theta, \frac{1}{\theta})$ where NWL is the new wighted Lindley distribution proposed by [3].*

Proposition 3.4. *The mode of the weighted Lindley exponential distribution can be obtained by solving the following equation*

$$m(x) = Be^{-x\theta}\Big[e^{-x\alpha\lambda}(1 + (x + 1)(\theta + \alpha\lambda)) + 1 - \theta(1 + x)\Big]. \tag{3.1}$$

To find the value of x that solves this equation, you would typically need to use numerical methods, such as the Newton-Raphson method or the bisection method to name a few.

4. Graphs

In order to comprehend the impact of each parameter on the overall shape of the WLE density, we present graphs with two fixed parameters and one variable parameter. Each curve has a unique line style to distinguish between the various values.

Figure 5 values illustrate the behavior of the Weighted Lindely Exponential density curve fore different values of theta. It shows that as θ increases, the peak of the density curve shifts to the left, and the tail of the distribution becomes thinner. For extremely large θ values (e.g., θ=40), the distribution is highly right-skewed, with a sharp peak near zero and a long right-extending tail. The peak shifts to the right and the tail becomes heavier as θ decreases (e.g., θ=20, 10, and 5), resulting in more dispersed distributions. At extremely small values of θ (e.g., θ= 0.005), the curve is relatively flat.This indicates that the distribution shape is highly dependent on the value of θ assuming all other parameters are constant and this observation supports the results (a) in properties 3.1 and 3.2.

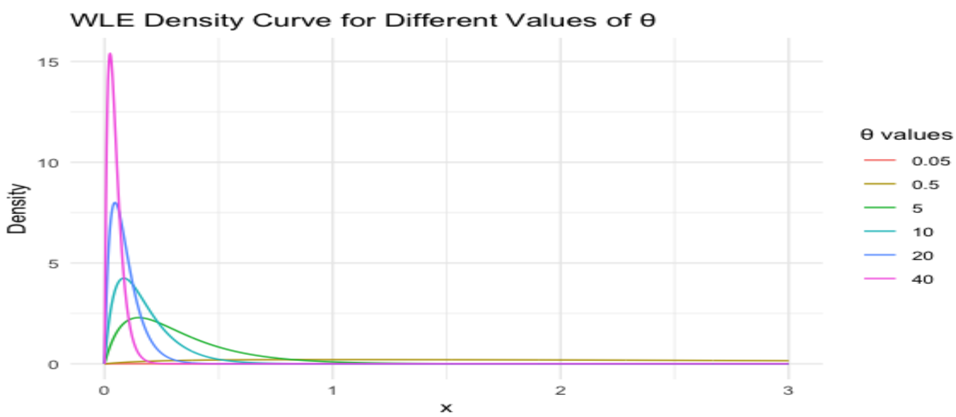


Figure 5. $WLE(\theta, \lambda = 3, \alpha = 2)$ density curves as the parameter θ varies.

Figure 6 illustrates the behavior of the weighted Lindely exponential density curves for different arbitrary values of the parameter (λ= 0.005, 0.05, 0.5, 1, 2, 10, and 50), while the other parameters,

θ and α, remain constant. Notice that as λ increases in value, the distribution becomes less dispersed and more concentrated around smaller x values. The distribution's peak shifts to the left, while the tail becomes thinner. For larger λ values (e.g., 10 and 50), the WLE density curve converges to the curve of Lindley distribution [1]. As λ decrease, the peak of the distribution becomes less sharp, and the tail of the distribution becomes thicker. For very small λ values (e.g., 0.005,0.05), the distribution curve converges to the curve of weighted Lindley distribution with two parameters 2 and θ proposed by [2]. The graphical illustrations of the behavior of the WLE density curve agrees with properties 3.1 and 3.2.

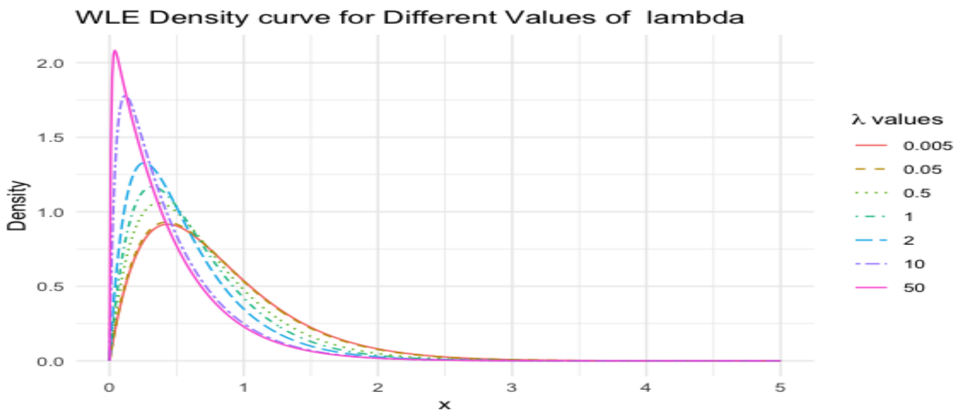


Figure 6. $WLE(\theta = 3, \lambda, \alpha = 2)$ density curves as the parameter λ varies.

Figure 7 illustrates the behavior of the Weighted Lindely Exponential density curves for different and arbitrary values of the parameter (α= 0.005, 0.05, 0.5, 1, 2, 10, and 50), while the other parameters, θ and λ, remain constant. Notice that as α increases in value, the distribution becomes less dispersed and more concentrated around smaller x values. The distribution's peak shifts to the left, while the tail becomes thinner. For larger α values (e.g., 10 and 50), the WLE density curve converges to the curve of Lindley distribution [1]. As α decrease, the peak of the distribution becomes less sharp, and the tail of the distribution becomes thicker. For very small α values (e.g., 0.005,0.05), the WLE distribution curve converges to the curve of weighted Lindley distribution with two parameters 2 and θ proposed by [2]. The graphical illustrations of the behavior of the WLE density curve agrees with properties 3.1 and 3.2. Notice that the behavior of the parameters λ and α are similar, assuming that the remaining two parameters are constant.

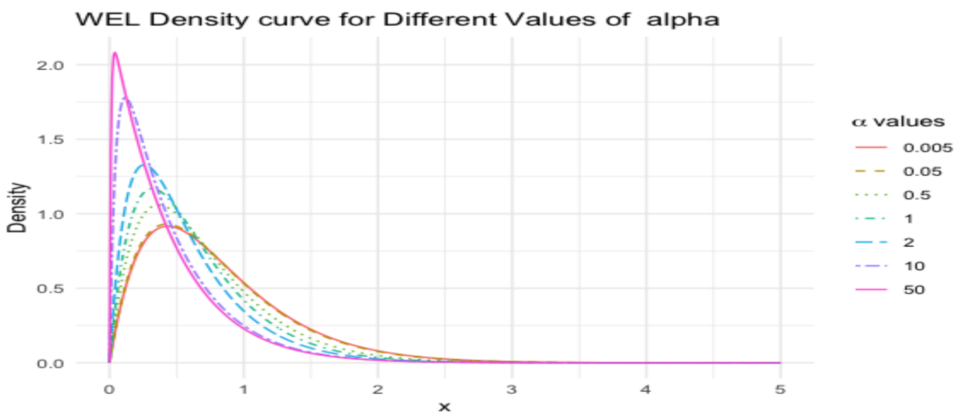


Figure 7. $WLE(\theta = 3, \lambda = 2, \alpha)$ density curves as the parameter α varies.

In summary, all three parameters (θ, λ, and α) have comparable effects on the WLE distribution, as depicted in Figures 5–7.

To examine the effect of each parameter on the shape of the density curve in more details, we created Figures 8–10.

Figures 8–10 facilitate the visual comprehension of the influence of each parameter on the WLE distribution studied on properties 3.1 and 3.2.

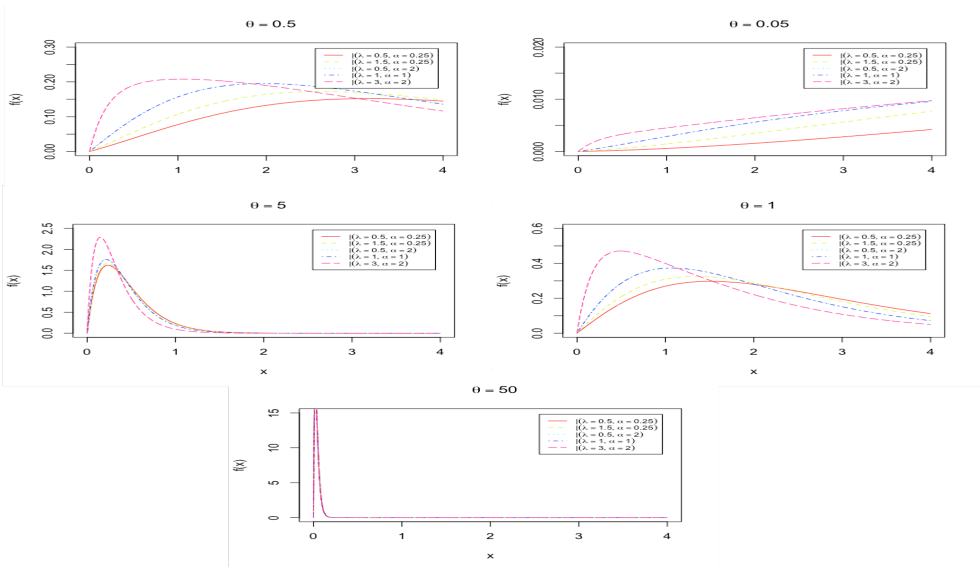


Figure 8. $WLE(\theta, \lambda, \alpha)$ density curves for different values of θ.

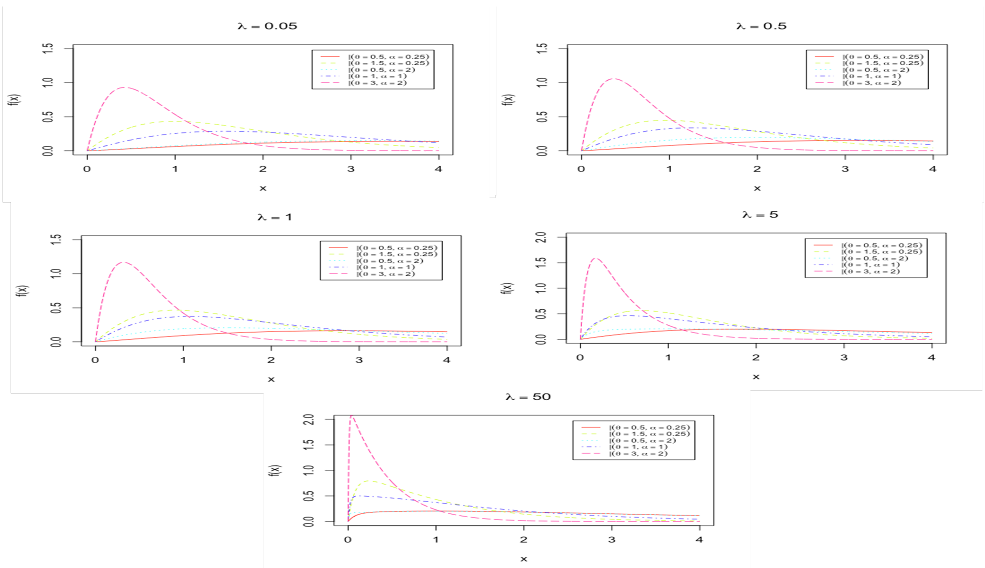


Figure 9. $WLE(\theta, \lambda, \alpha)$ density curves for different values of λ.

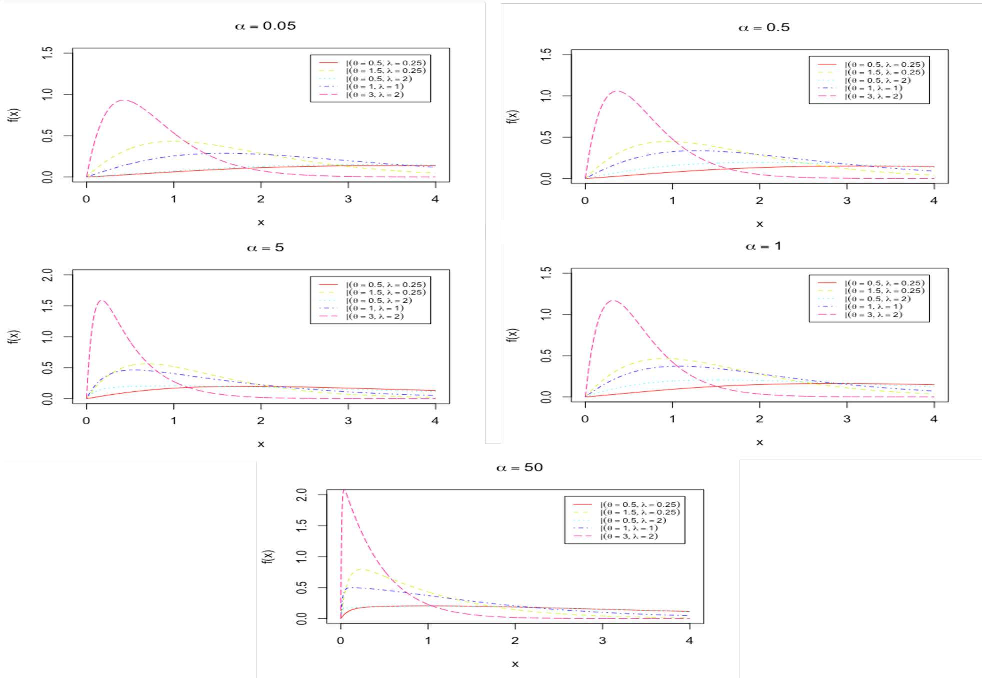


Figure 10. $WLE(\theta, \lambda, \alpha)$ density curves for different values of α.

Figure 11 provides a visual comparison between the WLE and NWL distributions with specific parameter values as specified by property 3.3, allowing us to observe the similarities in their shapes and behaviors.

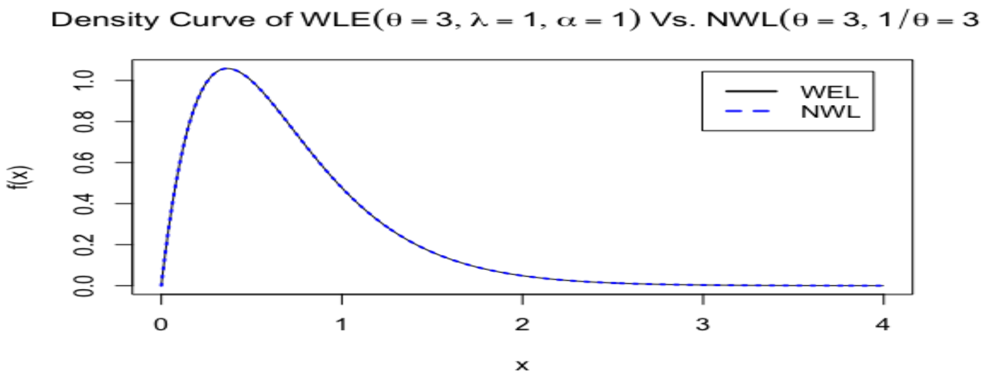


Figure 11. Density curve of $WLE(\theta = 3, \lambda = 1, \alpha = 1)$ Vs. $NWL(\theta = 3, \frac{1}{\theta=3})$.

Figure 12 provides a visual comparison between the WLE and WL distributions with specific parameter values as specified by property 3.2, allowing us to observe the similarities in their shapes and behaviors.

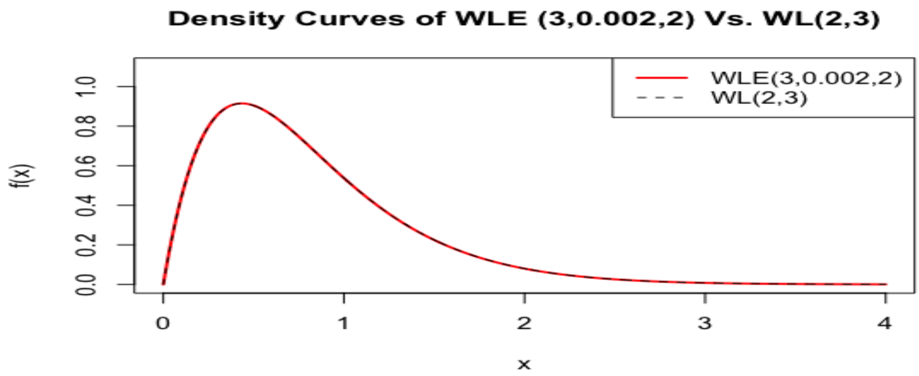


Figure 12. Density curve of $WLE(\theta = 3, \lambda = 0.002, \alpha = 2)$ Vs. $WL(2, \theta = 3)$.

5. Moments

In this section, explicit formulations for the moments of the random variable WLE are derived.

Proposition 5.1. *If X is a weighted Lindly exponential with parameters $\theta > 0, \alpha > 0$ and $\lambda > 0$ and for $n \geq 1$. The n^{th} central moment of X is given by:*

$$E(x^n) = Bn!\Big(\frac{n+1}{\theta^{(n+2)}} + \frac{1}{\theta^{n+1}} - \frac{(n+1)+\theta+\alpha\lambda}{(\theta+\alpha\lambda)^{n+2}}\Big). \tag{5.1}$$

Where B is a constant as defined in (2.1).

Using the formula in (5.1) we can derive explicit form of the first four central moments as follows.

$$E(X) = B(\frac{2}{\theta^3} + \frac{1}{\theta^2} - \frac{2+\theta+\alpha\lambda}{(\theta+\alpha\lambda)^3}). \tag{5.2}$$

$$E(X^2) = \frac{\theta^2(\theta+\alpha\lambda)^2(\frac{6}{\theta^4} + \frac{2}{\theta^3} - \frac{2(3+\theta+\alpha\lambda)}{(\theta+\alpha\lambda)^4})}{\alpha\lambda(\theta(2+\theta)+\alpha\lambda(1+\theta))}. \tag{5.3}$$

$$E(X^3) = \frac{6\theta^2(\theta+\alpha\lambda)^2(\frac{4}{\theta^5} + \frac{1}{\theta^4} - \frac{4+\theta+\alpha\lambda}{(\theta+\alpha\lambda)^5})}{\alpha\lambda(\theta(2+\theta)+\alpha\lambda(1+\theta))}. \tag{5.4}$$

$$E(X^4) = \frac{24\theta^2(\theta+\alpha\lambda)^2(\frac{5}{\theta^6} + \frac{1}{\theta^5} - \frac{5+\theta+\alpha\lambda}{(\theta+\alpha\lambda)^6})}{\alpha\lambda(\theta(2+\theta)+\alpha\lambda(1+\theta))}. \tag{5.5}$$

From (5.2) and (5.3) we concluded that the variance of the three parameters WLE random variable is given by the following explicit form.

$$Var(X) = B \cdot 2\left(\frac{3}{\theta^4} + \frac{1}{\theta^3} - \frac{3+\theta+\alpha\lambda}{(\theta+\alpha\lambda)^4}\right) - \left(B\left(\frac{2}{\theta^3} + \frac{1}{\theta^2} - \frac{2+\theta+\alpha\lambda}{(\theta+\alpha\lambda)^3}\right)\right)^2. \tag{5.6}$$

6. Maximum likelihood estimation

Maximum likelihood inference is a well-known topic whose notation is relatively conventional. The maximum likelihood estimators (MLEs) of the *WLE* parameters are provided in this section.

Proposition 6.1. *Assume that $\tilde{x} = \{x_1, x_2, ..., x_n\}$ is an observed sample of size n drawn from the density (2.1). Then, the likelihood function of $\tilde{x}$ is defined as*

$$L = \prod_{i=1}^{n} \frac{e^{-\theta x_i}(1 - e^{-\alpha\lambda x_i})(1 + x_i)\theta^2(\theta + \alpha\lambda)^2}{\alpha\lambda(\theta(2 + \theta) + \alpha\lambda(1 + \theta))}, \tag{6.1}$$

the corresponding log-likelihood function is given by

$$\begin{aligned} logL = \ell = -\theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log(1 - e^{-\alpha\lambda x_i}) + \sum_{i=1}^{n} \log(1 + x_i) \\ +2n\log\theta + 2n\log(\theta + \alpha\lambda) - n\log\alpha - n\log\lambda - n\log(\theta(2 + \theta) + \alpha\lambda(1 + \theta)). \end{aligned} \tag{6.2}$$

Now setting $\frac{\partial \ell}{\partial \theta} = 0$, $\frac{\partial \ell}{\partial \alpha} = 0$, $\frac{\partial \ell}{\partial \lambda} = 0$ respectively, we have

$$-\sum_{i=1}^{n} x_i + \frac{2n}{\theta} + \frac{2n}{\theta + \alpha\lambda} - \frac{n(2 + 2\theta + \alpha\lambda)}{\theta(2 + \theta) + \alpha\lambda(1 + \theta)} = 0 \tag{6.3}$$

$$\sum_{i=1}^{n} \frac{\lambda x_i e^{-\lambda\alpha x_i}}{1 - e^{-\lambda\alpha x_i}} + \frac{2n\lambda}{\theta + \alpha\lambda} - \frac{n}{\alpha} - \frac{n(1 + \theta)\lambda}{\theta(2 + \theta) + \alpha\lambda(1 + \theta)} = 0 \tag{6.4}$$

$$\sum_{i=1}^{n} \frac{\alpha x_i e^{-\lambda\alpha x_i}}{1 - e^{-\lambda\alpha x_i}} + \frac{2n\alpha}{\theta + \alpha\lambda} - \frac{n}{\lambda} - \frac{n\alpha(1 + \theta)}{\theta(2 + \theta) + \alpha\lambda(1 + \theta)} = 0. \tag{6.5}$$

Unfortunately, solving Eqs (6.3)–(6.5) for θ, λ and α, respectively, are quite difficult due to the non-linear nature of the equations. In practice, we would typically use a numerical method such as the Newton-Raphson method or a root-finding algorithm to find an approximate the solutions.

7. Simulation and application to real data

7.1. Simulation

In this section, we generate a random sample of the weighted Lindely exponential distribution using one of the standard techniques for random sampling. Given the probability density function and cumulative distribution function of the *WLE* distribution, the Inverse Transform Sampling method can be used to generate a random sample. We adhered to these steps:

(1) First, we defined the probability density function (2.1) and cumulative distribution function (2.2) of the *WLE* distribution with the parameters θ, λ and α.

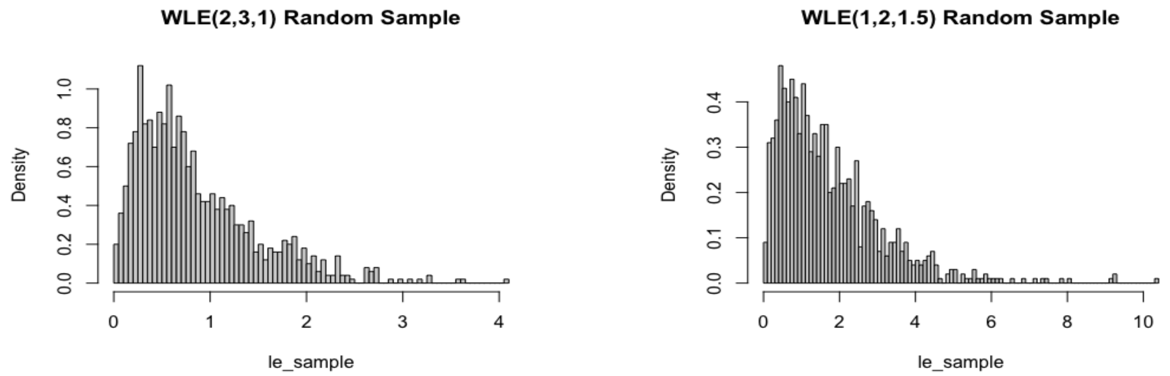
(2) Since the Inverse Transform Sampling technique requires the inverse of the cdf, we tried to seek an analytical solution for the inverse cdf. However, the given cdf was complex, and an analytical solution for its inverse could not be found. Therefore, a numerical method was chosen to approximate the inverse of the cdf.

(3) With a sample size of 1000, we generated a set of uniformly distributed random numbers $U(0, 1)$ with a uniform distribution. Using R's "runif" function, these random numbers were generated.

(4) We determined, for each uniformly distributed random number u_i, the value x_i such that $F(x_i) = u_i$. In R, the "uniroot" function was used to find the root of the equation $F(x) - u = 0$. We used the "uniroot" method which is a numerical method for locating the root of a function within a given interval.

(5) Step four yields a random sample from the *WLE* distribution with parameters θ, λ and α. Using the Inverse Transform Sampling method, we were able to successfully generate a random sample of size 1000 from the *WLE* distribution.

Figures 13 (a),(b) and 14 represent the histograms of three random samples with size 1000 simulated from $WLE(\theta, \lambda, \alpha)$ distribution using the inverse transform sampling method, with the parameter vectors $(\theta = 2, \lambda = 3, \alpha = 1)$, $(\theta = 1, \lambda = 2, \alpha = 1.5)$ and $(\theta = 3, \lambda = 0.5, \alpha = 2)$, respectively. Suggested from the preceding graphs, the *WLE* distribution has a right-skew unimodal density that is ideal for modeling highly skewed data. Changing the parameter values allows the distribution to model a diverse range of right-skewed data, regardless of whether the data has a thick or thin tail.



(a) Histogram of $WEL(\theta = 2, \lambda = 3, \alpha = 1)$. **(b)** Histogram of $WEL(\theta = 1, \lambda = 2, \alpha = 1.5)$.

Figure 13. Histogram for *WLE* random samples of size 500.

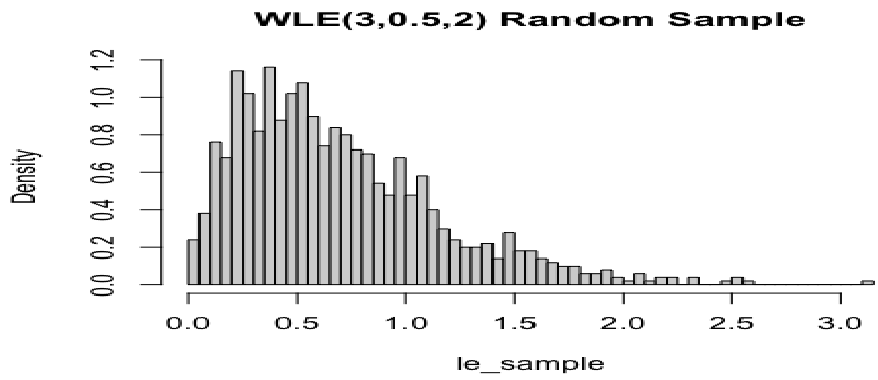


Figure 14. Histogram of $WEL(\theta = 3, \lambda = 0.5, \alpha = 2)$ random sample.

7.2. Application

Employing some discrimination criterion techniques based on the log-likelihood function estimated at the MLE, we demonstrate the superior performance of the WLE distribution presented here as compared with some of its sub-models. The Akaike Information Criterion (AIC), the Bayesian information criterion (BIC) and Hannan-Quinn Information Criterion (HQIC) were considered for model selection.

We present an example utilizing a widely recognized dataset to showcase the practicality of the suggested methodology. The table is utilized to present the estimated values of the three parameters $(\lambda, \alpha, \theta)$ for each model, together with the negative log-likelihood, AIC, BIC and HQIC values.

The relief time dataset illustrates the relief time (in minutes) of analgesic-treated patients. The data was first reported in [10] and also appeared in a number of lifetime distribution-related propositions [11] and [12] among others. The data set is given below: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.

Table 1 contains descriptive statistics for the data on relief time. It displays the number of observations, markers for the first four moments (mean, standard deviation, skewness, excess kurtosis), as well as the minimum and maximum value of the data. According to the descriptive statistics, the relief time data set is right-skewed which makes the proposed WLE model a good candidate to fit the data.

Table 1. Summary description of relief time data set.

N	Min.	Median	Mean	sd	Max.	skewness	kurtosis
20	1.1	1.7	1.9	0.704	4.1	1.59	2.34

Figure 15 shows the histogram and normal Q-Q plot corresponding to the relief time data set. The histogram indicates a sufficient quantity of tiny records. The data is skewed to the right, as seen by the histogram and the departure from normality in the Q-Q plot.

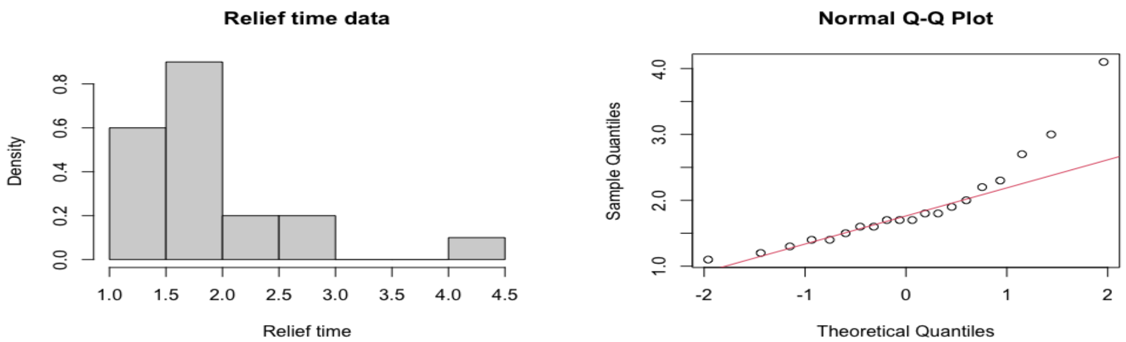


Figure 15. Histogram and Q-Q plot for relief time data set.

Table 2 compares three distribution models for the relief time data set. WLE distribution, Lindley distribution and exponential distribution are the models considered. The table provides parameter estimates for each distribution, as well as the negative log-likelihood ($-log(l)$) and the AIC, BIC and HQIC.

Table 2. Parameter estimations for the relief time data set.

Dist.	θ	λ	α	$-log(l)$	AIC	BIC	HQIC
WLE	1.3653	0.0024	0.0095	24.8593	55.7187	58.706	56.3019
$Lindley$	0.8161			30.2495	62.4991	63.4948	62.6935
$Exponential$		0.5263		32.8370	67.6742	68.6699	67.8685

The primary objective is to choose the optimal model using the AIC, BIC and HQIC selection techniques. Lower AIC, BIC and HQIC values indicate a better fit of the model to the data. Comparing the AIC, BIC and HQIC values for all three models, the WLE distribution has the lowest AIC, BIC and HQIC, indicating that it best fits the Relief time data set. The WLE distribution is the preferred model for this data set based on both the AIC, BIC and HQIC selection criteria.

8. Conclusions

The weighted Lindley exponential $WLE(\theta, \lambda, \alpha)$ distribution is a probability distribution that was created utilizing [4]'s skewing idea using two asymmetric distributions.

The probability density function of the WLE distribution is represented by Eq (2.1), where B is a normalization constant that ensures the density integrates to one over its support, which is the non-negative real line. It is a unimodel distribution which contains decreasing, increasing and upside-down bathtub classes of distribution that makes it a flexible class of distributions for modeling various lifetime data. The WLE distribution is a member of the family of decreasing failure rate distributions, as the hazard rate function has monotonically increased and the survival rate function has bathtub shapes. In addition, it is also a flexible distribution that can be used to model numerous real-world phenomena, including reliability engineering, finance and insurance. This WLE distribution contains the Lindley [1], the two parameters weighed Lindley [2] and the new weighted Lindley [3] distributions as special cases.

We investigated the mathematical properties of the proposed model. In addition to investigating central moments, we examine maximum likelihood estimators. Based on log-likelihood functions, different discrimination criterion methods have been used to demonstrate the superiority of our model.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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