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Research article

Traces of certain integral operators related to the Riemann hypothesis

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Abstract: We prove the existence of a nontrivial singular trace τ defined on an ideal $\mathcal{J}$ closed with respect to the logarithmic submajorization such that $\tau(A_\rho(\alpha)) = 0$, where $A_\rho(\alpha) : L^2(0,1) \to L^2(0,1)$, $[A_\rho(\alpha)f](\theta) = \int_0^1 \rho(\alpha\theta/x)f(x)dx$, $0 < \alpha \leq 1$. We also show that $\tau(A_\rho(\alpha)) = 0$ for every τ nontrivial singular trace on $\mathcal{J}$. Finally, we give a recursion formula from which we can evaluate all the traces $\mathrm{Tr}\,(A_\rho^r(\alpha))$, $r \in \mathbb{N}$, $r \geq 2$.

Keywords: singular trace; spectral trace; modified Fredholm determinant; Riemann hypothesis
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1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators on a separable complex Hilbert space H. The adjoint of an operator $T \in B(H)$ is denoted by T^* and the symbol I stands for identity maps. Denote by $\{s_n(T)\}_{n\geq 1}$ the sequence of singular values of a compact operator $T \in B(H)$. If $0 < p < \infty$ we say that $T \in S^p(H)$ if $\sum_{n=1}^{\infty} s_n^p(T) < \infty$; the set $S^p(H)$ is a two-side ideal in $B(H)$. The sets $S^1(H)$ and $S^2(H)$ will denote, respectively, the set of nuclear and Hilbert-Schmidt operators. By an ideal we mean a two-sided ideal in $B(H)$. A linear functional τ from the ideal J into $\mathbb{C}$ is said to be a trace if:

i) $\tau(U^*TU) = \tau(T)$ for every $T \in J$ and $U \in B(H)$ unitary. Equivalently, $\tau(ST) = \tau(TS)$ for every $T \in J$ and $S \in B(H)$.
ii) $\tau(T) \geq 0$ for every $T \in J$ with T a non-negative operator. We denote by $T \geq 0$ when T is a non-negative operator.

Then a trace is a positive unitarily invariant linear functional.

Let $T \in S^1(H)$, and let $\{\varphi_n\}_{n\geq 1}$ be an orthonormal system in H. By [13, p. 56], we have

$$\sum_{j=1}^{n} |\langle T\varphi_j, \varphi_j\rangle| \leq \sum_{j=1}^{n} s_j(T) \quad , \quad \forall n \geq 1.$$

Therefore,

$$\mathrm{Tr}\,(T) := \sum_{n=1}^{\infty} \langle T\varphi_n, \varphi_n \rangle$$

is well-defined, since the right-hand series converges absolutely, and its value does not depend on the choice of the orthonormal basis $\{\varphi_n\}_{n\geq 1}$. Clearly, this linear functional is a trace on the ideal $S^1(H)$. There is also the description of Tr as the sum of eigenvalues,

$$\mathrm{Tr}\,(T) = \sum_{n=1}^{\infty} \lambda_n(T), \tag{1.1}$$

where $\{\lambda_n(T)\}_{n\geq 1}$ is the sequence of nonzero eigenvalues of T, ordered in such a way that $|\lambda_n(T)| \geq |\lambda_{n+1}(T)|$, $\forall\, n \in \mathbb{N}$ and each one of them being counted according to its algebraic multiplicity. This result was shown by Von Neumann in [26] for self-adjoint operators and by Lidskii in [19] in general case. Formula (1.1) is called the Lidskii formula.

A natural question concerning the extension of the Lidskii formula to other ideals and traces on these ideals has been addressed in [6, 13, 20–22]. Thus, the notion of spectral traces arises. A trace τ on an ideal J is called spectral if for every $T \in J$, the value of $\tau(T)$ depends only on the eigenvalues of T and their multiplicities. For example, the classical trace Tr on the ideal $S^1(H)$ is spectral. Motivated by the problem of identifying spectral traces, it is shown in [12, Corollary 2.4] that every trace on a geometrically stable ideal is spectral. This result has been generalized in the setting of ideals closed with respect to the logarithmic submajorization (see [24]).

A trace τ on an ideal J will be called singular if it vanishes on the set $\mathcal{F}(H)$ of finite rank operators. This definition makes sense, since by the Calkin Theorem [9], each proper ideal in $B(H)$ contains the finite rank operators and is contained in the ideal $K(H)$ of the compact linear operators on H.

In 1966, J. Dixmier proved the existence of singular traces [11]. These traces are called Dixmier traces, and its importance is due to their applications in noncommutative geometry [10]. Other examples of singular traces appeared in [1, 17, 25].

The question whether an operator belongs to the domain of some singular trace was answered in [15]. Motivated by this, we give a nontrivial singular trace taking the value of zero on certain integral operator related to the Riemann hypothesis.

In [2], J. Alcántara-Bode has reformulated the Riemann hypothesis as a problem of functional analysis by means of the following theorem.

Theorem 1.1. *Let* $(A_\rho f)(\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)dx$*, where* ρ *is the fractional part function, be considered as an operator on* $L^2(0,1)$*. Then the Riemann hypothesis holds if and only if* $\mathrm{Ker}(A_\rho) = \{0\}$*, or if and only if* $h \notin \mathrm{Ran}\,(A_\rho)$ *where* $h(x) = x$*.*

It follows from Theorem 1.1 that the problem of verifying the condition $h \in \mathrm{Ran}\,(A_\rho)$ is ill posed in the sense of Hadamard [16, Definition 2.1.2]. This leads to regularizing the ill posed problem replacing A_ρ by

$$(A_\rho(\alpha)f)(\theta) = \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) f(x)dx\ ,\ 0 < \alpha \leq 1\ ,\ f \in L^2(0,1).$$

Observe that $A_\rho(1) = A_\rho$. Since A_ρ is nonnuclear (see [2, Theorem 6]), it was proved in [23, Theorem 4.3] that there exists a nontrivial singular trace τ with domain a geometrically stable ideal $\mathcal{J}$ such that $A_\rho \in \mathcal{J}$ and $\tau(A_\rho) = 0$. It follows from [24, Lemma 35] that $\mathcal{J}$ is closed with respect to the logarithmic submajorization.

Let $\mathcal{J}$ be as in the previous paragraph. We show that if $0 < \alpha \leq 1$ then $A_\rho(\alpha) \in \mathcal{J}$, and that each nontrivial singular trace on $\mathcal{J}$ takes the value of zero on $A_\rho(\alpha)$. More precisely, our first main result of the paper is the following theorem.

Theorem 1.2. *If* $0 < \alpha \leq 1$ *then* $\tau(A_\rho(\alpha)) = 0$ *for every* τ *nontrivial singular trace on* $\mathcal{J}$*, where* $\mathcal{J}$ *is the geometrically stable ideal in the above paragraph.*

The approach used to prove Theorem 1.2 is based on spectral traces and the fact that the operators $\alpha(A_\rho + Q_{f_1})$ and $A_\rho + Q_{f_\alpha}$, where $\alpha \in]0, 1[$, have the same nonzero eigenvalues with the same algebraic and geometric multiplicities. Here $Q_f g = \langle g, f\rangle h$, $h(x) = x$, $f_\alpha(x) = -\alpha^{-1}\rho\left(\dfrac{\alpha}{x}\right)$.

Finally, our second main result is a recursion formula to calculate the traces $\mathrm{Tr}\,(A^r_\rho(\alpha))$, $r \in \mathbb{N}$, $r \geq 2$.

Theorem 1.3. *If* $0 < \alpha \leq 1$ *then for every* $r \in \mathbb{N}$ *with* $r \geq 2$ *we have*

$$\mathrm{Tr}(A^{r+1}_\rho(\alpha)) = -(r+1)a_{r+1}(\alpha) - \alpha a_r(\alpha) - \sum_{k=1}^{r-1} a_{r-k}(\alpha)\mathrm{Tr}(A^{k+1}_\rho(\alpha)), \tag{1.2}$$

where

$$a_r(\alpha) = \begin{cases} -\alpha & , \ r = 1 \\ \dfrac{(-1)^r \alpha^{r(r+1)/2}}{r!r} \displaystyle\prod_{l=1}^{r-1} \zeta(l+1) & , \ r \geq 2. \end{cases}$$

In order to prove Theorem 1.3, we use the modified Fredholm determinant of $I - uA_\rho(\alpha)$ (see [3]).

2. Singular traces

Let l^∞ the space of all bounded sequence of complex numbers and w a dilation invariant extended limit on l^∞, that is, w is an extended limit on l^∞ and

$$w(\{x_1, x_2, \ldots\}) = w(\{x_1, x_1, x_2, x_2, \ldots\}) \text{ for all } x = \{x_1, x_2, \ldots\} \in l^\infty.$$

The Dixmier trace of $T \in M^{1,\infty}(H)$ with $T \geq 0$ is the number

$$\mathrm{Tr}_w(T) := w\left(\left\{\frac{1}{\log(n+1)} \sum_{k=1}^n s_k(T)\right\}_{n\geq 1}\right),$$

where

$$M^{1,\infty}(H) = \left\{T \in K(H) : \|T\|_{1,\infty} := \sup_{n\geq 1}\left\{\frac{1}{\log(n+1)} \sum_{k=1}^n s_k(T)\right\} < \infty\right\}.$$

It was shown in [11] that the weight Tr_w defines a positive, unitarily invariant, additive and positive homogeneous function on the positive cone of $M^{1,\infty}(H)$, that can uniquely be extended to a singular trace on all of $M^{1\infty}(H)$, i.e., for an arbitrary $T \in M^{1,\infty}(H)$, its Dixmier trace is defined by

$$\mathrm{Tr}_w(T) := w\left(\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n} s_k(T_1) - s_k(T_2) + is_k(T_3) - is_k(T_4)\right\}_{n\geq 1}\right),$$

where $T = T_1 - T_2 + iT_3 - iT_4$, $0 \leq T_j \in M^{1,\infty}(H)$, $j = 1, 2, 3, 4$. In addition to this, the Dixmier trace vanishes on the ideal $S^1(H)$ and is continuous in the norm $\|\cdot\|_{1,\infty}$.

The existence of a singular trace which is nontrivial on a compact operator T, i.e., on the two-sided ideal generated by T,

$$(T) = \bigcup_{r=1}^{\infty}\left\{\sum_{i=1}^{r} X_i T Y_i\,;\, X_i,\, Y_i \in B(H)\right\}$$

was studied by J. Varga [25], and it has been completely characterized in [1]. This leads to study irregular, eccentric and generalized eccentric operators.

Definition 2.1. *We say that a compact operator $T \in B(H)$ is*

a) regular if $\sum_{k=1}^{n} s_k(T) = \mathcal{O}(ns_n(T))$ $(n \to \infty)$;
b) irregular if it is not regular;
c) eccentric if it is irregular but not nuclear;
d) generalized eccentric if 1 is a limit point of the sequence $\left\{\frac{S_{2n}(T)}{S_n(T)}\right\}_{n\geq 1}$, where

$$S_n(T) = \begin{cases} \sum_{k=1}^{n} s_k(T) & ,\ T \notin S^1(H) \\ \sum_{k=1}^{n} s_k(T) - \mathrm{Tr}(|T|) & ,\ T \in S^1(H). \end{cases}$$

Remark 2.2. *By [25, Lemma 1], the class of generalized eccentric operators which are not nuclear coincides with the class of eccentric operators.*

By [1, Lemma 2.6], that an operator is generalized eccentric can be reformulated as follows.

Lemma 2.3. *Let $T \in B(H)$ be a compact operator. Then T is generalized eccentric if and only if there exists an increasing sequence of natural numbers $\{p_n\}_{n\geq 1}$ such that* $\lim_{k\to+\infty} \frac{S_{kp_k}(T)}{S_{p_k}(T)} = 1$.

In this context, the main result in [1] is the following.

Theorem 2.4. *Let $T \in B(H)$ be a compact operator. Then the following are equivalent:*

a) There exists a singular trace τ such that $0 < \tau(|T|) < \infty$.
b) T is generalized eccentric.

The process to construct the singular trace given by $a)$ is as follows:

We introduce a triple $\Omega = (T, w, \{n_k\}_{k\geq 1})$, where T is a generalized eccentric operator, w is an extended limit and $n_k = kp_k$, $k \in \mathbb{N}$, where $\{p_k\}_{k\geq 1}$ is the sequence given in Lemma 2.3. Associated with the triple Ω, on the positive part of the ideal (T), we defined the functional

$$\tau_\Omega(A) := w\left(\left\{\frac{S_{n_k}(A)}{S_{n_k}(T)}\right\}_{k\geq 1}\right)$$

and by [1, Theorem 2.11], this functional extends linearly to a singular trace on the ideal (T).

Remark 2.5.

a) *By Theorem 2.4, finite rank operators cannot by generalized eccentric.*

b) *The question whether an operator belongs to the domain of some singular trace is treated in [15]. By [15, Theorem 3.1 (i)], every compact operator A is in the kernel (hence in the domain) of some singular trace. The main idea for proving this theorem is the existence of a generalized eccentric operator B such that $A \in (B)_0 \subset (B)$. Here $(B)_0$ denotes the kernel of (B) (see [15]).*

Example 2.6. Let $V : L^2(0,1) \to L^2(0,1)$ be the integral operator

$$(Vf)(t) = 2i\int_0^t f(s)ds.$$

By [14, p. 250], $s_n(V) = \dfrac{4}{(2n-1)\pi}$, $n = 1, 2, \ldots$. Therefore, by Remark 2.2, V is a generalized eccentric operator. Let $\mathrm{Re}(V)$ and $\mathrm{Im}(V)$ be the real and imaginary parts of V, respectively. Then

$$(\mathrm{Re}(V)f)(t) = i\int_0^1 sign(t-s)f(s)ds$$

and

$$(\mathrm{Im}(T)f)(t) = \int_0^1 f(s)ds.$$

By [14, p. 172], $s_n(\mathrm{Re}(V)) = \dfrac{2}{(2n-1)\pi}$, $n = 1, 2, \ldots$, it follows that $\mathrm{Re}(V)$ is a generalized eccentric operator. However, the operator $\mathrm{Im}(V)$ has rank one, and by Remark 2.5 (a), $\mathrm{Im}(V)$ is not an eccentric operator.

3. Commutator subspace

Now we concentrate on the commutator subspace of geometrically stable ideals and ideals closed with respect to the logarithmic submajorization, terminologies used in [18, 24], respectively.

Definition 3.1. *Let J an ideal in $B(H)$. The subspace*

$$\mathrm{Com}(J) := \mathrm{span}\left\{[A, B] \,:\, A \in J\,,\, B \in B(H)\right\},$$

where $[A, B] = AB - BA$, is called the commutator subspace of J.

Definition 3.2. *An ideal J of $B(H)$ is called geometrically stable if a diagonal operator* $\mathrm{diag}\{s_1, s_2, \dots\} \in J$*, where* $s_1 \geq s_2 \geq \dots \geq 0$*, then* $\mathrm{diag}\{u_1, u_2, \dots\} \in J$*, where* $u_n = (s_1 s_2 \dots s_n)^{1/n}$.

The following theorem [18, Theorem 3.3] characterizes in terms of arithmetic means the commutator subspace of geometrically stable ideals.

Theorem 3.3. *Suppose that J is a geometrically stable ideal of $B(H)$. Then $T \in \mathrm{Com}(J)$ if and only if* $\mathrm{diag}\{\frac{1}{n}(\lambda_1(T) + \dots + \lambda_n(T))\} \in J$.

In order to extend the previous result, ideals closed with respect to the logarithmic submajorization are introduced in [24].

Definition 3.4. *If $A, C \in B(H)$, then the operator C is logarithmically submajorized by the operator A (written $C <<_{\log} A$) if*

$$\prod_{k=1}^{n} s_k(C) \leq \prod_{k=1}^{n} s_k(A) \quad , \quad n \geq 1.$$

Definition 3.5. *An ideal J is said to be closed with respect to the logarithmic submajorization if $C <<_{\log} A \in J$ implies $C \in J$.*

Remark 3.6. *By* [24, Lemma 35]*, every geometrically stable ideal is closed with respect to the logarithmic submajorization. However, by [24, Theorem 36 (c)], the converse assertion fails.*

The following theorem [24, Theorem 7] extends Theorem 3.3 to ideals closed with respect to the logarithmic submajorization.

Theorem 3.7. *Let an ideal J be closed with respect to the logarithmic submajorization and let $T \in J$. Then $T \in \mathrm{Com}(J)$ if and only if* $\mathrm{diag}\{\frac{1}{n}(\lambda_1(T) + \dots + \lambda_n(T))\} \in J$.

A Lidskii-type formula holds for every trace defined on ideals closed with respect to the logarithmic submajorization (see [24, Theorem 8]), the statement is the following.

Theorem 3.8. *Let J be an ideal in $B(H)$ and let τ be a trace on J. If J is closed with respect to the logarithmic submajorization, then τ is a spectral trace.*

Example 3.9.

a) Let $\mathcal{L}_{1,\infty}(H)$ be the ideal

$$\mathcal{L}_{1,\infty}(H) = \left\{ T \in K(H) : \sup_{n\geq 1}\{n s_n(T)\} < \infty \right\}.$$

Since $\mathcal{L}_{1,\infty}(H)$ is a quasi-Banach ideal with the complete quasi-norm $\|T\|_{\mathcal{L}_{1,\infty}(H)} = \sup_{n\geq 1}\{n s_n(T)\}$ and every quasi-Banach ideal is geometrically stable (see [18, Proposition 3.2]), then by Remark 3.6, $\mathcal{L}_{1,\infty}(H)$ is closed with respect to the logarithmic submajorization.

b) It can be shown that the integral operator $V : L^2(0,1) \to L^2(0,1)$, $(Vf)(t) = 2i \int_0^t f(s)ds$ has no eigenvalues ([14, p. 178]). Since $V \in \mathcal{L}_{1,\infty}(L^2(0,1))$, then by Theorem 3.7, $V \in \mathrm{Com}(\mathcal{L}_{1,\infty}(L^2(0,1)))$. Hence $\tau(V) = 0$ for every τ trace on $\mathcal{L}_{1,\infty}(L^2(0,1))$.

4. Modified Fredholm determinant

If $T \in S^2(H)$ and $\{\lambda_n(T)\}_{n\geq 1}$ is the sequence of non-zero eigenvalues of T, each repeated according to its algebraic multiplicity and ordered in such a way that $|\lambda_n(T)| \geq |\lambda_{n+1}(A)|$, $\forall n \in \mathbb{N}$ then

$$\det{}_2(I - \lambda T) = \prod_{n=1}^{\infty}[1 - \lambda\lambda_n(T)]e^{\lambda\lambda_n(T)}$$

is an entire function, known as the modified Fredholm determinant of $I - \lambda T$. By the formula of Plemelj-Smithies [13, p. 166], it follows that

$$\det{}_2(I - \lambda T) = \sum_{n=0}^{\infty} d_n \lambda^n,$$

where $d_0 = 1$ and

$$d_n = \frac{(-1)^n}{n!}\begin{vmatrix} 0 & n-1 & 0 & \dots & 0 & 0 \\ \sigma_2(T) & 0 & n-2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \sigma_{n-1}(T) & \sigma_{n-2}(T) & \sigma_{n-3}(T) & \dots & 0 & 1 \\ \sigma_n(T) & \sigma_{n-1}(T) & \sigma_{n-2}(T) & \dots & \sigma_2(T) & 0 \end{vmatrix}, \ n \geq 1$$

and $\sigma_n(T) = \mathrm{Tr}\,(T^n) = \sum_{j=1}^{\infty} \lambda_j^n(T)$, $n \geq 2$.

In particular, for Hilbert-Schmidt integral operators of the form

$$(Tf)(t) = \int_a^b k(t,s)f(s)ds,$$

where k is a measurable function and $\int_a^b \int_a^b |k(t,s)|^2 dtds < \infty$, the modified Fredholm determinant of $I - \lambda T$ is equal to its Hilbert-Carleman determinant (see [13, p. 176]). More precisely,

$$\det{}_2(I - \lambda T) = 1 + \sum_{n=2}^{\infty} b_n \lambda^n \tag{4.1}$$

where

$$b_n = \frac{1}{n!}\int_{(a,b)^n} \begin{vmatrix} 0 & k(t_1,t_2) & \dots & k(t_1,t_n) \\ k(t_2,t_1) & 0 & \dots & k(t_2,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ k(t_n,t_1) & k(t_n,t_2) & \dots & 0 \end{vmatrix} dt_1 \dots dt_n\ ,\ n \geq 2.$$

5. The integral operator A_ρ

To study the Riemann hypothesis, J. Alcántara-Bode [2] introduced the integral operator $A_\rho : L^2(0,1) \to L^2(0,1)$, $(A_\rho f)(\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)dx$. By Theorem 1.1, the Riemann hypothesis holds if and only if $\mathrm{Ker}(A_\rho) = \{0\}$, or if and only if $h \notin \mathrm{Ran}(A_\rho)$ where $h(x) = x$.

As we have seen in the introduction, the operators $A_\rho(\alpha)$ arise from the attempt to verify the condition $h \in \operatorname{Ran}(A_\rho)$.

We briefly summarize properties of $A_\rho(\alpha)$ established in [2, 3].

i) $A_\rho(\alpha)$, $0 < \alpha \leq 1$, is Hilbert-Schmidt, but neither nuclear, nor normal, nor monotone.

ii) $\lambda \in \sigma(A_\rho(\alpha)) \setminus \{0\}$ ($\sigma(A_\rho(\alpha))$ is the spectrum of $A_\rho(\alpha)$), $0 < \alpha \leq 1$, if and only if $T_\alpha(\lambda^{-1}) = 0$ where

$$T_\alpha(u) = 1 - \alpha u + \sum_{r=1}^{+\infty}(-1)^{r+1}\frac{\alpha^{(r+1)(r+2)/2}}{(r+1)!(r+1)}\prod_{l=1}^{r}\zeta(l+1)u^{r+1}.$$

iii) The modified Fredholm determinant of $I - uA_\rho(\alpha)$ is

$$\det{}_2(I - uA_\rho(\alpha)) = e^{\alpha u}T_\alpha(u)\ ,\ u \in \mathbb{C}.$$

6. Main results

For a given compact operator $T \notin S^1(H)$, where H is a separable complex Hilbert space, the following theorem shows the existence of a nontrivial singular trace defined on an ideal closed with respect to the logarithmic submajorization taking the value of zero on T.

Theorem 6.1. *For every compact operator $T \notin S^1(H)$ there exists an ideal $\mathcal{I}$ closed with respect to the logarithmic submajorization and a nontrivial singular trace τ on $\mathcal{I}$, such that $T \in \mathcal{I}$ and $\tau(T) = 0$.*

Proof. By Remark 2.5, there exists a generalized eccentric operator B such that $T \in (B)_0 \subset (B)$. As we explained in the construction of the singular trace in Theorem 2.4, we can take the triple $\Omega = (B, w, \{n_k\}_{k\geq 1})$. Associated with Ω, on the positive part of (B), we have the functional

$$t_\Omega(A) = w\left(\left\{\frac{S_{n_k}(A)}{S_{n_k}(B)}\right\}_{k\geq 1}\right)$$

that extends linearly to a singular trace on the ideal (B). We also denote this extension by t_Ω. Clearly $t_\Omega(|B|) = 1$. Since $T \notin S^1(H)$, it follows that $B \notin S^1(H)$ and then

$$t_\Omega(A) = w\left(\left\{\frac{\sum_{i=1}^{n_k} s_i(A)}{\sum_{i=1}^{n_k} s_i(B)}\right\}_{k\geq 1}\right). \tag{6.1}$$

It follows from (6.1) that t_Ω is bounded with respect to the norm $\|\cdot\|_B$, where

$$\|A\|_B = \sup_{n\geq 1}\left\{\frac{\sum_{k=1}^{n} s_k(A)}{\sum_{k=1}^{n} s_k(B)}\right\},$$

so it extends by continuity to a singular trace $\tilde{t}_\Omega : \overline{(B)}^{\|\cdot\|_B} \to \mathbb{C}$. Here $\overline{(B)}^{\|\cdot\|_B}$ denotes the closure of (B) with respect to $\|\cdot\|_B$. By [20, Lemma 5.5.9] and Remark 3.6 the ideal $\overline{(B)}^{\|\cdot\|_B}$ is closed with respect to the logarithmic submajorization. Finally, it follows from [15, Proposition 2.7] that $\tilde{\tau}_\Omega(T) = 0$. $\square$

Since $A_\rho \notin S^1(L^2(0,1))$, it follows from Theorem 6.1 that there exists an ideal $\mathcal{J}$ of $B(L^2(0,1))$ closed with respect to the logarithmic submajorization that contains A_ρ and a nontrivial singular trace τ on $\mathcal{J}$ such that $\tau(A_\rho) = 0$. The following theorem shows that every nontrivial singular trace on $\mathcal{J}$ take the value of zero on A_ρ.

Theorem 6.2. $\tau(A_\rho) = 0$ *for every* τ *nontrivial singular trace on* $\mathcal{J}$.

Proof. Let τ be a nontrivial singular trace on $\mathcal{J}$ (since $A_\rho \notin S^1(L^2(0,1))$, the existence of $\mathcal{J}$ and a nontrivial singular trace on $\mathcal{J}$ are guaranteed by Theorem 6.1. If $0 < \alpha < 1$, by [5], the operators $\alpha(A_\rho + Q_{f_1})$ and $A_\rho + Q_{f_\alpha}$ have the same nonzero eigenvalues with the same algebraic and geometric multiplicities, where $Q_f(g) = \langle g, f\rangle h$, $f_1(x) = -\rho(\frac{1}{x})$ and $f_\alpha(x) = -\alpha^{-1}\rho(\frac{\alpha}{x})$. It follows from Theorem 3.8 that τ is a spectral trace and, therefore,

$$\tau(\alpha(A_\rho + Q_{f_1})) = \tau(A_\rho + Q_{f_\alpha}). \tag{6.2}$$

Hence, by definition of singular trace, it follows from (6.2) that $\tau(A_\rho) = 0$. □

Remark 6.3.

i) *It follows from Theorem 3.8 that the singular trace given in Theorem 6.1 is spectral.*

ii) *If* $0 < \alpha < 1$, *by [5], the kernel of the operators* $\alpha(A_\rho + Q_{f_1})$ *and* $A_\rho + Q_{f_\alpha}$ *is trivial if and only if the Riemann hypothesis is true.*

iii) *It is easily checked that if* $0 < \alpha \le 1$ *and* $V_\alpha : L^2(0,1) \to L^2(0,1)$, $(V_\alpha f)(x) = f(\frac{x}{\alpha})\chi_{[0,\alpha]}(x)$, *then*

$$\begin{aligned}(V_\alpha^* f)(x) &= \alpha f(\alpha x)\\ A_\rho(\alpha) &= \frac{1}{\alpha} V_\alpha^* A_\rho\end{aligned} \tag{6.3}$$

$$A_\rho(\alpha) V_\alpha = \alpha A_\rho. \tag{6.4}$$

The operator V_α *was introduced in [4].*
We have from (6.4) *that* $A_\rho(\alpha) \in \mathcal{J}$ *for every* $\alpha \in \mathcal{J}$.

We are now ready to prove the first main result of the paper.

Proof of Theorem 1.2. First, we prove that if $0 < \alpha, \beta \le 1$ then

$$A_\rho(\alpha) = A_\rho V_\alpha^* + \alpha\left\langle \cdot, \chi_{[\alpha,1]}\frac{1}{h}\right\rangle h, \tag{6.5}$$

where $h(x) = x$. Indeed, by the Müntz-Szasz Theorem [8, Theorem 2.2], it is sufficient to verify (6.5) for h^r with $r \in \mathbb{N}$. To this end, we use the identity [7, p. 312]

$$\int_0^1 \rho\left(\frac{\theta}{x}\right) x^r dx = \frac{\theta}{r} - \frac{\zeta(r+1)}{r+1}\theta^{r+1}, \ \mathrm{Re}(r) > -1. \tag{6.6}$$

Evaluating the right-hand side of (6.5) we have

$$A_\rho V_\alpha^*(h^r)(\theta) + \alpha\left(\int_0^1 h^r(x)\chi_{[\alpha,1]}(x)\frac{1}{h(x)}dx\right)h(\theta)$$

$$= \alpha^{r+1} \int_0^1 \rho\left(\frac{\theta}{x}\right) x^r dx + \frac{\alpha}{r}(1-\alpha^r)\theta.$$

It follows from (6.6) that

$$\begin{aligned}\alpha^{r+1} \int_0^1 \rho\left(\frac{\theta}{x}\right) x^r dx + \frac{\alpha}{r}(1-\alpha^r)\theta &= \alpha^{r+1}\left(\frac{\theta}{r} - \frac{\zeta(r+1)}{r+1}\theta^{r+1}\right) \\ &\quad + \frac{\alpha}{r}(1-\alpha^r)\theta \\ &= (A_\rho(\alpha)h^r)(\theta).\end{aligned}$$

Hence, (6.5) is true for h^r.

Let τ be a nontrivial singular trace on $\mathcal{J}$. Applying τ in (6.5), we obtain

$$\tau(A_\rho(\alpha)) = \tau(A_\rho V_\alpha^*).$$

It follows from the definition of trace and (6.3) that

$$\tau(A_\rho(\alpha)) = \frac{1}{\alpha}\tau(V_\alpha^* A_\rho) = \frac{1}{\alpha}\tau(A_\rho V_\alpha^*).$$

Thus, for $0 < \alpha < 1$, we have $\tau(A_\rho V_\alpha^*) = 0$ and hence $\tau(A_\rho(\alpha)) = 0$. Since $\tau(A_\rho) = 0$ and $A_\rho(1) = A_\rho$, then $\tau(A_\rho(\alpha)) = 0$ for every $0 < \alpha \leq 1$. □

From now on, we concentrate on a recursion formula from which we can evaluate all the traces $\mathrm{Tr}\,(A_\rho^r(\alpha))$, $r \in \mathbb{N}$, $r \geq 2$ and $0 < \alpha \leq 1$. First, we require a preliminary lemma.

Lemma 6.4. $\mathrm{Tr}(A_\rho^2(\alpha)) = \alpha^2 - \frac{\zeta(2)}{2}\alpha^3$.

Proof. It follows from (4.1) that

$$\mathrm{Tr}\,(A_\rho^2(\alpha)) = \int_0^1 \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right)\rho\left(\frac{\alpha x}{\theta}\right) dx d\theta.$$

Since

$$\int_0^1 \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right)\rho\left(\frac{\alpha x}{\theta}\right) dx d\theta = \int_0^1 \int_0^\theta \rho\left(\frac{\alpha\theta}{x}\right)\rho\left(\frac{\alpha x}{\theta}\right) dx d\theta + \int_0^1 \int_\theta^1 \rho\left(\frac{\alpha\theta}{x}\right)\rho\left(\frac{\alpha x}{\theta}\right) dx d\theta$$

and

$$\begin{aligned}\rho\left(\tfrac{\alpha x}{\theta}\right) &= \tfrac{\alpha x}{\theta},\ 0 < x \leq \theta \\ \rho\left(\tfrac{\alpha\theta}{x}\right) &= \tfrac{\alpha\theta}{x},\ \theta < x \leq 1\end{aligned} \tag{6.7}$$

we get that

$$\mathrm{Tr}\,(A_\rho^2(\alpha)) = \int_0^1 \int_0^\theta \rho\left(\frac{\alpha\theta}{x}\right)\frac{\alpha x}{\theta} dx d\theta + \int_0^1 \int_\theta^1 \frac{\alpha\theta}{x}\rho\left(\frac{\alpha x}{\theta}\right) dx d\theta. \tag{6.8}$$

It follows from (6.8) that

$$\mathrm{Tr}\,(A_\rho^2(\alpha)) = \int_0^1 \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right)\frac{\alpha x}{\theta} dx d\theta - \int_0^1 \int_\theta^1 \rho\left(\frac{\alpha\theta}{x}\right)\frac{\alpha x}{\theta} dx d\theta +$$

$$\int_0^1 \int_0^1 \frac{\alpha\theta}{x}\rho\left(\frac{\alpha x}{\theta}\right) dxd\theta - \int_0^1 \int_0^\theta \frac{\alpha\theta}{x}\rho\left(\frac{\alpha x}{\theta}\right) dxd\theta.$$

Using (6.7) we obtain that

$$\mathrm{Tr}\left(A_\rho^2(\alpha)\right) = 2\int_0^1 \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right)\frac{\alpha x}{\theta} dxd\theta - \alpha^2.$$

Finally, by (6.6) we conclude that

$$\begin{aligned}\mathrm{Tr}\left(A_\rho^2(\alpha)\right) &= 2\alpha\int_0^1 \alpha - \frac{\zeta(2)}{2}\alpha^2\theta d\theta - \alpha^2\\ &= \alpha^2 - \frac{\zeta(2)}{2}\alpha^3.\end{aligned}$$

□

Proof of Theorem 1.3. By [3], we have

$$\det{}_2(I - uA_\rho(\alpha)) = e^{\alpha u}T_\alpha(u)\ ,\ u \in \mathbb{C} \tag{6.9}$$

where

$$T_\alpha(u) = 1 - \alpha u + \sum_{r=1}^{\infty}(-1)^{r+1}\frac{\alpha^{(r+1)(r+2)/2}}{(r+1)!(r+1)}\prod_{l=1}^{r}\zeta(l+1)u^{r+1}$$

is an entire function with an infinite number of zeros. It follows from [27, p. 349] that for u sufficiently small we have

$$\frac{\det{}_2'(I - uA_\rho(\alpha))}{\det{}_2(I - uA_\rho(\alpha))} = -\sum_{r=1}^{\infty}\mathrm{Tr}\left(A_\rho^{r+1}(\alpha)\right)u^r. \tag{6.10}$$

Replacing (6.9) in (6.10), we get that

$$T_\alpha'(u) = T_\alpha(u)\left(-\alpha - \sum_{r=1}^{\infty}\mathrm{Tr}\left(A_\rho^{r+1}(\alpha)\right)u^r\right). \tag{6.11}$$

Moreover, we obtain from (6.11) and the Cauchy product formula that for $r \geq 2$ we have

$$(r+1)a_{r+1}(\alpha) = -\alpha a_r(\alpha) - \sum_{k=1}^{r-1}a_{r-k}(\alpha)\mathrm{Tr}\left(A_\rho^{k+1}(\alpha)\right) - \mathrm{Tr}\left(A_\rho^{r+1}(\alpha)\right),$$

where

$$a_r(\alpha) = \begin{cases}-\alpha & ,\ r = 1\\ \dfrac{(-1)^r\alpha^{r(r+1)/2}}{r!r}\displaystyle\prod_{l=1}^{r-1}\zeta(l+1) & ,\ r \geq 2.\end{cases}$$

This proves the assertion. □

Remark 6.5. *Formula* (1.2) *is a recursion formula from which we can evaluate all the traces* $Tr\left(A_\rho^r(\alpha)\right)$, $r \geq 2$. *For example, it follows from Lemma 6.4 and Theorem 1.3 that*

$$\begin{aligned}\mathrm{Tr}(A_\rho^3(\alpha)) &= -3a_3(\alpha) - \alpha a_2(\alpha) - a_1(\alpha)\mathrm{Tr}(A_\rho^2(\alpha))\\ &= \frac{\alpha^6}{6}\zeta(2)\zeta(3) - \frac{3\alpha^4}{4}\zeta(2) + \alpha^3.\end{aligned}$$

7. Conclusions

In this article we have shown the existence of a nontrivial singular trace τ defined on an ideal $\mathcal{J}$ closed with respect to the logarithmic submajorization such that $\tau(A_\rho(\alpha)) = 0$, where $A_\rho(\alpha) : L^2(0,1) \to L^2(0,1)$, $[A_\rho(\alpha)f](\theta) = \int_0^1 \rho(\alpha\theta/x)f(x)dx$, $0 < \alpha \leq 1$. Based on the theory of spectral traces and the results of [5], we have also shown that $\tau(A_\rho(\alpha)) = 0$ for every τ nontrivial singular trace on $\mathcal{J}$.

Finally, using the modified Fredholm determinant of $I - uA_\rho(\alpha)$, a recursion formula is presented to calculate all the traces $\mathrm{Tr}\,(A_\rho^r(\alpha))$, $r \geq 2$.

Use of AI tools declaration

The author declares he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest.

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