AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 24957–24970.
DOI: 10.3934/math.20231273
Received: 25 June 2023
Accepted: 10 August 2023
Published: 28 August 2023

Research article

Hypersurfaces of revolution family supplying $\Delta \mathfrak{r} = \mathcal{A}\mathfrak{r}$ in pseudo-Euclidean space $\mathbb{E}^7_3$

Yanlin Li[1,2,*] **and Erhan Güler**[3]

[1] School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China

[2] Key Laboratory of Cryptography of Zhejiang Province, Hangzhou Normal University, Hangzhou 311121, China

[3] Department of Mathematics, Faculty of Sciences, Bartın University 74100, Bartın, Turkey

* **Correspondence:** Email: liyl@hznu.edu.cn.

Abstract: In this study, we introduce a family of hypersurfaces of revolution characterized by six parameters in the seven-dimensional pseudo-Euclidean space $\mathbb{E}^7_3$. These hypersurfaces exhibit intriguing geometric properties, and our aim is to analyze them in detail. To begin, we calculate the matrices corresponding to the fundamental form, Gauss map, and shape operator associated with this hypersurface family. These matrices provide essential information about the local geometry of the hypersurfaces, including their curvatures and tangent spaces. Using the Cayley-Hamilton theorem, we employ matrix algebra techniques to determine the curvatures of the hypersurfaces. This theorem allows us to express the characteristic polynomial of a matrix in terms of the matrix itself, enabling us to compute the curvatures effectively. In addition, we establish equations that describe the interrelation between the mean curvature and the Gauss-Kronecker curvature of the hypersurface family. These equations provide insights into the geometric behavior of the surfaces and offer a deeper understanding of their intrinsic properties. Furthermore, we investigate the relationship between the Laplace-Beltrami operator, a differential operator that characterizes the geometry of the hypersurfaces, and a specific 7×7 matrix denoted as $\mathcal{A}$. By studying this relation, we gain further insights into the geometric structure and differential properties of the hypersurface family. Overall, our study contributes to the understanding of hypersurfaces of revolution in $\mathbb{E}^7_3$, offering mathematical insights and establishing connections between various geometric quantities and operators associated with this family.

Keywords: pseudo-Euclidean 7-space; hypersurfaces of revolution family; Gauss map; curvature; shape operator; Laplace-Beltrami operator
Mathematics Subject Classification: 53A35, 53C42

1. Introduction

Chen [11–14] originally proposed the notion of submanifolds of finite order immersed in m-space $\mathbb{E}^m$ or pseudo-Euclidean m-space $\mathbb{E}^m_\nu$ employing a finite number of eigenfunctions of their Laplacian. This subject has subsequently undergone thorough investigation.

Takahashi established that a Euclidean submanifold is classified as 1-type if and only if it is minimal or minimal within a hypersphere of $\mathbb{E}^m$. The study of 2-type submanifolds on closed spheres was conducted by [9, 10, 12]. Garay further [26] examined Takahashi's theorem in $\mathbb{E}^m$. Cheng and Yau [18] focused on hypersurfaces with constant curvature, while Chen and Piccinni [15] concentrated on submanifolds with a Gauss map of finite type in $\mathbb{E}^m$. Dursun [22] introduced hypersurfaces with a pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$. Aminov [2] delved into the geometry of submanifolds. Within the domain of space forms, Chen et al. [16] dedicated four decades to the investigation of 1-type submanifolds and the 1-type Gauss map.

In $\mathbb{E}^3$, Takahashi [43] explored the concept of minimal surfaces, where spheres and minimal surfaces with $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$ are the exclusive types of surfaces. Ferrandez et al. [23] identified that surfaces $\Delta H = A_{3\times3}H$ are either the minimal sections of a sphere or a right circular cylinder. Choi and Kim [19] examined the minimal helicoid with a pointwise 1-type Gauss map of the first kind. Garay [25] derived a category of finite type surfaces that are revolution-based. Dillen et al. [20] investigated the unique surfaces characterized by $\Delta r = A_{3\times3}r + B_{3\times1}$, which include minimal surfaces, spheres, and circular cylinders. Stamatakis and Zoubi [42] established the properties of surfaces of revolution defined by $\Delta^{III}x = A_{3\times3}x$. Kim et al. [36] focused on the Cheng-Yau operator and the Gauss map of surfaces of revolution.

In $\mathbb{E}^4$, Moore [40, 41] conducted two studies on general rotational surfaces. Hasanis and Vlachos [35] examined hypersurfaces with a harmonic mean curvature vector field. Cheng and Wan [17] focused on complete hypersurfaces with constant mean curvature. Arslan et al. [3] explored the Vranceanu surface with a pointwise 1-type Gauss map. Arslan et al. [4] investigated generalized rotational surfaces and [5] introduced tensor product surfaces with a pointwise 1-type Gauss map. Yoon [44] established certain relations involving the Clifford torus. Güler et al. [30] delved into helicoidal hypersurfaces, while Güler et al. [29] studied the Gauss map and the third Laplace-Beltrami operator of rotational hypersurfaces. Güler [28] investigated rotational hypersurfaces characterized by $\Delta^I R = A_{4\times4}R$. Furthermore, Güler [27] obtained the fundamental form IV and curvature formulas of the hypersphere.

In Minkowski 4-space $\mathbb{E}^4_1$, Ganchev and Milousheva [24] explored the analogous surfaces to those in [40, 41]. Arvanitoyeorgos et al. [8] investigated the mean curvature vector field, where they established $\Delta H = \alpha H$ with a constant α. Arslan and Milousheva [6] focused on meridian surfaces of elliptic or hyperbolic type with a pointwise 1-type Gauss map. Arslan et al. [7] examined rotational λ-hypersurfaces in Euclidean spaces. Güler et al. [31–34] worked the concept of bi-rotational hypersurfaces. Li and Güler studied a family of hypersurfaces of revolution distinguished by four parameters in the five-dimensional pseudo-Euclidean space $\mathbb{E}^5_2$ [39].

The aim of this paper is to present a family of hypersurfaces of revolution in the seven-dimensional pseudo-Euclidean space $\mathbb{E}^7_3$. This family, denoted as $\mathfrak{r}$, is characterized by six parameters. The paper focuses on computing various matrices associated with $\mathfrak{r}$, including the fundamental form, Gauss map, and shape operator. The Cayley-Hamilton theorem is employed to determine the curvatures of $\mathfrak{r}$.

Furthermore, the paper establishes equations that describe the relationship between the mean curvature and Gauss-Kronecker curvature of $\mathfrak{r}$. Additionally, the paper explores the connection between the Laplace-Beltrami operator of $\mathfrak{r}$ and a 7×7 matrix.

In Section 2, we provide an explanation of the fundamental concepts of seven-dimensional pseudo-Euclidean geometry.

Section 3 is dedicated to presenting the curvature formulas of a hypersurface in $\mathbb{E}_3^7$.

In Section 4, we offer a comprehensive definition of the hypersurfaces of revolution family, focusing on their properties and characteristics.

In Section 5, we discuss the Laplace-Beltrami operator of a smooth function in $\mathbb{E}_3^7$ and utilize the previously discussed family to compute it.

Finally, we serve a conclusion in the last section.

2. Preliminaries

In this paper, we use the following notations, formulas, eqations, etc.

For clarity, $\mathbb{E}_v^m$ represents a pseudo-Euclidean m-space with coordinates denoted as $(x_1, x_2, \ldots, x_m)$ with index v. The canonical pseudo-Euclidean metric tensor on $\mathbb{E}_v^m$ is represented by $\widetilde{g}$ and defined as $\widetilde{g} = \langle \ , \ \rangle = -\sum_{i=1}^{v} dx_i^2 + \sum_{i=v+1}^{m} dx_i^2$. Let $\widetilde{M}$ be an m-dimensional semi-Riemannian submanifold, and is embedded in $\mathbb{E}_v^m$, and the Levi-Civita connections [38] associated with M are denoted as $\widetilde{\nabla}, \nabla$, respectively. We utilize X, Y, Z, and W to denote vector fields tangent to M, and ξ, ς to represent vector fields normal to M.

The Gauss formula and the Weingarten formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi,$$

where h represents the second fundamental form of M, A denotes the shape operator, and D corresponds to the normal connection of M. The shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at each point $p \in M$ for each $\xi \in T_p^\perp M$. The shape operator and the second fundamental form are related by the equation.

$$\langle h(X, Y), \xi \rangle = \left\langle A_\xi X, Y \right\rangle .$$

The Gauss equation is determined by

$$\langle R(X, Y,)Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle,$$

where R describes the curvature tensor associated with the Levi-Civita connection ∇, and h denotes the second fundamental form of M. The Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z),$$

where $\bar{\nabla} h$ denotes the covariant derivative of h w.r.t. the Levi-Civita connection ∇, and X, Y, Z represent tangent vector fields on M. The curvature tensor R^D associated with the normal connection D is not explicitly mentioned in the given equations. The covariant derivative of h is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

where D represents the normal connection of M.

Let M be an oriented hypersurface in $\mathbb{E}^{n+1}$ with its shape operator $\mathcal{S}$, and position vector x. Consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ consisting of principal directions of M coinciding with the principal curvature k_i for $i = 1, 2, \ldots, n$. Let the dual basis of this frame field be $\{f_1, f_2, \ldots, f_n\}$. Then, the first structural equation of Cartan is determined by

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \ldots, n,$$

where ω_{ij} indicates the connection forms coinciding with the chosen frame field. By the Codazzi equation, we derive the equations.

$$\begin{aligned} e_i(k_j) &= \omega_{ij}(e_j)(k_i - k_j), \\ \omega_{ij}(e_l)(k_i - k_j) &= \omega_{il}(e_j)(k_i - k_l) \end{aligned}$$

for different $i, j, l = 1, 2, \ldots, n$.

We let $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j denotes the j-th elementary symmetric function defined by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

We consider the notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

According to the given definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. The function s_k is referred to as the k-th mean curvature of the oriented hypersurface M. The mean curvature $H = \frac{1}{n} s_1$ is also defined, and the Gauss-Kronecker curvature of M is $K = s_n$. If $s_j \equiv 0$, the hypersurface M is known as j-minimal.

In Euclidean $(n+1)$-space, getting the i-th curvature formulas $\mathcal{K}_i$ (see [1, 37] for details), where $i = 0, \ldots, n$, we have the following characteristic polynomial equation $P_{\mathcal{S}}(\lambda) = 0$ of $\mathcal{S}$:

$$\sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = \det(\mathcal{S} - \lambda \mathcal{I}_n) = 0. \tag{2.1}$$

Here $i = 0, \ldots, n$, $\mathcal{I}_n$ indicates the identity matrix. Hence, we reveal the curvature formulas as $\binom{n}{i}\mathcal{K}_i = s_i$.

Let $\mathfrak{r} = \mathfrak{r}(u, v, w, \alpha, \beta, \gamma)$ be an immersion from $M^6 \subset \mathbb{E}^6$ to $\mathbb{E}^7_3$.

Definition 1. An inner product of $\upsilon^{\mathbf{1}} = (\upsilon^1_1, \upsilon^1_2, \ldots, \upsilon^1_7), \ldots, \upsilon^{\mathbf{2}} = (\upsilon^2_1, \upsilon^2_2, \ldots, \upsilon^2_7)$ of $\mathbb{E}^7_3$ is determined by

$$\left\langle \upsilon^{\mathbf{1}}, \upsilon^{\mathbf{2}} \right\rangle = \upsilon^1_1\upsilon^2_1 - \upsilon^1_2\upsilon^2_2 + \upsilon^1_3\upsilon^2_3 - \upsilon^1_4\upsilon^2_4 + \upsilon^1_5\upsilon^2_5 - \upsilon^1_6\upsilon^2_6 + \upsilon^1_7\upsilon^2_7.$$

Definition 2. A sixtuple vector product of $\upsilon^{\mathbf{1}} = (\upsilon^1_1, \upsilon^1_2, \ldots, \upsilon^1_7)$, $\upsilon^{\mathbf{2}} = (\upsilon^2_1, \upsilon^2_2, \ldots, \upsilon^2_7), \ldots, \upsilon^{\mathbf{6}} =$

$(\upsilon_1^6, \upsilon_2^6, \ldots, \upsilon_7^6)$ of $\mathbb{E}_3^7$ is defined by

$$\upsilon^{\mathbf{1}} \times \upsilon^{\mathbf{2}} \times \cdots \times \upsilon^{\mathbf{6}} = \det \begin{pmatrix} e_1 & -e_2 & e_3 & -e_4 & e_5 & -e_6 & e_7 \\ \upsilon_1^1 & \upsilon_2^1 & \upsilon_3^1 & \upsilon_4^1 & \upsilon_5^1 & \upsilon_6^1 & \upsilon_7^1 \\ \upsilon_1^2 & \upsilon_2^2 & \upsilon_3^2 & \upsilon_4^2 & \upsilon_5^2 & \upsilon_6^2 & \upsilon_7^2 \\ \upsilon_1^3 & \upsilon_2^3 & \upsilon_3^3 & \upsilon_4^3 & \upsilon_5^3 & \upsilon_6^3 & \upsilon_7^3 \\ \upsilon_1^4 & \upsilon_2^4 & \upsilon_3^4 & \upsilon_4^4 & \upsilon_5^4 & \upsilon_6^4 & \upsilon_7^4 \\ \upsilon_1^5 & \upsilon_2^5 & \upsilon_3^5 & \upsilon_4^5 & \upsilon_5^5 & \upsilon_6^5 & \upsilon_7^5 \\ \upsilon_1^6 & \upsilon_2^6 & \upsilon_3^6 & \upsilon_4^6 & \upsilon_5^6 & \upsilon_6^6 & \upsilon_7^6 \end{pmatrix}.$$

Definition 3. The product matrix $\left(\mathfrak{g}_{ij}\right)^{-1} \cdot \left(\mathfrak{h}_{ij}\right)$ describes the shape operator matrix $\mathcal{S}$ of hypersurface $\mathfrak{r}$ in pseudo-Euclidean 7-space $\mathbb{E}_3^7$, where, $\left(\mathfrak{g}_{ij}\right)_{6\times 6}$ and $\left(\mathfrak{h}_{ij}\right)_{6\times 6}$ describe the first and the second fundamental form matrices, respectively, and $\mathfrak{g}_{ij} = \left\langle \mathfrak{r}_i, \mathfrak{r}_j \right\rangle$, $\mathfrak{h}_{ij} = \left\langle \mathfrak{r}_{ij}, \mathcal{G} \right\rangle$, $i, j = 1, 2, \ldots, 6$, $\mathfrak{r}_u = \frac{\partial \mathfrak{r}}{\partial u}$ when $i = 1$, $\mathfrak{r}_{uv} = \frac{\partial^2 \mathfrak{r}}{\partial u \partial v}$ when $i = 1$ and $j = 2$, etc., e_k denotes the natural base elements of $\mathbb{E}^7$. Here,

$$\mathcal{G} = \frac{\mathfrak{r}_u \times \mathfrak{r}_v \times \mathfrak{r}_w \times \mathfrak{r}_\alpha \times \mathfrak{r}_\beta \times \mathfrak{r}_\gamma}{\left\| \mathfrak{r}_u \times \mathfrak{r}_v \times \mathfrak{r}_w \times \mathfrak{r}_\alpha \times \mathfrak{r}_\beta \times \mathfrak{r}_\gamma \right\|} \tag{2.2}$$

determines the Gauss map of the hypersurface $\mathfrak{r}$.

3. Curvatures in $\mathbb{E}_3^7$

In this section, we reveal the curvature formulas of any hypersurface $\mathfrak{r} = \mathfrak{r}(u, v, w, \alpha, \beta, \gamma)$ *in* $\mathbb{E}_3^7$.

Theorem 1. *A hypersurface* $\mathfrak{r}$ *in* $\mathbb{E}_3^7$ *has the following curvature formulas,* $\mathcal{K}_0 = 1$ *by definition,*

$$6\mathcal{K}_1 = -\frac{\mathfrak{a}_5}{\mathfrak{a}_6},\ 15\mathcal{K}_2 = \frac{\mathfrak{a}_4}{\mathfrak{a}_6},\ 20\mathcal{K}_3 = -\frac{\mathfrak{a}_3}{\mathfrak{a}_6},\ 15\mathcal{K}_4 = \frac{\mathfrak{a}_2}{\mathfrak{a}_6},\ 6\mathcal{K}_5 = -\frac{\mathfrak{a}_1}{\mathfrak{a}_6},\ \mathcal{K}_6 = \frac{\mathfrak{a}_0}{\mathfrak{a}_6}, \tag{3.1}$$

where $\mathfrak{a}_6\lambda^6 + \mathfrak{a}_5\lambda^5 + \mathfrak{a}_4\lambda^4 + \mathfrak{a}_3\lambda^3 + \mathfrak{a}_2\lambda^2 + \mathfrak{a}_1\lambda + \mathfrak{a}_0 = 0$ *denotes the characteristic polynomial equation* $P_{\mathcal{S}}(\lambda) = 0$ *of the shape operator matrix* $\mathcal{S}$*,* $\mathfrak{a}_6 = \det\left(\mathfrak{g}_{ij}\right)$*,* $\mathfrak{a}_0 = \det\left(\mathfrak{h}_{ij}\right)$*, and* $\left(\mathfrak{g}_{ij}\right)$*,* $\left(\mathfrak{h}_{ij}\right)$ *are the first, and the second fundamental form matrices, respectively.*

Proof. The solution matrix $\left(\mathfrak{g}_{ij}\right)^{-1} \cdot \left(\mathfrak{h}_{ij}\right)$ supplies the shape operator matrix $\mathcal{S}$ of hypersurface $\mathfrak{r}$ in pseudo-Euclidean 7-space $\mathbb{E}_3^7$. In $\mathbb{E}_3^7$, computing the curvature formula $\mathcal{K}_i$, where $i = 0, 1, \ldots, 6$, we reveal the characteristic polynomial equation $\det(\mathcal{S} - \lambda \mathcal{I}_6) = 0$ of $\mathcal{S}$. Then, we obtain

$$\begin{aligned} \binom{6}{0}\mathcal{K}_0 &= 1, \\ \binom{6}{1}\mathcal{K}_1 &= \sum_{i=1}^{6} k_i = -\frac{\mathfrak{a}_5}{\mathfrak{a}_6}, \\ \binom{6}{2}\mathcal{K}_2 &= \sum_{1=i_1<i_2}^{6} k_{i_1}k_{i_2} = \frac{\mathfrak{a}_4}{\mathfrak{a}_6}, \\ \binom{6}{3}\mathcal{K}_3 &= \sum_{1=i_1<i_2<i_3}^{6} k_{i_1}k_{i_2}k_{i_3} = -\frac{\mathfrak{a}_3}{\mathfrak{a}_6}, \end{aligned}$$

$$\binom{6}{4}\mathcal{K}_4 = \sum_{1=i_1<i_2<i_3<i_4}^{6} k_{i_1}k_{i_2}k_{i_3}k_{i_4} = \frac{\mathfrak{a}_2}{\mathfrak{a}_6},$$
$$\binom{6}{5}\mathcal{K}_5 = \sum_{1=i_1<i_2<i_3<i_4<i_5}^{6} k_{i_1}k_{i_2}k_{i_3}k_{i_4}k_{i_5} = -\frac{\mathfrak{a}_1}{\mathfrak{a}_6},$$
$$\binom{6}{6}\mathcal{K}_6 = \prod_{i=1}^{6} k_i = \frac{\mathfrak{a}_0}{\mathfrak{a}_6}.$$

Definition 4. *A space-like hypersurface* $\mathfrak{r}$ *is called j-maximal if* $\mathcal{K}_j = 0$*, where* $j = 1, ..., 6$.
Theorem 2. *A hypersurface* $\mathfrak{r} = \mathfrak{r}(u, v, w, \alpha, \beta, \gamma)$ *in* $\mathbb{E}^7_3$ *has the following relation*

$$\mathcal{K}_0\mathbb{VII} - 6\mathcal{K}_1\mathbb{VI} + 15\mathcal{K}_2\mathbb{V} - 20\mathcal{K}_3\mathbb{IV} + 15\mathcal{K}_4\mathbb{IIII} - 6\mathcal{K}_5\mathbb{II} + \mathcal{K}_6\mathbb{I} = \mathcal{O}_6,$$

where $\mathbb{I}, \mathbb{II}, \ldots, \mathbb{VII}$ *determines the fundamental form matrices,* $\mathcal{O}_6$ *represents the zero matrix having order* 6×6 *of the hypersurface.*
Proof. Regarding $n = 6$ in (2.1), it works.

4. Hypersurfaces of revolution family in $\mathbb{E}^7_3$

In this section, we define the hypersurfaces of revolution family (*HRF*), then find its differential geometric properties in pseudo-Euclidean 7-space $\mathbb{E}^7_3$. An *HR* in Riemannian space forms were given in [21].

An *HRF M* of Euclidean $(n + 1)$-space constructed by a hypersurface $\hbar$ around rotating axis ℓ does not meet $\hbar$ is acquired by taking the orbit of ℓ under the orthogonal transformations of $(n + 1)$-space.

To consctruct an *HRF*, we start with the generating hypersurface given by $\hbar = \hbar(u, v, w) = (\eta, 0, \psi, 0, \phi, 0, \varphi)$, and apply the rotation matrix $\Re = \operatorname{diag}\left(\mathcal{R}_\alpha, \mathcal{R}_\beta, \mathcal{R}_\gamma, 1\right)$ with the elements given by $\mathcal{R}_k = \begin{pmatrix} \cosh k & \sinh k \\ \sinh k & \cosh k \end{pmatrix}$, $k = \alpha, \beta, \gamma$, respectively, and $\Re{\cdot}\ell = \ell$, $\det \Re = 1$. Therefore, we state the *HRF* given by $\mathfrak{r} = \Re{\cdot}\hbar^T$ when $\hbar$ rotates about axis $\ell = \overrightarrow{e_7} = (0, 0, 0, 0, 0, 0, 1)$. We then present the following.
Definition 5. *An HRF is an immersion* $\mathfrak{r} : M^6 \subset \mathbb{E}^6 \longrightarrow \mathbb{E}^7_3$ *with rotating axis* $\overrightarrow{e_7}$*, defined by*

$$\mathfrak{r}(u, v, w, \alpha, \beta, \gamma) = (\eta \cosh \alpha, \eta \sinh \alpha, \psi \cosh\beta, \psi \sinh\beta, \phi \cosh \gamma, \phi \sinh \gamma, \varphi), \tag{4.1}$$

where $\eta, \psi, \phi, \varphi$ *denote the differentiable functions, depend on* $u, v, w \in \mathbb{R}$*,* $0 \le \alpha, \beta, \gamma < 2\pi$.

Considering the first derivatives of *HRF* given by Eq (4.1) w.r.t. $u, v, w, \alpha, \beta, \gamma$, respectively, we find the symmetical first fundamental form matrix

$$\left(\mathfrak{g}_{ij}\right) = \operatorname{diag}\left((\mathfrak{g}_{kl})_{3\times3}, \quad \mathfrak{g}_{44}, \quad \mathfrak{g}_{55}, \quad \mathfrak{g}_{66} \right), \tag{4.2}$$

where

$$\mathfrak{g}_{11} = \eta_u^2 + \psi_u^2 + \phi_u^2 + \varphi_u^2,$$
$$\mathfrak{g}_{12} = \eta_u\eta_v + \psi_u\psi_v + \phi_u\phi_v + \varphi_u\varphi_v,$$
$$\mathfrak{g}_{13} = \eta_u\eta_w + \psi_u\psi_w + \phi_u\phi_w + \varphi_u\varphi_w,$$

$$\begin{aligned}
\mathfrak{g}_{22} &= \eta_v^2+\psi_v^2+\phi_v^2+\varphi_v^2,\\
\mathfrak{g}_{23} &= \eta_v\eta_w+\psi_v\psi_w+\phi_v\phi_w+\varphi_v\varphi_w,\\
\mathfrak{g}_{33} &= \eta_w^2+\psi_w^2+\phi_w^2+\varphi_w^2,\\
\mathfrak{g}_{44} &= \eta^2,\ \mathfrak{g}_{55}=\psi^2,\ \mathfrak{g}_{66}=\phi^2,
\end{aligned}$$

and $\eta_u=\frac{\partial\eta}{\partial u}$, $\eta_v=\frac{\partial\eta}{\partial v}$, $\eta_u^2=\frac{\partial^2\eta}{\partial u^2}$, etc. Hence, $\hat{\mathbf{g}}=\det\left(\mathfrak{g}_{ij}\right)=\eta^2\psi^2\phi^2\mathcal{Q}$, where

$$\mathcal{Q}=(\mathcal{G}_1)^2+(\mathcal{G}_2)^2+(\mathcal{G}_3)^2+(\mathcal{G}_4)^2,$$

and

$$\begin{aligned}
\mathcal{G}_1 &= (\psi_v\phi_w-\psi_w\phi_v)\varphi_u+(\psi_w\phi_u-\psi_u\phi_w)\varphi_v+(\psi_u\phi_v-\psi_v\phi_u)\varphi_w,\\
\mathcal{G}_2 &= (\eta_v\phi_w-\eta_w\phi_v)\varphi_u+(\eta_w\phi_u-\eta_u\phi_w)\varphi_v+(\eta_u\phi_v-\eta_v\phi_u)\varphi_w,\\
\mathcal{G}_3 &= (\eta_v\psi_w-\eta_w\psi_v)\varphi_u+(\eta_w\psi_u-\eta_u\psi_w)\varphi_v+(\eta_u\psi_v-\eta_v\psi_u)\varphi_w,\\
\mathcal{G}_4 &= (\eta_w\psi_v-\eta_v\psi_w)\phi_u+(\eta_u\psi_w-\eta_w\psi_u)\phi_v+(\eta_v\psi_u-\eta_u\psi_v)\phi_w.
\end{aligned}$$

Since $\hat{\mathbf{g}}>0$, the *HRF* given by Eq (4.1) is a space-like hypersurface.

Using (2.2), we obtain the following Gauss map of the *HRF* determined by Eq (4.1):

$$\mathcal{G}=\mathcal{Q}^{-1/2}\left(\mathcal{G}_1\cosh\alpha,\mathcal{G}_1\sinh\alpha,\mathcal{G}_2\cosh\beta,\mathcal{G}_2\sinh\beta,\mathcal{G}_3\cosh\gamma,\mathcal{G}_3\sinh\gamma,\mathcal{G}_4\right). \tag{4.3}$$

With the help of the second derivatives w.r.t. $u, v, w, \alpha, \beta, \gamma$, of *HRF* described by Eq (4.1), and by using the Gauss map given by Eq (4.3), we reveal the following symmetical second fundamental form matrix

$$\left(\mathfrak{h}_{ij}\right)=\operatorname{diag}\left(\ (\mathfrak{h}_{kl})_{3\times3},\quad \mathfrak{h}_{44},\quad \mathfrak{h}_{55},\quad \mathfrak{h}_{66}\ \right), \tag{4.4}$$

where

$$\begin{aligned}
\mathfrak{h}_{11} &= \mathcal{Q}^{-1/2}\left(\mathcal{G}_1\eta_{uu}+\mathcal{G}_2\psi_{uu}+\mathcal{G}_3\phi_{uu}+\mathcal{G}_4\varphi_{uu}\right),\\
\mathfrak{h}_{12} &= \mathcal{Q}^{-1/2}\left(\mathcal{G}_1\eta_{uv}+\mathcal{G}_2\psi_{uv}+\mathcal{G}_3\phi_{uv}+\mathcal{G}_4\varphi_{uv}\right),\\
\mathfrak{h}_{13} &= \mathcal{Q}^{-1/2}\left(\mathcal{G}_1\eta_{uw}+\mathcal{G}_2\psi_{uw}+\mathcal{G}_3\phi_{uw}+\mathcal{G}_4\varphi_{uw}\right),\\
\mathfrak{h}_{22} &= \mathcal{Q}^{-1/2}\left(\mathcal{G}_1\eta_{vv}+\mathcal{G}_2\psi_{vv}+\mathcal{G}_3\phi_{vv}+\mathcal{G}_4\varphi_{vv}\right),\\
\mathfrak{h}_{23} &= \mathcal{Q}^{-1/2}\left(\mathcal{G}_1\eta_{vw}+\mathcal{G}_2\psi_{vw}+\mathcal{G}_3\phi_{vw}+\mathcal{G}_4\varphi_{vw}\right),\\
\mathfrak{h}_{33} &= \mathcal{Q}^{-1/2}\left(\mathcal{G}_1\eta_{ww}+\mathcal{G}_2\psi_{ww}+\mathcal{G}_3\phi_{ww}+\mathcal{G}_4\varphi_{ww}\right),\\
\mathfrak{h}_{44} &= \mathcal{Q}^{-1/2}\mathcal{G}_1\eta,\\
\mathfrak{h}_{55} &= \mathcal{Q}^{-1/2}\mathcal{G}_2\psi,\\
\mathfrak{h}_{66} &= \mathcal{Q}^{-1/2}\mathcal{G}_3\phi,
\end{aligned}$$

and $\eta_{uu}=\frac{\partial^2\eta}{\partial u^2}$, $\eta_{uv}=\frac{\partial^2\eta}{\partial u\partial v}$, ect.. By using (4.2) and (4.4), we compute the following shape operator matrix of (4.1):

$$\mathcal{S}=\operatorname{diag}\left(\ (\mathfrak{s}_{kl})_{3\times3},\quad \mathfrak{s}_{44},\quad \mathfrak{s}_{55},\quad \mathfrak{s}_{66}\ \right)$$

with the following components

$$\mathfrak{s}_{11} = \left[\left(\mathfrak{g}_{22}\mathfrak{g}_{33}-\mathfrak{g}_{23}^2\right)\mathfrak{h}_{11}+\left(\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{12}\mathfrak{g}_{33}\right)\mathfrak{h}_{12}+\left(\mathfrak{g}_{12}\mathfrak{g}_{23}-\mathfrak{g}_{13}\mathfrak{g}_{22}\right)\mathfrak{h}_{13}\right]/\mathcal{Q},$$

$$
\begin{aligned}
\mathfrak{s}_{12} &= \left[\left(\mathfrak{g}_{22}\mathfrak{g}_{33}-\mathfrak{g}_{23}^2\right)\mathfrak{h}_{12}+(\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{12}\mathfrak{g}_{33})\,\mathfrak{h}_{22}+(\mathfrak{g}_{12}\mathfrak{g}_{23}-\mathfrak{g}_{13}\mathfrak{g}_{22})\,\mathfrak{h}_{23}\right]/Q,\\
\mathfrak{s}_{13} &= \left[\left(\mathfrak{g}_{22}\mathfrak{g}_{33}-\mathfrak{g}_{23}^2\right)\mathfrak{h}_{13}+(\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{12}\mathfrak{g}_{33})\,\mathfrak{h}_{23}+(\mathfrak{g}_{12}\mathfrak{g}_{23}-\mathfrak{g}_{13}\mathfrak{g}_{22})\,\mathfrak{h}_{33}\right]/Q,\\
\mathfrak{s}_{21} &= \left[(\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{12}\mathfrak{g}_{33})\,\mathfrak{h}_{11}+\left(\mathfrak{g}_{11}\mathfrak{g}_{33}-\mathfrak{g}_{13}^2\right)\mathfrak{h}_{12}+(\mathfrak{g}_{12}\mathfrak{g}_{13}-\mathfrak{g}_{11}\mathfrak{g}_{23})\,\mathfrak{h}_{13}\right]/Q,\\
\mathfrak{s}_{22} &= \left[(\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{12}\mathfrak{g}_{33})\,\mathfrak{h}_{12}+\left(\mathfrak{g}_{11}\mathfrak{g}_{33}-\mathfrak{g}_{13}^2\right)\mathfrak{h}_{22}+(\mathfrak{g}_{12}\mathfrak{g}_{13}-\mathfrak{g}_{11}\mathfrak{g}_{23})\,\mathfrak{h}_{23}\right]/Q,\\
\mathfrak{s}_{23} &= \left[(\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{12}\mathfrak{g}_{33})\,\mathfrak{h}_{13}+\left(\mathfrak{g}_{11}\mathfrak{g}_{33}-\mathfrak{g}_{13}^2\right)\mathfrak{h}_{23}+(\mathfrak{g}_{12}\mathfrak{g}_{13}-\mathfrak{g}_{11}\mathfrak{g}_{23})\,\mathfrak{h}_{33}\right]/Q,\\
\mathfrak{s}_{31} &= \left[(\mathfrak{g}_{12}\mathfrak{g}_{23}-\mathfrak{g}_{13}\mathfrak{g}_{22})\,\mathfrak{h}_{11}+(\mathfrak{g}_{12}\mathfrak{g}_{13}-\mathfrak{g}_{11}\mathfrak{g}_{23})\,\mathfrak{h}_{12}+\left(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2\right)\mathfrak{h}_{13}\right]/Q,\\
\mathfrak{s}_{32} &= \left[(\mathfrak{g}_{12}\mathfrak{g}_{23}-\mathfrak{g}_{13}\mathfrak{g}_{22})\,\mathfrak{h}_{12}+(\mathfrak{g}_{12}\mathfrak{g}_{13}-\mathfrak{g}_{11}\mathfrak{g}_{23})\,\mathfrak{h}_{22}+\left(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2\right)\mathfrak{h}_{23}\right]/Q,\\
\mathfrak{s}_{33} &= \left[(\mathfrak{g}_{12}\mathfrak{g}_{23}-\mathfrak{g}_{13}\mathfrak{g}_{22})\,\mathfrak{h}_{13}+(\mathfrak{g}_{12}\mathfrak{g}_{13}-\mathfrak{g}_{11}\mathfrak{g}_{23})\,\mathfrak{h}_{23}+\left(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2\right)\mathfrak{h}_{33}\right]/Q,\\
\mathfrak{s}_{44} &= \frac{\mathfrak{h}_{44}}{\mathfrak{g}_{44}},\quad \mathfrak{s}_{55}=\frac{\mathfrak{h}_{55}}{\mathfrak{g}_{55}},\quad \mathfrak{s}_{66}=\frac{\mathfrak{h}_{66}}{\mathfrak{g}_{66}}.
\end{aligned}
$$

Finally, using (3.1), with (4.2), (4.4), respectively, we find the curvatures of the *HRF* defined by Eq (4.1) as follows.

Theorem 3. *Let* $\mathfrak{r}$ *be an HRF determined by Eq* (4.1) *in* $\mathbb{E}_3^7$. $\mathfrak{r}$ *contains the following curvatures*

$$
\mathcal{K}_1 = (\mathfrak{s}_{11}+\mathfrak{s}_{22}+\mathfrak{s}_{33}+\mathfrak{s}_{44}+\mathfrak{s}_{55}+\mathfrak{s}_{66})/6,
$$

$$
\mathcal{K}_6 = \left((\mathfrak{s}_{11}\mathfrak{s}_{13}+\mathfrak{s}_{12}\mathfrak{s}_{23})\,\mathfrak{s}_{13}+(\mathfrak{s}_{12}\mathfrak{s}_{13}+\mathfrak{s}_{22}\mathfrak{s}_{23})\,\mathfrak{s}_{23}-(\mathfrak{s}_{11}+\mathfrak{s}_{22})\left(\mathfrak{s}_{13}^2+\mathfrak{s}_{23}^2\right)+\left(\mathfrak{s}_{11}\mathfrak{s}_{22}-\mathfrak{s}_{12}^2\right)\mathfrak{s}_{33}\right)\mathfrak{s}_{44}\mathfrak{s}_{55}\mathfrak{s}_{66}.
$$

Here, $\mathcal{K}_1$ *represents the mean curvature,* $\mathcal{K}_6$ *denotes the Gauss-Kronecker curvature.*

Proof. By using the Cayley-Hamilton theorem, we reveal the following characteristic polynomial equation $P_{\mathcal{S}}(\lambda)=0$ of $\mathcal{S}$:

$$
\mathcal{K}_0\lambda^6-6\mathcal{K}_1\lambda^5+15\mathcal{K}_2\lambda^4-20\mathcal{K}_3\lambda^3+15\mathcal{K}_4\lambda^2-6\mathcal{K}_5\lambda+\mathcal{K}_6=0.
$$

The curvatures $\mathcal{K}_1$ and $\mathcal{K}_6$ of $\mathfrak{r}$ are obtained by the above equation.

Corollary 1. *Let* $\mathfrak{r}$ *be an HRF defined by Eq* (4.1) *in* $\mathbb{E}_3^7$. $\mathfrak{r}$ *is a* 1*-maximal (i.e., has zero mean curvature) iff the following partial differential equation appears*

$$
\begin{aligned}
&(\mathfrak{g}_{44}\mathfrak{g}_{55}\mathfrak{h}_{66}+\mathfrak{g}_{44}\mathfrak{h}_{55}\mathfrak{g}_{66}+\mathfrak{h}_{44}\mathfrak{g}_{55}\mathfrak{g}_{66})\,Q\\
&-2\mathfrak{g}_{44}\mathfrak{g}_{55}\mathfrak{g}_{66}(\mathfrak{g}_{11}\mathfrak{g}_{23}\mathfrak{h}_{23}-\mathfrak{g}_{12}\mathfrak{g}_{13}\mathfrak{h}_{23}+\mathfrak{g}_{12}\mathfrak{h}_{12}\mathfrak{g}_{33}-\mathfrak{g}_{12}\mathfrak{g}_{23}\mathfrak{h}_{13}+\mathfrak{g}_{13}\mathfrak{g}_{22}\mathfrak{h}_{13}\\
&-\mathfrak{g}_{13}\mathfrak{h}_{12}\mathfrak{g}_{23}+\mathfrak{g}_{11}\mathfrak{g}_{22}\mathfrak{h}_{33}+\mathfrak{g}_{11}\mathfrak{h}_{22}\mathfrak{g}_{33}+\mathfrak{h}_{11}\mathfrak{g}_{22}\mathfrak{g}_{33}-\mathfrak{h}_{11}\mathfrak{g}_{23}^2-\mathfrak{g}_{13}^2\mathfrak{h}_{22}-\mathfrak{g}_{12}^2\mathfrak{h}_{33})=0.
\end{aligned}
$$

Corollary 2. *Let* $\mathfrak{r}$ *be a HRF given by Eq* (4.1) *in* $\mathbb{E}_3^7$. $\mathfrak{r}$ *is a* 6*-maximal (i.e., has zero Gauss-Kronecker curvature) iff the following partial differential equation occurs*

$$
\begin{aligned}
&[(\mathfrak{s}_{11}\mathfrak{s}_{13}+\mathfrak{s}_{12}\mathfrak{s}_{23})\,\mathfrak{s}_{13}+(\mathfrak{s}_{12}\mathfrak{s}_{13}+\mathfrak{s}_{22}\mathfrak{s}_{23})\,\mathfrak{s}_{23}\\
&-(\mathfrak{s}_{11}+\mathfrak{s}_{22})\left(\mathfrak{s}_{13}^2+\mathfrak{s}_{23}^2\right)+\left(\mathfrak{s}_{11}\mathfrak{s}_{22}-\mathfrak{s}_{12}^2\right)\mathfrak{s}_{33}]\mathfrak{s}_{44}\mathfrak{s}_{55}\mathfrak{s}_{66}=0.
\end{aligned}
$$

Corollary 3. *Let* $\mathfrak{r}$ *be a HRF defined by Eq* (4.1) *in* $\mathbb{E}_3^7$. $\mathfrak{r}$ *has umbilical point (i.e.,* $(\mathcal{K}_1)^6=\mathcal{K}_6$*) iff the following partial differential equation holds*

$$
(\mathfrak{s}_{11}+\mathfrak{s}_{22}+\mathfrak{s}_{33}+\mathfrak{s}_{44}+\mathfrak{s}_{55}+\mathfrak{s}_{66})^6
$$

$$-46\,656\left\{\begin{array}{c}(\mathfrak{s}_{11}\mathfrak{s}_{13}+\mathfrak{s}_{12}\mathfrak{s}_{23})\,\mathfrak{s}_{13}+(\mathfrak{s}_{12}\mathfrak{s}_{13}+\mathfrak{s}_{22}\mathfrak{s}_{23})\,\mathfrak{s}_{23}\\-(\mathfrak{s}_{11}+\mathfrak{s}_{22})\left(\mathfrak{s}_{13}^2+\mathfrak{s}_{23}^2\right)+\left(\mathfrak{s}_{11}\mathfrak{s}_{22}-\mathfrak{s}_{12}^2\right)\mathfrak{s}_{33}\end{array}\right\}\mathfrak{s}_{44}\mathfrak{s}_{55}\mathfrak{s}_{66}=0.$$

Hence, we find the following.

Example 1. *Let* $\mathfrak{r}$ *be an HRF determined by Eq* (4.1) *in* $\mathbb{E}_3^7$. *When the profile hypersurface* γ *of* $\mathfrak{r}$ *is parametrized by the unit hypersphere:* $\eta=\cos u\cos v\cos w$, $\psi=\sin u\cos v\cos w$, $\phi=\sin v\cos w$, $\varphi=\sin w$, *then* $\mathcal{S}=\mathcal{I}_6$ *and the HRF has the following curvatures* $\mathcal{K}_i=1$, *where* $i=0,1,...,6$.

Example 2. *Assume* $\mathfrak{r}$ *be an HRF denoted by Eq* (4.1) *in* $\mathbb{E}_3^7$. *While the profile hypersurface* γ *of* $\mathfrak{r}$ *is parametrized by the rational unit hypersphere:* $\eta=\frac{1-u^2}{1+u^2}\frac{1-v^2}{1+v^2}\frac{1-w^2}{1+w^2}$, $\psi=\frac{2u}{1+u^2}\frac{1-v^2}{1+v^2}\frac{1-w^2}{1+w^2}$, $\phi=\frac{2v}{1+v^2}\frac{1-w^2}{1+w^2}$, $\varphi=\frac{2w}{1+w^2}$, *the HRF has the same results determined by Example 1.*

Example 3. *Let* $\mathfrak{r}$ *be an HRF defined by Eq* (4.1) *in* $\mathbb{E}_3^7$. *When the generating hypersurface* γ *of* $\mathfrak{r}$ *is parametrized by the Riemann hypersphere:* $\eta=\frac{2u}{u^2+v^2+w^2+1}$, $\psi=\frac{2v}{u^2+v^2+w^2+1}$, $\phi=\frac{2w}{u^2+v^2+w^2+1}$, $\varphi=\frac{u^2+v^2+w^2-1}{u^2+v^2+w^2+1}$, *the HRF has* $\mathcal{S}=-\mathcal{I}_6$, *and has the following curvatures* $\mathcal{K}_i=(-1)^i$, *where* $i=0,1,...,6$.

Example 4. *Considering the pseudo-hypersphere* $\mathbb{S}_3^6(\rho):=\left\{\mathbf{p}\in\mathbb{E}_3^7\mid\langle\mathbf{p},\mathbf{p}\rangle=\rho^2\right\}$, *radius* $\rho>0$, *parametrized by*

$$\mathbf{p}(u,v,w,\alpha,\beta,\gamma)=\begin{pmatrix}\rho\cos u\cos v\cos w\cosh\alpha\\\rho\cos u\cos v\cos w\sinh\alpha\\\rho\sin u\cos v\cos w\cosh\beta\\\rho\sin u\cos v\cos w\sinh\beta\\\rho\sin v\cos w\cosh\gamma\\\rho\sin v\cos w\sinh\gamma\\\rho\sin w\end{pmatrix},\tag{4.5}$$

we compute $\mathcal{S}=\frac{1}{\rho}\mathcal{I}_6$. *Hence, we find the following curvatures* $\mathcal{K}_i=\frac{1}{\rho^i}$, *where* $i=0,1,...,6$. *Then, the hypersurface* $\mathbf{p}$ *described by Eq* (4.5) *is an umbilical hypersphere (i.e., it supplies* $(\mathcal{K}_1)^6=\mathcal{K}_6$*) of* $\mathbb{E}_3^7$.

5. Hypersurfaces of revolution family satisfying $\Delta\mathfrak{r}=\mathcal{A}\mathfrak{r}$ in $\mathbb{E}_3^7$

In this section, our focus is on the Laplace-Beltrami operator of a smooth function in $\mathbb{E}_3^7$. We will proceed to compute it utilizing the *HRF*, which is defined by Eq (4.1).

Definition 6. *The Laplace-Beltrami operator of a smooth function* $f=f(x^1,x^2,...,x^6)\mid_{\mathcal{D}}$ $(\mathcal{D}\subset\mathbb{R}^6)$ *of class* C^6 *depends on the first fundamental form* $\left(\mathfrak{g}_{ij}\right)$ *of a hypersurface* $\mathfrak{r}$, *and is the operator defined by*

$$\Delta f=\frac{1}{\hat{\mathbf{g}}^{1/2}}\sum_{i,j=1}^{6}\frac{\partial}{\partial x^i}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{ij}\frac{\partial f}{\partial x^j}\right),\tag{5.1}$$

where $\left(\mathfrak{g}^{ij}\right)=(\mathfrak{g}_{kl})^{-1}$ *and* $\hat{\mathbf{g}}=\det\left(\mathfrak{g}_{ij}\right)$.

By using the inverse matrix of the first fundamental form matrix $\left(\mathfrak{g}_{ij}\right)_{6\times6}$, we have the following.

For an *HRF* given by Eq (4.1), $\mathfrak{g}_{ij}=0$ when $i\neq j$ except for $i,j<4$. Therefore, the Laplace-Beltrami operator of the *HRF* $\mathfrak{r}=\mathfrak{r}(u,v,w,\alpha,\beta,\gamma)$ is given by

$$\Delta\mathfrak{r}\;=\;\frac{1}{\hat{\mathbf{g}}^{1/2}}[\frac{\partial}{\partial u}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{11}\frac{\partial\mathfrak{r}}{\partial u}\right)+\frac{\partial}{\partial u}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{12}\frac{\partial\mathfrak{r}}{\partial v}\right)+\frac{\partial}{\partial u}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{13}\frac{\partial\mathfrak{r}}{\partial w}\right)\tag{5.2}$$

$$+\frac{\partial}{\partial v}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{21}\frac{\partial \mathfrak{r}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{22}\frac{\partial \mathfrak{r}}{\partial v}\right)+\frac{\partial}{\partial v}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{23}\frac{\partial \mathfrak{r}}{\partial w}\right)$$
$$+\frac{\partial}{\partial w}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{31}\frac{\partial \mathfrak{r}}{\partial u}\right)+\frac{\partial}{\partial w}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{32}\frac{\partial \mathfrak{r}}{\partial v}\right)+\frac{\partial}{\partial w}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{33}\frac{\partial \mathfrak{r}}{\partial w}\right)$$
$$+\frac{\partial}{\partial \alpha}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{44}\frac{\partial \mathfrak{r}}{\partial \alpha}\right)+\frac{\partial}{\partial \beta}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{55}\frac{\partial \mathfrak{r}}{\partial \beta}\right)+\frac{\partial}{\partial \gamma}\left(\hat{\mathbf{g}}^{1/2}\mathfrak{g}^{66}\frac{\partial \mathfrak{r}}{\partial \gamma}\right).$$

By using the derivatives of the functions in (5.2), w.r.t. $u, v, w, \alpha, \beta, \gamma$, resp., we obtain the following.

Theorem 4. *The Laplace-Beltrami operator of the HRF* $\mathfrak{r}$ *denoted by Eq* (4.1) *is given by* $\Delta\mathfrak{r} = 6\mathcal{K}_1\mathcal{G}$*, where* $\mathcal{K}_1$ *denotes the mean curvature,* $\mathcal{G}$ *represents the Gauss map of* $\mathfrak{r}$.

Proof. By directly computing (5.2), we obtain $\Delta\mathfrak{r}$.

Theorem 5. *Let* $\mathfrak{r}$ *be an HRF defined by Eq* (4.1). $\Delta\mathfrak{r} = \mathcal{A}\mathfrak{r}$*, where* $\mathcal{A}$ *denotes the square matrix of order* 7 *iff* $\mathfrak{r}$ *has* $\mathcal{K}_1 = 0$*, i.e., it is a* 1*-maximal hypersurface.*

Proof. We found $6\mathcal{K}_1\mathcal{G} = \mathcal{A}\mathfrak{r}$**,** and then we have

$$\begin{aligned}&a_{11}\eta\cosh\alpha + a_{12}\eta\sinh\alpha + a_{13}\psi\cosh\beta + a_{14}\psi\sinh\beta + a_{15}\phi\cosh\gamma + a_{16}\phi\sinh\gamma + a_{17}\varphi\\ =\;& \Upsilon\eta\psi\phi\left[(\psi_v\phi_w - \psi_w\phi_v)\varphi_u + (\psi_w\phi_u - \psi_u\phi_w)\varphi_v + (\psi_u\phi_v - \psi_v\phi_u)\varphi_w\right]\cosh\alpha,\end{aligned}$$

$$\begin{aligned}&a_{21}\eta\cosh\alpha + a_{22}\eta\sinh\alpha + a_{23}\psi\cosh\beta + a_{24}\psi\sinh\beta + a_{25}\phi\cosh\gamma + a_{26}\phi\sinh\gamma + a_{27}\varphi\\ =\;& \Upsilon\eta\psi\phi\left[(\psi_v\phi_w - \psi_w\phi_v)\varphi_u + (\psi_w\phi_u - \psi_u\phi_w)\varphi_v + (\psi_u\phi_v - \psi_v\phi_u)\varphi_w\right]\sinh\alpha,\end{aligned}$$

$$\begin{aligned}&a_{31}\eta\cosh\alpha + a_{32}\eta\sinh\alpha + a_{33}\psi\cosh\beta + a_{34}\psi\sinh\beta + a_{35}\phi\cosh\gamma + a_{36}\phi\sinh\gamma + a_{37}\varphi\\ =\;& \Upsilon\eta\psi\phi\left[(\eta_v\phi_w - \eta_w\phi_v)\varphi_u + (\eta_w\phi_u - \eta_u\phi_w)\varphi_v + (\eta_u\phi_v - \eta_v\phi_u)\varphi_w\right]\cosh\beta,\end{aligned}$$

$$\begin{aligned}&a_{41}\eta\cosh\alpha + a_{42}\eta\sinh\alpha + a_{43}\psi\cosh\beta + a_{44}\psi\sinh\beta + a_{45}\phi\cosh\gamma + a_{46}\phi\sinh\gamma + a_{47}\varphi\\ =\;& \Upsilon\eta\psi\phi\left[(\eta_v\phi_w - \eta_w\phi_v)\varphi_u + (\eta_w\phi_u - \eta_u\phi_w)\varphi_v + (\eta_u\phi_v - \eta_v\phi_u)\varphi_w\right]\sinh\beta,\end{aligned}$$

$$\begin{aligned}&a_{51}\eta\cosh\alpha + a_{52}\eta\sinh\alpha + a_{53}\psi\cosh\beta + a_{54}\psi\sinh\beta + a_{55}\phi\cosh\gamma + a_{56}\phi\sinh\gamma + a_{57}\varphi\\ =\;& \Upsilon\eta\psi\phi\left[(\eta_v\psi_w - \eta_w\psi_v)\varphi_u + (\eta_w\psi_u - \eta_u\psi_w)\varphi_v + (\eta_u\psi_v - \eta_v\psi_u)\varphi_w\right]\cosh\gamma,\end{aligned}$$

$$\begin{aligned}&a_{61}\eta\cosh\alpha + a_{62}\eta\sinh\alpha + a_{63}\psi\cosh\beta + a_{64}\psi\sinh\beta + a_{65}\phi\cosh\gamma + a_{66}\phi\sinh\gamma + a_{67}\varphi\\ =\;& \Upsilon\eta\psi\phi\left[(\eta_v\psi_w - \eta_w\psi_v)\varphi_u + (\eta_w\psi_u - \eta_u\psi_w)\varphi_v + (\eta_u\psi_v - \eta_v\psi_u)\varphi_w\right]\sinh\gamma,\end{aligned}$$

$$\begin{aligned}&a_{71}\eta\cosh\alpha + a_{72}\eta\sinh\alpha + a_{73}\psi\cosh\beta + a_{74}\psi\sinh\beta + a_{75}\phi\cosh\gamma + a_{76}\phi\sinh\gamma + a_{77}\varphi\\ =\;& \Upsilon\eta\psi\phi\left[(\eta_w\psi_v - \eta_v\psi_w)\phi_u + (\eta_u\psi_w - \eta_w\psi_u)\phi_v + (\eta_v\psi_u - \eta_u\psi_v)\phi_w\right],\end{aligned}$$

where $\mathcal{A} = \left(a_{ij}\right)$ is the 7×7 matrix, $\Upsilon = 6\mathcal{K}_1\hat{\mathbf{g}}^{-1/2}$, where $\hat{\mathbf{g}} = \eta^2\psi^2\phi^2\mathcal{Q}$. Derivating above ODEs twice w.r.t. α, we obtain the following $a_{i7} = 0$, $\Upsilon = 0$, where $i = 1, 2, ..., 7$. Then, we get $(a_{i1}\cosh\alpha + a_{i2}\sinh\alpha)\,\eta = 0$, where $i = 1, 2, ..., 7$. The functions cosh and sinh are linear independent on α, then all the components of the matrix $\mathcal{A}$ are 0. Since $\Upsilon = 6\mathcal{K}_1\hat{\mathbf{g}}^{-1/2}$, then $\mathcal{K}_1 = 0$. This means, $\mathfrak{r}$ is a 1-maximal *HRF*.

Therefore, we give the following.

Example 5. *Let* $\mathfrak{r}$ *be an HRF given by Eq* (4.1), *and let the generating hypersurface* γ *of* $\mathfrak{r}$ *be parametrized by the unit hypersphere determined by Example 1. Then, an HRF* $\mathfrak{r}$ *supplies* $\Delta\mathfrak{r} = \mathcal{A}\mathfrak{r}$, *where* $\mathcal{A} = -6\mathcal{I}_7$, $\mathcal{I}_7$*denotes identity matrix.*

Example 6. *Let* $\mathfrak{r}$ *be an HRF denoted by Eq* (4.1), *and let the generating hypersurface* γ *of* $\mathfrak{r}$ *be parametrized by the Riemann hypersphere defined by Example 3. An HRF* $\mathfrak{r}$ *has the same results denoted by Example 5.*

6. Conclusions

This research has presented a detailed analysis of a family of hypersurfaces of revolution $\mathfrak{r}$ is characterized by six parameters in the seven-dimensional pseudo-Euclidean space $\mathbb{E}^7_3$, and its geometric properties have been thoroughly explored.

The main focus of the paper was on computing and investigating various matrices associated with $\mathfrak{r}$. The fundamental form, Gauss map, and shape operator matrices were derived, providing essential information about the local geometry of the hypersurfaces. By utilizing the Cayley-Hamilton theorem, the curvatures of $\mathfrak{r}$ were determined, facilitating a comprehensive understanding of their intrinsic curvature properties. Moreover, the paper established equations that describe the relationship between the mean curvature and Gauss-Kronecker curvature of $\mathfrak{r}$. These equations shed light on the geometric behavior of the hypersurfaces and offer valuable insights into their intrinsic properties. Additionally, the paper investigated the connection between the Laplace-Beltrami operator of $\mathfrak{r}$ and a specific 7×7 matrix. This exploration further deepened our understanding of the geometric structure and differential properties of the hypersurface family.

In summary, this research contributes to the understanding of hypersurfaces of revolution in $\mathbb{E}^7_3$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

References

1. L. J. Alias, N. Gürbüz, An extension of Takashi theorem for the linearized operators of the highest order mean curvatures, *Geometriae Dedicata*, **121** (2006), 113–127. https://doi.org/10.1007/s10711-006-9093-9

2. Y. Aminov, *The geometry of submanifolds*, Amsterdam: Gordon and Breach Sci. Pub., 2001.

3. K. Arslan, B. K. Bayram, B. Bulca, Y. H. Kim, C. Murathan, G. Öztürk, Vranceanu surface in $\mathbb{E}^4$ with pointwise 1-type Gauss map, *Indian J. Pure Appl. Math.,* **42** (2011), 41–51. https://doi.org/10.1007/s13226-011-0003-y

4. K. Arslan, B. K. Bayram, B. Bulca, G. Öztürk, Generalized rotation surfaces in $\mathbb{E}^4$, *Results Math.*, **61** (2012), 315–327. https://doi.org/10.1007/s00025-011-0103-3

5. K. Arslan, B. Bulca, B. Kılıç, Y. H. Kim, C. Murathan, G. Öztürk, Tensor product surfaces with pointwıse 1-type Gauss map, *Bull. Korean Math. Soc.*, **48** (2011), 601–609. https://doi.org/10.4134/BKMS.2011.48.3.601

6. K. Arslan, V. Milousheva, Meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map in Minkowski 4-space, *Taiwan. J. Math.*, **20** (2016), 311–332. https://doi.org/10.11650/tjm.20.2016.5722

7. K. Arslan, A. Sütveren, B. Bulca, Rotational λ-hypersurfaces in Euclidean spaces, *Creat. Math. Inform.*, **30** (2021), 29–40.

8. A. Arvanitoyeorgos, G. Kaimakamis, M. Magid, Lorentz hypersurfaces in $\mathbb{E}^4_1$ satisfying $\Delta H = \alpha H$, *Illinois J. Math.*, **53** (2009), 581–590. https://doi.org/10.1215/IJM/1266934794

9. M. Barros, B. Y. Chen, Stationary 2-type surfaces in a hypersphere, *J. Math. Soc. Jap.*, **39** (1987), 627–648. https://doi.org/10.2969/jmsj/03940627

10. M. Barros, O. J. Garay, 2-type surfaces in S^3, *Geometriae Dedicata*, **24** (1987), 329–336. https://doi.org/10.1007/BF00181605

11. B. Y. Chen, On submanifolds of finite type, *Soochow J. Math.*, **9** (1983), 65–81.

12. B. Y. Chen, *Total mean curvature and submanifolds of finite type*, Singapore: World Scientific, 1984.

13. B. Y. Chen, *Finite type submanifolds and generalizations*, Rome: University of Rome, 1985.

14. B. Y. Chen, Finite type submanifolds in pseudo-Euclidean spaces and applications, *Kodai Math. J.*, **8** (1985), 358–374. https://doi.org/10.2996/kmj/1138037104

15. B. Y. Chen, P. Piccinni, Submanifolds with finite type Gauss map, *Bull. Austral. Math. Soc.*, **35** (1987), 161–186. https://doi.org/10.1017/S0004972700013162

16. B. Y. Chen, E. Güler, Y. Yaylı, H. H. Hacısalihoğlu, Differential geometry of 1-type submanifolds and submanifolds with 1-type Gauss map, *Int. Elec. J. Geom.*, **16** (2023), 4–49. https://doi.org/10.36890/iejg.1216024

17. Q. M. Cheng, Q. R. Wan, Complete hypersurfaces of $\mathbb{R}^4$ with constant mean curvature, *Monatsh. Math.*, **118** (1994), 171–204. https://doi.org/10.1007/BF01301688

18. S. Y. Cheng, S. T. Yau, Hypersurfaces with constant scalar curvature, *Math. Ann.*, **225** (1977), 195–204. https://doi.org/10.1007/BF01425237

19. M. Choi, Y. H. Kim, Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map, *Bull. Korean Math. Soc.*, **38** (2001), 753–761.

20. F. Dillen, J. Pas, L. Verstraelen, On surfaces of finite type in Euclidean 3-space, *Kodai Math. J.*, **13** (1990), 10–21. https://doi.org/10.2996/kmj/1138039155

21. M. Do Carmo, M. Dajczer, Rotation hypersurfaces in spaces of constant curvature, *Trans. Amer. Math. Soc.*, **277** (1983), 685–709. https://doi.org/10.1090/S0002-9947-1983-0694383-X

22. U. Dursun, Hypersurfaces with pointwise 1-type Gauss map, *Taiwan. J. Math.*, **11** (2007), 1407–1416. https://doi.org/10.11650/twjm/1500404873

23. A. Ferrandez, O. J. Garay, P. Lucas, *On a certain class of conformally at Euclidean hypersurfaces*, In Global Analysis and Global Differential Geometry, Springer: Berlin, Germany, 1990, 48–54.

24. G. Ganchev, V. Milousheva, General rotational surfaces in the 4-dimensional Minkowski space, *Turkish J. Math.*, **38** (2014), 883–895. https://doi.org/10.3906/mat-1312-10

25. O. J. Garay, On a certain class of finite type surfaces of revolution, *Kodai Math. J.*, **11** (1988), 25–31. https://doi.org/10.2996/kmj/1138038815

26. O. J. Garay, An extension of Takahashi's theorem, *Geometriae Dedicata*, **34** (1990), 105–112. https://doi.org/10.1007/BF00147319

27. E. Güler, Fundamental form IV and curvature formulas of the hypersphere, *Malaya J. Mat.*, **8** (2020), 2008–2011. https://doi.org/10.26637/MJM0804/0116

28. E. Güler, Rotational hypersurfaces satisfying $\Delta^{I}R = AR$ in the four-dimensional Euclidean space, *J. Polytech.*, **24** (2021), 517–520. https://doi.org/10.2339/POLITEKNIK.670333

29. E. Güler, H. H. Hacısalihoğlu, Y. H. Kim, The Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface in 4-space, *Symmetry*, **10** (2018), 1–12. https://doi.org/10.3390/sym10090398

30. E. Güler, M. Magid, Y. Yaylı, Laplace-Beltrami operator of a helicoidal hypersurface in four-space, *J. Geom. Symmetry. Phys.*, **41** (2016), 77–95. https://doi.org/10.7546/jgsp-41-2016-77-95

31. E. Güler, Y. Yaylı, H. H. Hacısalihoğlu, Bi-rotational hypersurface with $\Delta x = Ax$ in 4-space, *Facta Universitatis (Nis) Ser. Math. Inform.*, **37** (2022), 917–928.

32. E. Güler, Y. Yaylı, H. H. Hacısalihoğlu, Bi-rotational hypersurface and the second Laplace-Beltrami operator in the four dimensional Euclidean space $\mathbb{E}^4$, *Turkish J. Math.*, **46** (2022), 2167–2177. https://doi.org/10.55730/1300-0098.3261

33. E. Güler, Y. Yaylı, H. H. Hacısalihoğlu, Bi-rotational hypersurface satisfying $\Delta^{III}x = Ax$ in 4-space, *Honam Math. J.*, **44** (2022), 219–230.

34. E. Güler, Y. Yaylı, H. H. Hacısalihoğlu, Bi-rotational hypersurface satisfying $\Delta x = Ax$ in pseudo-Euclidean space $\mathbb{E}^4_2$, *TWMS J. Pure Appl. Math.*, Preprint.

35. T. Hasanis, T. Vlachos, Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field, *Math. Nachr.*, **172** (1995), 145–169. https://doi.org/10.1002/mana.19951720112

36. D. S. Kim, J. R. Kim, Y. H. Kim, Cheng-Yau operator and Gauss map of surfaces of revolution, *Bull. Malays. Math. Sci. Soc.*, **39** (2016), 1319–1327. https://doi.org/10.1007/s40840-015-0234-x

37. W. Kühnel, *Differential geometry. Curves-surfaces-manifolds*, 3 Eds., Translated from the 2013 German ed. AMS, Providence, RI, 2015.

38. T. Levi-Civita, Famiglie di superficie isoparametriche nellordinario spacio euclideo, *Rend. Acad. Lincei*, **26** (1937), 355–362.

39. Y. Li, E. Güler, A hypersurfaces of revolution family in the five-dimensional Pseudo-Euclidean space $\mathbb{E}^5_2$, *Mathematics*, **11** (2023), 3427. https://doi.org/10.3390/math11153427

40. C. Moore, Surfaces of rotation in a space of four dimensions, *Ann. Math.*, **21** (1919), 81–93. https://doi.org/10.2307/2007223

41. C. Moore, Rotation surfaces of constant curvature in space of four dimensions, *Bull. Amer. Math. Soc.*, **26** (1920), 454–460.

42. S. Stamatakis, H. Zoubi, Surfaces of revolution satisfying $\Delta^{III} x = Ax$, *J. Geom. Graph.*, **14** (2010), 181–186.

43. T. Takahashi, Minimal immersions of Riemannian manifolds, *J. Math. Soc. Japan*, **18** (1966), 380–385. https://doi.org/10.2969/jmsj/01840380

44. D. W. Yoon, Some properties of the Clifford torus as rotation surfaces, *Indian J. Pure Appl. Math.*, **34** (2003), 907–915.