Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 24947–24956
DOI: 10.3934/math.20231272
Received: 02 July 2023
Revised: 14 August 2023
Accepted: 16 August 2023
Published: 28 August 2023

Research article

A rigidity result for 2-dimensional λ-translators

Jin Liu and Botao Wang*

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China

* **Correspondence:** Email: btwang2022@126.com.

Abstract: In this paper, we will develop a different technique to study the rigidity of complete λ-translators $x : M^2 \to \mathbb{R}^3$ with the non-zero constant gauss curvature in the Euclidean space $\mathbb{R}^3$.

Keywords: second fundamental form; λ-translator; Gauss curvature; non-umbilic points; mean curvature
Mathematics Subject Classification: 53C40, 53E10

1. Introduction

A λ-translating soliton (or, simply, a λ-translator) to the mean curvature flow is an immersed hypersurface $x : M^n \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$ satisfying the equation

$$H + \langle \nu, e_{n+1} \rangle = \lambda. \tag{1.1}$$

Here, and in what follows, H is the mean curvature, ν is a non-zero constant unit vector and e_{n+1} denotes the unit normal vector field.

A special case of (1.1) is when $\lambda = 0$. In such a case the immersion x is called a translating soliton of the mean curvature flow, or, simply, a translator [22]. Translators play an important role in the study of mean curvature flow. On one hand, a translating soliton is a solution of the mean curvature flow that evolves purely by translations along the direction T. On the other hand, they arise as blow-up solutions for a mean convex flow under type II singularities [9, 11]. For instance, Huisken and Sinestrari [10] proved that, under the condition of a type II singularity of an MCF (i.e., an MCF with a mean convex solution), there exists a blow-up solution which is a convex translating solution. As we know, translating solitons have been widely studied and various interesting results have been obtained in recent years. For more information about translating solitons, please refer to the literature ([1, 4, 6, 7, 12, 16, 8, 17, 18, 19, 23, 25]).

In 2018, López classified, in [14], all λ-translators in $\mathbb{R}^3$ that are invariant by a one-parameter group of translations and a one-parameter group of rotations. He also studied the shape of a compact λ-

translator of $\mathbb{R}^3$ in terms of its boundary in [15]. Inspired by the work of Cheng and Wei [3], Li et al. [13] classified 2-dimensional complete λ-translators in the Euclidean space $\mathbb{R}^3$ and the Minkowski space $\mathbb{R}^3_1$ with constant squared norm S of the second fundamental form, which use the generalized maximum principle from [2]. Recently, Yang et al. [26] developed a new technique to study the rigidity of self-shrinkers. In this paper, we will use a similar method to study the classification theorem for 2-dimensional complete λ-translators with the non-zero constant Gauss curvature. The specific conclusions are as follows:

Theorem 1.1. *There are no complete λ-translators with the non-zero constant Gauss curvature in $\mathbb{R}^3$.*

2. Preliminaries

Let $x : M^2 \to \mathbb{R}^3$ be an isometric immersion of a surface of the 3-dimensional Euclidean space $\mathbb{R}^3$. Denote the Levi-Civita connections of M^2 and $\mathbb{R}^3$ by ∇ and D, respectively. Around each point of M^2, we choose a local orthonormal frame field $\{e_A\}_{A=1}^3$ in $\mathbb{R}^3$ with the dual coframe field $\{\omega^A\}_{A=1}^3$ such that, restricted to M^2, e_1, e_2 are tangent on M^2 and e_3 is a normal vector field. The gauss and weingarten formulae are given, respectively, by

$$D_{e_i}e_j = \nabla_{e_i}e_j + h(e_i, e_j), \quad D_{e_i}e_3 = -A_{e_3}e_i, \tag{2.1}$$

where h and A are the second fundamental form and the shape operator, respectively. As we know, the second fundamental form h and the shape operator A are related by

$$h_{ij} = \langle h(e_i, e_j), e_3\rangle = \langle A_{e_3}e_i, e_j\rangle = A(e_i, e_j).$$

Let

$$H = \sum_i h_{ii}, \quad S = \sum_{i,j}(h_{ij})^2$$

be the mean curvature and the squared norm of the second fundamental form. For those points $p \in M^2$, near which we could take a principal frame $\{e_1, e_2\}$ such that

$$A(e_i, e_j) = h_{ij} = \lambda_i\delta_{ij},$$

the mean curvature H, the squared norm of the second fundamental form S and the Gauss curvature K are given, respectively, by

$$H = \lambda_1 + \lambda_2, \quad S = \lambda_1^2 + \lambda_2^2, \quad K = \lambda_1\lambda_2, \quad H^2 - S = 2K.$$

For any fixed i, j, k, since $\langle e_j, e_j\rangle = 1$ and $\langle e_i, e_j\rangle = 0$ $(i \neq j)$, we have

$$\begin{cases} 0 = e_i(\langle e_j, e_j\rangle) = 2\langle \nabla_{e_i}e_j, e_j\rangle, \quad \nabla_{e_i}e_j = \sum\limits_{k\neq j}\Gamma_{ji}^k e_k, \\ 0 = e_k(\langle e_i, e_j\rangle) = \langle \nabla_{e_k}e_i, e_j\rangle + \langle e_i, \nabla_{e_k}e_j\rangle = \Gamma_{ik}^j + \Gamma_{jk}^i \end{cases}$$

for some smooth functions Γ_{ji}^k near p. Then,

$$\Gamma_{1i}^1 = \Gamma_{2i}^2 = 0, \quad \Gamma_{2i}^1 + \Gamma_{1i}^2 = 0, \quad i = 1, 2. \tag{2.2}$$

It follows from $A(e_k, e_k) = \lambda_k$ and $A(e_k, e_m) = 0$ $(k \neq m)$ that

$$\begin{aligned} e_i(\lambda_k) = e_i(A(e_k, e_k)) =& (\nabla_{e_i} A)(e_k, e_k) + 2A(\nabla_{e_i} e_k, e_k) \\ =& (\nabla_{e_i} A)(e_k, e_k) + 2\sum_{j\neq k} \Gamma_{ki}^j A(e_j, e_k) \\ =& (\nabla_{e_i} A)(e_k, e_k) = h_{kki} \end{aligned}$$

and

$$\begin{aligned} 0 = e_i(A(e_k, e_m)) =& (\nabla_{e_i} A)(e_k, e_m) + A(\nabla_{e_i} e_k, e_m) + A(e_k, \nabla_{e_i} e_m) \\ =& (\nabla_{e_i} A)(e_k, e_m) + \sum_{j\neq k} \Gamma_{ki}^j A(e_j, e_m) + \sum_{j\neq m} \Gamma_{mi}^j A(e_k, e_j) \\ =& h_{kmi} + \Gamma_{ki}^m \lambda_m + \Gamma_{mi}^k \lambda_k. \end{aligned}$$

Thus,

$$e_i(\lambda_k) = h_{kki}, \quad h_{kim} = h_{kmi} = \Gamma_{ki}^m(\lambda_k - \lambda_m).$$

That is,

$$e_1(\lambda_2) = \Gamma_{22}^1(\lambda_2 - \lambda_1), \quad e_2(\lambda_1) = \Gamma_{11}^2(\lambda_1 - \lambda_2). \tag{2.3}$$

Since covariant differentiation is torsion free, by calculating the Lie bracket $[e_1, e_2]$, it can be obtained that

$$e_1 \cdot e_2 - e_2 \cdot e_1 = [e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 = \Gamma_{21}^1 e_1 - \Gamma_{12}^2 e_2. \tag{2.4}$$

Suppose that the given hypersurface $x : M^n \to \mathbb{R}^3$ is a λ-translator with a translating vector ν. For a tangent C^1-vector field V on M^2, define a differential operator

$$\Delta_V(\cdot) = \Delta(\cdot) + \langle V, \nabla(\cdot)\rangle,$$

where Δ and ∇ denote the Laplacian and the gradient operator, respectively. Under the local orthonormal frame field $\{e_1, e_2\}$, from the definition (1.1) of the translator in $\mathbb{R}^3$, by a direct computation, we have the following basic formulas:

$$\begin{cases} e_1(\langle \nu, e_1\rangle) = \Gamma_{11}^2\langle \nu, e_2\rangle + \lambda_1(\lambda - H), \\ e_1(\langle \nu, e_2\rangle) = -\Gamma_{11}^2\langle \nu, e_1\rangle, \\ e_2(\langle \nu, e_1\rangle) = -\Gamma_{22}^1\langle \nu, e_2\rangle, \\ e_2(\langle \nu, e_2\rangle) = \Gamma_{22}^1\langle \nu, e_1\rangle + \lambda_2(\lambda - H), \\ e_1(H) = \lambda_1\langle \nu, e_1\rangle, \\ e_2(H) = \lambda_2\langle \nu, e_2\rangle, \end{cases} \tag{2.5}$$

and

$$\Delta_{-V} H = -S(H - \lambda), \tag{2.6}$$

where $V = \nu^\top$, i.e., the tangent component of the translating vector ν when restricted to M^2.

3. The proof of Theorem 1.1

In this section, by differentiating the eigenvalues of the shape operator on the set of umbilic points (cf. [5, 20, 21, 24]), we are able to prove that the mean curvature H and the principal curvature are each constant. First, we assume that λ_1 is not constant on an open set $\Theta \subset M^2$ which is composed of non-umbilic points. That is, $\lambda_1^2 - K \neq 0$ on Θ. Next, we will prove the following several important propositions.

Proposition 3.1. *For the function λ_1 defined on $\Theta \subset M^2$, we have*

$$K\lambda_1(\lambda_1^2 - K)e_1 \cdot e_1(\lambda_1) - K(\lambda_1^2 - 3K)e_1^2(\lambda_1) - \lambda_1^4 e_2^2(\lambda_1) + K\lambda_1^4(\lambda_1^2 - \lambda\lambda_1 + K) = 0, \tag{3.1}$$

$$\lambda_1(\lambda_1^2 - K)e_2 \cdot e_1(\lambda_1) + 3Ke_1(\lambda_1)e_2(\lambda_1) = 0, \tag{3.2}$$

$$\lambda_1(\lambda_1^2 - K)e_1 \cdot e_2(\lambda_1) + (\lambda_1^2 + 2K)e_1(\lambda_1)e_2(\lambda_1) = 0, \tag{3.3}$$

$$\lambda_1^3(\lambda_1^2 - K)e_2 \cdot e_2(\lambda_1) + \lambda_1^2(\lambda_1^2 + K)e_2^2(\lambda_1) - K^2 e_1^2(\lambda_1) + K^2\lambda_1^2(\lambda_1^2 - \lambda\lambda_1 + K) = 0. \tag{3.4}$$

Proof. For the convenience of calculation, assume that $a = \langle \nu, e_1 \rangle$ and $b = \langle \nu, e_2 \rangle$. Recall that $\lambda_2 = \frac{K}{\lambda_1}$ and $H = \lambda_1 + \frac{K}{\lambda_1}$; by the fifth and sixth equations of (2.5), we have

$$\lambda_1 a = e_1(H) = \frac{\lambda_1^2 - K}{\lambda_1^2} e_1(\lambda_1), \quad \lambda_2 b = e_2(H) = \frac{\lambda_1^2 - K}{\lambda_1^2} e_2(\lambda_1).$$

Thus,

$$\begin{cases} a = \dfrac{\lambda_1^2 - K}{\lambda_1^3} e_1(\lambda_1), \quad e_1(a) = \dfrac{\lambda_1^2 - K}{\lambda_1^3} e_1 \cdot e_1(\lambda_1) - \dfrac{\lambda_1^2 - 3K}{\lambda_1^4} e_1^2(\lambda_1), \\ e_2(a) = \dfrac{\lambda_1^2 - K}{\lambda_1^3} e_2 \cdot e_1(\lambda_1) - \dfrac{\lambda_1^2 - 3K}{\lambda_1^4} e_1(\lambda_1)e_2(\lambda_1), \\ b = \dfrac{\lambda_1^2 - K}{K\lambda_1} e_2(\lambda_1), \quad e_1(b) = \dfrac{\lambda_1^2 - K}{K\lambda_1} e_1 \cdot e_2(\lambda_1) + \dfrac{\lambda_1^2 + K}{K\lambda_1^2} e_1(\lambda_1)e_2(\lambda_1), \\ e_2(b) = \dfrac{\lambda_1^2 - K}{K\lambda_1} e_2 \cdot e_2(\lambda_1) + \dfrac{\lambda_1^2 + K}{K\lambda_1^2} e_2^2(\lambda_1). \end{cases} \tag{3.5}$$

We will use (2.3) to get

$$\Gamma_{22}^1 = \frac{K}{\lambda_1(\lambda_1^2 - K)} e_1(\lambda_1), \quad \Gamma_{11}^2 = \frac{\lambda_1}{\lambda_1^2 - K} e_2(\lambda_1). \tag{3.6}$$

Substituting (3.5) and (3.6) into (2.5), we obtain (3.1)–(3.4). □

Proposition 3.2. *There exists a point $p \in \Theta \subset M^2$ such that $e_1(\lambda_1) \neq 0$ at p.*

Proof. Assume that $e_1(\lambda_1) = 0$ on Θ. Since λ_1 is not constant on Θ, then there is a point $p \in \Theta$ such that $e_2(\lambda_1) \neq 0$.
It follows from (3.1) and (3.4) that

$$e_2^2(\lambda_1) = K(\lambda_1^2 - \lambda\lambda_1 + K) \tag{3.7}$$

and

$$\lambda_1(\lambda_1^2 - K)e_2 \cdot e_2(\lambda_1) + (\lambda_1^2 + K)e_2^2(\lambda_1) + K^2(\lambda_1^2 - \lambda\lambda_1 + K) = 0. \tag{3.8}$$

Differentiating (3.7) with respect to e_2 yields

$$e_2 \cdot e_2(\lambda_1) = \frac{1}{2}K(2\lambda_1 - \lambda). \tag{3.9}$$

Substituting (3.7) and (3.9) into (3.8), we know

$$4\lambda_1^4 - 3\lambda\lambda_1^3 + 4K\lambda_1^2 - 3\lambda K\lambda_1 + 4K^2 = 0.$$

It is obvious that λ_1 is a constant function on Θ which contradicts the fact that there is a point $p \in \Theta$ such that $e_2(\lambda_1) \neq 0$. □

Proposition 3.3. *For the non-constant function λ_1 on $\Theta \subset M^2$, the following two differential equations hold*

$$\begin{aligned}&\lambda_1^2(\lambda_1^2 - K)^2 e_1 \cdot e_1 \cdot e_1(\lambda_1) - \lambda_1(\lambda_1^2 - K)(\lambda_1^2 - 13K)e_1(\lambda_1)e_1 \cdot e_1(\lambda_1)\\ &- 12K(\lambda_1^2 - 2K)e_1^3(\lambda_1) + \lambda_1^4(4\lambda_1^4 + 4K\lambda_1^2 - 3\lambda\lambda_1^3 - 3\lambda K\lambda_1\\ &+ 4K^2)e_1(\lambda_1) = 0.\end{aligned} \tag{3.10}$$

$$\begin{aligned}&2\lambda_1(\lambda_1^2 - K)(\lambda_1^2 + K)e_1(\lambda_1)e_1 \cdot e_1(\lambda_1) + 6K(\lambda_1^2 + K)e_1^3(\lambda_1)\\ &+ \lambda_1^4(4\lambda_1^4 + 4K\lambda_1^2 - 3\lambda\lambda_1^3 - 3\lambda K\lambda_1 + 4K^2)e_1(\lambda_1) = 0.\end{aligned} \tag{3.11}$$

Proof. In this part, we will consider the differential problems on a neighborhood U of p such that $e_1(\lambda_1) \neq 0$.

It follows from (3.1)–(3.4) and $\lambda_1^2 - K \neq 0$ that

$$\begin{cases} e_2^2(\lambda_1) = \dfrac{1}{\lambda_1^4}\Big(K\lambda_1(\lambda_1^2 - K)e_1 \cdot e_1(\lambda_1) - K(\lambda_1^2 - 3K)e_1^2(\lambda_1)\\ \quad + K\lambda_1^4(\lambda_1^2 - \lambda\lambda_1 + K)\Big),\\ e_2 \cdot e_1(\lambda_1) = -\dfrac{3K}{\lambda_1(\lambda_1^2 - K)}e_1(\lambda_1)e_2(\lambda_1),\\ e_1 \cdot e_2(\lambda_1) = -\dfrac{\lambda_1^2 + 2K}{\lambda_1(\lambda_1^2 - K)}e_1(\lambda_1)e_2(\lambda_1),\\ e_2 \cdot e_2(\lambda_1) = -\dfrac{1}{\lambda_1^5(\lambda_1^2 - K)}\Big(K\lambda_1(\lambda_1^2 - K)(\lambda_1^2 + K)e_1 \cdot e_1(\lambda_1)\\ \quad - K(\lambda_1^4 - K\lambda_1^2 - 3K^2)e_1^2(\lambda_1) + K\lambda_1^4(\lambda_1^2 + 2K)(\lambda_1^2 - \lambda\lambda_1 + K)\Big). \end{cases} \tag{3.12}$$

Differentiating (3.1) with respect to e_1 leads to

$$\begin{aligned}&K\lambda_1(\lambda_1^2 - K)e_1 \cdot e_1 \cdot e_1(\lambda_1) + K(\lambda_1^2 + 5K)e_1(\lambda_1)e_1 \cdot e_1(\lambda_1)\\ &- 2K\lambda_1 e_1^3(\lambda_1) - 4\lambda_1^3 e_1(\lambda_1)e_2^2(\lambda_1) - 2\lambda_1^4 e_2(\lambda_1)e_1 \cdot e_2(\lambda_1)\\ &+ K\lambda_1^3(6\lambda_1^2 - 5\lambda\lambda_1 + 4K)e_1(\lambda_1) = 0.\end{aligned} \tag{3.13}$$

Substituting the first and third equations of (3.12) into (3.13) gives

$$\begin{aligned}
&K\lambda_1(\lambda_1^2-K)e_1\cdot e_1\cdot e_1(\lambda_1)-K(\lambda_1^2-13K)e_1(\lambda_1)e_1\cdot e_1(\lambda_1)\\
&-\frac{12K^2(\lambda_1^2-2K)}{\lambda_1(\lambda_1^2-K)}e_1^3(\lambda_1)+\frac{K\lambda_1^3}{\lambda_1^2-K}(4\lambda_1^4+4K\lambda_1^2-3\lambda\lambda_1^3\\
&-3\lambda K\lambda_1+4K^2)e_1(\lambda_1)=0.
\end{aligned}$$

Especially,

$$\begin{aligned}
&\lambda_1^2(\lambda_1^2-K)^2e_1\cdot e_1\cdot e_1(\lambda_1)-\lambda_1(\lambda_1^2-K)(\lambda_1^2-13K)e_1(\lambda_1)e_1\cdot e_1(\lambda_1)\\
&-12K(\lambda_1^2-2K)e_1^3(\lambda_1)+\lambda_1^4(4\lambda_1^4+4K\lambda_1^2-3\lambda\lambda_1^3-3\lambda K\lambda_1\\
&+4K^2)e_1(\lambda_1)=0.
\end{aligned}$$

Differentiating (3.4) with respect to e_1 yields

$$\begin{aligned}
&\lambda_1^3(\lambda_1^2-K)e_1\cdot e_2\cdot e_2(\lambda_1)+\lambda_1^2(5\lambda_1^2-3K)e_1(\lambda_1)e_2\cdot e_2(\lambda_1)\\
&+2\lambda_1(2\lambda_1^2+K)e_1(\lambda_1)e_2^2(\lambda_1)+2\lambda_1^2(\lambda_1^2+K)e_2(\lambda_1)e_1\cdot e_2(\lambda_1)\\
&-2K^2e_1(\lambda_1)e_1\cdot e_1(\lambda_1)+K^2\lambda_1(4\lambda_1^2-3\lambda\lambda_1+2K)e_1(\lambda_1)=0.
\end{aligned}\tag{3.14}$$

Substituting the third equation of (3.12) into (3.14), we have

$$\begin{aligned}
&\lambda_1^3(\lambda_1^2-K)^2e_1\cdot e_2\cdot e_2(\lambda_1)+\lambda_1^2(\lambda_1^2-K)(5\lambda_1^2-3K)e_1(\lambda_1)e_2\cdot e_2(\lambda_1)\\
&+2\lambda_1(\lambda_1^4-4K\lambda_1^2-3K^2)e_1(\lambda_1)e_2^2(\lambda_1)-2K^2(\lambda_1^2-K)e_1(\lambda_1)e_1\cdot e_1(\lambda_1)\\
&+K^2\lambda_1(\lambda_1^2-K)(4\lambda_1^2-3\lambda\lambda_1+2K)e_1(\lambda_1)=0.
\end{aligned}\tag{3.15}$$

It follows from (2.4), (3.6) and the third equation of (3.12) that

$$\begin{aligned}
&e_1\cdot e_2\cdot e_2(\lambda_1)\\
=&e_2\cdot e_1\cdot e_2(\lambda_1)-\Gamma_{11}^2e_1\cdot e_2(\lambda_1)+\Gamma_{22}^1e_2\cdot e_2(\lambda_1)\\
=&e_2\cdot e_1\cdot e_2(\lambda_1)+\frac{\lambda_1^2+2K}{(\lambda_1^2-K)^2}e_1(\lambda_1)e_2^2(\lambda_1)+\frac{K}{\lambda_1(\lambda_1^2-K)}e_1(\lambda_1)e_2\cdot e_2(\lambda_1).
\end{aligned}\tag{3.16}$$

Differentiating (3.3) with respect to e_2 leads to

$$\begin{aligned}
&\lambda_1(\lambda_1^2-K)e_2\cdot e_1\cdot e_2(\lambda_1)+(3\lambda_1^2-K)e_2(\lambda_1)e_1\cdot e_2(\lambda_1)+2\lambda_1e_1(\lambda_1)e_2^2(\lambda_1)\\
&+(\lambda_1^2+2K)e_2(\lambda_1)e_2\cdot e_1(\lambda_1)+(\lambda_1^2+2K)e_1(\lambda_1)e_2\cdot e_2(\lambda_1)=0.
\end{aligned}\tag{3.17}$$

Substituting the second and third equations of (3.12) into (3.17), we know that

$$\begin{aligned}
&\lambda_1^2(\lambda_1^2-K)^2e_2\cdot e_1\cdot e_2(\lambda_1)-(\lambda_1^4+10K\lambda_1^2+4K^2)e_1(\lambda_1)e_2^2(\lambda_1)\\
&+\lambda_1(\lambda_1^2-K)(\lambda_1^2+2K)e_1(\lambda_1)e_2\cdot e_2(\lambda_1)=0.
\end{aligned}\tag{3.18}$$

Combining (3.16) with (3.18), we obtain

$$\begin{aligned}
&\lambda_1^2(\lambda_1^2-K)^2e_1\cdot e_2\cdot e_2(\lambda_1)\\
=&(2\lambda_1^4+12K\lambda_1^2+4K^2)e_1(\lambda_1)e_2^2(\lambda_1)-\lambda_1(\lambda_1^2-K)(\lambda_1^2+K)e_1(\lambda_1)e_2\cdot e_2(\lambda_1).
\end{aligned}$$

And, by (3.15), we get

$$
\begin{aligned}
&4\lambda_1^2(\lambda_1^2-K)^2e_1(\lambda_1)e_2\cdot e_2(\lambda_1)+2\lambda_1(2\lambda_1^4+2K\lambda_1^2-K^2)e_1(\lambda_1)e_2^2(\lambda_1)\\
&-2K^2(\lambda_1^2-K)e_1(\lambda_1)e_1\cdot e_1(\lambda_1)+K^2\lambda_1(\lambda_1^2-K)(4\lambda_1^2-3\lambda\lambda_1\\
&+2K)e_1(\lambda_1)=0,
\end{aligned}
\tag{3.19}
$$

Substituting the first and fourth equations of (3.12) into (3.19) gives

$$
\begin{aligned}
&2\lambda_1K^2(\lambda_1^2-K)(\lambda_1^2+K)e_1(\lambda_1)e_1\cdot e_1(\lambda_1)+6K^3(\lambda_1^2+K)e_1^3(\lambda_1)\\
&+K^2\lambda_1^4(4\lambda_1^4+4K\lambda_1^2-3\lambda\lambda_1^3-3\lambda K\lambda_1+4K^2)e_1(\lambda_1)=0.
\end{aligned}
$$

□

Proposition 3.4. *For a 2-dimensional complete λ-translator $x : M^2 \to \mathbb{R}^3$ with the non-zero constant Gauss curvature K, the mean curvature H is constant.*

Proof. Since the Gauss curvature K is a non-zero constant on M^2, we know that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ on the whole of M^2. We hereby declare that λ_1 is a constant on M^2. Then, λ_2 and the mean curvature H must be a constant. In fact, let us assume that there is an open subset $\Theta \subset M^2$ such that λ_1 is non-constant and $\lambda_1 \neq \lambda_2$. This implies that $\lambda_1^2 - K \neq 0$. For the convenience of calculation, take $e_1(\lambda_1) = \lambda_1'$, $e_1\cdot e_1(\lambda_1) = \lambda_1''$ and $e_1\cdot e_1\cdot e_1(\lambda_1) = \lambda_1'''$. Differentiating (3.11) with respect to e_1, we obtain

$$
\begin{aligned}
&2\lambda_1(\lambda_1^2-K)(\lambda_1^2+K)\lambda_1'\lambda_1'''+2\lambda_1(\lambda_1^2-K)(\lambda_1^2+K)(\lambda_1'')^2\\
&+2(5\lambda_1^4+9K\lambda_1^2+8K^2)(\lambda_1')^2\lambda_1''+12K\lambda_1(\lambda_1')^4+\lambda_1^4(4\lambda_1^4+4K\lambda_1^2\\
&-3\lambda\lambda_1^3-3\lambda K\lambda_1+4K^2)\lambda_1''+\lambda_1^3(32\lambda_1^4+24K\lambda_1^2-21\lambda\lambda_1^3-15\lambda K\lambda_1\\
&+16K^2)(\lambda_1')^2=0.
\end{aligned}
\tag{3.20}
$$

It follows from (3.10) and (3.20) that

$$
\begin{aligned}
&2\lambda_1^2(\lambda_1^2-K)^2(\lambda_1^2+K)(\lambda_1'')^2+2\lambda_1(\lambda_1^2-K)(6\lambda_1^4-3K\lambda_1^2\\
&-5K^2)(\lambda_1')^2\lambda_1''+12K(3\lambda_1^4-3K\lambda_1^2-4K^2)(\lambda_1')^4+\lambda_1^5(\lambda_1^2-K)(4\lambda_1^4\\
&+4K\lambda_1^2-3\lambda\lambda_1^3-3\lambda K\lambda_1+4K^2)\lambda_1''+3\lambda_1^4(8\lambda_1^6-8K\lambda_1^4-8K^2\lambda_1^2\\
&-5\lambda\lambda_1^5+6\lambda K\lambda_1^3+7\lambda K^2\lambda_1-8K^3)(\lambda_1')^2=0.
\end{aligned}
\tag{3.21}
$$

Making use of (3.11) and (3.21), we obtain

$$
\begin{aligned}
&4\lambda_1(\lambda_1^2-K)(3\lambda_1^4-3K\lambda_1^2-4K^2)(\lambda_1')^3\lambda_1''+12K(3\lambda_1^4-3K\lambda_1^2-4K^2)(\lambda_1')^5\\
&+3\lambda_1^4(8\lambda_1^6-8K\lambda_1^4-8K^2\lambda_1^2-5\lambda\lambda_1^5+6\lambda K\lambda_1^3+7\lambda K^2\lambda_1-8K^3)(\lambda_1')^3=0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&4\lambda_1(\lambda_1^2-K)(3\lambda_1^4-3K\lambda_1^2-4K^2)\lambda_1''+12K(3\lambda_1^4-3K\lambda_1^2-4K^2)(\lambda_1')^2\\
&+3\lambda_1^4(8\lambda_1^6-8K\lambda_1^4-8K^2\lambda_1^2-5\lambda\lambda_1^5+6\lambda K\lambda_1^3+7\lambda K^2\lambda_1-8K^3)=0
\end{aligned}
\tag{3.22}
$$

since $\lambda_1' \neq 0$.

Finally, using (3.11) and (3.22) again, it can be obtained that

$$
\lambda_1^4(\lambda_1^2-K)(3\lambda\lambda_1^5+6\lambda K\lambda_1^3-16K^2\lambda_1^2+3\lambda K^2\lambda_1-8K^3)=0.
$$

It is obvious that λ_1 is a constant function. It is a contradiction. □

The proof of Theorem 1.1. It follows from Proposition 3.4 that the mean curvature H is constant and each principal curvature is constant. Since the Gauss curvature K is a non-zero constant, it follows from (1.1) and (2.6) that

$$\lambda = H, \quad \langle \nu, e_3 \rangle = 0.$$

So, the non-zero constant vector $\nu = \nu^{\top}$ is tangent to $x(M^2)$ at each point of M^2. It is obvious that $x : M^2 \to \mathbb{R}^3$ is, locally, a plane or a cylinder. This is impossible since the Gauss curvature K is a non-zero constant. Theorem 1.1 is proved.

4. Conclusions

One concern is that $h_{ij} = A(e_i, e_j)$ might not be differentiable in a local eigen frame if some positive principal curvatures repeat. However, in this article, we mainly study the principal curvature eigenvalues of the second fundamental form in the locally open set Θ composed of non-umbilic points. let $\{e_1, e_2\}$ be the adapted moving frame around a point p in Θ. Then, for any eigenvalue λ_i ($i = 1, 2$) of multiplicity one and at the point p, it follows that principal curvatures are differentiable. By differentiating the eigenvalues of the shape operator on the set of umbilic points, we have that the mean curvature H and each principal curvature are constant.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Acknowledgments

We would like to thank the anonymous referees for many valuable suggestions. The first author was supported by 2023 Teacher Education Curriculum Reform in Henan Province (Grant No. 2023-JSJYYB-023).

Conflict of interest

The authors declare no conflict of interest.

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