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Research article

Periodic solutions for chikungunya virus dynamics in a seasonal environment with a general incidence rate

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Abstract: The chikungunya virus (CHIKV) infects macrophages and adherent cells and it can be transmitted via a direct contact with the virus or with an already infected cell. Thus, the CHIKV infection can have two routes. Furthermore, it can exhibit seasonal peak periods. Thus, in this paper, we consider a dynamical system model of the CHIKV dynamics under the conditions of a seasonal environment with a general incidence rate and two routes of infection. In the first step, we studied the autonomous system by investigating the global stability of the steady states with respect to the basic reproduction number. In the second step, we establish the existence, uniqueness, positivity and boundedness of a periodic orbit for the non-autonomous system. We show that the global dynamics are determined by using the basic reproduction number denoted by $\mathcal{R}_0$ and they are calculated using the spectral radius of an integral operator. We show the global stability of the disease-free periodic solution if $\mathcal{R}_0 < 1$ and we also show the persistence of the disease if $\mathcal{R}_0 > 1$ where the trajectories converge to a limit cycle. Finally, we display some numerical investigations supporting the theoretical findings.

Keywords: CHIKV epidemic model; seasonal environment; periodic solution; Lyapunov stability; uniform persistence; extinction; basic reproduction number
Mathematics Subject Classification: 34K13, 34K20, 34D23, 37B25, 49K40, 92D30

1. Introduction

The chikungunya virus (CHIKV) is an arbovirus (i.e., a virus transmitted by arthropods) whose vectors are female mosquitoes of the genus Aedes which are identifiable by the presence of black and white stripes. The two implicated species are Aedes aegypti and Aedes albopictus. Aedes albopictus is present in the south of France and Aedes aegypti in areas including Antilles, Guyana, French Polynesia and New Caledonia. These two mosquitoes are also implicated in the transmission of other arboviruses, including dengue, yellow fever and Zika virus. This disease has been prevalent on the

African and Asian continents for more than 50 years. Over the past 30 years, mathematical modeling has been applied to study several epidemic diseases [1–4]. Recently, the study of vector-borne diseases has gained considerable attention and mathematics have become a useful tool for such studies; also, several temporal deterministic models have been proposed for diseases like dengue, malaria, CHIKV, etc. Several mathematical models have been developed and studied to explain a variety of features influencing the transmission of CHIKV [5–12].

However, several infectious diseases including all diseases caused by the transmission of a pathogenic agent exhibit seasonal peak periods. Studying the population behaviors associated with a seasonal environment becomes a necessity for predicting the risk of transmission of such a disease . In [13], the authors discuss the periodic "SIR" epidemic model. In [14], the authors study an "SEIRS" epidemic model with periodic fluctuations. They calculated the basic reproduction number $\mathcal{R}_0$ by using the time-averaged system and proved a sufficient but unnecessary condition such that the disease could not persist. In [15], the authors consider a class of SIQRS models with periodic behavior of the contact rate; they proved the existence of periodic trajectories. The authors of [16, 17] studied an "SEIRS" epidemic model in a seasonal environment and proved some sufficient conditions for both the persistence and the extinction of the disease. In [18], the authors give the definition of the basic reproduction number for seasonal environments. The basic reproduction number $\mathcal{R}_0$ is defined in [19] for several compartmental periodic epidemic models; the authors show that $\mathcal{R}_0$ is a threshold value to prove the stability of the disease-free periodic solution. In [20], the authors propose an extension of the "SVEIR" model by taking into account the seasonal environment.

The aim of this work is to propose a new class of dynamical systems that models CHIKV dynamics under the conditions of a seasonal environment with a general incidence rate, and where the adherent cells are the main target for CHIKV. We will establish the existence, uniqueness, positivity and boundedness of a periodic solution. Therefore, we will study the global dynamics with respect to the basic reproduction number that will be calculated by using the spectral radius of an integral operator. The global stability of the disease free periodic solution will be proved for $\mathcal{R}_0 < 1$; however, the persistence of the disease will be proved for $\mathcal{R}_0 > 1$ by proving that the trajectories will converge to a limit cycle. Finally, some numerical simulations will be given to confirm the theoretical results.

2. Proposed mathematical model

CHIKV is spread as follows. Mosquitoes contract the virus by biting animals or humans infected with it. They then spread the virus by biting uninfected people. CHIKV infects macrophages and adherent cells and it can be transmitted via a direct contact with the virus or with an already infected cell. Thus, the CHIKV infection can have two routes, i.e., CHIKV-to-cell and cell-to-cell infections (see Figure 1). We adopt a general non-linear incidence rate for both routes of infection. Furthermore, CHIKV dynamics can exhibit seasonal peak periods, which is why all parameters of the models are T-periodic functions where $T > 0$ is the period. The considered epidemic model for the CHIKV dynamics in a seasonal environment with a general form of the incidence rate is given by the following four-dimensional ordinary differential equation system which generalizes the models given in [21, 22].

$$\begin{cases} \dot{S}(t) &= m(t)S_{in}(t) - m(t)S(t) - \beta_1(t)P(t)f(S(t)) - \beta_2(t)I(t)f(S(t)), \\ \dot{I}(t) &= \beta_1(t)P(t)f(S(t)) + \beta_2(t)I(t)f(S(t)) - m(t)I(t), \\ \dot{P}(t) &= \delta(t)I(t) - m_p(t)P(t) - r(t)A(t)P(t), \\ \dot{A}(t) &= m_a(t)A_{in}(t) + k(t)r(t)A(t)P(t) - m_a(t)A(t), \end{cases} \tag{2.1}$$

with the positive initial condition $(S^0, I^0, P^0, A^0) \in \mathbb{R}^4_+$. $S(t)$, $I(t)$, $P(t)$, and $A(t)$ describe susceptible cells, infected cells, CHIKV and antibodies, respectively. The uninfected cells are generated by a periodic rate $m(t)S_{in}(t)$, die at a periodic rate $m(t)s(t)$ and become infected by the virus and infected cells at a periodic rate $\beta_1(t)P(t)f(S(t)) + \beta_2(t)I(t)f(S(t))$, where $\beta_1(t)$ and $\beta_2(t)$ are the periodic incidence rates. The periodic variables $m(t)$, $m_p(t)$, and $m_a(t)$ represent, respectively, the periodic mortality rates for the infected cells, CHIKV and antibodies. $\delta(t)$ is the rate of periodic production of CHIKV from infected cells. Antibodies attack the CHIKV at a periodic rate of $r(t)A(t)P(t)$. Once an antigen is encountered, the antibodies expand at a periodic rate of $m_a(t)A_{in}(t)$ and proliferate at a periodic rate of $k(t)r(t)A(t)P(t)$. All of the parameters of the model are positive periodic functions. We give in Table 1 more epidemiological significance of the model parameters.

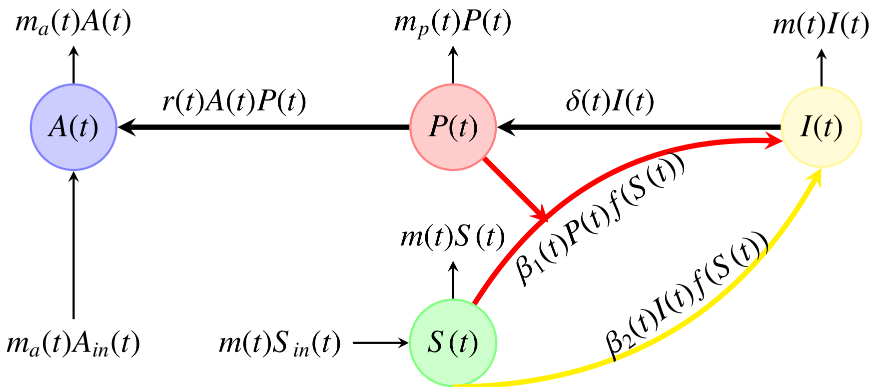

Figure 1. Diagram describing an epidemic compartmental model that takes into consideration a seasonal environment for the CHIKV dynamics (inspired from [23, Figure 2]). Compartments S, I, P and A are described by circles; transition rates between compartments are described by arrows and labels.

Throughout the rest of the paper, we will use the following assumptions:

Assumption 1. *(1) f is an increasing, non-negative $C^1(\mathbb{R}_+)$ concave function such that $f(0) = 0$.*
(2) $S_{in}(t), A_{in}(t), m(t), m_p(t), m_a(t), \delta(t), k(t), \beta_1(t), \beta_2(t)$ and $r(t)$ are non-negative continuous bounded T-periodic functions.
(3) $m_a(t) \leq m_p(t), \ \forall t \geq 0$.

Assumption 1 expresses that the CHIKV-to-cell and cell-to-cell incidence rates increase with increasing number of susceptible cells. Assumption 1 affirms also that no CHIKV-to-cell or cell-to-cell infection can take place in the absence of susceptible cells. All of the model parameters are

T-periodic functions influenced by the seasonal environment. We assume also that the instantaneous CHIKV mortality rate is greater than the instantaneous antibody loss rate.

Lemma 1. *The incidence rate f satisfies $f'(x)x \leq f(x) \leq f'(0)x,\ \forall x > 0$.*

Proof. Let $x, x_1 \in \mathbb{R}_+$, and the function $g_1(x) = f(x) - xf'(x)$. Since $f'(x) \geq 0$ (f is an increasing function) and $f''(x) \leq 0$ (f is concave), $g_1'(x) = -xf''(x) \geq 0$ and $g_1(x) \geq g_1(0) = 0$. Therefore, $f(x) \geq xf'(x)$. Similarly, let $g_2(x) = f(x) - xf'(0)$; then, $g_2'(x) = f'(x) - f'(0) \leq 0$ when f is a concave function. Thus $g_2(x) \leq g_2(0) = 0$ and $f(x) \leq xf'(0)$. □

Table 1. Significance of the variables and parameters of the proposed model (2.1).

Parameter	Description
$m(t)S_{in}(t)$	Instantaneous uninfected cell recruitment rate
$m_a(t)A_{in}(t)$	Instantaneous antibody expansion rate
$m(t)$	Instantaneous cell mortality rate
$m_p(t)$	Instantaneous CHIKV mortality rate
$m_a(t)$	Instantaneous antibody loss rate
$\delta(t)$	Rate of instantaneous production of the virus from infected cells
$r(t)$	Rate of instantaneous attack of the virus by the antibodies
$k(t)$	Instantaneous proliferation rate for antibodies
$\beta_1(t), \beta_2(t)$	Instantaneous incidence coefficients

3. Mathematical analysis for the autonomous system

As a first step, we consider the autonomous form of the model (2.1) where its right-hand side is independent of t i.e., all parameters are constants.

$$\begin{cases} \dot{S} &= mS_{in} - mS - \beta_1 Pf(S) - \beta_2 If(S), \\ \dot{I} &= \beta_1 Pf(S) + \beta_2 If(S) - mI, \\ \dot{P} &= \delta I - m_p P - rAP, \\ \dot{A} &= m_a A_{in} + krAP - m_a A. \end{cases} \tag{3.1}$$

In this section, we will use the following additional assumption on the incidence rate.

Assumption 2. $f(S_{in}) < \dfrac{m}{\beta_2}$.

The given condition of Assumption 2 is a mathematical artifice and has no biological meaning; here, it is only used to prove the existence and uniqueness of the positive steady state.

3.1. Basic properties

In this subsection, we give some basic properties for the model (3.1) such as the existence, positivity and boundedness of the trajectories of (3.1), as well as the existence of an invariance set of all solutions of system (3.1).

Lemma 2.

$$\Omega = \left\{(S, I, P, A) \in \mathbb{R}_+^4 : \ S + I = S_{in}, kP + A \leq A_{in} + \frac{\delta k S_{in}}{m_a}\right\}$$

is a positively invariant bounded set for system (3.1).

Proof. Note that if $S = 0$ then $\dot{S} = mS_{in} > 0$; if $I = 0$ then $\dot{I} = \beta_1 P f(S) \geq 0$; if $P = 0$ then $\dot{P} = \delta I \geq 0$; if $A = 0$ then $\dot{S} = m_a A_{in} > 0$. Consider that $F_1(t) = S(t) + I(t) - S_{in}$ and $F_2(t) = kP(t) + A(t) - \frac{\delta k S_{in}}{m_a} - A_{in}$. Then, one has that $\dot{F}_1(t) = mS_{in} - mS(t) - mI(t) = -mF_1(t)$. Hence $F_1(t) = F_1(0)e^{mt} = (S(0) + I(0) - S_{in})e^{-mt}$. Then, $F_1(t) = 0$ if $F_1(0) = 0$. Furthermore, one has

$$\dot{F}_2(t) = \delta k I(t) - m_p k P(t) + m_a A_{in} - m_a A(t) \leq \delta k S_{in} + m_a A_{in} - m_a\Big(kP(t) + A(t)\Big) = -m_a F_2(t).$$

Then $F_2(t) \leq F_2(0)e^{-m_a t}$. Hence $F_2(t) \leq 0$ if $F_2(0) \leq 0$. Thus, Ω is invariant for the model (2.1) since all variables are non-negative. □

3.2. Basic reproduction number and steady states

When studying a disease, an important health question is whether the disease is spreading in the population. The response can be obtained by calculating the average number of people an infectious person infects while they are contagious. This average is known as the basic reproduction number whose calculation is complex. It can be determined by using the values of the model parameters; thus, we can determine if the disease spreads. The basic reproduction number can be calculated by using several methods. For example, in the case of a single infected compartment, the basic reproduction number is simply the product of the infection rate and its mean duration [24]. It can be calculated by using graph, or network, theory [25, 26]. If there are several compartments representing infectious individuals as in our case here, the next-generation matrix method introduced by Diekmann et al. [24] and developed later in [27, 28], divides the population into two compartments whose first compartment represents the infected individuals. The goal is therefore to see the rate of change in the population established in each of these compartments. The matrix approach calculating the basic reproduction number explains the relationship between compartmental models and population matrix models. For the dynamics given by (3.1), we shall calculate the basic reproduction number by using the next generation matrix method. Therefore, we will calculate the steady states by proving their existence and uniqueness.

$$F = \begin{pmatrix} \beta_2 f(S_{in}) & \beta_1 f(S_{in}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} m & 0 & 0 \\ -\delta & rA_{in} + m_p & 0 \\ 0 & -krA_{in} & m_a \end{pmatrix}.$$

The determinant of V is given by $\det(V) = m_a m(rA_{in} + m_p) > 0$; thus,

$$V^{-1} = \frac{1}{\det(V)} \begin{pmatrix} m_a(rA_{in} + m_p) & 0 & 0 \\ m_a\delta & m_a m & 0 \\ \delta k r A_{in} & mkrA_{in} & m(rA_{in} + m_p) \end{pmatrix}$$

and the next-generation matrix is given by

$$FV^{-1} = \frac{1}{m_a m(rA_{in}+m_p)}\begin{pmatrix} m_a(rA_{in}+m_p)\beta_2 f(S_{in}) + m_a\delta\beta_1 f(S_{in}) & m_a m\beta_1 f(S_{in}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the basic reproduction number (i.e., spectral radius of FV^{-1}) is given by:

$$\begin{aligned} \mathcal{R}_0 &= \frac{m_a(rA_{in}+m_p)\beta_2 f(S_{in}) + m_a\delta\beta_1 f(S_{in})}{m_a m(rA_{in}+m_p)} \\ &= \frac{\beta_2 f(S_{in})}{m} + \frac{\delta\beta_1 f(S_{in})}{m(rA_{in}+m_p)}. \end{aligned} \tag{3.2}$$

Lemma 3. • *If $\mathcal{R}_0 \le 1$, then (3.1) admits only $E_0 = (S_{in}, 0, 0, A_{in})$ as an equilibrium point.*

• *If $\mathcal{R}_0 > 1$, then (3.1) admits two equilibrium points, i.e., E_0 and an endemic equilibrium $E^* = (S_*, I_*, P_*, A_*)$.*

Proof. Let $E = (S, I, P, A)$ be an equilibrium point satisfying the following

$$\begin{aligned} 0 &= mS_{in} - mS - \beta_1 P f(S) - \beta_2 I f(S), \\ 0 &= \beta_1 P f(S) + \beta_2 I f(S) - mI, \\ 0 &= \delta I - m_p P - rAP, \\ 0 &= m_a A_{in} + krAP - m_a A. \end{aligned} \tag{3.3}$$

From Eq (3.3) we obtain the CHIKV-free steady state $E_0 = (S_{in}, 0, 0, A_{in})$. Furthermore, we have

$$\begin{cases} A = \dfrac{m_a A_{in}}{m_a - krP}, \\ I = \dfrac{m_p P + rAP}{\delta} = \dfrac{m_p P}{\delta} + \dfrac{r m_a A_{in} P}{\delta(m_a - krP)}, \\ S = \dfrac{mS_{in} - mI}{m} = S_{in} - \dfrac{m m_p P}{\delta m} - \dfrac{r m_a A_{in} m P}{\delta m(m_a - krP)}, \\ \beta_1 P f(S) = (m - \beta_2 f(S))I. \end{cases} \tag{3.4}$$

We define the function

$$\begin{aligned} g(P) &= \beta_1 f(S) + (\beta_2 f(S) - m)\frac{I}{P} \\ &= \beta_1 f\Big(S_{in} - \frac{m m_p P}{\delta m} - \frac{r m_a A_{in} m P}{\delta m(m_a - krP)}\Big) \\ &\quad + \Big(\beta_2 f\Big(S_{in} - \frac{m m_p P}{\delta m} - \frac{r m_a A_{in} m P}{\delta m(m_a - krP)}\Big) - m\Big)\Big(\frac{m_p}{\delta} + \frac{r m_a A_{in}}{\delta(m_a - krP)}\Big). \end{aligned} \tag{3.5}$$

Then, we obtain

$$\begin{aligned} g(0) &= \beta_1 f(S_{in}) + (\beta_2 f(S_{in}) - m)\frac{m_a m_p + r m_a A_{in}}{\delta m_a} \\ &= m\frac{m_a m_p + r m_a A_{in}}{\delta m_a}\Big(\frac{\beta_2 f(S_{in})}{m} + \frac{\delta m_a \beta_1 f(S_{in})}{m(m_a m_p + r m_a A_{in})} - 1\Big) \ lll \\ &= m\frac{m_a m_p + r m_a A_{in}}{\delta m_a}(\mathcal{R}_0 - 1) > 0 \quad \text{if} \quad \mathcal{R}_0 > 1. \end{aligned} \tag{3.6}$$

Now, we have

$$\lim_{P\to(m_a/kr)^-}\Big(S_{in}-\frac{m_pmP}{\delta m}-\frac{rm_aA_{in}mP}{\delta m(m_a-krP)}\Big)=-\infty$$

then,

$$\lim_{P\to(m_a/kr)^-}\beta_1 f\Big(S_{in}-\frac{m_pmP}{\delta m}-\frac{rm_aA_{in}mP}{\delta m(m_a-krP)}\Big)<0$$

and

$$\lim_{P\to(m_a/kr)^-}\beta_2 f\Big(S_{in}-\frac{m_pmP}{\delta m}-\frac{rm_aA_{in}mP}{\delta m(m_a-krP)}\Big)<0.$$

One deduces, therefore, that

$$\lim_{P\to(m_a/kr)^-} g(P)<0. \tag{3.7}$$

The derivative of the function g is given by

$$\begin{aligned}
g'(P)=&-\beta_1\Big(\frac{m_pm}{m\delta}+\frac{mrm_aA_{in}}{m\delta}\frac{m_a}{(m_a-krp)^2}\Big)f'\Big(S_{in}-\frac{m_pmP}{\delta m}-\frac{rm_aA_{in}mP}{\delta m(m_a-krP)}\Big)\\
&-\beta_2\Big(\frac{m_p}{\delta}+\frac{rm_aA_{in}}{\delta(m_a-krP)}\Big)\Big(\frac{m_pm}{m\delta}+\frac{mrm_aA_{in}}{m\delta}\frac{m_a}{(m_a-krP)^2}\Big)\\
&\times f'\Big(S_{in}-\frac{m_pmP}{\delta m}-\frac{rm_aA_{in}mP}{\delta m(m_a-krP)}\Big)\\
&+\frac{m_aA_{in}kr^2}{\delta(m_a-krP)^2}\Big(\beta_2 f\Big(S_{in}-\frac{m_pmP}{\delta m}-\frac{rm_aA_{in}mP}{\delta m(m_a-krP)}\Big)-m\Big)\\
&\leq-\beta_1\Big(\frac{m_pm}{m\delta}+\frac{mrm_aA_{in}}{m\delta}\frac{m_a}{(m_a-krp)^2}\Big)f'(S)\\
&-\beta_2\Big(\frac{m_p}{\delta}+\frac{rm_aA_{in}}{\delta(m_a-krP)}\Big)\Big(\frac{m_pm}{m\delta}+\frac{mrm_aA_{in}}{m\delta}\frac{m_a}{(m_a-krP)^2}\Big)f'(S)\\
&+\frac{m_aA_{in}kr^2}{\delta(m_a-krP)^2}\Big(\beta_2 f(S_{in})-m\Big).
\end{aligned} \tag{3.8}$$

By Assumption 1, we have that $g'(P)\leq 0\ \forall\ P\in(0,\frac{m_a}{kr})$. Then, the function $g(P)$ admits a unique root $P_*\in(0,\frac{m_a}{kr})$. Thus, we obtain

$$A_*=\frac{m_aA_{in}}{m_a-krP_*}, \tag{3.9}$$

$$I_*=\frac{m_pP_*+\dfrac{rP_*m_aA_{in}}{m_a-krP_*}}{\delta}=\frac{m_pm_aP_*-krm_pP_*{}^2+rP_*m_aA_{in}}{\delta(m_a-krP_*)}, \tag{3.10}$$

$$S_*=S_{in}-\frac{m}{m}\frac{m_pm_aP_*-krm_pP_*{}^2+rP_*m_aA_{in}}{\delta(m_a-krP_*)}\leq S_{in}. \tag{3.11}$$

Thus, the infected equilibrium $E^*=(S_*,I_*,P_*,A_*)$ exists if $\mathcal{R}_0>1$.

From the equilibrium point conditions of E^*, we have that $mS_{in} = mS_* + \beta_1 P_* f(S_*) + \beta_2 I_* f(S_*) \rightarrow mS_* + mI_* = mS_{in}$; then, $S_* + I_* = S_{in}$. Furthermore, we have that $m_p k P_* = \delta k I_* + m_a A_{in} - m_a A_*$; then. $m_a k P_* + m_a A_* \leq m_p k P_* + m_a A_* = \delta k I_* + m_a A_{in} < \delta k S_{in} + m_a A_{in}$, which means that $kP_* + A_* < A_{in} + \dfrac{\delta k S_{in}}{m_a}$. Thus, $E^* \in \overset{\circ}{\Omega}$. □

3.3. Local stability

In this subsection, we aim to study the local stability of the steady states of system (3.1) by using the linearization method with the Jacobian matrix.

Theorem 1. *If $\mathcal{R}_0 < 1$, then E_0 is locally asymptotically stable, and if $\mathcal{R}_0 > 1$, it is unstable.*

Proof. The Jacobian matrix at point E_0 is given by:

$$J_0 = \begin{pmatrix} -m & -\beta_2 f(S_{in}) & -\beta_1 f(S_{in}) & 0 \\ 0 & \beta_2 f(S_{in}) - m & \beta_1 f(S_{in}) & 0 \\ 0 & \delta & -(m_p + rA_{in}) & 0 \\ 0 & 0 & krA_{in} & -m_a \end{pmatrix}.$$

J_0 admits four eigenvalues; $\lambda_1 = -m < 0$ and $\lambda_2 = -m_a < 0$. λ_3 and λ_4 are eigenvalues of the sub-matrix

$$S_{j_0} := \begin{pmatrix} \beta_2 f(S_{in}) - m & \beta_1 f(S_{in}) \\ \delta & -(m_p + rA_{in}) \end{pmatrix}.$$

The trace of S_{j_0} is given by

$$\begin{aligned} \mathrm{Tr}(S_{j_0}) &= \beta_2 f(S_{in}) - m - (m_p + rA_{in}) \\ &\leq -(m_p + rA_{in}) - m\Big(1 - \frac{\beta_2 f(S_{in})}{m} - \frac{\delta\beta_1 f(S_{in})}{m(m_p + rA_{in})}\Big) \\ &\leq -(m_p + rA_{in}) - m\Big(1 - \mathcal{R}_0\Big) \end{aligned}$$

and the determinant of S_{j_0} is given by

$$\begin{aligned} \mathrm{Det}(S_{j_0}) &= -(m_p + rA_{in})\Big(\beta_2 f(S_{in}) - m\Big) - \delta\beta_1 f(S_{in}) \\ &= -m(m_p + rA_{in})\Big(\frac{\beta_2 f(S_{in})}{m} - 1 + \frac{\delta\beta_1 f(S_{in})}{m(m_p + rA_{in})}\Big) \\ &= -m(m_p + rA_{in})\Big(\mathcal{R}_0 - 1\Big) \\ &= m(m_p + rA_{in})\Big(1 - \mathcal{R}_0\Big). \end{aligned}$$

Then, E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$, and it is unstable if $\mathcal{R}_0 > 1$. □

Theorem 2. *If $\mathcal{R}_0 > 1$, then E^* is locally asymptotically stable.*

Proof. The Jacobian matrix at a point $E^* = (S_*, I_*, P_*, A_*)$ is given by:

$$J^* = \begin{pmatrix} -m - \beta_1 P_* f'(S_*) - \beta_2 I_* f'(S_*) & -\beta_2 f(S_*) & -\beta_1 f(S_*) & 0 \\ \beta_1 P_* f'(S_*) + \beta_2 I_* f'(S_*) & \beta_2 f(S_*) - m & \beta_1 f(S_*) & 0 \\ 0 & \delta & -(m_p + rA_*) & -rP_* \\ 0 & 0 & krA_* & krP_* - m_a \end{pmatrix}.$$

The characteristic polynomial is then given by:

$$\begin{aligned}
P(X) &= \begin{vmatrix} -X-m-\beta_1 P_* f'(S_*)-\beta_2 I_* f'(S_*) & -\beta_2 f(S_*) & -\beta_1 f(S_*) & 0 \\ \beta_1 P_* f'(S_*)+\beta_2 I_* f'(S_*) & -X+\beta_2 f(S_*)-m & \beta_1 f(S_*) & 0 \\ 0 & \delta & -X-(m_p+rA_*) & -rP_* \\ 0 & 0 & krA_* & -X+krP_*-m_a \end{vmatrix} \\
&= \begin{vmatrix} -(X+m) & -(X+m) & 0 & 0 \\ \beta_1 P_* f'(S_*)+\beta_2 I_* f'(S_*) & -X+\beta_2 f(S_*)-m & \beta_1 f(S_*) & 0 \\ 0 & \delta & -X-(m_p+rA_*) & -rP_* \\ 0 & 0 & krA_* & -X+krP_*-m_a \end{vmatrix} \\
&= -(X+m)\begin{vmatrix} -X+\beta_2 f(S_*)-m & \beta_1 f(S_*) & 0 \\ \delta & -X-(m_p+rA_*) & -rP_* \\ 0 & krA_* & -X+krP_*-m_a \end{vmatrix} \\
&\quad +(X+m)\begin{vmatrix} \beta_1 P_* f'(S_*)+\beta_2 I_* f'(S_*) & \beta_1 f(S_*) & 0 \\ 0 & -X-(m_p+rA_*) & -rP_* \\ 0 & krA_* & -X+krP_*-m_a \end{vmatrix} \\
&= (X+m)\Big[(X+m-\beta_2 f(S_*))\Big((X+m_p+rA_*)(X+m_a-krP_*)+kr^2P_*A_*\Big) \\
&\quad -\delta\beta_1 f(S_*)(X+m_a-krP_*)\Big] + (\beta_1 P_* f'(S_*)+\beta_2 I_* f'(S_*))(X+m) \\
&\quad \Big((X+m_p+rA_*)(X+m_a-krP_*)+kr^2P_*A_*\Big).
\end{aligned}$$

The characteristic polynomial $P(X) = 0$ if, and only if

$$\begin{aligned}
&\Big[(X+m)(X+m-\beta_2 f(S_*)) + (\beta_1 P_* f'(S_*)+\beta_2 I_* f'(S_*))(X+m)\Big] \\
&\Big((X+m_p+rA_*)(X+m_a-krP_*)+kr^2P_*A_*\Big) \\
= \;&\delta\beta_1 f(S_*)(X+m)(X+m_a-krP_*)
\end{aligned}$$

or if

$$\begin{aligned}
&\Big[(X+m)(X+m-\beta_2 f(S_*)) + (\beta_1 P_* f'(S_*)+\beta_2 I_* f'(S_*))(X+m)\Big] \\
=&\frac{\delta\beta_1 f(S_*)(X+m)(X+m_a-krP_*)}{\Big((X+m_p+rA_*)(X+m_a-krP_*)+kr^2P_*A_*\Big)}.
\end{aligned}$$

Suppose that X is an eigenvalue with $Re(X) \geq 0$; then, since $(m-\beta_2 f(S_*)) = \dfrac{\beta_1 P_* f(S_*)}{I_*}$ and $\dfrac{P_*}{I_*} = \dfrac{\delta}{(m_p+rA_*)}$, the left-hand side satisfies the following condition:

$$\begin{aligned}
&\Big|(X+m)(X+m-\beta_2 f(S_*)) + (\beta_1 P_* f'(S_*)+\beta_2 I_* f'(S_*))(X+m)\Big| \\
>&(m-\beta_2 f(S_*))|X+m| = \frac{\beta_1 P_* f(S_*)}{I_*}|X+m| = \frac{\delta\beta_1 f(S_*)}{m_p+rA_*}|X+m| \qquad (3.12)
\end{aligned}$$

and the right-hand side satisfies the following condition:

$$\begin{aligned}
&\Big|\frac{\delta\beta_1 f(S_*)(X+m)(X+m_a-krP_*)}{(X+m_p+rA_*)(X+m_a-krP_*)+kr^2P_*A_*}\Big| \\
< &\Big|\frac{\delta\beta_1 f(S_*)(X+m)(X+m_a-krP_*)}{(X+m_p+rA_*)(X+m_a-krP_*)}\Big| \\
= &\ \delta\beta_1 f(S_*)\Big|\frac{(X+m)}{(X+m_p+rA_*)}\Big| \\
\leq &\ \frac{\delta\beta_1 f(S_*)}{m_p+rA_*}|X+m|.
\end{aligned} \tag{3.13}$$

This is impossible; then, $Re(X) < 0$ and E^* is locally asymptotically stable. □

3.4. Global stability

In this subsection, we aim to study the global stability of the steady states of system (3.1) by using the Lyapunov theory. Let us define the function $G(z) = z - 1 - \ln z$ that will be used throughout this subsection.

Theorem 3. *E_0 is globally asymptotically stable once $\mathcal{R}_0 \leq 1$.*

Proof. Consider the following Lyapunov function $U_0(S, I, P, A)$:

$$U_0(S,I,P,A) = S - S_{in} - \int_{S_{in}}^{S} \frac{f(S_{in})}{f(v)}dv + I + \frac{\beta_1 f(S_{in})}{m_p + rA_{in}}\Big(P + \frac{A_{in}}{k}G\Big(\frac{A}{A_{in}}\Big)\Big).$$

Note that $U_0(S, I, P, A) > 0$ for all $S, I, P, A > 0$ and $U_0(S_{in}, 0, 0, A_{in}) = 0$. Furthermore, we have

$$\begin{aligned}
\dot{U}_0 = &\ (1 - \frac{f(S_{in})}{f(S)})(mS_{in} - mS - \beta_1 Pf(S) - \beta_2 If(S)) + \beta_1 Pf(S) + \beta_2 If(S) - mI \\
&+ \frac{\beta_1 f(S_{in})}{m_p + rA_{in}}\Big(\delta I - m_pP - rAP + \frac{1}{k}(1 - \frac{A_{in}}{A})(m_aA_{in} + krAP - m_aA)\Big) \\
= &\ (1 - \frac{f(S_{in})}{f(S)})(mS_{in} - mS) + \beta_1 Pf(S_{in}) + \beta_2 If(S_{in}) - mI \\
&+ \frac{\beta_1 f(S_{in})}{m_p + rA_{in}}\Big(\delta I + \frac{1}{k}(1 - \frac{A_{in}}{A})(m_aA_{in} - m_aA) - m_pP - rAP + r(1 - \frac{A_{in}}{A})AP\Big) \\
\leq &\ m(1 - \frac{f(S_{in})}{f(S)})(S_{in} - S) + \beta_1 Pf(S_{in}) + \beta_2 If(S_{in}) - mI \\
&+ \frac{\beta_1 f(S_{in})}{m_p + rA_{in}}\Big(\delta I + \frac{1}{k}(1 - \frac{A_{in}}{A})(m_aA_{in} - mx) - P(m_p + rA_{in})\Big) \\
\leq &\ m(1 - \frac{f(S_{in})}{f(S)})(S_{in} - S) + \beta_2 If(S_{in}) - mI \\
&+ \frac{\beta_1 f(S_{in})}{m_p + rA_{in}}\Big(\delta I + \frac{1}{k}(1 - \frac{A_{in}}{A})(m_aA_{in} - m_aA)\Big) \\
\leq &\ -m\frac{(f(S) - f(S_{in}))}{f(S)}(S - S_{in}) + m\Big(\frac{\beta_2 f(S_{in})}{m} \\
&+ \frac{\delta\beta_1 f(S_{in})}{m(m_p + rA_{in})} - 1\Big)I - \frac{rm_a\beta_1 f(S_{in})}{kr(m_p + rA_{in})}\frac{(A - A_{in})^2}{A} \\
\leq &\ -m\frac{(f(S) - f(S_{in}))}{f(S)}(S - S_{in}) - \frac{rm_a\beta_1 f(S_{in})}{kr(m_p + rA_{in})}\frac{(A - A_{in})^2}{A} + m(\mathcal{R}_0 - 1)I.
\end{aligned}$$

If $\mathcal{R}_0 \le 1$, then $\dot{U}_0 \le 0$ for all $S, I, P, A > 0$. Let $W_0 = \{(S, I, P, A) : \dot{U}_0 = 0\} = \{E_0\}$. By LaSalle's invariance principle [29], E_0 is globally asymptotically stable once $\mathcal{R}_0 \le 1$. □

Theorem 4. *For system (3.1), if $\mathcal{R}_0 > 1$, then E^* is globally asymptotically stable.*

Proof. Let a function $U_1(S, I, P, A)$ be defined as:

$$U_1(S,I,P,A) = S - S_* - \int_{S_*}^{S} \frac{f(S_*)}{f(v)} dv + I_* G(\frac{I}{I_*}) + \frac{\beta_1 P_* f(S_*)}{\delta I_*} P_* G(\frac{P}{P_*}) + \frac{rP_*\beta_1 f(S_*)}{kr\delta I_*} A_* G(\frac{A}{A_*}).$$

Clearly, $U_1(S, I, P, A) > 0$ for all non-negative variables $S, I, P, A > 0$; also, $U_1(S_*, I_*, P_*, A_*) = 0$. Furthermore, we have

$$\begin{aligned}
\dot{U}_1 =&(1 - \frac{f(S_*)}{f(S)})(mS_{in} - mS - \beta_1 Pf(S) - \beta_2 If(S)) + (1 - \frac{I_*}{I})(\beta_1 Pf(S) + \beta_2 If(S) - mI) \\
&+ \frac{\beta_1 P_* f(S_*)}{\delta I_*}(1 - \frac{P_*}{P})(\delta I - m_p P - rAP) + \frac{r\beta_1 P_* f(S_*)}{kr\delta I_*}(1 - \frac{A_*}{A})(m_a A_{in} + krAP - mx) \\
=&(1 - \frac{f(S_*)}{f(s)})(mS_{in} - ms) - \beta_1 Pf(S) - \beta_2 If(S) + \beta_1 Pf(S_*) + \beta_2 If(S_*) + \beta_1 Pf(S) \\
&+ \beta_2 If(S) - mI - \beta_1 Pf(S)\frac{I_*}{I} - \beta_2 I_* f(S) + mI_* + \beta_1 P_* f(S_*)\frac{I}{I_*} - \beta_1 P_* f(S_*)\frac{P_* I}{PI_*} \\
&- \beta_1 P_* f(S_*)\frac{m_p P}{\delta I_*} + \beta_1 P_* f(S_*)\frac{m_p P_*}{\delta I_*} - \beta_1 P_* f(S_*)\frac{rAP}{\delta I_*} + \beta_1 P_* f(S_*)\frac{rAP_*}{\delta I_*} \\
&+ \beta_1 P_* f(S_*)\frac{rAP}{\delta I_*} - \beta_1 P_* f(S_*)\frac{rA_* P}{\delta I_*} + \frac{r\beta_1 P_* f(S_*)}{kr\delta I_*}(1 - \frac{A_*}{A})(m_a A_{in} - m_a A) \\
=&(1 - \frac{f(S_*)}{f(S)})(mS_{in} - mS) + \beta_1 Pf(S_*) + \beta_2 If(S_*) - mI - \beta_1 Pf(S)\frac{I_*}{I} - \beta_2 I_* f(S) + mI_* \\
&+ \beta_1 P_* f(S_*)\frac{I}{I_*} - \beta_1 P_* f(S_*)\frac{P_* I}{PI_*} - \beta_1 P_* f(S_*)\frac{m_p P}{\delta I_*} + \beta_1 P_* f(S_*)\frac{m_p P_*}{\delta I_*} \\
&+ \beta_1 P_* f(S_*)\frac{rAP_*}{\delta I_*} - \beta_1 P_* f(S_*)\frac{rA_* P}{\delta I_*} + \frac{r\beta_1 P_* f(S_*)}{kr\delta I_*}(1 - \frac{A_*}{A})(m_a A_{in} - m_a A).
\end{aligned}$$

Since E^*: $mS_{in} = mS_* + \beta_1 P_* f(S_*) + \beta_2 I_* f(S_*)$, $mI_* = \beta_1 P_* f(S_*) + \beta_2 I_* f(S_*)$, $m_p P_* + rA_* P_* = \delta I_*$ and $m_a A_{in} + krA_* P_* = m_a A_*$, we get

$$\begin{aligned}
\dot{U}_1 =& - m\frac{(S - S_*)(f(S) - f(S_*))}{f(S)} + \Big(1 - \frac{f(S_*)}{f(S)}\Big)\Big(\beta_1 P_* f(S_*) + \beta_2 I_* f(S_*)\Big) + \beta_1 Pf(S_*) \\
&+ \beta_2 If(S_*) - \beta_1 P_* f(S_*)\frac{I}{I_*} - \beta_2 If(S_*) - \beta_1 Pf(S)\frac{I_*}{I} - \beta_2 I_* f(S) + \beta_1 P_* f(S_*) \\
&+ \beta_2 I_* f(S_*) + \beta_1 P_* f(S_*)\frac{I}{I_*} - \beta_1 P_* f(S_*)\frac{P_* I}{PI_*} - \beta_1 P_* f(S_*)\frac{p(\delta I_* - rA_* P_*)}{\delta P_* I_*} \\
&+ \beta_1 P_* f(S_*)\frac{(\delta I_* - rA_* P_*)}{\delta I_*} + \beta_1 P_* f(S_*)\frac{rAP_*}{\delta I_*} - \beta_1 P_* f(S_*)\frac{rA_* P}{\delta I_*} \\
&+ \frac{r\beta_1 P_* f(S_*)}{kr\delta I_*}(1 - \frac{A_*}{A})(m_a A_* - krA_* P_* - m_a A) \\
=& - m\frac{(S - S_*)(f(S) - f(S_*))}{f(S)} + \Big(1 - \frac{f(S_*)}{f(S)}\Big)\Big(\beta_1 P_* f(S_*) + \beta_2 I_* f(S_*)\Big) + \beta_1 pf(S_*)
\end{aligned}$$

$$
\begin{aligned}
&-\beta_1 P_* f(S_*)\frac{I}{I_*} - \beta_1 P f(S)\frac{I_*}{I} - \beta_2 I_* f(S) + \beta_1 P_* f(S_*) + \beta_2 I_* f(S_*) + \beta_1 P_* f(S_*)\frac{I}{I_*} \\
&-\beta_1 P_* f(S_*)\frac{P_* I}{P I_*} - \beta_1 P f(S_*) + \beta_1 P_* f(S_*)\frac{rPA_*}{\delta I_*} + \beta_1 P_* f(S_*) - \beta_1 P_* f(S_*)\frac{rA_* P_*}{\delta I_*} \\
&+\beta_1 P_* f(S_*)\frac{rAP_*}{\delta I_*} - \beta_1 P_* f(S_*)\frac{rA_* P}{\delta I_*} - m_a\frac{r\beta_1 P_* f(S_*)}{kr\delta I_*}\frac{(A-A_*)^2}{A} - \beta_1 P_* f(S_*)\frac{rA_* P_*}{\delta I_*} \\
&+\beta_1 P_* f(S_*)\frac{r{A_*}^2 P_*}{\delta A I_*} \\
=&-m\frac{(S-S_*)(f(S)-f(S_*))}{f(S)} + \Big(1-\frac{f(S_*)}{f(S)}\Big)\Big(\beta_1 P_* f(S_*) + \beta_2 I_* f(S_*)\Big) - \beta_1 P f(S)\frac{I_*}{I} \\
&-\beta_2 I_* f(S) + \beta_1 P_* f(S_*) + \beta_2 I_* f(S_*) - \beta_1 P_* f(S_*)\frac{P_* I}{P I_*} + \beta_1 P_* f(S_*) \\
&-\beta_1 P_* f(S_*)\frac{rA_* P_*}{\delta I_*} + \beta_1 P_* f(S_*)\frac{rAP_*}{\delta I_*} - m_a\frac{r\beta_1 P_* f(S_*)}{kr\delta I_*}\frac{(A-A_*)^2}{A} \\
&-\beta_1 P_* f(S_*)\frac{rA_* P_*}{\delta I_*} + \beta_1 P_* f(S_*)\frac{r{A_*}^2 P_*}{\delta A I_*} \\
=&-m\frac{(S-S_*)(f(S)-f(S_*))}{f(S)} + \beta_1 P_* f(S_*)\Big(3-\frac{f(S_*)}{f(S)}-\frac{PI_* f(S)}{P_* I f(S_*)}-\frac{P_* I}{P I_*}\Big) \\
&+\beta_2 I_* f(S_*)\Big(2-\frac{f(S_*)}{f(S)}-\frac{f(S)}{f(S_*)}\Big) - \beta_1 P_* f(S_*)\frac{rA_* P_*}{\delta I_*}\Big(2-\frac{A}{A_*}-\frac{A_*}{A}\Big) \\
&-m_a\frac{r\beta_1 P_* f(S_*)}{kr\delta I_*}\frac{(A-A_*)^2}{A} \\
=&-m\frac{(S-S_*)(f(S)-f(S_*))}{f(S)} - \frac{\beta_1 f(S_*) P_*}{\delta I_*}\frac{rm_a A_{in}}{krA_*}\frac{(A-A_*)^2}{A} \\
&+\beta_2 I_* f(S_*)\Big(2-\frac{f(S_*)}{f(S)}-\frac{f(S)}{f(S_*)}\Big) + \beta_1 P_* f(S_*)\Big(3-\frac{f(S_*)}{f(S)}-\frac{PI_* f(S)}{P_* I f(S_*)}-\frac{P_* I}{P I_*}\Big).
\end{aligned}
$$

Using the rule that

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}, \tag{3.14}$$

we get

$$\frac{1}{2}\Big(\frac{f(S_*)}{f(S)}+\frac{f(S)}{f(S_*)}\Big) \geq 1$$

and

$$\frac{1}{3}\Big(\frac{f(S_*)}{f(S)}+\frac{PI_* f(S)}{P_* I f(S_*)}+\frac{P_* I}{P I_*}\Big) \geq 1.$$

Therefore, $\dot{U}_1 \leq 0$ for all $S, I, P, A > 0$ and $\dot{U}_1 = 0$ if, and only if $S = S_*, I = I_*, P = P_*$ and $A = A_*$. We deduce that E^* is globally stable by LaSalle's invariance principle [29]. □

4. Periodic system

Let us reconsider the periodic dynamics given by (2.1) where our aim is to prove that the system admits a bounded positive T-periodic solution.

For a continuous, positive T-periodic function $g(t)$, we set $g^u = \max_{t\in[0,T)} g(t)$ and $g^l = \min_{t\in[0,T)} g(t)$.

4.1. Preliminary

Let $(\mathbb{R}^m, \mathbb{R}^m_+)$ be the ordered m-dimensional Euclidean space associated with the norm $\|\cdot\|$. For $X_1, X_2 \in \mathbb{R}^m$, we establish that $X_1 \geq X_2$ if $X_1 - X_2 \in \mathbb{R}^m_+$. We establish that $X_1 > X_2$ if $X_1 - X_2 \in \mathbb{R}^m_+\backslash\{0\}$. We establish that $X_1 \gg X_2$ if $X_1 - X_2 \in \mathrm{Int}(\mathbb{R}^m_+)$. Consider a T-periodic $m\times m$ matrix function denoted by $C(t)$ which is continuous, irreducible and cooperative. Let us denote by $\phi_C(t)$ the fundamental matrix, which is the solution of the following system

$$\dot{x}(t) = C(t)x(t). \tag{4.1}$$

Let us denote the spectral radius of the matrix $\phi_C(T)$ by $r(\phi_C(T))$. Therefore, all entries of $\phi_C(t)$ are positive for each $t > 0$. Let us apply the theorem of Perron-Frobenius to deduce that $r(\phi_C(T))$ is the principal eigenvalue of $\phi_C(T)$ (simple and admits an eigenvector $y^* \gg 0$). For the rest of the paper, the following lemma will be useful.

Lemma 4. *[30]. There exists a positive T-periodic function $y(t)$ such that $x(t) = y(t)e^{kt}$ is a solution of system (4.1) where $k = \dfrac{1}{T}\ln(r(\phi_C(T)))$.*

Let us start by proving the existence (and uniqueness) of the disease free periodic trajectory of model (2.1). Let us consider the following subsystem

$$\begin{aligned} \dot{S}(t) &= m(t)S_{in}(t) - m(t)S(t), \\ \dot{A}(t) &= m_a(t)A_{in}(t) - m_a(t)A(t), \end{aligned} \tag{4.2}$$

with the initial condition $(S^0, A^0) \in \mathbb{R}^2_+$. Equation (4.2) admits a unique T-periodic solution $(S_*(t), A_*(t))$ with $S_*(t) > 0$ and $A_*(t) > 0$ which is globally attractive in $\mathbb{R}^2_+$; thus, system (2.1) has a unique disease-free periodic solution $(S_*(t), 0, 0, A_*(t))$.

Let us introduce the following result.

Proposition 1.

$$\Omega^u = \left\{(S, I, P, A) \in \mathbb{R}^4_+ \;/\; S + I \leq S^u_{in}; kP + A \leq A^u_{in} + \frac{\delta^u k^u S^u_{in}}{m^l_a}\right\}$$

is a positively invariant, compact and attractor set for model (2.1). Furthermore, we have

$$\begin{aligned} &\lim_{t\to\infty} S(t) + I(t) - S_*(t) = 0, \\ &\lim_{t\to\infty} k(t)P(t) + A(t) - A_*(t) = 0. \end{aligned} \tag{4.3}$$

Proof. From (2.1), we have

$$\begin{aligned} \dot{S}(t) + \dot{I}(t) &= m(t)S_{in}(t) - m(t)(S(t) + I(t)) \\ &\leq m(t)S^u_{in} - m(t)(S(t) + I(t)) \leq 0, \text{ if } S(t) + I(t) \geq S^u_{in}, \end{aligned}$$

and

$$
\begin{aligned}
k(t)\dot{P}(t) + \dot{A}(t) =& k(t)(\delta(t)I(t) - m_p(t)P(t)) + m_a(t)A_{in}(t) - m_a(t)A(t) \\
=& k(t)\delta(t)I(t) - k(t)m_p(t)P(t) + m_a(t)A_{in}(t) - m_a(t)A(t) \\
& \leq k(t)\delta(t)I(t) - k(t)m_a(t)P(t) + m_a(t)A_{in}(t) - m_a(t)A(t) \\
& \leq k(t)\delta(t)I(t) + m_a(t)(A_{in}(t) - (k(t)P(t) + A(t))) \\
& \leq \delta^u k^u S^u_{in} + m^l_a(A^u_{in} - (k(t)P(t) + A(t))) \text{ if } S(t) + I(t) \geq A^u_{in} + \frac{\delta^u k^u S^u_{in}}{m^l_a} \\
=& (A^u_{in} + \frac{\delta^u k^u S^u_{in}}{m^l_a} - (k(t)P(t) + A(t))) \\
& \leq 0, \text{ if } kP(t) + A(t) \geq A^u_{in} + \frac{\delta^u k^u S^u_{in}}{m^l_a},
\end{aligned} \tag{4.4}
$$

which implies that Ω^u is a forward invariant compact absorbing set for (2.1). Let $N_1(t) = S(t) + I(t)$ and $N_2(t) = k(t)P(t) + A(t)$ be the sub-population sizes at time t. Next, let $y_1(t) = N_1(t) - S_*(t), t \geq 0$. Then, it follows that $\dot{y}_1(t) = -m(t)y_1(t)$, which implies that $\lim_{t\to\infty} y_1(t) = \lim_{t\to\infty}(N_1(t) - S_*(t)) = 0$. Similarly, let $y_2(t) = N_2(t) - A_*(t), t \geq 0$. Then, it follows that $\dot{y}_2(t) = -m(t)y_2(t)$, which implies that $\lim_{t\to\infty} y_2(t) = \lim_{t\to\infty}(N_2(t) - A_*(t)) = 0$. □

Next, in Subsection 4.2, we define $\mathcal{R}_0$, the basic reproduction number and we will prove that the disease free periodic trajectory $(0, 0, S_*(t), A_*(t))$ is globally asymptotically stable (and therefore, that the disease dies out) once $\mathcal{R}_0 < 1$. Then, in Subsection 4.3, we will prove that $I(t)$ and $P(t)$ exhibit uniform persistence (i.e., the disease persists) once $\mathcal{R}_0 > 1$. Therefore, we deduce that $\mathcal{R}_0$ is the threshold parameter between the uniform persistence and the extinction of the disease.

4.2. Disease free periodic solution

We start by giving the definition of the basic reproduction number for model (2.1) by using the theory given in [19] where

$$
\mathcal{F}(t, X) = \begin{pmatrix} \beta_1(t)P(t)f(S(t)) + \beta_2(t)I(t)f(S(t)) \\ \delta(t)I(t) \\ 0 \\ 0 \end{pmatrix},
$$

$$
\mathcal{V}^-(t, X) = \begin{pmatrix} m(t)I(t) \\ m_p(t)P(t) + r(t)A(t)P(t) \\ m(t)S(t) + \beta_1(t)P(t)f(S(t)) + \beta_2(t)I(t)f(S(t)) \\ m_a(t)A(t) \end{pmatrix},
$$

and

$$
\mathcal{V}^+(t, X) = \begin{pmatrix} 0 \\ 0 \\ m(t)S_{in}(t) \\ m_a(t)A_{in}(t) + k(t)r(t)A(t)P(t) \end{pmatrix}
$$

with $X = \begin{pmatrix} I \\ P \\ S \\ A \end{pmatrix}$.

Our aim is to check the conditions (A1)–(A7) in [19, Section 1]. Note that system (2.1) can have the following form

$$\dot{X} = \mathcal{F}(t,X) - \mathcal{V}(t,X) = \mathcal{F}(t,X) - \mathcal{V}^{-}(t,X) + \mathcal{V}^{+}(t,X). \tag{4.5}$$

The first five conditions (A1)–(A5) are fulfilled.

The system (4.5) admits a disease free periodic trajectory $X^{*}(t) = \begin{pmatrix} 0 \\ 0 \\ S_{*}(t) \\ A_{*}(t) \end{pmatrix}$. Let $f(t,X(t)) = \mathcal{F}(t,X) - \mathcal{V}^{-}(t,X) + \mathcal{V}^{+}(t,X)$ and $M(t) = \left(\frac{\partial f_i(t,X^{*}(t))}{\partial X_j}\right)_{3\le i,j\le 4}$ where $f_i(t,X(t))$ and X_i are the i-th components of $f(t,X(t))$ and X, respectively. By an easy calculation, we get that $M(t) = \begin{pmatrix} -m(t) & 0 \\ 0 & -m_a(t) \end{pmatrix}$ and then that $r(\phi_M(T)) < 1$. Therefore $X^{*}(t)$ is linearly asymptotically stable in the subspace $\Gamma_s = \left\{(0,0,S,A) \in R^4_+\right\}$. Thus, the condition (A6) in [19, Section 1] is satisfied.

Now, let us define $\mathbf{F}(t)$ and $\mathbf{V}(t)$ to be two by two matrices given by $\mathbf{F}(t) = \left(\frac{\partial \mathcal{F}_i(t,X^{*}(t))}{\partial X_j}\right)_{1\le i,j\le 2}$ and $\mathbf{V}(t) = \left(\frac{\partial \mathcal{V}_i(t,X^{*}(t))}{\partial X_j}\right)_{1\le i,j\le 2}$ where $\mathcal{F}_i(t,X)$ and $\mathcal{V}_i(t,X)$ are the i-th components of $\mathcal{F}(t,X)$ and $\mathcal{V}(t,X)$, respectively. By an easy calculation, we obtain the following from system (4.5):

$$\mathbf{F}(t) = \begin{pmatrix} \beta_2(t)f(S_{*}(t)) & \beta_1(t)f(S_{*}(t)) \\ \delta(t) & 0 \end{pmatrix}, \mathbf{V}(t) = \begin{pmatrix} m(t) & 0 \\ 0 & m_p(t) + r(t)A_{*}(t) \end{pmatrix}.$$

Consider $Z(t_1,t_2)$ to be the two by two matrix solution of the system $\frac{d}{dt}Z(t_1,t_2) = -\mathbf{V}(t_1)Z(t_1,t_2)$ for any $t_1 \ge t_2$, with $Z(t_1,t_1) = I$, i.e., the two by two identity matrix. Thus, condition (A7) is satisfied.

Let us define C_T to be the ordered Banach space of T-periodic functions defined on $\mathbb{R} \mapsto \mathbb{R}^2$, associated with the maximum norm $\|.\|_\infty$ and the positive cone $C_T^{+} = \{\psi \in C_T : \psi(s) \ge 0,$ for any $s \in \mathbb{R}\}$. Define the linear operator $K : C_T \to C_T$ by

$$(K\psi)(s) = \int_0^\infty Z(s,s-w)\mathbf{F}(s-w)\psi(s-w)dw, \quad \forall s \in \mathbb{R}, \psi \in C_T. \tag{4.6}$$

Let us now define the basic reproduction number, $\mathcal{R}_0$, of model (2.1) by using $\mathcal{R}_0 = r(K)$.

Therefore, we conclude the local asymptotic stability of the disease free periodic solution $\mathcal{E}_0(t) = (S_{*}(t),0,0,A_{*}(t))$ for (2.1) to be as follows.

Theorem 5. *[19, Theorem 2.2]*

- $\mathcal{R}_0 < 1 \;\Leftrightarrow\; r(\phi_{F-V}(T)) < 1$.

- $\mathcal{R}_0 = 1 \;\Leftrightarrow\; r(\phi_{F-V}(T)) = 1$.
- $\mathcal{R}_0 > 1 \;\Leftrightarrow\; r(\phi_{F-V}(T)) > 1$.

Therefore, $\mathcal{E}_0(t)$ is unstable if $\mathcal{R}_0 > 1$ and it is asymptotically stable if $\mathcal{R}_0 < 1$.

Theorem 6. *$\mathcal{E}_0(t)$ is globally asymptotically stable if $\mathcal{R}_0 < 1$. It is unstable if $\mathcal{R}_0 > 1$.*

Proof. Using Theorem 5, we have that $\mathcal{E}_0(t)$ is locally stable once $\mathcal{R}_0 < 1$ and it is unstable once $\mathcal{R}_0 > 1$. Therefore, we need to prove the global attractivity of $\mathcal{E}_0(t)$ when $\mathcal{R}_0 < 1$.

Consider the case in which $\mathcal{R}_0 < 1$. Using the limits given by (4.3) in Proposition 1, for any $\delta_1 > 0$, there exists $T_1 > 0$ satisfying $S(t) + I(t) \le S_*(t) + \delta_1$ and $k(t)P(t) + A(t) \le A_*(t) + \delta_1$ for $t > T_1$. Then $S(t) \le S_*(t) + \delta_1$ and $A(t) \le A_*(t) + \delta_1$; also, we deduce that

$$\begin{cases} \dot{I}(t) & \le & \beta_1(t)P(t)f(S_*(t) + \delta_1) + \beta_2(t)I(t)f(S_*(t) + \delta_1) - m(t)I(t), \\ \dot{P}(t) & = & \delta(t)I(t) - m_p(t)P(t) \end{cases} \tag{4.7}$$

for $t > T_1$. Let $M_2(t)$ be the following 2×2 matrix function

$$M_2(t) = \begin{pmatrix} \beta_2(t)f(S_*(t) + \delta_1) & \beta_1(t)f(S_*(t) + \delta_1) \\ \delta(t) & 0 \end{pmatrix}. \tag{4.8}$$

By Theorem 5, we have that $r(\varphi_{F-V}(T)) < 1$. Let us choose $\delta_1 > 0$ such that $r(\varphi_{F-V+\delta_1 M_2}(T)) < 1$. Consider the following system hereafter,

$$\begin{cases} \dot{\bar{I}}(t) & = & \beta_1(t)\bar{P}(t)f(S_*(t) + \delta_1) + \beta_2(t)\bar{I}(t)f(S_*(t) + \delta_1) - m(t)\bar{I}(t), \\ \dot{\bar{P}}(t) & = & \delta(t)\bar{I}(t) - m_p(t)\bar{P}(t). \end{cases} \tag{4.9}$$

Applying Lemma 4 and using the standard comparison principle, we deduce that there exists a positive T-periodic function $y_1(t)$ satisfying $x(t) \le y_1(t)e^{k_1 t}$ where $x(t) = \begin{pmatrix} I(t) \\ P(t) \end{pmatrix}$ and $k_1 = \frac{1}{T}\ln\left(r(\varphi_{F-V+\delta_1 M_2}(T)\right) < 0$. Thus, $\lim_{t\to\infty} I(t) = 0$ and $\lim_{t\to\infty} P(t) = 0$. Furthermore, we have that $\lim_{t\to\infty} S(t) - S_*(t) = \lim_{t\to\infty} N_1(t) - I(t) - S_*(t) = 0$ and $\lim_{t\to\infty} A(t) - A_*(t) = \lim_{t\to\infty} N_2(t) - k(t)P(t) - A_*(t) = 0$. Then, we deduce that the disease free periodic solution $\mathcal{E}_0(t)$ is globally attractive which completes the proof. □

For the following subsection, we consider only the case in which $\mathcal{R}_0 > 1$.

4.3. *Endemic periodic solution*

From Proposition 1, system (2.1) admits a positively invariant compact set Ω^u.

Let us define the function $Q : \mathbb{R}_+^4 \to \mathbb{R}_+^4$ to be the Poincaré map associated with system (2.1) such that $X_0 \mapsto u(T, X^0)$, where $u(t, X^0)$ is the unique solution of the system (2.1) with the initial condition $u(0, X^0) = X^0 \in \mathbb{R}_+^4$.

Let us define

$$\Gamma = \left\{(S, I, P, A) \in \mathbb{R}_+^4\right\},\ \Gamma_0 = Int(\mathbb{R}_+^4) \text{ and } \partial\Gamma_0 = \Gamma \setminus \Gamma_0.$$

Note that from Proposition 1, both Γ and Γ_0 are positively invariant. P is point dissipative. Define

$$M_\partial = \left\{(S^0, I^0, P^0, A^0) \in \partial\Gamma_0 : Q^n(S^0, I^0, P^0, A^0) \in \partial\Gamma_0,\ \text{for any } n \ge 0\right\}.$$

In order to apply the theory of uniform persistence detailed in [31] (also in [30, Theorem 2.3]), we prove that

$$M_\partial = \{(S,0,0,A),\ S \geq 0, A \geq 0\}. \tag{4.10}$$

Note that $M_\partial \supseteq \{(S,0,0,A),\ S \geq 0, A \geq 0\}$. To show that $M_\partial \setminus \{(S,0,0,A),\ S \geq 0, A \geq 0\} = \emptyset$. Let us consider $(S^0, I^0, P^0, A^0) \in M_\partial \setminus \{(S,0,0,A),\ S \geq 0, A \geq 0\}$. If $P^0 = 0$ and $0 < I^0$, then $I(t) > 0$ for any $t > 0$. Then, it holds that $\dot{P}(t)_{|t=0} = \delta(0)I^0 > 0$. If $P^0 > 0$ and $I^0 = 0$, then $P(t) > 0$ and $S(t) > 0$ for any $t > 0$. Therefore, for any $t > 0$, we have

$$I(t) = \Big[I^0 + \int_0^t \beta(\omega)(\beta_1 P(\omega) f(S(\omega)) + \beta_2 I(\omega) f(S(\omega))) e^{\int_0^\omega m(u)du} d\omega\Big] e^{-\int_0^t m(u)du} > 0$$

for all $t > 0$. This means that $(S(t), I(t), P(t), A(t)) \notin \partial\Gamma_0$ for $0 < t \ll 1$. Therefore, Γ_0 is positively invariant from which we deduce (4.10). Using the previous discussion, we deduce that there exists one fixed point $(S^*(0), 0, 0, A^*(0))$ of P in M_∂. We deduce, therefore, the uniform persistence of the disease as follows.

Theorem 7. *Consider the case in which $\mathcal{R}_0 > 1$. System (2.1) admits at least one positive periodic trajectory and $\exists\, \gamma > 0$ satisfying $\forall\ (S^0, I^0, P^0, A^0) \in \mathbb{R}_+ \times Int(\mathbb{R}_+^2) \times \mathbb{R}_+$ and*

$$\liminf_{t\to\infty} I(t) \geq \gamma > 0.$$

Proof. Let us start by proving that P is uniformly persistent with respect to $(\Gamma_0, \partial\Gamma_0)$, which will prove that the trajectory of the system (2.1) is uniformly persistent with respect to $(\Gamma_0, \partial\Gamma_0)$ by using [31, Theorem 3.1.1]. Recall that using Theorem 5, we obtain that $r(\varphi_{F-V}(T)) > 1$. Therefore, there exists $\eta > 0$ small enough and satisfying that $r(\varphi_{F-V-\eta M_2}(T)) > 1$. Let us consider the following perturbed equation

$$\begin{cases} \dot{S}_\alpha(t) &= m(t)S_{in}(t) - m(t)S_\alpha(t) - \alpha(\beta_1(t) + \beta_2(t)) f(S_\alpha(t)), \\ \dot{A}_\alpha(t) &= m_a(t)A_{in}(t) + \alpha k(t) r(t) A_\alpha(t) - m_a(t)A_\alpha(t). \end{cases} \tag{4.11}$$

The function Q associated with the perturbed system (4.11) has a unique positive fixed point $(\bar{S}_\alpha^0, \bar{A}_\alpha^0)$ that it is globally attractive in $\mathbb{R}_+^2$. We apply the implicit function theorem to deduce that $(\bar{S}_\alpha^0, \bar{A}_\alpha^0)$ is continuous with respect to α. Therefore, we can choose $\alpha > 0$ small enough and satisfying $\bar{S}_\alpha(t) > \bar{S}(t) - \eta$, and $\bar{A}_\alpha(t) > \bar{A}(t) - \eta$, $\forall\ t > 0$. Let $M_1 = (\bar{S}^0, 0, 0, \bar{A}^0)$. Since the trajectory is continuous with respect to the initial condition, there exists α^* satisfying $(S^0, I^0, P^0, A^0) \in \Gamma_0$ with $\|(S^0, I^0, P^0, A^0) - u(t, M_1)\| \leq \alpha^*$; it holds that

$$\|u(t, (S^0, I^0, P^0, A^0)) - u(t, M_1)\| < \alpha \text{ for } 0 \leq t \leq T.$$

We prove by contradiction that

$$\limsup_{n\to\infty}\ d(Q^n(S^0, I^0, P^0, A^0), M_1) \geq \alpha^* \ \forall\ (S^0, I^0, P^0, A^0) \in \Gamma_0. \tag{4.12}$$

Suppose that $\limsup_{n\to\infty} d(Q^n(S^0, I^0, P^0, A^0), M_1) < \alpha^*$ for some $(S^0, I^0, P^0, A^0) \in \Gamma_0$. We can assume that $d(Q^n(S^0, I^0, P^0, A^0), M_1) < \alpha^* \ \forall\ n > 0$. Therefore

$$\|u(t, Q^n(S^0, I^0, P^0, A^0)) - u(t, M_1)\| < \alpha \ \forall\ n > 0 \text{ and } 0 \le t \le T.$$

For all $t \ge 0$, let $t = nT + t_1$, with $t_1 \in [0, T)$ and $n = [\frac{t}{T}]$ (greatest integer $\le \frac{t}{T}$). Then, we get

$$\|u(t, (S^0, I^0, P^0, A^0)) - u(t, M_1)\| = \|u(t_1, Q^n(S^0, I^0, P^0, A^0)) - u(t_1, M_1)\| < \alpha \text{ for all } t \ge 0.$$

Set $(S(t), I(t), P(t), A(t)) = u(t, (S^0, I^0, P^0, A^0))$. Therefore $0 \le I(t), P(t) \le \alpha, t \ge 0$ and

$$\begin{cases} \dot{S}(t) & \ge & m(t)S_{in}(t) - m(t)S(t) - \alpha(\beta_1(t) + \beta_2(t))f(S(t)), \\ \dot{A}(t) & \ge & m_a(t)A_{in}(t) - m_a(t)A(t). \end{cases} \tag{4.13}$$

The fixed point $\bar{S}^0_\alpha$ of the function Q associated with the perturbed system (4.11) is globally attractive such that $\bar{S}_\alpha(t) > \bar{S}(t) - \eta$, and $\bar{A}_\alpha(t) > \bar{A}(t) - \eta$; then, there exists $T_2 > 0$ large enough and satisfying the condition that $S(t) > \bar{S}(t) - \eta$ and $A(t) > \bar{A}(t) - \eta$ for $t > T_2$. Therefore, for $t > T_2$,

$$\begin{cases} \dot{I}(t) & \ge & \beta_1(t)P(t)f(\bar{S}(t) - \eta) + \beta_2(t)I(t)f(\bar{S}(t) - \eta) - m(t)I(t), \\ \dot{P}(t) & = & \delta(t)I(t) - m_p(t)P(t). \end{cases} \tag{4.14}$$

Note that we have the condition that $r(\varphi_{F-V-\eta M_2}(T)) > 1$. Applying Lemma 4 and the comparison principle, there exists a positive T-periodic trajectory $y_2(t)$ satisfying the condition that $J(t) \ge e^{k_2 t} y_2(t)$ with $k_2 = \frac{1}{T} \ln \ r\Big(\varphi_{F-V-\eta M_2}(T)\Big) > 0$, which implies that $\lim_{t\to\infty} I(t) = \infty$ which is impossible since the trajectories are bounded. Therefore, the inequality (4.12) is satisfied and Q is weakly uniformly persistent with respect to $(\Gamma_0, \partial\Gamma_0)$. By applying Proposition 1, Q has a global attractor. We deduce that $M_1 = (\bar{S}^0, 0, 0, \bar{A}^0)$ is an isolated invariant set inside X and that $W^s(M_1) \cap \Gamma_0 = \emptyset$. All trajectories inside M_∂ converges to M_1 which is acyclic in M_∂. Applying [31, Theorem 1.3.1 and Remark 1.3.1], we deduce that Q is uniformly persistent with respect to $(\Gamma_0, \partial\Gamma_0)$. Furthermore, using [31, Theorem 1.3.6], Q admits a fixed point $(\tilde{S}^0, \tilde{I}^0, \tilde{P}^0, \tilde{A}^0) \in \Gamma_0$. Note that

$$(\tilde{S}^0, \tilde{I}^0, \tilde{P}^0, \tilde{A}^0) \in R_+ \times Int(R^2_+) \times R_+.$$

We prove also by contradiction that $\tilde{S}^0 > 0$. Assume that $\tilde{S}^0 = 0$. Using the first equation of the system (2.1), $\tilde{S}(t)$ verifies that

$$\dot{\tilde{S}}(t) \ge m(t)S_{in}(t) - m(t)\tilde{S}(t) - (\beta_1(t)\tilde{P}(t) + \beta_2(t)\tilde{I}(t))f(\tilde{S}(t)), \tag{4.15}$$

with $\tilde{S}^0 = \tilde{S}(pT) = 0, p = 1, 2, 3, \cdots$. Applying Proposition 1, $\forall\ \delta_3 > 0$; there exists $T_3 > 0$ large enough and satisfying the condition that $\tilde{I}(t) \le S^u_{in} + \delta_3$ and $\tilde{P}(t) \le \frac{A^u_{in}}{k^l} + \frac{\delta^u k^u S^u_{in}}{k^l m^l_a} + \delta_3$ for $t > T_3$. Then, by Lemma 1, we obtain

$$\dot{\tilde{S}}(t) \ge m(t)S_{in}(t) - m(t)\tilde{S}(t) - \Big(\beta_1(t)\Big(\frac{A^u_{in}}{k^l} + \frac{\delta^u k^u S^u_{in}}{k^l m^l_a} + \delta_3\Big) + \beta_2(t)S^u_{in}\Big)f'(0)\tilde{S}(t), \text{ for } t \ge T_3. \tag{4.16}$$

There exists $\bar{p}$ large enough and satisfying the condition that $pT > T_3$ for all $p > \bar{p}$. Applying the comparison principle, we deduce the following:

$$\tilde{S}(pT) = \Big[\tilde{S}^0 + \int_0^{pT} m(\omega) S_{in}(\omega) e^{\int_0^{\omega} \left(\left(\beta_1(u)\left(\frac{A^u_{in}}{k^l} + \frac{\delta^u k^u S^u_{in}}{k^l m^l_a} + \delta_3\right) + \beta_2(u) S^u_{in}\right) f'(0) + m(u) \right) du} d\omega \Big] \times e^{-\int_0^{pT} \left(\left(\beta_1(u)\left(\frac{A^u_{in}}{k^l} + \frac{\delta^u k^u S^u_{in}}{k^l m^l_a} + \delta_3\right) + \beta_2(u) S^u_{in}\right) f'(0) + m(u) \right) du} > 0$$

for any $p > \bar{p}$ which is impossible. Therefore, $\tilde{S}^0 > 0$ and $(\tilde{S}^0, \tilde{I}^0, \tilde{P}^0, \tilde{A}^0)$ is a positive T-periodic trajectory of the system (2.1). □

5. Applications and numerical results

For all of our numerical results, we will apply a nonlinear Monod-type function (or, also, a Holling type-II function) as a typical example that describes the incidence rate and satisfies Assumptions 1 and 2:

$$f(S) = \frac{\eta S}{\kappa + S}.$$

Here η and κ are non-negative constants known as Monod constants. The periodic functions are given by

$$\begin{cases} m(t) &=& m^0(1 + m^1 \cos(2\pi(t+\phi))), \\ m_p(t) &=& m_p^0(1 + m_p^1 \cos(2\pi(t+\phi))), \\ m_a(t) &=& m_a^0(1 + m_a^1 \cos(2\pi(t+\phi))), \\ \delta(t) &=& \delta^0(1 + \delta^1 \cos(2\pi(t+\phi))), \\ S_{in}(t) &=& S_{in}^0(1 + S_{in}^1 \cos(2\pi(t+\phi))), \\ \beta_1(t) &=& \beta_1^0(1 + \beta_1^1 \cos(2\pi(t+\phi))), \\ \beta_2(t) &=& \beta_2^0(1 + \beta_2^1 \cos(2\pi(t+\phi))), \\ r(t) &=& r^0(1 + r^1 \cos(2\pi(t+\phi))), \\ k(t) &=& k^0(1 + k^1 \cos(2\pi(t+\phi))), \end{cases} \tag{5.1}$$

with $|m^1|$, $|m_p^1|$, $|m_a^1|$, $|\delta^1|$, $|S_{in}^1|$, $|\beta_1^1|$, $|\beta_2^1|$, $|r^1|$ and $|k^1|$ denoting the frequencies of seasonal cycles, also, ϕ is the phase shift. The values of m^0, m_p^0, m_a^0, δ^0, S_{in}^0, β_1^0, β_2^0, r^0 and k^0 are given in Table 2. However, the values of m^1, m_p^1, m_a^1, δ^1, S_{in}^1, β_1^1, β_2^1, r^1 and k^1 are given in Table 3.

Table 2. Used values for m^0, m_p^0, m_a^0, δ^0, S_{in}^0, β_1^0, β_2^0, r^0 and k^0.

Parameter	m^0	m_p^0	m_a^0	δ^0	S_{in}^0	β_1^0	β_2^0	r^0	k^0
Value	0.8	0.8	10	2	1	0.8	10	2	1

Table 3. Used values for m^1, m_p^1, m_a^1, δ^1, S_{in}^1, β_1^1, β_2^1, r^1 and k^1.

Parameter	m^1	m_p^1	m_a^1	δ^1	S_{in}^0	β_1^1	β_2^1	r^1	k^1
Value	0.8	0.8	10	2	1	0.8	10	2	1

We will consider three cases. The first case applies the autonomous system (constant parameters) to confirm the global stability of the equilibrium points $\mathcal{E}_0$ and $\mathcal{E}^*$. The second case applies the partially non-autonomous system (only $\beta(t)$ is a periodic function). The third case considers all parameters as periodic functions (i.e., a totally non-autonomous system).

5.1. Case in which all parameters are constants

In the first step, we performed numerical simulations for the system (3.1) when all parameters are constant. Thus, the model is given by

$$\begin{cases} \dot{S}(t) &= m^0 S_{in}^0(t) - m^0 S(t) - \dfrac{\eta\beta_1^0 P(t)S(t)}{\kappa + S(t)} - \dfrac{\eta\beta_2^0 I(t)S(t)}{\kappa + S(t)}, \\ \dot{I}(t) &= \dfrac{\eta\beta_1^0 P(t)S(t)}{\kappa + S(t)} + \dfrac{\eta\beta_2^0 I(t)S(t)}{\kappa + S(t)} - m^0 I(t), \\ \dot{P}(t) &= \delta^0 I(t) - m_p^0 P(t) - r^0 A(t)P(t), \\ \dot{A}(t) &= m_a^0 A_{in}^0 + k^0 r^0 A(t)P(t) - m_a^0 A(t), \end{cases} \tag{5.2}$$

with the positive initial condition $(S_0, E_0, I_0, R_0) \in \mathbb{R}_+^4$.

We give the results of some numerical simulations confirming the stability of the steady states of system (5.2).

In Figure 2, the approximated solution of system (5.2) approaches asymptotically to $\mathcal{E}^*$ once $\mathcal{R}_0 > 1$. In Figure 3, the approximated solution of the given model (5.2) approaches the equilibrium $\mathcal{E}_0$, which confirms that $\mathcal{E}_0$ is globally asymptotically stable once $\mathcal{R}_0 \leq 1$.

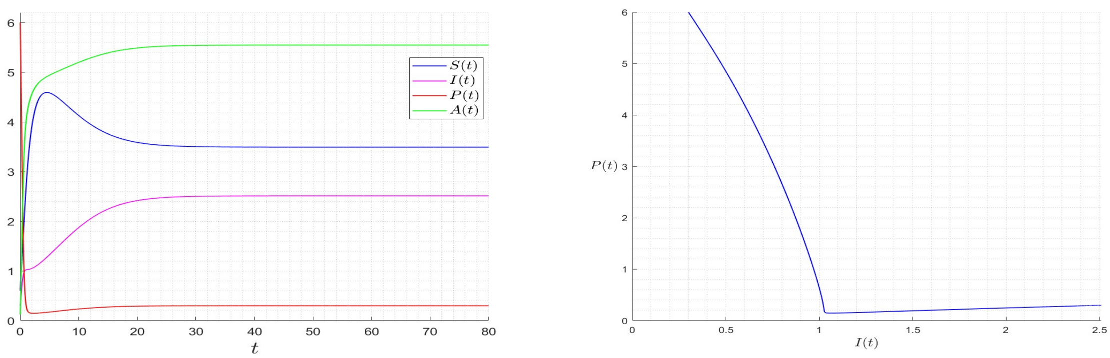

Figure 2. Behavior of the solution of system (2.1) for $\eta = 0.9$ and $\kappa = 2$; $\mathcal{R}_0 \approx 1.22 > 1$.

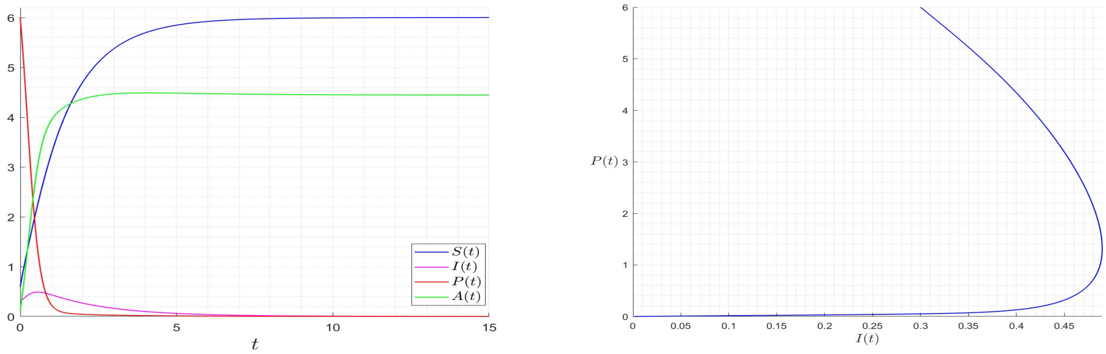

Figure 3. Behavior of the solution of system (2.1) for $\eta = 0.25$ and $\kappa = 1$; $\mathcal{R}_0 \approx 0.46 < 1$.

5.2. Case in which all parameters are constant with a periodic seasonally forced function

In the second step, we performed numerical simulations for the system (2.1) where only the T-periodic seasonally forced functions β_1 and β_2 are dependent on time. The other parameters were set to be constant. Thus the model is given by

$$\begin{cases} \dot{S}(t) &= m^0 S^0_{in}(t) - m^0 S(t) - \dfrac{\eta\beta_1(t)P(t)S(t)}{\kappa + S(t)} - \dfrac{\eta\beta_2(t)I(t)S(t)}{\kappa + S(t)}, \\ \dot{I}(t) &= \dfrac{\eta\beta_1(t)P(t)S(t)}{\kappa + S(t)} + \dfrac{\eta\beta_2(t)I(t)S(t)}{\kappa + S(t)} - m^0 I(t), \\ \dot{P}(t) &= \delta^0 I(t) - m^0_p P(t) - r^0 A(t)P(t), \\ \dot{A}(t) &= m^0_a A^0_{in} + k^0 r^0 A(t)P(t) - m^0_a A(t), \end{cases} \tag{5.3}$$

with the positive initial condition $(S^0, I^0, P^0, A^0) \in \mathbb{R}^4_+$.

We give the results of some numerical simulations confirming the stability of the steady states of system (5.3). The basic reproduction number $\mathcal{R}_0$ was approximated by using the time-averaged system.

In Figure 4, the approximated solution of system (5.3) asymptotically approaches the periodic solution with the persistence of the disease. In Figure 5, we show a magnified view of the limit cycle when $\mathcal{R}_0 > 1$. In Figure 6, the approximated solution of system (5.3) approaches the disease-free trajectory once $\mathcal{R}_0 < 1$.

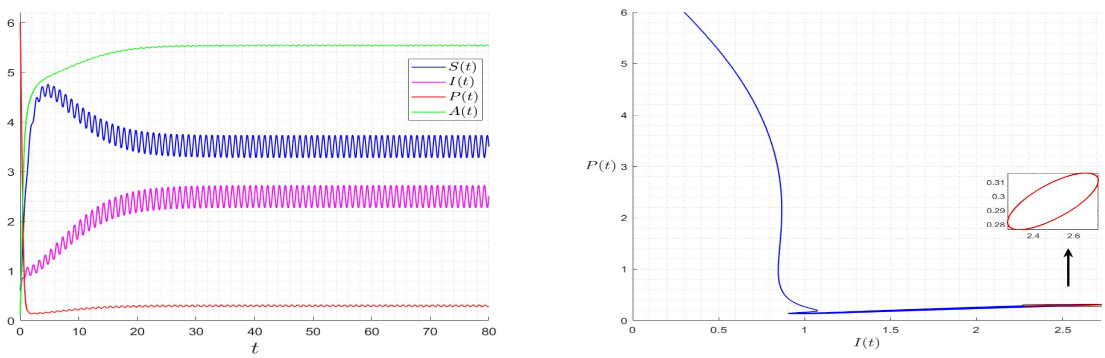

Figure 4. Behavior of the solution of system (2.1) for $\eta = 0.75$ and $\kappa = 2$; $\mathcal{R}_0 \approx 0.7187 < 1$.

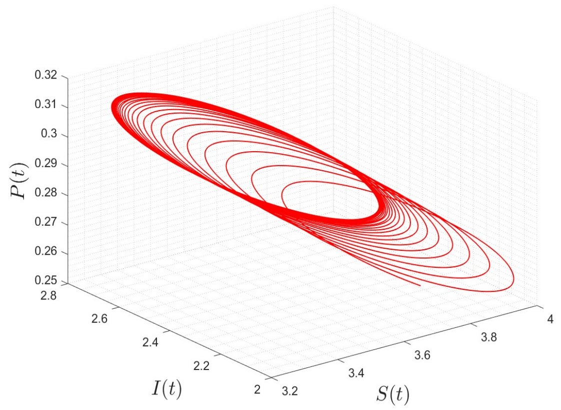

Figure 5. Enlarged view of the behavior of the solution of system (2.1) for $\eta = 0.9$ and $\kappa = 2$; $\mathcal{R}_0 \approx 1.22 > 1$.

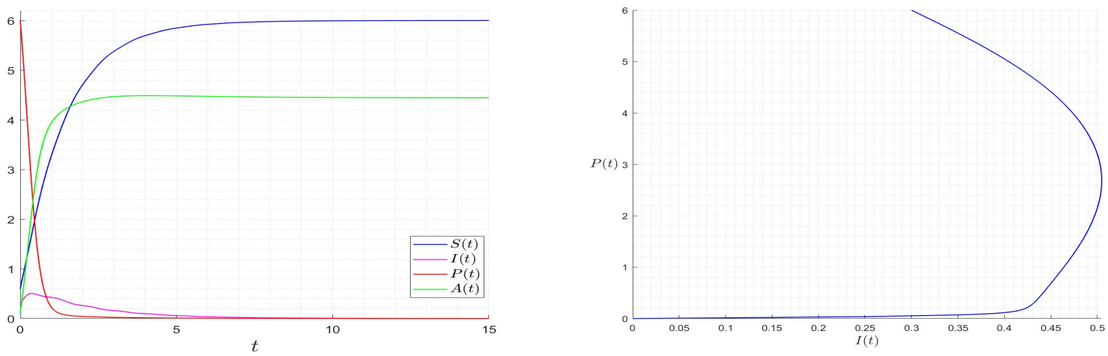

Figure 6. Behavior of the solution of system (2.1) for $\eta = 0.25$ and $\kappa = 1$; $\mathcal{R}_0 \approx 0.46 < 1$.

5.3. *Case in which all parameters are periodic functions*

In the third step, we performed numerical simulations for the system (2.1) where all parameters were set as T-periodic functions. Thus the model is given by

$$\begin{cases} \dot{S}(t) &= m(t)S_{in}(t) - m(t)S(t) - \dfrac{\eta\beta_1(t)P(t)S(t)}{\kappa + S(t)} - \dfrac{\eta\beta_2(t)I(t)S(t)}{\kappa + S(t)}, \\ \dot{I}(t) &= \dfrac{\eta\beta_1(t)P(t)S(t)}{\kappa + S(t)} + \dfrac{\eta\beta_2(t)I(t)S(t)}{\kappa + S(t)} - m(t)I(t), \\ \dot{P}(t) &= \delta(t)I(t) - m_p(t)P(t) - r(t)A(t)P(t), \\ \dot{A}(t) &= m_a(t)A_{in}(t) + k(t)r(t)A(t)P(t) - m_a(t)A(t), \end{cases} \tag{5.4}$$

with the positive initial condition $(S^0, I^0, P^0, A^0) \in \mathbb{R}^4_+$.

We give the results of some numerical simulations confirming the stability of the steady states of system (5.4). The basic reproduction number $\mathcal{R}_0$ was approximated by using the time-averaged system.

In Figure 7, the approximated solution of system (5.4) approaches asymptotically to a periodic solution with the persistence of the disease once $\mathcal{R}_0 > 1$. In Figure 8, we provide a magnified view of the limit cycle when $\mathcal{R}_0 > 1$. In Figure 9, the approximated solution of system (5.4) approaches the disease-free periodic trajectory $\mathcal{E}_0(t) = (S^*(t), 0, 0, A^*(t))$ once $\mathcal{R}_0 \leq 1$.

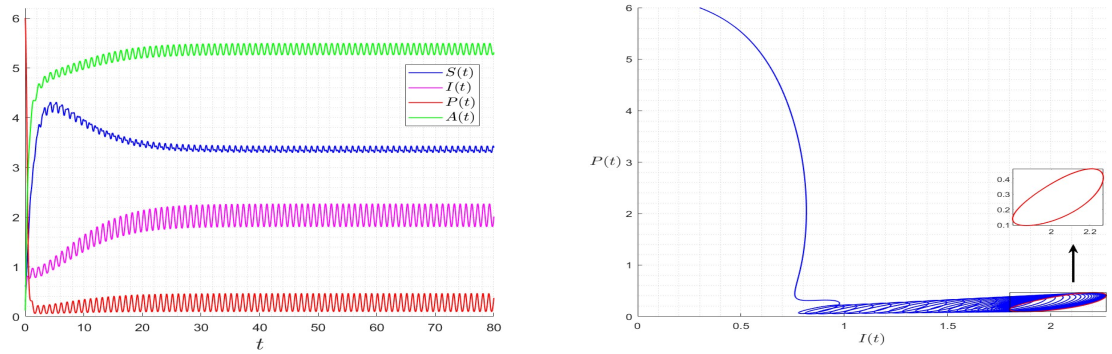

Figure 7. Behavior of the solution of system (2.1) for $\eta = 0.75$ and $\kappa = 2$; $\mathcal{R}_0 \approx 0.7187 < 1$.

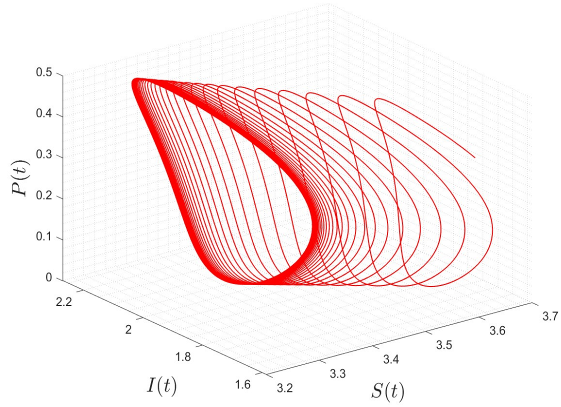

Figure 8. Magnified view of the behaviour of the solution of system (2.1) for $\eta = 0.9$ and $\kappa = 2$; $\mathcal{R}_0 \approx 1.22 > 1$.

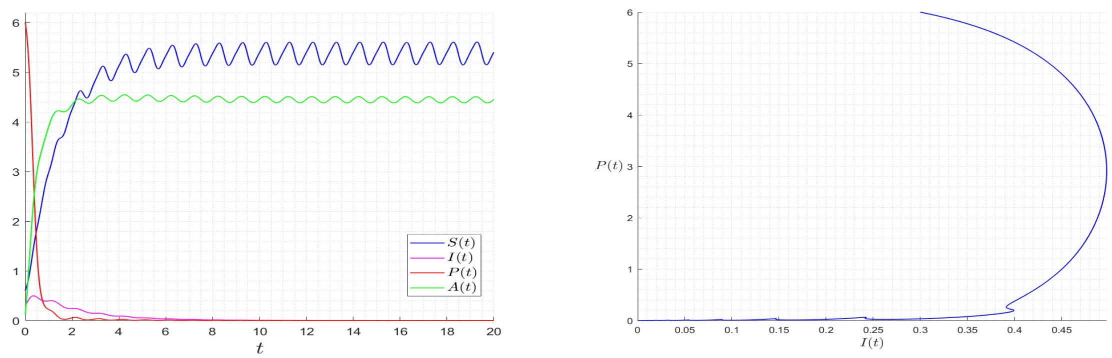

Figure 9. Behavior of the solution of system (2.1) for $\eta = 0.25$ and $\kappa = 1$; $\mathcal{R}_0 \approx 0.46 < 1$.

6. Conclusions

When studying the CHIKV dynamics, it is important to consider that the contamination of uninfected cells can be realized via contact with CHIKV (CHIKV-to-cell transmission) and by contact with infected cells (cell-to-cell transmission). Moreover, disease spread can have seasonal peak periods and it is important to consider this when modelling the dynamics. In this paper, we have proposed an extension of the CHIKV epidemic model already considered in [5, 21, 22] by taking into account the seasonal environment. In the first step, we studied the case of an autonomous system where all parameters are supposed to be constants. We calculated the basic reproduction number and the steady states of the system. We gave the existence and stability conditions for these steady states. In the second step, we considered the non-autonomous system, gave some theoretical results and defined the basic reproduction number, $\mathcal{R}_0$ through the use of an integral operator. We show that if $\mathcal{R}_0 \leq 1$, all trajectories converge to the disease-free periodic solution; however, the disease persists once $\mathcal{R}_0$ is greater than 1. Finally, we gave some numerical examples that support the theoretical findings, including those for the autonomous system, the partially non-autonomous system and the fully non-autonomous system. It has been deduced that if the system is autonomous, the trajectories converge to one of the equilibriums of the system (2.1) according to Theorems 3 and 4. However, if at least one of the model parameters is periodic, the trajectories converge to a limit cycle according to Theorems 6 and 7.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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