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Research article

Existence of solutions to mixed local and nonlocal anisotropic quasilinear singular elliptic equations

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Abstract: In this paper, we consider the existence of positive solutions to mixed local and nonlocal singular quasilinear singular elliptic equations

$$\begin{cases} -\Delta_{\vec{p}}u(x) + (-\Delta)_p^s u(x) = \frac{f(x)}{u(x)^\delta}, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N (N > 2)$, $-\Delta_{\vec{p}}u$ is an anisotropic p-Laplace operator, $\vec{p} = (p_1, p_2, ..., p_N)$ with $2 \le p_1 \le p_2 \le \cdots \le p_N$, $(-\Delta)_p^s$ is the fractional p-Laplace operator. The major results shows the interplay between the summability of the datum $f(x)$ and the power exponent δ in singular nonlinearities.

Keywords: mixed local and nonlocal; anisotropic p-Laplace equation; singular elliptic operator
Mathematics Subject Classification: 35J67, 35R11

1. Introduction

Our main purpose of this study is to investigate the existence of positive solutions to the following mixed local and nonlocal quasilinear singular elliptic equation

$$\begin{cases} -\Delta_{\vec{p}}u(x) + (-\Delta)_p^s u(x) = \frac{f(x)}{u(x)^\delta}, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N (N > 2)$, $\Delta_{\vec{p}}u$ is an anisotropic version of the p-Laplace operator, which is sometimes referred as the pseudo p-Laplace operator,

$$\Delta_{\vec{p}}u(x) = \sum_{i=1}^{N} \partial_i \left[|\partial_i u(x)|^{p_i-2} \partial_i u(x) \right],$$

where $\vec{p} = (p_1, p_2, ..., p_N)$, $p_i \geq 2$ for all $i = 1, 2, ..., N$. The fractional p-Laplace operator $(-\Delta)_p^s$, $(s \in (0,1), p \geq 1)$ is defined by

$$(-\Delta)_p^s u(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy,$$

where P.V. denotes the Cauchy principal value.

Recently, there has been increasing attention focused on the study of elliptic operators that involve mixed local and nonlocal operators.These equations often arise spontaneously in the study of plasma physics and population dynamics [1, 2]. For some other related results of mixed local and nonlocal equation, see [3–8] and the references therein. In the nonlocal case $(0 < s < 1)$, Barrios et al. [9] investigated the existence and uniqueness results of positive solutions to the following problem with $p = 2$,

$$\begin{cases} (-\Delta)_p^s u(x) = \frac{f(x)}{u(x)^\delta}, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

In the case $\delta > 0$, the existence of solutions to problem (1.2) obtained by the range of δ to the summability of f. In case $0 < \delta < 1$ and $1 < \delta$, Youssfi and Mahmoud [10] studied the existence of solutions to problem (1.2) with $p = 2$ under some suitable assumptions on the datum f. For further information, readers may refer to the related work [11–13] and references therein.

In the local case, Boccardo and Orsina [14] used the method of approximation to prove the existence of solutions to following the problem with $p = 2$,

$$\begin{cases} -\Delta_p u(x) = \frac{f(x)}{u(x)^\delta}, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

They also studied the summability of these solutions when $\delta \in (0, \infty)$. Giacomoni and Schindler [15] employ variational methods proved the existence of solution to quasilinear elliptic problem for $p_i = p \in (1, \infty)$ with $\delta \in (0, 1)$. During the past few years, there has been a vast amount of literature devoted to studying the anisotropic operator, which has numerous applications in fluid dynamics and physical phenomena, (we refer readers to [16–19] and references therein). Miri [19] further extended some results of [14] to an anisotropic quasilinear singular elliptic problem with variable exponent $\delta(x)$, and obtained existence of a solution to this problem. Bal and Garain [20, Theorems 2.7 and Theorems 2.9] established existence and uniqueness of solutions to the following mixed singular problems

$$\begin{cases} -L_1 u(x) = f_1(x)u(x)^{-\delta} + g_1(x)u(x)^{-\gamma}, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

and

$$\begin{cases} -L_2u(x) = f_2(x)u(x)^{-\delta} + g_2(x)u(x)^{-\gamma}, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded smooth subset of $\mathbb{R}^N$, $N > 2$, $\delta > 0$, $\gamma > 1$, f_j, g_j $(j = 1, 2)$ are nonnegative integrable functions,

$$L_1u(x) = \operatorname{div}\left[w(x)|\nabla u(x)|^{p-2}\nabla u(x)\right], \;\; L_2u(x) = \sum_{i=1}^{N} \partial_i \left[|\partial_i u(x)|^{p_i-2}\, \partial_i u(x)\right].$$

When $g_j = 0$ $(j = 1, 2)$, they obtained a solution to problems (1.4) and (1.5) associated with the following minimizing problems

$$v_1(\Omega) := \inf_{u \in W_0^{1,p}(\Omega,\omega)} \left\{ \int_\Omega |\nabla u|^p \omega dx : \int_\Omega |u|^{1-\delta} f_1 dx = 1 \right\},$$

and

$$v_2(\Omega) := \inf_{u \in W_0^{1,\vec{p}}(\Omega)} \left\{ \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i}\, dx : \int_\Omega |u|^{1-\delta} f_2 dx = 1 \right\}.$$

Garain and Ukhlov [21, Theorems 2.13] proved the existence of solution to the following problem

$$\begin{cases} -\Delta_p u(x) + (-\Delta)_p^s\, u(x) = \frac{f(x)}{u(x)^\delta}, & x \in \Omega, \\ \qquad\qquad\qquad u(x) > 0, & x \in \Omega, \\ \qquad\qquad\qquad u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.6}$$

It has been shown that problem (1.6) has a weak solution $u \in W_0^{1,p}(\Omega)$ when $\delta \in (0, 1]$ and $f \in L^m(\Omega) \setminus \{0\}$ with $m = (\frac{p^*}{1-\delta})'$, where $p^* = \frac{Np}{N-p}$, while if $\delta \in (1, \infty)$ and $f \in L^1(\Omega) \setminus \{0\}$, then problem (1.6) has a weak solution $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ with $u^{\frac{p+\delta-1}{p}} \in W_0^{1,p}(\Omega)$. Moreover, they proved that mixed Sobolev inequalities are both necessary and sufficient for the existence of weak solutions to such singular problems. For related results about mixed local and nonlocal elliptic operators see [22–30] and references therein.

Motivated by the results of the above cited papers, especially [20, 21], the our purpose of this study is to establish the existence of solutions to problem (1.1) according to the range of the power exponent δ and to the summability of datum $f(x)$. The main results as follows:

Theorem 1.1. *Let* $0 < \delta < 1$ *and* $1 < \bar{p} < N$. *Suppose that* $f > 0$, $f \in L^m(\Omega)$ *with* $m \geq 1$. *Then there exists a weak solution u to problem* (1.1) *such that*

(i) $u \in L^\infty(\Omega)$ *if* $m > \frac{N\bar{p}}{N\bar{p}-(N-\bar{p})p_N}$, *where* $\bar{p}$ *satisfies*

$$\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}.$$

(ii) $u \in L^t(\Omega)$ *if* $\bar{m} < m < \frac{N\bar{p}}{N\bar{p}-(N-\bar{p})p_N}$, *where*

$$\bar{m} = \frac{N\bar{p}}{N\bar{p} - p_i(N-\bar{p}) - (1-\delta-p_i)(N-\bar{p})}, \quad t = \frac{m(1-\delta-p_i)N\bar{p}}{N\bar{p}(m-1) - mp_i(N-\bar{p})}.$$

(iii) $u \in W_0^{1,q}(\Omega)$ *if* $1 \le m < \bar{m}$, *where*

$$q = \frac{p_i m(1-\delta-p_i)N\bar{p}}{m(1-\delta)\left[N\bar{p} - (N-\bar{p})p_i\right] - p_i N\bar{p}}.$$

Remark 1.2. Notice that when $p_i = 2$, the range of corresponding m values is exactly the summability of solutions obtained in [14].

When $p_i = p$, then problem (1.1) reduces to problem (1.6). Therefore

(i) If $f \in L^m(\Omega)$ with $\frac{Np}{Np-(1-\delta)(N-p)} < m < \frac{N}{p}$, then the solutions u to problem (1.6) satisfies $u \in L^t(\Omega)$ with

$$t = \frac{mN(1-\delta-p_i)}{mp-N}.$$

(ii) If $f \in L^m(\Omega)$ with $1 \le m < \frac{Np}{Np-(1-\delta)(N-p)}$, then the solutions u to problem (1.6) satisfies $u \in W_0^{1,q}(\Omega)$ with $q = \frac{mN(1-\delta-p)}{m(1-\delta)-N}$.

Theorem 1.3. *Suppose that* $\delta = 1$ *and* $1 < \bar{p} < N$, $f > 0$, $f \in L^m(\Omega)$ *with* $m > 1$. *Then there exists a weak solution u to problem* (1.1) *such that*

(i) $u \in L^\infty(\Omega)$ *if* $m > \frac{N\bar{p}}{N\bar{p}-(N-\bar{p})p_N}$.

(ii) $u \in L^t(\Omega)$ *if* $1 \le m < \frac{N\bar{p}}{N\bar{p}-(N-\bar{p})p_N}$, *where*

$$t = \frac{mp_iN\bar{p}}{mp_i(N-\bar{p}) - N\bar{p}(m-1)}.$$

Theorem 1.4. *Let* $\delta > 1$ *and* $1 < \bar{p} < N$. *Suppose that* $f > 0$, $f \in L^m(\Omega)$ *with* $m > 1$. *Then there exists a weak solution u to problem* (1.1) *such that*

(i) $u \in L^\infty(\Omega)$ *if* $m > \frac{N\bar{p}}{N\bar{p}-(N-\bar{p})p_N}$.

(ii) $u \in L^t(\Omega)$ *if* $1 \le m < \frac{N\bar{p}}{N\bar{p}-(N-\bar{p})p_N}$, *where*

$$t = \frac{m(1-\delta-p_i)N\bar{p}}{N\bar{p}(m-1) - mp_i(N-\bar{p})}.$$

The order of the article is organized as follows: In Section 2, we provide basic notations and algebraic inequalities needed in this paper, as well as some definitions and useful lemmas. In Section 3, we present the proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.4.

2. Preliminaries and auxiliary results

2.1. Preliminaries

In this article, we will use the following notations:

For any v, we denote by $v^+ = \max\{v, 0\}$, $v^- = \max\{-v, 0\}$. For $p > 1$, we denote by $p' = \frac{p}{p-1}$ to mean the conjugate exponent of p.

Definition 2.1. Let $p > 1$, $\Omega \subset \mathbb{R}^N$ with $N > 2$. The fractional Sobolev space $\mathcal{W}^{s,p}(\Omega)$ is defined by

$$\mathcal{W}^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

with

$$\|u\|_{\mathcal{W}^{s,p}(\Omega)} = \left(\int_\Omega |u(x)|^p dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

The space $\mathcal{W}^{s,p}(\mathbb{R}^N)$ and $\mathcal{W}^{s,p}_{\text{loc}}(\Omega)$ are defined analogously. The space $\mathcal{W}^{s,p}_0(\Omega)$ is defined as

$$\mathcal{W}^{s,p}_0(\Omega) = \left\{ u \in \mathcal{W}^{s,p}\left(\mathbb{R}^N\right) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega \right\}.$$

Both $\mathcal{W}^{s,p}(\Omega)$ and $\mathcal{W}^{s,p}_0(\Omega)$ are reflexive Banach spaces [31].

Recall that the Lebesgue space $L^{p_i}(E)$ is defined as the space of p_i-integrable functions $u : E \to \mathbb{R}$ with the finite norm

$$\|u\|_{L^{p_i}(E)} = \left(\int_E |u(x)|^{p_i} dx \right)^{\frac{1}{p_i}},$$

where $p_i \in (1, +\infty)$ for all $i = 1, 2, ..., N$.

The anisotropic Sobolev space is defined as follows:

$$W^{1,p_i}(\Omega) = \left\{ u \in W^{1,1}(\Omega) : \partial_i u \in L^{p_i}(\Omega) \right\},$$

and

$$W^{1,p_i}_0(\Omega) = \left\{ u \in W^{1,1}_0(\Omega) : \partial_i u \in L^{p_i}(\Omega) \right\},$$

endowed with the norm

$$\|u\|_{W^{1,p_i}_0(\Omega)} = \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

Definition 2.2. A function $\mathcal{J}$: $\mathcal{W}^{1,p_i}_0 \to \mathbb{R}$ is defined to be weakly lower semi-continuous if

$$\mathcal{J}(u) \leq \liminf_{n\to\infty} \mathcal{J}(u_n),$$

for any sequence u_n approaching $u \in \mathcal{W}^{1,p_i}_0$ in the weak topology on $\mathcal{W}^{1,p_i}_0$.

The zero Dirichlet boundary condition in this paper is defined as follows:

Definition 2.3. We say that $u \leq 0$ in $\mathbb{R}^N \backslash \Omega$ if $u = 0$ in $\mathbb{R}^N \backslash \Omega$ and for any $\epsilon > 0$, we have

$$(u-\epsilon)^+ \in \mathcal{W}_0^{1,p}(\Omega).$$

We say that $u = 0$ on $\mathbb{R}^N \backslash \Omega$, if u is nonnegative and $u \leq 0$ in $\mathbb{R}^N \backslash \Omega$.

The definition of weak solution in this paper is defined as

Definition 2.4. A positive function $u \in W^{1,p_i}_{\text{loc}}(\Omega) \cap L^{p_i-1}(\mathbb{R}^N)$ is a weak solution to problem (1.1) if

$$u > 0 \text{ in } \Omega,\ u = 0 \text{ in } \mathbb{R}^N \backslash \Omega,\ \frac{f(x)}{u^\delta} \in L^1_{\text{loc}}(\Omega),$$

for every $\phi \in C^1_c(\Omega)$, we have that

$$\sum_{i=1}^N \int_\Omega |\partial_i u(x)|^{p_i-2}\, \partial_i u(x) \partial_i \phi dx + \int\int_{\mathcal{D}(\Omega)} \mathcal{K}u(x,y)(\phi(x)-\phi(y))d\mu = \int_\Omega \frac{f(x)}{u(x)^\delta}\phi dx, \tag{2.2}$$

where

$$\mathcal{D}(\Omega) = \mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^c \times \Omega^c),$$

and

$$\mathcal{K}u(x,y) = |u(x)-u(y)|^{p-2}(u(x)-u(y)), \quad d\mu = |x-y|^{-N-ps}dxdy.$$

Lemma 2.5. *[19, Theorem 1.2] There exists a positive constant C, such that for every $u \in W^{1,p_i}(\Omega)$, we have*

$$||u||^{p_N}_{L^{\bar{p}^*}(\Omega)} \leq C \sum_{i=1}^N ||\partial_i u||^{p_i}_{L^{p_i}(\Omega)}, \tag{2.3}$$

where

$$\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i},$$

and

$$\bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}.$$

Lemma 2.6. *[32, Lemma 2.1] Let $1 < p_i < \infty$. Then for $\xi, \eta \in \mathbb{R}^N$, there exists a constant $C = C(p_i) > 0$ such that*

$$\left\langle |\xi|^{p_i-2}\xi - |\eta|^{p_i-2}\eta, \xi-\eta \right\rangle \geq \begin{cases} c_{p_i}|\xi-\eta|^{p_i}, & \text{if } p_i \geq 2, \\ c_{p_i}\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p_i}}, & \text{if } 1 < p_i < 2. \end{cases} \tag{2.4}$$

Lemma 2.7. *[33, Lemma 2.1] Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that*

$$|\psi(h)| \leq \frac{M\psi(k)^\alpha}{|h-k|^\beta}\ for\ all\ h > k > 0,$$

where $M > 0$, $\alpha > 1$ and $\beta > 0$. Then $\psi(d) = 0$, where $d^\beta = C\psi(0)^{\alpha-1}2^{\frac{\alpha\beta}{(\alpha-1)}}$.

2.2. Auxiliary results

For $n \in \mathbb{N}$, $f(x) \in L^1(\Omega)$ and $f(x) > 0$, let $f_n(x) := \min\{f(x), n\}$ and we consider the following approximated problem

$$\begin{cases} -\Delta_{\vec{p}} u_n(x) + (-\Delta)_p^s u_n(x) = \frac{f_n(x)}{(u^+ + \frac{1}{n})^\delta}, & x \in \Omega, \\ u_n(x) > 0, & x \in \Omega, \\ u_n(x) = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.5}$$

First, we consider the following useful result.

Lemma 2.8. *Let $g(x) \in L^\infty(\Omega)$, $g(x) \geq 0$. Then the following elliptic problem*

$$\begin{cases} -\Delta_{\vec{p}} u(x) + (-\Delta)_p^s u(x) = g(x), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.6}$$

has a unique positive weak solution $u \in \mathcal{W}_0^{1,p_i}(\Omega)$.

Proof. **Existence** : Define the energy functional $\mathcal{J} : \mathcal{W}_0^{1,p_i}(\Omega) \to \mathbb{R}$ as

$$\mathcal{J}(u) := \mathcal{J}_1(u) + \mathcal{J}_2(u) - \mathcal{J}_3(u),$$

where

$$\mathcal{J}_1(u) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u(x)|^{p_i}\, dx,$$

$$\mathcal{J}_2(u) = \frac{1}{p} \int\int_{\mathcal{D}(\Omega)} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy,$$

and

$$\mathcal{J}_3(u) = \int_\Omega g(x)u(x)dx.$$

(i) By the Sobolev embedding theorem and $g \in L^\infty(\Omega)$, we have

$$\mathcal{J}(v) \geq \frac{1}{p_i}\|v\|^{p_i}_{\mathcal{W}_0^{1,p_i}(\Omega)} - |\Omega|^{\frac{p-1}{p}}\|g\|_{L^\infty(\Omega)}\|v\|_{L^p(\Omega)} \;\to\; \infty \text{ as } \|v\|^{p_i}_{\mathcal{W}_0^{1,p_i}(\Omega)} \to \infty,$$

which implies the $\mathcal{J}$ is coercive.

(ii)$\mathcal{J}(v)$ is weakly lower semi-continuous in $\mathcal{W}_0^{1,p_i}(\Omega)$.

It easy to see that $\mathcal{J}_1(u)$ is differentiable, according to [34, Lemma 3.4], we know that $\mathcal{J}_i(u), i = 1, 2$ are weakly lower semi-continuous. On the other hand, we know that $\mathcal{J}_3(u)$ is a bounded linear functional. Thus $\mathcal{J}_3(u)$ is continuous. Therefore, $\mathcal{J}$ is weakly lower semi-continuous and

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{J}(u_n) =& \liminf_{n\to\infty} \mathcal{J}_1(u_n) + \liminf_{n\to\infty} \mathcal{J}_2(u_n) - \lim_{n\to\infty} \mathcal{J}_3(u_n) \\ \geq& \mathcal{J}_1(u) + \mathcal{J}_2 - \mathcal{J}_3(u) \\ =& \mathcal{J}(u). \end{aligned}$$

Combining the above properties of $\mathcal{J}$, we know that there exists a minimizer $u \in \mathcal{W}_0^{1,p_i}(\Omega)$ and which is also a critical point of $\mathcal{J}$, which also is the solution to Eq (2.6).

Uniqueness : Let $u_1, u_2 \in \mathcal{W}_0^{1,p_i}(\Omega)$ be two solutions to problem (2.6). Thus, for any $\varphi \in \mathcal{W}_0^{1,p_i}(\Omega)$, we have

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u_1|^{p_i-2} \partial_i u_1 \partial_i \varphi dx + \int\int_{\mathcal{D}(\Omega)} \mathcal{K}u_1(x,y)(\varphi(x) - \varphi(y))d\mu = \int_{\Omega} g\varphi dx, \tag{2.7}$$

and

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u_2|^{p_i-2} \partial_i u_2 \partial_i \varphi dx + \int\int_{\mathcal{D}(\Omega)} \mathcal{K}u_2(x,y)(\varphi(x) - \varphi(y))d\mu = \int_{\Omega} g\varphi dx. \tag{2.8}$$

Choosing $\varphi = u_1 - u_2$ and then subtracting (2.7) and (2.8), we obtain

$$\begin{aligned}&\sum_{i=1}^{N} \int_{\Omega} \left(|\partial_i u_1|^{p_i-2} \partial_i u_1 - |\partial_i u_2|^{p_i-2} \partial_i u_2\right)(\partial_i u_1 - \partial_i u_2)\, dx \\ &+ \int\int_{\mathcal{D}(\Omega)} (\mathcal{K}u_1(x,y) - \mathcal{K}u_2(x,y))\, [(u_1 - u_2)(x) - (u_1 - u_2)(y)]\, d\mu = 0. \end{aligned} \tag{2.9}$$

Using Lemma 2.6, we get the first term of the left hand side of (2.9) is nonnegative. On the other hand, by the monotonicity of the function $f(t) = t^{p-1} (p > 1)$, we have

$$\begin{aligned}&[\mathcal{K}u_1(x,y) - \mathcal{K}(u_2(x,y))]\, [(u_1 - u_2)(x) - (u_1 - u_2)(y)] \\ =&[|u_1(x) - u_1(y)|^{p-2}(u_1(x) - u_1(y)) - |u_2(x) - u_2(y)|^{p-2}(u_2(x) - u_2(y))]\cdot \\ &[(u_1(x) - u_1(x)) - (u_2(y) - u_2(y))] \\ \geq&0. \end{aligned} \tag{2.10}$$

Consequently,

$$\sum_{i=1}^{N} \int_{\Omega} \left(|\partial_i u_1|^{p_i-2} \partial_i u_1 - |\partial_i u_2|^{p_i-2} \partial_i u_2\right)(\partial_i u_1 - \partial_i u_2)\, dx = 0. \tag{2.11}$$

Therefore, $u_1(x) - u_2(x) = C$ for all $x \in \mathbb{R}^N$. Note that $u_1 - u_2 = 0$ on $\mathbb{R}^N \backslash \Omega$ since $u_i(x) = 0$ for $x \in \mathbb{R}^N \backslash \Omega$. Thus $u_1(x) \equiv u_2(x)$, which implies that the solution of (2.6) is unique.

Boundedness : For any $k > 1$, decompose $\mathbb{R}^N$ as $\mathbb{R}^N = A_k \cup A_k^c$, where

$$\begin{aligned} A_k &= \{x \in \Omega : u(x) \geq k\}, \\ A_k^c &= \{x \in \Omega : 0 < u(x) < k\}. \end{aligned}$$

Taking $G_k(u) := (u - k)^+ = \max\{u - k, 0\}$ as a test function in (2.6), we have

$$\begin{aligned}&\sum_{i=1}^{N} \int_{\Omega} |\partial_i u(x)|^{p_i-2} \partial_i u(x) \partial_i G_k(u(x)) dx + \int\int_{\mathcal{D}(\Omega)} \mathcal{K}u(x,y)\, [G_k(u(x)) - G_k(u(y))]\, d\mu \\ =& \int_{\Omega} g(x) G_k(u(x)) dx. \end{aligned} \tag{2.12}$$

Obviously,

$$
\begin{aligned}
&\mathcal{K}u(x,y)\left[G_k(u(x))-G_k(u(y))\right]\\
=&\,|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[(u(x)-k)^+-(u(y)-k)^+\right]\\
=&\begin{cases}
|u(x)-u(y)|^p, & \text{if} \quad u(x)>k, u(y)>k,\\
|u(y)-u(x)|^{p-1}(u(y)-k), & \text{if} \quad u(y)>k\geq u(x),\\
|u(x)-u(y)|^{p-1}(u(x)-k), & \text{if} \quad u(x)>k\geq u(y),\\
0, & \text{if} \quad u(x)\leq k, u(y)\leq k,
\end{cases}\\
\geq&\, 0.
\end{aligned} \tag{2.13}
$$

Therefore, combining (2.13) and (2.12) with Sobolev embedding theorem, we have

$$
\begin{aligned}
\sum_{i=1}^N\int_\Omega |\partial_i G_k(u)|^{p_i}\,dx &= \sum_{i=1}^N\int_{A_k} |\partial_i G_k(u)|^{p_i}\,dx + \sum_{i=1}^N\int_{A_k^c} |\partial_i G_k(u)|^{p_i}\,dx\\
&= \sum_{i=1}^N\int_{A_k} |\partial_i G_k(u)|^{p_i}\,dxdx\\
&\leq \int_\Omega g(x)G_k(u)dx\\
&\leq \|g\|_{L^\infty(\Omega)}\left(\int_\Omega G_k(u)^{\bar{p}^*}dx\right)^{\frac{1}{\bar{p}^*}}|A(k)|^{\frac{\bar{p}^*-1}{\bar{p}^*}}\\
&\leq C\|g\|_{L^\infty(\Omega)}\left(\sum_{i=1}^N\int_\Omega |\partial_i G_k(u)|^{p_i}\,dx\right)^{\frac{1}{p_N}}|A(k)|^{\frac{\bar{p}^*-1}{\bar{p}^*}}.
\end{aligned}
$$

Therefore

$$
\sum_{i=1}^N\int_\Omega |\partial_i G_k(u)|^{p_i}\,dx \leq C\|g\|_{L^\infty(\Omega)}^{\frac{p_N}{p_N-1}}|A(k)|^{\frac{p_N(\bar{p}^*-1)}{\bar{p}^*(p_N-1)}}. \tag{2.14}
$$

For every $1\leq k<h$ we know that $A(h)\subset A(k)$ and $u(x)-k\geq(h-k)$ in $A(h)$, we get

$$
\begin{aligned}
&(h-k)^{p_N}|A(h)|^{\frac{p_N}{\bar{p}^*}}\\
\leq&\left(\int_{A(h)} G_k(u)^{\bar{p}^*}dx\right)^{\frac{p_N}{\bar{p}^*}}\\
\leq&\left(\int_{A(k)} G_k(u)^{\bar{p}^*}dx\right)^{\frac{p_N}{\bar{p}^*}}\\
\leq& C\sum_{i=1}^N\int_\Omega |\partial_i G_k(u)|^{p_i}\,dx\\
\leq& C\|g\|_{L^\infty(\Omega)}^{\frac{p_N}{p_N-1}}|A(k)|^{\frac{p_N(\bar{p}^*-1)}{\bar{p}^*(p_N-1)}}.
\end{aligned}
$$

Hence, we have

$$
|A(h)|\leq \frac{C\|g\|_{L^\infty(\Omega)}^{\frac{\bar{p}^*}{p_N-1}}}{(h-k)^{\bar{p}^*}}|A(k)|^{\frac{\bar{p}^*-1}{p_N-1}}.
$$

Obviously,

$$\bar{p}^{*} > p_N.$$

Hence, using Lemma 2.7 we obtain

$$\|u\|_{L^\infty(\Omega)} \le C.$$

Positivity : First, taking $u_-(x) := \min\{u(x), 0\}$ as a test function in (2.6) and using $g \ge 0$, we have

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_-(x)|^{p_i}\, dx + \int\int_{\mathbb{R}^{2N}} \mathcal{K}u(x,y)\,(u_-(x) - u_-(y))\, d\mu = \int_\Omega g u_- dx \le 0, \tag{2.15}$$

where $\mathcal{K}u(x,y) = |u(x) - u(y)|^{p-2}(u(x) - u(y))$. Rewrite

$$\mathbb{R}^N \times \mathbb{R}^N = \cup_{i=1}^{4} A_i.$$

Denote,

$$\begin{aligned} A_1 &= \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x) \ge 0, u(y) \ge 0\right\}, \\ A_2 &= \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x) \ge 0, u(y) < 0\right\}, \\ A_3 &= \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x) < 0, u(y) \ge 0\right\}, \\ A_4 &= \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x) < 0, u(y) < 0\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathcal{K}u(x,y)\,(u_-(x) - u_-(y)) \\ &= |u(x) - u(y)|^{p-2}(u(x) - u(y))\,(u_-(x) - u_-(y)) \\ &= \begin{cases} 0, & \text{if} \quad (x,y) \in A_1, \\ |u(y)|^p, & \text{if} \quad (x,y) \in A_2, \\ |u(x)|^p, & \text{if} \quad (x,y) \in A_3, \\ |u(x) - u(y)|^p, & \text{if} \quad (x,y) \in A_4, \end{cases} \\ &\ge 0. \end{aligned}$$

Obviously,

$$|u(x) - u(y)|^{p-2}(u(x) - u(y))\,(u_-(x) - u_-(y)) \ge 0. \tag{2.16}$$

Using (2.16) in (2.15) we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_-|^{p_i}\, dx = 0.$$

Therefore, $u_- = C$ for all $x \in \mathbb{R}^N$. Note that $u_- = 0$ on $\mathbb{R}^N \backslash \Omega$ since $u_- := \min\{u, 0\}$. Thus $u \ge 0$ in Ω.

Second, assume that there exists a point $x_0 \in \Omega$ such that $u(x_0) = \inf_{x \in \Omega} u(x) = 0$, thus

$$\sum_{i=1}^{N}\int_\Omega \partial_i\left[|\partial_i u(x_0)|^{p_i-2}\,\partial_i u(x_0)\right]+\int\int_{\mathcal{D}(\Omega)}\frac{|u(x_0)-u(y)|^{p-2}[u(x_0)-u(y)]}{|x-y|^{N+ps}}dy$$
$$=\int\int_{\mathcal{D}(\Omega)}\frac{|-u(y)|^{p-2}[-u(y)]}{|x-y|^{N+ps}}dy$$
$$=-\int\int_{\mathbb{R}^{2N}}\frac{|u(y)|^{p-1}}{|x-y|^{N+ps}}dy$$
$$<0.$$

This is a contradiction since $g(x_0)\geq 0$. Hence, $u>0$ in Ω. □

Lemma 2.9. *For any $n\in\mathbb{N}$, there exists a unique positive solution $u_n\in \mathcal{W}_0^{1,p_i}(\Omega)\cap L^\infty(\Omega)$ to problem* (2.5)*. Moreover, The sequence $\{u_n\}$ is increasing with respect to n and*

$$u_n(x)\geq C_K>0 \ \ for\ \ K\Subset\Omega.$$

Proof. **Step1**. (**Existence**) Let $n\in\mathbb{N}$. By Lemma 2.8, for every $u\in \mathcal{W}_0^{1,p_i}(\Omega)\cap L^\infty(\Omega)$, there exists a unique $v\in \mathcal{W}_0^{1,p_i}(\Omega)\cap L^\infty(\Omega)$ such that

$$\begin{cases} -\Delta_{\vec{p}}v(x)+(-\Delta)_p^s\,v(x)=\frac{f_n(x)}{(u+\frac{1}{n})^\delta}, & x\in\Omega,\\ v(x)>0, & x\in\Omega,\\ v(x)=0, & x\in\mathbb{R}^N\setminus\Omega. \end{cases} \tag{2.17}$$

Define the operator $\mathcal{T}:u\mapsto v=\mathcal{T}(u)$, where v is the unique solution to (2.17). Choosing v as a test function in (2.17), using Sobolev imbedding theorem, we obtain

$$\sum_{i=1}^{N}\int_\Omega|\partial_i v(x)|^{p_i}\,dx\leq\int_\Omega n^{\delta+1}v(x)dx\leq Cn^{\delta+1}|\Omega|^{\frac{\bar{p}^*-1}{\bar{p}^*}}\left(\sum_{i=1}^{N}\int_\Omega|\partial_i v(x)|^{p_i}\,dx\right)^{\frac{1}{p_N}}.$$

Thus

$$\sum_{i=1}^{N}\int_\Omega|\partial_i v(x)|^{p_i}\,dx\leq Cn^{\frac{p_N(\delta+1)}{p_N-1}}|\Omega|^{\frac{p_N(\bar{p}^*-1)}{\bar{p}^*(p_N-1)}}:=R, \tag{2.18}$$

which implies that the ball with radius R in $\mathcal{W}_0^{1,p_i}(\Omega)$ remains unchanged under $\mathcal{T}$.

Now, we have to prove the continuity and compactness of $\mathcal{T}$, which is an operator from $\mathcal{W}_0^{1,p_i}(\Omega)$ to $\mathcal{W}_0^{1,p_i}(\Omega)$.

(i) **Continuity of $\mathcal{T}$:** In order to do this, we have to show that $\lim_{k\to\infty}\|v_k-v\|_{\mathcal{W}_0^{1,p_i}(\Omega)}=0$ if $\lim_{k\to\infty}\|u_k-u\|_{\mathcal{W}_0^{1,p_i}(\Omega)}=0$, where $v_k=\mathcal{T}(u_k)$ and $v=\mathcal{T}(u)$.

Choosing $\bar{v}_k(x)=v_k(x)-v(x)$ as a test function of the equations of v_k and v respectively, using (2.10), we get

$$\int_\Omega|\partial_i\bar{v}_k(x)|^{p_i}\,dx$$
$$\leq\sum_{i=1}^{N}\int_\Omega|\partial_i(v_k(x)-v(x))|^{p_i}\,dx$$

$$\leq \sum_{i=1}^{N} \int_{\Omega} \left[|\partial_i v_k(x)|^{p_i-2} \, \partial_i v_k(x) - |\partial_i v(x)|^{p_i-2} \, \partial_i v(x) \right] [\partial_i v_k(x) - \partial_i v(x)] \, dx$$
$$+ \int \int_{\mathcal{D}(\Omega)} [\mathcal{K} v_k(x,y) - \mathcal{K} v(x,y)] \, [(v_k - v)(x) - (v_k - v)(y)] \, d\mu$$
$$= \int_{\Omega} \left[\frac{f_n(x)}{\left(v_k + \frac{1}{n}\right)^{\delta}} - \frac{f(x)}{v^{\delta}} \right] [v_k(x) - v(x)] \, dx, \quad \text{if } \ p_i \geq 2. \tag{2.19}$$

Using Hölder and Sobolev inequalities we infer that

$$\left| \int_{\Omega} \left[\frac{f_n(x)}{\left(u_k + \frac{1}{n}\right)^{\delta}} - \frac{f(x)}{u^{\delta}} \right] [v_k(x) - v(x)] dx \right|$$
$$\leq \left[\int_{\Omega} \left| \frac{f_n(x)}{\left(u_k + \frac{1}{n}\right)^{\delta}} - \frac{f(x)}{u^{\delta}} \right|^{p_i^{*\prime}} dx \right]^{\frac{1}{p_i^{*\prime}}} \|\bar{v}_k\|_{L^{p_i^*}(\Omega)}$$
$$\leq C \left[\int_{\Omega} \left| \frac{f_n(x)}{\left(u_k + \frac{1}{n}\right)^{\delta}} - \frac{f(x)}{u^{\delta}} \right|^{p_i^{*\prime}} dx \right]^{\frac{1}{p_i^{*\prime}}} \|\partial_i \bar{v}_k\|_{L^{p_i}(\Omega)}. \tag{2.20}$$

By (2.19) and (2.20), using [10, Lemma 2.2], we find

$$\|\partial_i \bar{v}_k\|_{L^{p_i}(\Omega)}$$
$$\leq C n^{\frac{1}{p_i-1}} \left[\int_{\Omega} \left| \frac{1}{\left(u_k + \frac{1}{n}\right)^{\delta}} - \frac{1}{u^{\delta}} \right|^{p_i^{*\prime}} dx \right]^{\frac{1}{(p_i-1)p_i^{*\prime}}}$$
$$\leq C n^{\frac{1}{p_i-1}} \left[\int_{\Omega} \left| \frac{u^{\delta} - \left(u_k + \frac{1}{n}\right)^{\delta}}{\left(u_k + \frac{1}{n}\right)^{\delta} u^{\delta}} \right|^{p_i^{*\prime}} dx \right]^{\frac{1}{(p_i-1)p_i^{*\prime}}}$$
$$\leq C n^{\frac{1}{p_i-1}+\delta} \left[\int_{\Omega} |u - u_k|^{p_i^{*\prime}} \, dx \right]^{\frac{1}{(p_i-1)p_i^{*\prime}}}, \tag{2.21}$$

since the pointwise convergence of $u_k \to u$ in $\mathcal{W}_0^{1,p_i}(\Omega)$. we get

$$\lim_{k \to +\infty} \|v_k - v\|_{\mathcal{W}_0^{1,p_i}(\Omega)} = 0.$$

Therefore, in the case $p_i \geq 2$, the operator $\mathcal{T}$ is continuous from $\mathcal{W}_0^{1,p_i}(\Omega)$ to $\mathcal{W}_0^{1,p_i}(\Omega)$.

(ii) **Compactness of $\mathcal{T}$:** To achieve this, we have to show that, for some $v \in W_0^{1,p_i}(\Omega)$, it holds

$$\lim_{k \to +\infty} \|v_k - v\|_{W_0^{1,p_i}(\Omega)} = 0.$$

Let u_k be a bounded sequence in $\mathcal{W}_0^{1,p_i}(\Omega)$ and $v_k := \mathcal{T}(u_k)$. Then we have

$$u_k \rightharpoonup u \text{ in } \mathcal{W}_0^{1,p_i}(\Omega), \;\; u_k \to u \text{ in } L^t(\Omega), \; 1 < t < \bar{p}^*.$$

According to (2.18), we have $\|\mathcal{T}(u_k)\|_{\mathcal{W}_0^{1,p_i}(\Omega)} \le C$. Therefore there exists a subsequence, still denoted by $\{u_k\}$, such that

$$\mathcal{T}(u_k) \rightharpoonup v \in \mathcal{W}_0^{1,p_i}(\Omega), \;\; \mathcal{T}(u_k) \to v \in L^t(\Omega), \; 1 < t < \bar{p}^*.$$

For any $\varphi \in \mathcal{W}_0^{1,p_i}(\Omega)$,

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega |\partial_i v_k(x)|^{p_i-2} \partial_i v_k(x) \partial_i \varphi dx + \int\int_{\mathcal{D}(\Omega)} \frac{|v_k(x)-v_k(y)|^{p-2}(v_k(x)-v_k(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dxdy \\ &= \int_\Omega \frac{f_n(x)}{\left(u_k+\frac{1}{n}\right)^\delta} \varphi dx. \end{aligned} \tag{2.22}$$

Now, we show that as $k \to \infty$, (2.22) converges to

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega |\partial_i v(x)|^{p_i-2} \partial_i v(x) \partial_i \varphi dx + \int\int_{\mathcal{D}(\Omega)} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dxdy \\ &= \int_\Omega \frac{f_n(x)}{\left(u+\frac{1}{n}\right)^\delta} \varphi dx. \end{aligned} \tag{2.23}$$

By the dominated convergence theorem, we have

$$\lim_{k\to\infty} \int_\Omega \frac{f_n(x)}{\left(u_k+\frac{1}{n}\right)^\delta} \varphi dx = \int_\Omega \frac{f_n(x)}{\left(u+\frac{1}{n}\right)^\delta} \varphi dx,$$

and

$$\sum_{i=1}^{N} \partial_i v_k \to \sum_{i=1}^{N} \partial_i v \;\text{ pointwise almost everywhere in } \Omega.$$

Therefore, for every $\varphi \in C_c^1(\Omega)$, we have

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i v_k(x)|^{p_i-2} \partial_i v_k(x) \partial_i \varphi dx = \sum_{i=1}^{N} \int_\Omega |\partial_i v(x)|^{p_i-2} \partial_i v(x) \partial_i \varphi dx.$$

Since $\varphi \in C_c^1(\Omega)$ and v_k is uniformly bounded in $\mathcal{W}_0^{1,p_i}(\Omega)$,

$$\left\{ \frac{|v_k(x)-v_k(y)|^{p-2}(v_k(x)-v_k(y))}{|x-y|^{\frac{N+ps}{p'}}} \right\}_{n\in\mathbb{N}} \in L^{p'}\left(\mathbb{R}^N \times \mathbb{R}^N\right),$$

by the pointwise convergence of $v_k(x)$ to $v(x)$

$$\frac{|v_k(x)-v_k(y)|^{p-2}\left(v_k(x)-v_k(y)\right)}{|x-y|^{\frac{N+ps}{p'}}} \to \frac{|v(x)-v(y)|^{p-2}\left(v(x)-v(y)\right)}{|x-y|^{\frac{N+ps}{p'}}} \text{ a.e. in } \mathbb{R}^{2N}.$$

Since

$$\frac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\mathbb{R}^{2N}),$$

we get that the (2.22) converges to the (2.23). Similarly, combining (2.21) and (2.18), we have

$$\lim_{k\to+\infty} \|\mathcal{T}(u_k)-\mathcal{T}(u)\|_{\mathcal{W}_0^{1,p_i}(\Omega)} = 0.$$

Therefore, the operator $\mathcal{T}$ is continuous from $\mathcal{W}_0^{1,p_i}(\Omega)$ to $\mathcal{W}_0^{1,p_i}(\Omega)$. Then, Schauder fixed point theorem implies the existence of a fixed points u_n such that $u_n = \mathcal{T}(u_n)$, which is a weak solution to approximated problem (2.5).

Step2. (**Monotonicity**) Since u_n and u_{n+1} are positive solutions to problem (2.8), for any $\varphi \in \mathcal{W}_0^{1,p_i}(\Omega)$, we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega |\partial_i u_n(x)|^{p_i-2}\,\partial_i u_n(x)\partial_i\varphi dx + \int\int_{\mathcal{D}(\Omega)} \mathcal{K}u_n(x,y)(\varphi(x)-\varphi(y))d\mu\\ &= \int_\Omega \frac{f_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^\delta}\varphi dx,\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega |\partial_i u_{n+1}(x)|^{p_i-2}\,\partial_i u_{n+1}(x)\partial_i\varphi dx + \int\int_{\mathcal{D}(\Omega)} \mathcal{K}u_{n+1}(x,y)(\varphi(x)-\varphi(y))d\mu\\ &= \int_\Omega \frac{f_{n+1}(x)}{\left(u_{n+1}(x)+\frac{1}{n+1}\right)^\delta}\varphi dx.\end{aligned} \tag{2.25}$$

Taking $\varphi = (u_n(x)-u_{n+1}(x))^+$ as test function in (2.24) and (2.25), we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega |\partial_i u_n(x)|^{p_i-2}\,\partial_i u_n(x)\partial_i\left(u_n(x)-u_{n+1}(x)\right)^+ dx\\ &+\int\int_{\mathcal{D}(\Omega)} \mathcal{K}u_n(x,y)\left[(u_n(x)-u_{n+1}(x))^+-(u_n(y)-u_{n+1}(y))^+\right]d\mu\\ &= \int_\Omega \frac{f_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^\delta}\left(u_n(x)-u_{n+1}(x)\right)^+ dx,\end{aligned} \tag{2.26}$$

and

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}|\partial_i u_{n+1}(x)|^{p_i-2}\,\partial_i u_{n+1}(x)\partial_i\,(u_n(x)-u_{n+1}(x))^{+}dx\\
&+\int\int_{\mathcal{D}(\Omega)}\mathcal{K}u_{n+1}(x,y)\left[(u_n(x)-u_{n+1}(x))^{+}-(u_n(y)-u_{n+1}(y))^{+}\right]d\mu\\
=&\int_{\Omega}\frac{f_{n+1}(x)}{\left(u_{n+1}(x)+\frac{1}{n+1}\right)^{\delta}}\,(u_n(x)-u_{n+1}(x))^{+}\,dx. \qquad (2.27)
\end{aligned}
$$

Since $f_n(x)\leq f_{n+1}(x)$ for $x\in\Omega$, we have

$$
\begin{aligned}
&\int_{\Omega}\left[\frac{f_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^{\delta}}-\frac{f_{n+1}(x)}{\left(u_{n+1}(x)+\frac{1}{n+1}\right)^{\delta}}\right](u_n(x)-u_{n+1}(x))^{+}\,dx\\
\leq&\int_{\Omega}f_{n+1}(x)\left[\frac{1}{\left(u_n(x)+\frac{1}{n}\right)^{\delta}}-\frac{1}{\left(u_{n+1}(x)+\frac{1}{n+1}\right)^{\delta}}\right](u_n(x)-u_{n+1}(x))^{+}\,dx\\
=&\int_{\Omega}f_{n+1}(x)\left[\frac{\left(u_{n+1}(x)+\frac{1}{n+1}\right)^{\delta}-\left(u_n(x)+\frac{1}{n}\right)^{\delta}}{\left(u_n(x)+\frac{1}{n}\right)^{\delta}\left(u_{n+1}(x)+\frac{1}{n+1}\right)^{\delta}}\right](u_n(x)-u_{n+1}(x))^{+}\,dx\\
\leq&0. \qquad (2.28)
\end{aligned}
$$

Subtracting (2.26) with (2.27) and using the (2.28), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\left[|\partial_i u_n(x)|^{p_i-2}\,\partial_i u_n(x)-|\partial_i u_{n+1}(x)|^{p_i-2}\,\partial_i u_{n+1}(x)\right]\partial_i\,(u_n(x)-u_{n+1}(x))^{+}dx\\
+&\int\int_{\mathcal{D}(\Omega)}(\mathcal{K}u_n(x,y)-\mathcal{K}u_{n+1}(x,y))\left[(u_n(x)-u_{n+1}(x))^{+}-(u_n(y)-u_{n+1}(y))^{+}\right]d\mu\leq 0. \qquad (2.29)
\end{aligned}
$$

Following the argument in the proof of [35, Lemma 9], we obtain

$$
\int\int_{\mathcal{D}(\Omega)}(\mathcal{K}u_n(x,y)-\mathcal{K}u_{n+1}(x,y))\left[(u_n(x)-u_{n+1}(x))^{+}-(u_n(y)-u_{n+1}(y))^{+}\right]d\mu\geq 0. \qquad (2.30)
$$

Therefore, applying (2.30) in (2.29) we get

$$
\sum_{i=1}^{N}\int_{\Omega}\left[|\partial_i u_n(x)|^{p_i-2}\,\partial_i u_n(x)-|\partial_i u_{n+1}(x)|^{p_i-2}\,\partial_i u_{n+1}(x)\right]\partial_i\,(u_n(x)-u_{n+1}(x))^{+}dx\leq 0.
$$

Using Lemma 2.6 we obtain

$$
(u_n(x)-u_{n+1}(x))^{+}=C\ \text{ for all }\ x\in\mathbb{R}^N.
$$

Note that $u_n(x)=u_{n+1}(x)=0$ on $\mathbb{R}^N\backslash\Omega$ thus $C=0$, which implies that $u_{n+1}(x)\geq u_n(x)$ in Ω.

Step3. (**Uniform Positivity**) Let u_1 solves (2.6). By Lemma 2.8, for every $K\Subset\Omega$, there exists a constant $C_K>0$ such that $u_1\geq C_K>0$ in K. Again, since the monotonicity of u_n, we have $u_n\geq u_1$ in K. Therefore, for any $K\Subset\Omega$,

$$
u_n(x)\geq C_K>0,\quad\text{for }\ x\in K.
$$

□

3. Proof of main results

In order to prove the existence of positive solution to (1.1), we use the sequence of solutions u_n of problem (2.5). Then we need a priori estimates on u_n.

3.1. Auxiliary lemma

Lemma 3.1. *Let* $0 < \delta < 1$ *and* $1 < \bar{p} < N$*. Suppose that* $f > 0$*,* $f \in L^m(\Omega)$ *with*

$$m > \bar{m} = \frac{N\bar{p}}{N\bar{p} - p_i(N - \bar{p}) - (1 - \delta - p_i)(N - \bar{p})}.$$

Then, the sequence solutions u_n *to the approximate problem* (2.5) *such that*

(*i*) $u_n \in L^\infty(\Omega)$ *if* $m > \frac{N\bar{p}}{N\bar{p} - p_N(N-\bar{p})}$.

(*ii*) $u_n \in L^t(\Omega)$*, where* $t = \frac{m(1-\delta-p_i)N\bar{p}}{N\bar{p}(m-1) - p_i m(N-\bar{p})}$ *if*

$$\frac{N\bar{p}}{N\bar{p} - p_i(N - \bar{p}) - (1 - \delta - p_i)(N - \bar{p})} < m < \frac{N\bar{p}}{N\bar{p} - p_N(N - \bar{p})}.$$

Proof. (*i*) Let $A_k = \{x \in \Omega : u_n(x) \geq k\}$. Choosing $G_k(u) := (u - k)^+ \in \mathcal{W}_0^{1,p_i}(\Omega)$ as a test function in (2.5), we get

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n(x)|^{p_i - 2} \partial_i u_n(x) \partial_i G_k(u_n(x)) dx + \int\int_{\mathcal{D}(\Omega)} \mathcal{K} u_n(x, y) \left[G_k(u_n(x)) - G_k(u_n(y))\right] d\mu$$
$$= \int_\Omega \frac{f_n(x)}{\left(u_n(x) + \frac{1}{n}\right)^\delta} G_k(u_n(x)) dx. \tag{3.1}$$

Foe any $k > 1$, by (2.13) we know that,

$$\mathcal{K} u_n(x, y) \left[G_k(u_n(x)) - G_k(u_n(y))\right] \geq 0.$$

By Hölder inequality, Sobolev embedding theorem, $f_n(x) \leq f(x)$ and (3.1), we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u_n)|^{p_i} \, dx \\
&= \sum_{i=1}^{N} \int_{A_k} |\partial_i G_k(u_n)|^{p_i} \, dx \\
&\leq \int_\Omega \frac{f_n(x)}{\left(u_n(x) + \frac{1}{n}\right)^\delta} G_k(u_n) dx \\
&\leq \int_{A_k} f(x) G_k^{1-\delta}(u_n) dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_{A_k} f(x)^m dx \right)^{\frac{1}{m}} \left(\int_{\Omega} G_k(u_n)^{\bar{p}^*} dx \right)^{\frac{1-\delta}{\bar{p}^*}} |A(k)|^{1-\frac{1}{m}-\frac{1-\delta}{\bar{p}^*}} \\
&\leq C \left(\int_{A_k} f(x)^m dx \right)^{\frac{1}{m}} \left(\sum_{i=1}^{N} \int_{\Omega} |\partial_i G_k(u_n)|^{p_i} \, dx \right)^{\frac{1-\delta}{p_N}} |A(k)|^{1-\frac{1}{m}-\frac{1-\delta}{\bar{p}^*}}. \qquad (3.2)
\end{aligned}
$$

Hence

$$
\sum_{i=1}^{N} \int_{\Omega} |\partial_i G_k(u_n)|^{p_i} \, dx \leq C \left(\int_{A_k} f(x)^m dx \right)^{\frac{p_N}{m(p_N+\delta-1)}} |A(k)|^{\left(1-\frac{1}{m}-\frac{1-\delta}{\bar{p}^*}\right)\frac{p_N}{p_N+\delta-1}}.
$$

Let $h > k \geq 1$, we know that $A_h \subset A_k$ and $G_k(u_n) \geq h - k$ for in Ω, we have that

$$
\begin{aligned}
&|h-k|^{p_N} |A_h|^{\frac{p_N}{\bar{p}^*}} \\
&\leq \left(\int_{A(h)} G_k(u_n)^{\bar{p}^*} dx \right)^{\frac{p_N}{\bar{p}^*}} \leq \left(\int_{A(k)} G_k(u_n)^{\bar{p}^*} dx \right)^{\frac{p_N}{\bar{p}^*}} \\
&\leq C \sum_{i=1}^{N} \int_{A(k)} |\partial_i G_k(u_n)|^{p_i} \, dx \leq C \|f\|_{L^m(\Omega)}^{\frac{p_N}{p_N+\delta-1}} |A(k)|^{\left(1-\frac{1}{m}-\frac{1-\delta}{\bar{p}^*}\right)\frac{p_N}{p_N+\delta-1}}.
\end{aligned}
$$

Therefore

$$
|A_h| \leq C \frac{\|f\|_{L^m(\Omega)}^{\frac{\bar{p}^*}{p_N+\delta-1}} |A_k|^{\left(1-\frac{1}{m}-\frac{1-\delta}{\bar{p}^*}\right)\frac{\bar{p}^*}{p_N+\delta-1}}}{|h-k|^{\bar{p}^*}}.
$$

Note that

$$
\left(1 - \frac{1}{m} - \frac{1-\delta}{\bar{p}^*}\right) \frac{\bar{p}^*}{p_N + \delta - 1} > 1,
$$

if $m > \frac{N\bar{p}}{N\bar{p} - p_N(N-\bar{p})}$. Hence, apply Lemma 2.7 with

$$
M = C\|f\|_{L^m(\Omega)}^{\frac{\bar{p}^*}{p_N+\delta-1}} > 0, \alpha = \left(1 - \frac{1}{m} - \frac{1-\delta}{\bar{p}^*}\right) \frac{\bar{p}^*}{p_N + \delta - 1} > 1, \beta = \bar{p}^* > 0 \text{ and } \psi(k) = |A_k|,
$$

there exists k_0 such that $\psi(k) \equiv 0$ for all $k \geq k_0$. Thus,

$$
\operatorname{ess\,sup}_{\Omega} u \leq k_0.
$$

(ii) Choose $u_n^{p_i(\gamma-1)+1}$ $(\gamma > 1)$ as test function in (2.17), we have

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n|^{p_i-2} \, \partial_i u_n \partial_i u_n^{p_i(\gamma-1)+1} dx + \int \int_{\mathcal{D}(\Omega)} \mathcal{K} u_n(x,y) \left[u_n(x)^{p_i(\gamma-1)+1} - u_n(y)^{p_i(\gamma-1)+1} \right] d\mu \\
&= \int_{\Omega} \frac{f_n(x)}{\left(u_n(x) + \frac{1}{n}\right)^{\delta}} u_n(x)^{p_i(\gamma-1)+1} dx. \qquad (3.3)
\end{aligned}
$$

According to [10, Lemma 2.2], we have

$$\begin{aligned}
&\mathcal{K}u_n(x,y)\left[u_n(x)^{p_i(\gamma-1)+1}-u_n(y)^{p_i(\gamma-1)+1}\right]\\
=&|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\left[u_n(x)^{p_i(\gamma-1)+1}-u_n(y)^{p_i(\gamma-1)+1}\right]\\
\geq& C\left[u_n(x)+u_n(y)\right]^{p_i(\gamma-1)}|u_n(x)-u_n(y)|^p\\
\geq& 0.
\end{aligned}\tag{3.4}$$

Combining (3.4) and (3.3), and using Hölder inequality, we get

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega|\partial_i u_n|^{p_i-2}\partial_i u_n\partial_i u_n^{p_i(\gamma-1)+1}dx\\
=&\sum_{i=1}^N[p_i(\gamma-1)+1]\int_\Omega|\partial_i u_n|^{p_i}u_n^{p_i(\gamma-1)}dx\\
\leq&\int_\Omega\frac{f(x)}{\left(u_n+\frac{1}{n}\right)^\delta}u_n^{p_i(\gamma-1)+1}dx\\
\leq&\int_\Omega f(x)u_n^{p_i(\gamma-1)+1-\delta}dx\\
\leq&\|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{[p_i(\gamma-1)+1-\delta]m'}dx\right)^{\frac{1}{m'}}.
\end{aligned}$$

By Sobolev inequality,

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega[p_i(\gamma-1)+1]|\partial_i u_n|^{p_i}u_n^{p_i(\gamma-1)}dx\\
=&\sum_{i=1}^N\int_\Omega[p_i(\gamma-1)+1]\left(\frac{1}{\gamma}\right)^{p_i}\left|\partial_i u_n^\gamma\right|^{p_i}dx\\
\geq& C\left(\int_\Omega u_n^{\gamma\bar{p}^*}\right)^{\frac{p_N}{\bar{p}^*}}.
\end{aligned}$$

Therefore,

$$\left(\int_\Omega u_n^{\gamma\bar{p}^*}\right)^{\frac{p_N}{\bar{p}^*}}\leq C\|f\|_{L^m(\Omega)}\left(\int_\Omega u_n(x)^{[p_i(\gamma-1)+1-\delta]m'}\right)^{\frac{1}{m'}}.\tag{3.5}$$

Now we choose γ such that

$$\gamma\bar{p}^*=\left[p_i(\gamma-1)+1-\delta\right]m',$$

that is

$$\gamma=\frac{m(1-\delta-p_i)(N-\bar{p})}{N\bar{p}(m-1)-p_i m(N-\bar{p})}.$$

Since $\gamma > 1$, we know

$$\frac{N\bar{p}}{N\bar{p} - p_i(N - \bar{p}) - (1 - \delta - p_i)(N - \bar{p})} < m.$$

Thus $\frac{p_N}{\bar{p}^*} > \frac{1}{m'}$ gives

$$\left(\int_\Omega u_n(x)^{\gamma\bar{p}^*}\right)^{\frac{p_N}{\bar{p}^*} - \frac{1}{m'}} \leq C\|f\|_{L^m(\Omega)}.$$

Therefore, u_n is uniformly bounded in $L^t(\Omega)$ with $t = \gamma\bar{p}^*$. □

Lemma 3.2. *Let* $0 < \delta < 1$ *and* $1 < \bar{p} < N$*. Suppose that* $f > 0$ *and* $f \in L^m(\Omega)$ *with*

$$1 \leq m < \frac{N\bar{p}}{N\bar{p} - p_i(N - \bar{p}) - (1 - \delta - p_i)(N - \bar{p})}.$$

Then, the sequence solutions $\{u_n\}$ *to the approximate problem* (2.5) *are uniformly bounded in* $\mathcal{W}_0^{1,q}(\Omega)$ *with*

$$q = \frac{p_i m(1 - \delta - p_i)N\bar{p}}{m(1 - \delta)\left[N\bar{p} - (N - \bar{p})p_i\right] - p_i N\bar{p}}.$$

Proof. Similar to above taking $u_n^{p_i(\gamma-1)+1}$ as test function in (2.5) with $\frac{\delta+p_i-1}{p_i} \leq \gamma < 1$. However, this option is not acceptable, since the gradient of such a test function will be singular where $u_n(x) = 0$. Hence, for n fixed, choose $(u_n + \varepsilon)^{p_i(\gamma-1)+1} - \varepsilon^{p_i(\gamma-1)+1}$ $(0 < \varepsilon < \frac{1}{n})$ as test function in (2.5), we get

$$\sum_{i=1}^{N}[p_i(\gamma - 1) + 1]\int_\Omega |\partial_i u_n|^{p_i}(u_n + \varepsilon)^{p_i(\gamma-1)}dx \leq \int_\Omega \frac{f(x)[(u_n + \varepsilon)^{p_i(\gamma-1)+1} - \varepsilon^{p_i(\gamma-1)+1}]}{\left(u_n + \frac{1}{n}\right)^\delta}dx.$$

By $f_n(x) \leq f(x)$ and $\varepsilon < \frac{1}{n}$, we have

$$\sum_{i=1}^{N}[p_1(\gamma - 1) + 1]\int_\Omega |\partial_i u_n|^{p_i}(u_n + \varepsilon)^{p_i(\gamma-1)}dx \leq \int_\Omega f(x)(u_n + \varepsilon)^{p_i(\gamma-1)+1-\delta}dx. \tag{3.6}$$

By Sobolev inequality,

$$\begin{aligned}
&\sum_{i=1}^{N}[p_i(\gamma - 1) + 1]\int_\Omega |\partial_i u_n|^{p_i}(u_n + \varepsilon)^{p_i(\gamma-1)}dx\\
=&\sum_{i=1}^{N}[p_i(\gamma - 1) + 1]\int_\Omega \left(\frac{1}{\gamma}\right)^{p_i}|\partial_i\left[(u_n + \varepsilon)^\gamma - \varepsilon^\gamma\right]|^{p_i}dx\\
\geq &C\sum_{i=1}^{N}\int_\Omega |\partial_i\left[(u_n + \varepsilon)^\gamma - \varepsilon^\gamma\right]|^{p_i}dx\\
\geq &C\left(\int_\Omega\left[(u_n + \varepsilon)^\gamma - \varepsilon^\gamma\right]^{\bar{p}^*}dx\right)^{\frac{p_N}{\bar{p}^*}}.
\end{aligned} \tag{3.7}$$

Hence, by (3.6) and (3.7), we get

$$\left(\int_\Omega [(u_n+\varepsilon)^\gamma-\varepsilon^\gamma]^{\bar{p}^*}\,dx\right)^{\frac{p_N}{\bar{p}^*}} \le C\int_\Omega f(x)(u_n+\varepsilon)^{p_i(\gamma-1)+1-\delta}dx.$$

Let $\varepsilon\to 0$, we have

$$\left(\int_\Omega u_n^{\gamma\bar{p}^*}dx\right)^{\frac{\bar{p}}{\bar{p}^*}} \le C\int_\Omega f(x)u_n^{p_i(\gamma-1)+1-\delta}dx. \tag{3.8}$$

If $m=1$, we choose $\gamma=\frac{p_i+\delta-1}{p_i}$ in the (3.8), so that $u_n\in L^{\frac{(\delta+p_i-1)N\bar{p}}{p_i(N-\bar{p})}}(\Omega)$.
If $m>1$, from the proof of Lemma 3.1, we get that $u_n(x)\in L^t(\Omega)$ with

$$t=\frac{m(1-\delta-p_i)N\bar{p}}{N\bar{p}(m-1)-p_im(N-\bar{p})}.$$

Since $\gamma<1$, by (3.6), we have

$$\sum_{i=1}^N\int_\Omega \frac{|\partial_iu_n|^{p_i}}{(u_n+\varepsilon)^{p_i-p_i\gamma}}dx=\sum_{i=1}^N\int_\Omega |\partial_iu_n|^{p_i}\,(u_n+\varepsilon)^{p_i\gamma-p_i}dx \le C.$$

We can apply Hölder inequality (since $q<p_i$),

$$\begin{aligned}
&\int_\Omega |\partial_iu_n|^q\,dx\\
=&\int_\Omega \frac{|\partial_iu_n|^q}{(u_n+\varepsilon)^{(1-\gamma)q}}(u_n+\varepsilon)^{(1-\gamma)q}dx\\
=&\int_\Omega \left[\frac{|\partial_iu_n|^{p_i}}{(u_n+\varepsilon)^{(1-\gamma)p_i}}\right]^{\frac{q}{p_i}}(u_n+\varepsilon)^{(1-\gamma)q}dx\\
\le&\left(\int_\Omega \frac{|\partial_iu_n|^{p_i}}{(u_n+\varepsilon)^{(1-\gamma)p_i}}dx\right)^{\frac{q}{p_i}}\left(\int_\Omega (u_n+\varepsilon)^{(1-\gamma)q\frac{p_i}{p_i-q}}dx\right)^{1-\frac{q}{p_i}}\\
\le& C\left(\int_\Omega (u_n+\varepsilon)^{\frac{(1-\gamma)p_iq}{p_i-q}}dx\right)^{1-\frac{q}{p_i}}.
\end{aligned}$$

Choice γ and q such that

$$\frac{(1-\gamma)p_iq}{p_i-q}=t.$$

Therefore, $u_n\in \mathcal{W}_0^{1,q}(\Omega)$ with

$$q=\frac{p_im(1-\delta-p_i)N\bar{p}}{m(1-\delta)\left[N\bar{p}-(N-\bar{p})p_i\right]-p_iN\bar{p}}.$$

□

Lemma 3.3. *Suppose that $\delta = 1$ and $1 < \bar{p} < N$, $f > 0$, $f \in L^m(\Omega)$ with $m > 1$. Then there exists a weak solution u_n to problem* (2.5) *such that*

(i) If $m > \frac{N\bar{p}}{N\bar{p}-p_N(N-\bar{p})}$, Then $u_n \in L^\infty(\Omega)$;

(ii) If $1 \le m < \frac{N\bar{p}}{N\bar{p}-p_N(N-\bar{p})}$, Then $u_n \in L^t(\Omega)$, where

$$t = \frac{mp_iN\bar{p}}{p_im(N-\bar{p}) - N\bar{p}(m-1)}.$$

Proof. The proof of (i) is identical to that of Lemma 3.1, so we will omit it.

As for (ii), observe that if $m = 1$, then $t = \frac{N\bar{p}}{N-\bar{p}} = \bar{p}^*$. If $m > 1$, similar to Lemma 3.1, Choosing $u_n^{p_i(\gamma-1)+1}$ as test function in (2.5), we know that there is

$$\left(\int_\Omega u_n^{\gamma\bar{p}^*} dx\right)^{\frac{\bar{p}}{\bar{p}^*}} \le C\|f\|_{L^m(\Omega)} \left(\int_\Omega u_n(x)^{p_i(\gamma-1)m'}\right)^{\frac{1}{m'}}.$$

Choose γ such that

$$\gamma\bar{p}^* = [p_i(\gamma-1)]\, m'.$$

Obviously

$$\gamma = \frac{mp_i(N-\bar{p})}{mp_i(N-\bar{p}) - N\bar{p}(m-1)}.$$

Since $\gamma > 1$, we arrive at $1 < m$. Thus $\frac{p_N}{\bar{p}^*} > \frac{1}{m'}$ being

$$m < \frac{N\bar{p}}{N\bar{p} - p_N(N-\bar{p})},$$

so that $u_n \in L^t(\Omega)$ with $t = \gamma\bar{p}^*$. □

Simple modifications to the proof of Lemma 3.1 enable us to demonstrate Lemma 3.4.

Lemma 3.4. *Suppose that $\delta > 1$ and $1 < \bar{p} < N$, $f > 0$, $f \in L^m(\Omega)$ with $m > 1$. Then there exists a weak solution u_n to problem* (2.5) *such that*

(i) If $m > \frac{N\bar{p}}{N\bar{p}-p_N(N-\bar{p})}$, then $u_n \in L^\infty(\Omega)$.

(ii) If $1 \le m < \frac{N\bar{p}}{N\bar{p}-p_N(N-\bar{p})}$, then $u_n \in L^t(\Omega)$ with

$$t = \frac{m(1-\delta-p_i)N\bar{p}}{N\bar{p}(m-1) - p_im(N-\bar{p})}.$$

Proof. The proof of (i) is identical to that given in Lemma 3.1, so we omit it.

For (ii), by [21, Lemma 3.7], we known, if $m = 1$, the sequence $u_n^{\frac{\delta+p_i-1}{p_i}}$ is uniformly bounded in $\mathcal{W}_0^{1,p_i}(\Omega)$, This also gives u_n is bounded in $\mathcal{W}_{\mathrm{loc}}^{1,p_i}(\Omega)$.

If $1 < m < \frac{N\bar{p}}{N\bar{p}-p_N(N-\bar{p})}$, similar to Lemma 3.1, taking $u_n^{p_i(\gamma-1)+1}$ as test function in (2.5), this time with $\gamma > 1$ since $\gamma > \frac{\delta+p_i-1}{p_i}$, we have

$$\left(\int_\Omega u_n^{\gamma\bar{p}^*}dx\right)^{\frac{\bar{p}}{\bar{p}^*}} \leq C\|f\|_{L^m(\Omega)}\left(\int_\Omega u_n(x)^{[p_i(\gamma-1)+1-\delta]m'}\right)^{\frac{1}{m'}}.$$

Choosing γ in such a way that

$$\gamma\bar{p}^* = [p_i(\gamma-1)+1-\delta]\,m',$$

since $\gamma > \frac{\delta+p_i-1}{p_i}$ gives $m > 1$, and by $\frac{p_N}{\bar{p}^*} > \frac{1}{m'}$ being

$$m < \frac{N\bar{p}}{N\bar{p}-p_N(N-\bar{p})}.$$

Therefore, u_n is uniformly bounded in $L^t(\Omega)$ as well. □

3.2. *Proof of main theorem*

In this section, we give the proof of Theorem 1.1 by the approximate method.

Proof of Theorem 1.1. Let $f \in L^m(\Omega)$. By Lemma 3.2 and 3.1, we know that the solutions u_n to problem (2.5) are bounded in $\mathcal{W}_0^{1,p_i}(\Omega)$. Then, the pointwise limit u in $\mathcal{W}_0^{1,p_i}(\Omega)\cap L^{p_i-1}(\Omega)$. For any $\varphi\in\mathcal{W}_0^{1,p_i}(\Omega)$,

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega|\partial_i u_n(x)|^{p_i-2}\,\partial_i u_n(x)\partial_i\varphi dx + \int\int_{\mathcal{D}(\Omega)}\mathcal{K}u_n(x,y)(\varphi(x)-\varphi(y))d\mu\\ &=\int_\Omega\frac{f_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^\delta}\varphi dx.\end{aligned}\tag{3.9}$$

Then, for any $\varphi\in C_c^1(\Omega)$, we get

$$\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega|\partial_i u_n(x)|^{p_i-2}\,\partial_i u_n(x)\partial_i\varphi dx = \sum_{i=1}^N\int_\Omega|\partial_i u(x)|^{p_i-2}\,\partial_i u(x)\partial_i\varphi dx.\tag{3.10}$$

Since $\{u_n\}$ is uniformly bounded in $\mathcal{W}_0^{1,p_i}(\Omega)$,

$$\frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))}{|x-y|^{\frac{N+ps}{p'}}}\in L^{p'}\left(\mathbb{R}^N\times\mathbb{R}^N\right).$$

By point-wise convergence of $u_n(x)$ to $u(x)$

$$\frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))}{|x-y|^{\frac{N+ps}{p'}}}\to\frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{\frac{N+ps}{p'}}}\ \text{ a.e. in }\ \mathbb{R}^{2N}.$$

Then, we have

$$\lim_{n\to\infty}\int\int_{\mathcal{D}(\Omega)}\mathcal{K}u_n(x,y)(\varphi(x)-\varphi(y))d\mu=\int\int_{\mathcal{D}(\Omega)}\mathcal{K}u(x,y)(\varphi(x)-\varphi(y))d\mu. \tag{3.11}$$

By Lemma 2.9, for any $K\Subset\Omega$, $u_n(x)\geq C_K>0$ with $\mathrm{supp}(\varphi)=C_K>0$. Therefore, for any $\varphi\in C_c^1(\Omega)$ such that

$$\left|\frac{f_n(x)}{\left(u_n(x)+\frac{1}{n}\right)^{\delta}}\varphi\right|\leq\frac{\|\varphi\|_{L^\infty(\Omega)}}{C_K^{\delta}}|f| \text{ in } \Omega.$$

We conclude that

$$\lim_{n\to\infty}\int_\Omega\frac{f_n}{\left(u_n(x)+\frac{1}{n}\right)^{\delta}}\varphi dx=\int_\Omega\frac{f(x)}{u(x)^{\delta}}\varphi dx. \tag{3.12}$$

Finally, passing to the limit in (3.9), we conclude that

$$\sum_{i=1}^{N}\int_\Omega|\partial_i u(x)|^{p_i-2}\,\partial_i u(x)\partial\varphi dx+\int\int_{\mathcal{D}(\Omega)}\mathcal{K}u(x,y)(\varphi(x)-\varphi(y))d\mu=\int_\Omega\frac{f(x)}{\left(u(x)+\frac{1}{n}\right)^{\delta}}\varphi dx,$$

for all $\varphi\in C_c^1(\Omega)$, which shows that u is a solution to problem (1.1) and $u\in\mathcal{W}_0^{1,q}(\Omega)$. □

Proof of Theorem 1.3 and Theorem 1.4. The proof of Theorem 1.3 and 1.4 are similar, here we omit the details. □

4. Conclusions

The manuscript establishes the existence of solutions to mixed local and nonlocal anisotropic quasilinear singular elliptic eqautions. The interplay between the integrability and the singularity power is investigated. This results generalizes and complements the existing results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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