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Research article

Class of crosscap two graphs arising from lattices-II

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Abstract: In this series of papers, we study the crosscap two embedding of a class of multipartite graphs, namely, annihilating-ideal graphs of a lattice. In Part 1 of the series [Class of crosscap two graphs arising from lattices-I, *Mathematics*, **11** (2023), 1–26], we classified lattices with the number of atoms less than or equal to 4, whose annihilating-ideal graph can be embedded in the Klein bottle. In this paper, which is Part 2 of the series, we classify all finite lattices with at least 5 atoms whose annihilating-ideal graph is embedded in crosscap two surfaces. These characterizations help us to identify classes of multipartite graphs, which are embedded in the Klein bottle.

Keywords: atom of a lattice; multipartite graphs; Klein bottle; crosscap; annihilating-ideal graph
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1. Introduction

Let $\mathcal{L}$ be a finite lattice with a least element 0 and $A(\mathcal{L})$ be the set of all atoms. Before reading the paper, to familiarize with the notation and concepts used here, we strongly recommend the readers to read the first part of this work [3]. The annihilating-ideal graph of a lattice $\mathcal{L}$, denoted by $\mathbb{AG}(\mathcal{L})$, and defined by the graph whose vertex set is the set of all non-trivial ideals of $\mathcal{L}$ and two distinct vertices I and J being adjacent if and only if $I \wedge J = 0$, which was introduced by Afkhami et al. [1]. Note that the graph $\mathbb{AG}(\mathcal{L})$ is an r-partite graph for some $r \in \mathbb{N}$.

One of the most important topological properties of a graph is its genus. The genus of graphs associated with algebraic structures has been studied by many authors, see [2, 4, 5, 11]. The planar and crosscap one annihilating-ideal graph of lattices were characterized by Shahsavar [13] and Parsapour et al. [10], respectively. Also, whether the line graph associated with the annihilating-ideal graph of a lattice is planar or projective was characterized by Parsapour et al. [12]. Moreover, the authors of [9] characterized all lattices $\mathcal{L}$ whose line graph of $\mathbb{AG}(\mathcal{L})$ is toroidal. Recently, Asir et al. [3], provided the classification of lattices with the number of atoms less than or equal to 4 whose annihilating-ideal graph can be embedded in the non-orientable surface of crosscap two.

Note that a graph is planar if and only if it does not contain either of two forbidden graphs K_5 and $K_{3,3}$. An analogous characterization for embeddings of graphs on surfaces is known for the projective plane, which has 103 forbidden subgraphs, see [6]. For surfaces in general, it is known that the set of forbidden minors is finite and an explicit upper bound can be given, see [14]. In this paper, we have identified a class of minimal r-partite graphs that are not crosscap two graphs. So, these graphs may be realized as forbidden subgraphs for crosscap two.

The aim of this paper is to find the lattices with at least 5 atoms whose annihilating-ideal graph has non-orientable genus two embedding. This lead to the addition of r-partite graphs, where $r \geq 5$, to the family of crosscap two graphs. First of all, we observe that, by Proposition 3.3 [3], $|A(\mathcal{L})| \leq 6$ whenever the crosscap of $\mathbb{AG}(\mathcal{L})$ is two. Therefore, the lattices under consideration has either 5 or 6 atoms, and the corresponding classifications are done in Theorems 2.2 and 3.1. The reader can find an interesting connection between $\mathbb{AG}(\mathcal{L})$ and multipartite graphs in Examples 2.1 and 3.1.

Before moving into our main results, we have collected the crosscap lower bound of some graphs which will be used in the subsequent sections. In what follows, the notations $K_4 - e$ and $K_{4,5} - e$ denote graphs with an arbitrary edge removed from K_4 and $K_{4,5}$ respectively. Also, $K_8 - 3e$ denotes a graph with three arbitrary edges removed from K_8. Moreover, $K_{6,3} \cup (K_4 - e)$, denotes a graph, that includes the vertices and edges of $K_{6,3}$ with partition (X, Y), $|X| = 6$, as well as the edges of $K_4 - e$, a subgraph induced by any 4 arbitrary vertices in the partition X.

Proposition 1.1. *Let G be a graph.*

(i) *[3, Proof of Theorem 3] If G is isomorphic to $K_{6,3} \cup (K_4 - e)$ or $K_{4,5} - e$, then $\tilde{\gamma}(G) \geq 3$.*

(ii) *If G is isomorphic to $K_8 - 3e$ or $K_{2,2,2,2,2}$, then $\tilde{\gamma}(G) \geq 3$.*

Proof. (ii) The non-embeddability of K_8-3e in the Klein bottle directly follows from Euler's polyhedral equation. The non-embeddability of $K_{2,2,2,2,2}$ in the Klein bottle is a straightforward consequence of the characterization [8] of the graphs that triangulate both the torus and Klein bottle; note that it is well-known that $K_{2,2,2,2}$ triangulates the torus, see [7]. □

Let us directly move on to the lattices with 5 atoms.

2. The case when $|A(\mathcal{L})| = 5$

Before going into the characterization of crosscap two $\mathbb{AG}(\mathcal{L})$ for number of atoms of size 5, we rectify missing cases in projective characterization given in [10]. In particular, the authors have not discussed the sets of the form U_{ijk} for $1 \leq i \neq j \neq k \leq 5$ in Theorem 2.7. To be precise, let $|\bigcup_{n=1}^{5} U_n| = 5$. From the proof of Theorem 2.7(i) [10], the following two possibilities should be considered.

Case 1. $|U_{ij}| = 2$ for unique $1 \leq i \neq j \leq 5$. Then, Theorem 2.7(i) of [10] says that $\mathbb{AG}(\mathcal{L})$ is projective whenever $|\bigcup_{k\neq i,j} (U_{ik} \cup U_{jk})| \leq 2$ with $|U_{ik}|, |U_{jk}| \leq 1$. But consider any one of $U_{(ij)^c}, U_{(ik)^c}, U_{(jk)^c}$ as non-empty. More generally, let us assume $U_{(pq)^c} \neq \emptyset$ for some $U_{pq} \neq \emptyset$ with $1 \leq p, q \leq 5$. Then the sets $X = U_i \cup U_j \cup U_{ij} \cup \{[I_{pq}, I_{(pq)^c}]\}$ and $Y = \bigcup_{k \neq i,j} U_k$ form $K_{5,3}$ in $\mathbb{AG}(\mathcal{L})$ and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 2$. Therefore, $U_{(pq)^c} = \emptyset$ whenever $U_{pq} \neq \emptyset$.

Case 2. For some fixed i, $|\bigcup_{j=1}^{5} U_{ij}| \leq 3$, $|U_{ij}| \leq 1$ and $|U_{k\ell}| \leq 1$ for at most one pair k, ℓ with $1 \leq i \neq j \neq k \neq \ell \leq 5$ such that every vertex of $U_{k\ell}$ is adjacent to a maximum of one vertex from $\bigcup_{j=1}^{5} U_{ij}$.

If $|U_{(pq)^c}| \geq 2$ for some $U_{pq} \neq \emptyset$ and $1 \leq p, q \leq 5$, then the sets $X = U_p \cup U_q \cup U_{pq}$ and $Y = \bigcup_{r \neq p,q} U_r \cup U_{(pq)^c}$ form $K_{3,5}$ in $\mathbb{AG}(\mathcal{L})$, a contradiction. Also if $|U_{(pq)^c}|, |U_{(p_1q_1)^c}| = 1$ for some $|U_{pq}|, |U_{p_1q_1}| \neq \emptyset$ and $1 \leq p, q, p_1, q_1 \leq 5$, then the sets $X = U_p \cup U_q \cup U_{pq} \cup \{[I_{p_1q_1}, I_{(p_1q_1)^c}]\}$ and $Y = \bigcup_{r \neq p,q} U_r \cup U_{(pq)^c}$ form $K_{4,4} - e$ which has crosscap two, a contradiction. Therefore, $|\bigcup U_{(pq)^c}| \leq 1$, where the union is taken over all $U_{pq} \neq \emptyset$.

Suppose $|U_{k\ell}| = 1$ for some $1 \leq k, \ell \leq 5$ with a vertex in $U_{k\ell}$ adjacent to exactly one vertex of $\bigcup_{j=1}^{5} U_{ij}$, say a vertex in U_{ij}. We now claim that $\bigcup U_{(pq)^c} = \emptyset$, where the union is taken over all $U_{pq} \neq \emptyset$. In order to prove the claim, when either $U_{(ij)^c} \neq \emptyset$ or $U_{(k\ell)^c} \neq \emptyset$, let us take $U_{(ij)^c} \neq \emptyset$, then $X = U_i \cup U_j \cup U_{ij}$ and $Y = \bigcup_{m \neq i,j} U_m \cup U_{k\ell} \cup U_{(ij)^c}$ form $K_{3,5}$. Further, if $U_{(ij')^c} \neq \emptyset$ for some $1 \leq j' \neq j \leq 5$, then, $X = U_i \cup U_j \cup U_{ij} \cup [I_{ij'}, I_{(ij')^c}]$ and $Y = \bigcup_{m \neq i,j} U_m \cup U_{k\ell}$ form $K_{4,4} - e$. Thus, $\bigcup U_{(pq)^c} = \emptyset$.

Based on the addition of above mentioned cases in projective characterization, we can summarize it as follows.

Theorem 2.1. *Let $\mathcal{L}$ be a lattice with $|A(\mathcal{L})| = 5$ and let $|\bigcup_{n=1}^{5} U_n| = 5$. Then, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 1$ if and only if $|U_{ij}| \leq 2$ for all $1 \leq i \neq j \leq 5$ and one of the following conditions hold:*

(i) *There is a unique U_{ij} such that $|U_{ij}| = 2$ with $U_{i'j'}, U_{(ij)^c} = \emptyset$ for $i', j' \in \{1, \ldots, 5\} \setminus \{i, j\}$, $|\bigcup_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq}| \leq 2$ and $\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c} = \emptyset$. Moreover, if $U_{p_1q_1}, U_{p_2q_2} \neq \emptyset$, then $\{p_1, q_1\} \cap \{p_2, q_2\} \neq \emptyset$.*

(ii) *$|U_{ij}| \leq 1$ for all $1 \leq i \neq j \leq 5$. For some fixed $i \in \{1, \ldots, 5\}$, $|\bigcup_{j=1}^{5} U_{ij}| \leq 3$, atmost one of the sets $U_{k\ell}$ such that $|U_{k\ell}| = 1$, where $1 \leq i \neq k \neq \ell \leq 5$ and $|\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c}| \leq 1$. Moreover, if $U_{k\ell}$ has a vertex, then it is adjacent to atmost one vertex in $\bigcup_{j=1}^{5} U_{ij}$ and if such adjacency exists, then $\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c} = \emptyset$.*

We are now in a position to state and prove the main result of this section.

Theorem 2.2. *Let $\mathcal{L}$ be a lattice with $|A(\mathcal{L})| = 5$ and let $1 \leq i \neq j \neq k \neq l \neq m \leq 5$. Then, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if and only if one of the following conditions hold:*

(i) *$|\bigcup_{n=1}^{5} U_n| = 8$, there exists U_i with $|U_i| = 4$ and $U_{pq} = U_{jkl} = U_{jklm} = \emptyset$ for all $1 \leq p \neq q \leq 5$.*

(ii) *$|\bigcup_{n=1}^{5} U_n| = 7$, one of the following cases is satisfied:*

[a] *There exists U_i such that $|U_i| = 3$ with $|\bigcup_{j \neq i} U_{ij}| \leq 2$ and $|\bigcup_{k,\ell,m,n \neq i} U_{k\ell} \cup U_{k\ell m} \cup U_{k\ell mn}| \leq 1$. Moreover, if $|\bigcup_{k,\ell,m,n \neq i} U_{k\ell} \cup U_{k\ell m} \cup U_{k\ell mn}| = 1$, then $\bigcup_{j \neq i} U_{ij} = \emptyset$ and if $\bigcup_{k,\ell,m,n \neq i} U_{k\ell} \cup U_{k\ell m} \cup U_{k\ell mn} = \emptyset$, then $|\bigcup_{j \neq i} U_{ij}| \in \{1, 2\}$.*

[b] *There exist* U_i *and* U_j *such that* $|U_i| = |U_j| = 2$ *with* $|\bigcup_{k,\ell\notin\{i,j\}} U_{k\ell}| \leq 1$. *Moreover:*

[b1] *If* $|\bigcup_{k,\ell\notin\{i,j\}} U_{k\ell}| = 1$, *then* $\bigcup_{m=i,j;pq\neq ij}(U_{mn} \cup U_{pqr}) \cup U_{(ij)^c} \cup U_{ij} \cup U_{(i)^c} \cup U_{(j)^c} = \emptyset$.

[b2] *If* $\bigcup_{k,\ell\notin\{i,j\}} U_{k\ell} = \emptyset$, *then* $|\bigcup_{m=i,j;mn\neq ij} U_{mn} \cup U_{(ij)^c}| \leq 1$. *Also, if* $|\bigcup_{m=i,j;mn\neq ij} U_{mn} \cup U_{(ij)^c}| = 1$, *then* $|U_{ij}| \leq 1$ *and* $\bigcup_{pq\neq ij;pqr\neq(ij)^c} U_{pqr} \cup U_{(i)^c} \cup U_{(j)^c} = \emptyset$. *Moreover, if* $\bigcup_{m=i,j;mn\neq ij} U_{mn} \cup U_{(ij)^c} = \emptyset$, *then* $|U_{ij}| \leq 2$. *In addition, if* $|U_{ij}| = 2$, *then* $\bigcup_{pq\neq ij;pqr\neq(ij)^c} U_{pqr} \cup U_{(i)^c} \cup U_{(j)^c} = \emptyset$ *and if* $|U_{ij}| \leq 1$, *then* $|\bigcup_{p=i,j;pq\neq ij} U_{pqr}| \leq 2$ *together with* $|U_{pqr}| \leq 1$, *and either* $\bigcup_{p=i;q\neq j} U_{pqr} \cup U_{(j)^c} = \emptyset$ *or* $\bigcup_{p=j} U_{pqr} \cup U_{(i)^c} = \emptyset$.

(iii) $|\bigcup_{n=1}^{5} U_n| = 6$. *There exists* U_i *such that* $|U_i| = 2$ *with* $|\bigcup_{j,k\neq i} U_{jk}| \leq 2$ *and* $U_{j'k'} = \emptyset$ *when* $U_{jk} \neq \emptyset$ *for all* $j', k' \notin \{j, k\}$. *Moreover:*

[a] *There is* U_{jk} *such that* $|U_{jk}| = 2$ *for* $j, k \neq i$ *with* $\bigcup_{\ell\notin\{j,k\}} U_{i\ell} = \emptyset$, $|\bigcup_{m\in\{j,k\};p,q,r\neq i}(U_{im} \cup U_{pqr})| \leq 2$ *in which* $|U_{im}|, |U_{pqr}| \leq 1$ *and* $U_{(st)^c} = \emptyset$ *if* $U_{st} \neq \emptyset$ *for all* $1 \leq s \neq t \leq 5$.

[b] *There exist* $U_{jk}, U_{j_1k_1}$ *such that* $|U_{jk}| = |U_{j_1k_1}| = 1$ *where* $j, k, j_1, k_1 \neq i$ *and* $|\{j,k\}\cap\{j_1,k_1\}| = 1$ *with* $\bigcup_{\ell\notin\{j,k\}\cap\{j_1,k_1\}} U_{i\ell} = \emptyset$, $|\bigcup_{m=\{j,k\}\cap\{j_1,k_1\};p,q,r\neq i}(U_{im} \cup U_{pqr})| \leq 2$ *in which* $|U_{im}|, |U_{pqr}| \leq 1$ *and* $U_{(st)^c} = \emptyset$ *if* $U_{st} \neq \emptyset$ *for all* $1 \leq s \neq t \leq 5$.

[c] *There is a unique* U_{jk} *such that* $|U_{jk}| = 1$ *for all* $j, k \neq i$ *with* $|\bigcup_{\ell\notin\{j,k\}} U_{i\ell} \cup U_{(jk)^c}| \leq 1$. *Also, if* $|\bigcup_{\ell\notin\{j,k\}} U_{i\ell} \cup U_{(jk)^c}| = 1$, *then* $|\bigcup_{m\in\{j,k\}} U_{im}| \leq 2$ *in which each* $|U_{im}| \leq 1$ *and* $\bigcup_{p,q,r,s\neq i} U_{pqr} \cup U_{pqrs} = \emptyset$. *Furthermore, one of the following is satisfied in the case of* $\bigcup_{\ell\notin\{j,k\}} U_{i\ell} \cup U_{(jk)^c} = \emptyset$*:*

[c1] *If* $|\bigcup_{m\in\{j,k\}} U_{im}| = 3$ *or* 4, *then exactly one of the sets* U_{im}, *for* $m = j, k$, *has more than one element and* $\bigcup_{p,q,r,s\neq i} U_{pqr} \cup U_{pqrs} = \emptyset$.

[c2] *If* $|U_{im}| = 2$ *and* $U_{im'} = \emptyset$ *for* $m \neq m' \in \{j, k\}$, *then* $U_{(im)^c} = \emptyset$ *and* $|\bigcup_{p,q,r\neq i} U_{pqr}| \leq 1$. *Also,* $U_{pqrs} = \emptyset$ *for* $p, q, r, s \neq i$ *whenever* $|\bigcup_{p,q,r\neq i} U_{pqr}| = 1$.

[c3] *If* $|\bigcup_{m\in\{j,k\}} U_{im}| \leq 2$ *in which each* $|U_{im}| \leq 1$, *then* $U_{(im)^c} = \emptyset$ *and* $1 \leq |\bigcup_{m\in\{j,k\};p,q,r\neq i}(U_{im} \cup U_{pqr})| \leq 3$. *Also,* $U_{pqrs} = \emptyset$ *for* $p, q, r, s \neq i$ *whenever* $|\bigcup_{m\in\{j,k\};p,q,r\neq i}(U_{im} \cup U_{pqr})| = 3$ *together with* $|\bigcup_{p,q,r\neq i} U_{pqr}| = 2$.

[d] $\bigcup_{j,k\neq i} U_{jk} = \emptyset$ *and* $|\bigcup_{m\neq i} U_{im}| \leq 4$ *in which* $|U_{im}| \leq 3$. *Also, if two of* U_{im}*'s has 2 elements, then* $\bigcup_{p,q,r,s\neq i} U_{pqr} \cup U_{pqrs} = \emptyset$, *and if exactly one set* U_{im} *has more than 1 element, then* $|\bigcup_{p,q,r\neq i} U_{pqr}| \leq 1$ *together with* $\bigcup_{U_{im}\neq\emptyset} U_{(im)^c} = \emptyset$. *Furthermore, one of the following is satisfied in case of* $|U_{im}| \leq 1$ *for all* $m \neq i$*:*

[d1] *If* $|\bigcup_{m\neq i} U_{im}| = 4$, *then* $\bigcup_{U_{im}\neq\emptyset} U_{(im)^c} = \emptyset$.

[d2] *If* $|\bigcup_{m\neq i} U_{im}| \leq 3$, *then* $|\bigcup_{U_{im}\neq\emptyset} U_{(im)^c}| \leq 1$. *Also, if* $|\bigcup_{U_{im}\neq\emptyset} U_{(im)^c}| = 1$, *then* $\bigcup_{p\neq i,pqr\neq(im)^c} U_{pqr} = \emptyset$.

If $\bigcup_{U_{im}\neq\emptyset} U_{(im)^c} = \emptyset$, *then* $|\bigcup_{p,q,r\neq i;pqr\neq(im)^c} U_{pqr}| \leq 2$ *whenever* $|\bigcup_{m\neq i} U_{im}| = 3$, *and* $2 \leq |\bigcup_{p,q,r\neq i;pqr\neq(im)^c} U_{pqr}| \leq 4$ *with atmost one* $|U_{pqr}| \in \{2,3\}$ *whenever* $|\bigcup_{m\neq i} U_{im}| \leq 2$. *Further, in the last part, if* $|\bigcup_{p,q,r\neq i;pqr\neq(im)^c} U_{pqr}| = 2$, *then exactly one non-empty set exists in the collection* $\{U_{pqr} : p,q,r \neq i; pqr \neq (im)^c\}$.

(iv) $|\bigcup_{n=1}^{5} U_n| = 5$ *and one of the following cases is satisfied:*

[a] *There is a* U_{ij} *such that* $|U_{ij}| = 4$, $\bigcup_{\ell,m\notin\{i,j\}} U_{\ell m} \cup U_{(ij)^c} = \emptyset$, $|\bigcup_{p\in\{i,j\};q\notin\{i,j\}} U_{pq}| \leq 2$ *in which* $|U_{pq}| \leq 1$ *and* $U_{p'q'}, U_{(pq)^c} = \emptyset$ *when* $|U_{pq}| = 1$ *where* $p', q' \notin \{p, q\}$.

[b] *There is a* U_{ij} *such that* $|U_{ij}| = 3$, $\bigcup_{\ell,m\notin\{i,j\}} U_{\ell m} \cup U_{(ij)^c} = \emptyset$, $|\bigcup_{p\in\{i,j\};q\notin\{i,j\}} U_{pq}| \leq 3$, *where the choice i or j for p is placed at most two times in the union, in which at most one of* U_{pq}*'s has two elements and* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \leq 1$. *Further, if* $|U_{pq}| = 2$ *for some* $p \in \{i, j\}; q \notin \{i, j\}$, *then* $\bigcup_{p',q'\notin\{p,q\}} U_{p'q'} \cup \bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$, *and if* $|\bigcup_{p\in\{i,j\};q\notin\{i,j\}} U_{pq}| = 3$ *with* $|U_{pq}| \leq 1$, *then the three choices for q is not distinct. Moreover, if* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| = 1$, *then* $|\bigcup_{p\in\{i,j\};q\notin\{i,j\}} U_{pq}| \leq 2$ *with the choice for two pairs of* p, q*'s are not mutually disjoint.*

[c] *There is a* U_{ij} *such that* $|U_{ij}| = 2$ *with* $|\bigcup_{\ell,m\notin\{i,j\}} U_{\ell m} \cup U_{(ij)^c}| \leq 1$. *Further, if* $|\bigcup_{\ell,m\notin\{i,j\}} U_{\ell m} \cup U_{(ij)^c}| = 1$, *then* $|\bigcup_{p\in\{i,j\};q\notin\{i,j\}} U_{pq}| \leq 2$ *with* $|U_{pq}| \leq 1$ *and* $U_{p'q'}, U_{(pq)^c} = \emptyset$ *when* $|U_{pq}| = 1$, *where* $p', q' \notin \{p, q\}$. *Moreover, if* $\bigcup_{\ell,m\notin\{i,j\}} U_{\ell m} \cup U_{(ij)^c} = \emptyset$, *then* $2 \leq |\bigcup_{p\in\{i,j\};q\notin\{i,j\}} U_{pq}| \leq 4$ *in which at most one of the sets* U_{pq} *has two elements, where the choice i or j for p is placed at most once in the union, and one of the following is satisfied:*

[c1] *If* $|U_{rs}| = 2$ *for some* $r \in \{i, j\}$, $s \notin \{i, j\}$, *then* $\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$ *and at most one of the sets* U_{tu} *is non-empty with the property that* $\{r, s\} \cap \{t, u\} = \emptyset$.

[c2] *If* $|U_{pq}| \leq 1$ *for all* $p \in \{i, j\}, q \notin \{i, j\}$, *then* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \leq 1$. *Also, if* $|U_{(pq)^c}| = 1$ *for some* $|U_{pq}| = 1$, *then every non-empty set* U_{rs} *should have the property that* $\{r, s\} \cap \{p, q\} \neq \emptyset$.

[d] $|U_{ij}| \leq 1$ *for all* $1 \leq i \neq j \leq 5$.

[d1] *If* $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| = 5$, *then* $\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$, *and at least one of the sets* $U_{p_1q_1}, U_{p_2q_2}, U_{p_3q_3}$ *or* $U_{p_4q_4}$ *must be empty whenever the indices satisfy the condition* $\{p_1, q_1\} \cap \{p_2, q_2\} = \emptyset$ *and* $\{p_3, q_3\} \cap \{p_4, q_4\} = \emptyset$.

[d2] *If* $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| = 4$, *then* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \leq 1$. *Moreover, if* $\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$, *then the subgraph induced by the set* $\bigcup_{1\leq p\neq q\leq 5} U_{pq}$ *has more than one edge and if* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| = 1$, *say* $|U_{(rs)^c}| = 1$, *then the vertex in* U_{rs} *is adjacent to at most two vertices of* $\bigcup_{U_{pq}\neq\emptyset;pq\neq rs} U_{pq}$. *Further, if there is an adjacency between the vertex of* U_{rs} *and a vertex of* $\bigcup_{U_{pq}\neq\emptyset;pq\neq rs} U_{pq}$, *then the subgraph induced by the set* $\bigcup_{U_{pq}\neq\emptyset;pq\neq rs} U_{pq}$ *is an empty graph.*

[d3] *If* $|\bigcup_{1\le p\neq q\le 5} U_{pq}| \in \{2,3\}$, *then* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \le 3$. *Moreover:*

• *If* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| = 3$, *then* $|U_{(pq)^c}| = 3$ *for some* $1 \le p \neq q \le 5$ *and no non-empty set* U_{rs} *exist with* $\{r,s\} \cap \{p,q\} = \emptyset$.

• *Let* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| = 2$. *If a unique set* $U_{(pq)^c} \neq \emptyset$ *for* $1 \le p \neq q \le 5$, *then at most one non-empty set* U_{rs} *exists with* $\{r,s\}\cap\{p,q\} = \emptyset$. *If two sets* $U_{(p_1q_1)^c}, U_{(p_2q_2)^c} \neq \emptyset$ *for* $1 \le p_1, q_1, p_2, q_2 \le 5$, *then no non-empty set* U_{rs} *exists with* $\{r,s\} \cap \{p_f, q_f\} = \emptyset$ *for* $1 \le f \le 2$.

• *If* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| = 1$, *then exactly one non-empty set* U_{rs} *exists with* $\{r,s\} \cap \{p,q\} = \emptyset$.

[d4] *If* $|\bigcup_{1\le p\neq q\le 5} U_{pq}| = 1$, *then* $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \in \{2,3\}$.

Proof. If $|\bigcup_{n=1}^5 U_n| \ge 9$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{5,4}$ as a subgraph so that $|\bigcup_{n=1}^5 U_n| \le 8$.

Case 1. Let $|\bigcup_{n=1}^5 U_n| = 8$. Suppose $|U_1| = 4$. If $\bigcup_{k,p\neq 1} U_{ij} \cup U_{k\ell m} \cup U_{pqrs} \neq \emptyset$ for some $1 \le i < j \le 5$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{4,5} - e$ and by Proposition 1.1, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Therefore, $\bigcup_{k,p\neq 1} U_{ij} \cup U_{k\ell m} \cup U_{pqrs} = \emptyset$. Now the graph $\mathbb{AG}(\mathcal{L})$ (except the vertices of degree one and two) is a subgraph of H_1 (as given in [3, Figure 1(a)]) and so by [3, Lemma 3.5], we get $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$. If $|U_1| = 3$, then the subgraph induced by the sets $X = U_1 \cup U_5$ and $Y = U_2 \cup U_3 \cup U_4$ contains H_4 in $\mathbb{AG}(\mathcal{L})$ and so by [3, Lemma 3.6], $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Also, if $|U_1| = 2$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{2,2,2,2}$ as a subgraph and by Proposition 1.1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$.

Case 2. Let $|\bigcup_{n=1}^5 U_n| = 7$.

Case 2.1. Suppose $|U_1| = 3$. If the subgraph induced by $\left\langle V(\mathbb{AG}(\mathcal{L})) - \{\bigcup_{n=1}^5 U_n\}\right\rangle$ has an edge (I,J), then the vertices $I_1, I_1', I_1'', I_2, I_3, I_4, I_5, [I,J]$ form $K_8 - 3e$ and so by Proposition 1.1, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Therefore each vertex of U_{1mn} is adjacent to exactly two vertices in $\mathbb{AG}(\mathcal{L})$ which are also adjacent. Also if $I, J \in \bigcup_{i,p,s\neq 1} U_{ij} \cup U_{pqr} \cup U_{stuv}$, then the subgraph induced by $I_1, [I_1', I], [I_1'', J], I_2, I_3, I_4$ and I_5 form K_7 in $\mathbb{AG}(\mathcal{L})$, a contradiction. Thus $|\bigcup_{i,p,s\neq 1} U_{ij} \cup U_{pqr} \cup U_{stuv}| \le 1$ and among all the remaining sets we have to examine only those sets of the form U_{1k}.

Let $I \in \bigcup_{i,p,s\neq 1} U_{ij} \cup U_{pqr} \cup U_{stuv}$. If $U_{1k} \neq \emptyset$ for some $k \neq 1$, then, the subgraph $G_{21} = \mathbb{AG}(\mathcal{L}) - \{I, (I_k, I_\ell), (I_k, I_m), (I_k, I_n)\}$ contains $K_{4,4} - e$ with partite sets $X = U_1 \cup U_{1k}$ and $Y = U_k \cup U_\ell \cup U_m \cup U_n$ where $\ell, m, n \in \{2,3,4,5\} \setminus \{k\}$ and $e = (I_{1k}, I_k)$. Note that any N_2-embedding of $K_{4,4} - e$ has one hexagonal and six rectangular faces. Since I is adjacent to three vertices I_1, I_1' and I_1'' of X, the vertex I must be inserted into the hexagonal face of the N_2-embedding of $K_{4,4} - e$. If I_k is in the hexagonal face, then I_k is in exactly two distinct rectangular faces so that the three edges incident with I_k, namely $(I_k, I_\ell), (I_k, I_m), (I_k, I_n)$, cannot be drawn without edge crossing, a contradiction. If not, $I_{1k} \in X$ must be in the hexagonal face. Therefore, the hexagonal face does not contain all the three vertices of X namely I_1, I_1' and I_1''. Thus, I cannot be embedded, a contradiction. Hence, $U_{1k} = \emptyset$ for all $2 \le k \le 5$.

Assume $\bigcup_{i,p,s\neq 1} U_{ij} \cup U_{pqr} \cup U_{stuv} = \emptyset$. Suppose $|\bigcup_{k=2}^5 U_{1k}| \ge 3$ and let $I_{1\ell}, I_{1m}, I_{1n} \in \bigcup_{k=2}^5 U_{1k}$. Then, the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{1m}, I_{1n}\}$ contains $K_{4,4} - e$ with partitions $X = U_1 \cup U_{1\ell}$ and $Y = U_2 \cup U_3 \cup U_4 \cup U_5$. Clearly, any N_2-embedding of $K_{4,4} - e$ has one hexagonal and six rectangular faces. The vertices I_{1m} and I_{1n} are adjacent to three vertices of Y in $\mathbb{AG}(\mathcal{L})$. So it requires at least two hexagonal faces in a N_2-embedding of $K_{4,4} - e$, a contradiction. Thus, $|\bigcup_{k=2}^5 U_{1k}| \le 2$.

Further, $\mathbb{AG}(\mathcal{L})$ is projective whenever $\bigcup_{1\leq i<j\leq 5} U_{ij} = \emptyset$ with $\bigcup_{p,s\neq 1} U_{pqr} \cup U_{stuv} = \emptyset$. Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if $|\bigcup_{i,p,s\neq 1} U_{ij} \cup U_{pqr} \cup U_{stuv}| = 1$ with $\bigcup_{k=2}^{5} U_{1k} = \emptyset$ or $\bigcup_{i,p,s\neq 1} U_{ij} \cup U_{pqr} \cup U_{stuv} = \emptyset$ with $|\bigcup_{k=2}^{5} U_{1k}| \in \{1,2\}$.

Case 2.2. Suppose $|U_1| = 2$. Then, $|U_2|$ must be 2.

If $|U_{ij}| \geq 2$ for some $i \neq 1,2$, then the contraction of $\mathbb{AG}(\mathcal{L})$ induced by the set $\{I_1, [I'_1, I_{ij}], I_2, [I'_2, I'_{ij}], I_3, I_4, I_5\}$ form K_7. So, $|U_{ij}| \leq 1$ for all $i, j \notin \{1,2\}$.

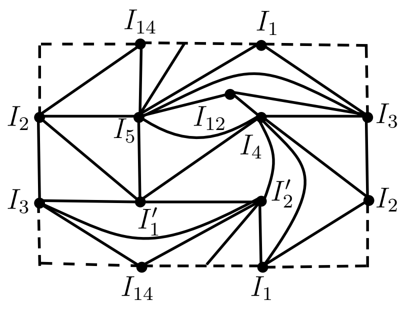
Suppose $|U_{ij}| = 1$ for some $i \neq 1,2$. If $I \in \bigcup_{k=1,2;pq\neq 12} (U_{k\ell} \cup U_{pqr}) \cup U_{(ij)^c} \cup U_{1345} \cup U_{2345}$, then I is adjacent to one of I_1, I_2 or I_{ij}. The latter case, that is $(I, I_{ij}) \in E(\mathbb{AG}(\mathcal{L}))$, is not possible because $\left\langle \bigcup_{n=1}^{5} U_n \cup \{[I, I_{ij}]\} \right\rangle \cong K_8 - 2e$. Also if either $(I, I_1) \in E(\mathbb{AG}(\mathcal{L}))$ or $(I, I_2) \in E(\mathbb{AG}(\mathcal{L}))$, then we can merge such an edge so that $K_8 - 3e$ is a minor subgraph of $\mathbb{AG}(\mathcal{L})$. Thus, in this case, $V(\mathbb{AG}(\mathcal{L})) \setminus \{\bigcup_{n=1}^{5} U_n \cup U_{12i} \cup U_{12j}\} = \emptyset$.

Suppose $U_{ij} = \emptyset$ for all $i \neq 1,2$. Let $I, J \in \bigcup_{k=1,2;k\ell\neq 12} U_{k\ell} \cup U_{345}$. If $I, J \in U_{345}$, then the partition sets $\{I, J, I_3, I_4, I_5\}$ and $\{U_1 \cup U_2\}$ form $K_{5,4}$ in $\mathbb{AG}(\mathcal{L})$. If not, we have $|U_{k\ell}| \geq 1$ for some $k \in \{1,2\}$ and $k\ell \neq 12$ so that the partition sets $U_k \cup U_\ell \cup \{I, J\}$ and $\bigcup_{m\neq k,\ell} U_m$ form $K_{5,4} - e$ in $\mathbb{AG}(\mathcal{L})$. Thus, $|\bigcup_{k=1,2;k\ell\neq 12} U_{k\ell} \cup U_{345}| \leq 1$.

Let $I \in \bigcup_{k=1,2;k\ell\neq 12} U_{k\ell} \cup U_{345}$.

- In the case of $|U_{12}| \geq 2$, note that I is adjacent to either the vertices of U_1 or U_2, say U_1. Here, the contraction of $\mathbb{AG}(\mathcal{L})$ contains $K_{6,3} \cup (K_4 - e)$ with partite sets $\{I_1, [I'_1, I], I_2, I'_2, I_{12}, I'_{12}\}$ and $\bigcup_{m\neq 1,2} U_m$.
- In the case of $\bigcup_{pq\neq 12;pqr\neq 345} U_{pqr} \cup U_{1345} \cup U_{2345} \neq \emptyset$, by contracting a single edge in $\mathbb{AG}(\mathcal{L})$, we get the contraction of $\mathbb{AG}(\mathcal{L})$ contains H_4.

Thus, $|\bigcup_{k=1,2;k\ell\neq 12} U_{k\ell} \cup U_{345}| = 1$, $|U_{12}| \leq 1$ and $\bigcup_{pq\neq 12;pqr\neq 345} U_{pqr} \cup U_{1345} \cup U_{2345} = \emptyset$. For this case, with the help of Figure 1, we get $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.



$|\bigcup_{k=1,2;pq\neq 12} U_{k\ell} \cup U_{345}| = 1, |U_{12}| \leq 1$ and $\bigcup_{pq\neq 12;pqr\neq 345} U_{pqr} \cup U_{1345} \cup U_{2345} = \emptyset.$

Figure 1. $|\bigcup_{n=1}^{5} U_n| = 7$ with $|U_1| = |U_2| = 2$.

Let $\bigcup_{k=1,2;k\ell\neq 12} U_{k\ell} \cup U_{345} = \emptyset$.

- In the case of $|U_{12}| \geq 3$, $\mathbb{AG}(\mathcal{L})$ contains $K_{3,7}$ with partite sets $U_1 \cup U_2 \cup U_{12}$ and $U_3 \cup U_4 \cup U_5$.
- In the case of $|U_{12}| = 2$, we have $\bigcup_{pq\neq 12;pqr\neq 345} U_{pqr} \cup U_{1345} \cup U_{2345} = \emptyset$. If not, there exists some $J \in \bigcup_{pq\neq 12;pqr\neq 345} U_{pqr} \cup U_{1345} \cup U_{2345}$, then J is adjacent to either I_1 or I_2, say $(J, I_1) \in E(\mathbb{AG}(\mathcal{L}))$ so that the contraction of $\mathbb{AG}(\mathcal{L})$ contains $K_{6,3} \cup (K_4 - e)$ with partite sets $\{[J, I_1], I'_1, I_2, I'_2, I_{12}, I'_{12}\}$ and $U_3 \cup U_4 \cup U_5$.
- In the case of $|U_{12}| \leq 1$:

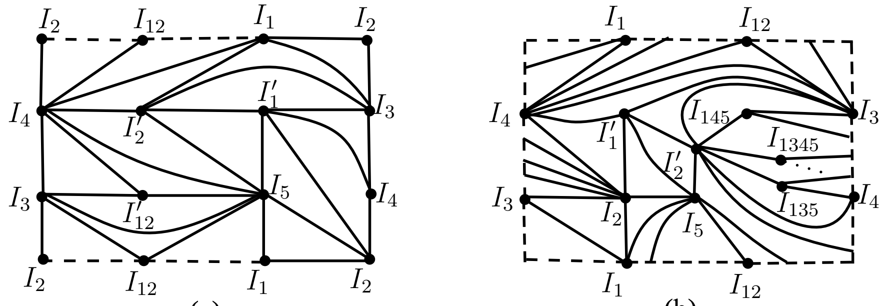
  (a) If $|U_{pqr}| \geq 2$ for $pq \neq 12$ and $pqr \neq 345$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,6} \cup (K_4 - e)$ with partite sets $U_p \cup U_q \cup U_r \cup U_{pqr}$ and $\bigcup_{m\neq p,q,r} U_m$. Therefore, $|U_{pqr}| \leq 1$ for $pq \neq 12$ and $pqr \neq 345$.

  (b) If $J \in \bigcup_{p=1;q\neq 2} U_{pqr} \cup U_{1345}$ and $K \in \bigcup_{p=2} U_{pqr} \cup U_{2345}$, then the contraction of $\mathbb{AG}(\mathcal{L})$ induced by the set $\{I_1, [I'_1, K], I_2, [I'_2, J], I_3, I_4, I_5\}$ form K_7. Therefore, either $\bigcup_{p=1;q\neq 2} U_{pqr} \cup U_{1345} = \emptyset$ or $\bigcup_{p=2} U_{pqr} \cup U_{2345} = \emptyset$.

  (c) If $|\bigcup_{p=1;q\neq 2} U_{pqr}| \geq 3$, that is $|U_{134}| = |U_{135}| = |U_{145}| = 1$, then consider the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{135}, I_{145}, (I_1, I_4), (I'_1, I_4), (I_3, I_4), (I_1, I_3), (I'_1, I_3)\}$ which contains $K_{5,3}$ with partite sets $X = U_1 \cup U_3 \cup U_4 \cup U_{134}$ and $Y = U_2 \cup U_5$. Notice that any N_2-embedding of $K_{5,3}$ has one hexagonal face and six rectangular faces. Also in $\mathbb{AG}(\mathcal{L})$, the vertex I_4 is adjacent to the vertices of $\{I_1, I'_1, I_3\} \subseteq X$ and I_{135} is adjacent to I_4 as well as $\{I_2, I'_2\} \subseteq Y$. Since $deg_{K_{5,3}}(I_4) = 3$, three rectangular faces cannot adopt all these edges incident with I_3 together with the edges incident with I_{135}. Therefore, I_4 must be in the hexagonal face. A similar technique also proves that I_3 is a part of the hexagonal face. To an extent, the hexagonal face can adopt the vertices I_{135} and I_{145} with its edges together with an edge incident to either I_4 or I_3. We let the edge (I_1, I_4) be embedded in the hexagonal face. Here the two other edges incident with I_4, namely (I'_1, I_4) and (I_3, I_4) can be embedded in two rectangle faces that contains I_4. Now we have to embed two more edges incident with I_3, namely (I_1, I_3) and (I'_1, I_3) but we are left-out with only one rectangular face that contains I_3, a contradiction. Therefore, $|\bigcup_{p=1;q\neq 2} U_{pqr}| \leq 2$; likewise $|\bigcup_{p=2} U_{pqr}| \leq 2$.

Thus, in the case of $\bigcup_{k=1,2;k\ell\neq 12} U_{k\ell} \cup U_{345} = \emptyset$, either $|U_{12}| = 2$ with $\bigcup_{pq\neq 12;pqr\neq 345} U_{pqr} \cup U_{1345} \cup U_{2345} = \emptyset$ or $|U_{12}| \leq 1$ with $|\bigcup_{p=1,2;pq\neq 12} U_{pqr}| \leq 2$ and $|U_{pqr}| \leq 1$ together with either $\bigcup_{p=1;q\neq 2} U_{pqr} \cup U_{1345} = \emptyset$ or $\bigcup_{p=2} U_{pqr} \cup U_{2345} = \emptyset$. For all these cases, by using Figure 2, we get $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if $|\bigcup_{i\neq 1,2} U_{ij}| = 1$ with $\bigcup_{k=1,2;pq\neq 12} (U_{k\ell} \cup U_{pqr}) \cup U_{(ij)^c} \cup U_{12} \cup U_{1345} \cup U_{2345} = \emptyset$ or $\bigcup_{i\neq 1,2} U_{ij} = \emptyset$ with $|\bigcup_{k=1,2;k\ell\neq 12;pq\neq 12} U_{k\ell} \cup U_{pqr}| \leq 1$. Also, if $|\bigcup_{k=1,2;k\ell\neq 12;pq\neq 12} U_{k\ell} \cup U_{pqr}| = 1$, then $|U_{12}| \leq 1$ and $U_{1345} = U_{2345} = \emptyset$. Moreover, if $\bigcup_{k=1,2;k\ell\neq 12;pq\neq 12} U_{k\ell} \cup U_{pqr} = \emptyset$, then $|U_{12}| \leq 2$. In addition, $U_{1345} = U_{2345} = \emptyset$ whenever $|U_{12}| = 2$ and either $U_{1345} = \emptyset$ or $U_{2345} = \emptyset$ whenever $|U_{12}| \leq 1$.



(a) (b)

(a) $\bigcup_{k=1,2;k\ell\neq 12} U_{k\ell} \cup U_{345} = \emptyset, |U_{12}| = 2$ and $\bigcup_{pq\neq 12} U_{pqr} = U_{1345} = U_{2345} = \emptyset$.

(b) $\bigcup_{k=1,2;k\ell\neq 12} U_{k\ell} \cup U_{345} = \emptyset, |U_{12}| = |U_{135}| = |U_{145}| = 1$ and $U_{1345} \neq \emptyset$.

Figure 2. $|\bigcup_{n=1}^{5} U_n| = 7$ with $|U_1| = |U_2| = 2$.

Case 3. Let $|\bigcup_{n=1}^{5} U_n| = 6$. Then $|U_1| = 2$. If $|\bigcup_{i\neq 1} U_{ij}| \geq 3$, then the graph G_{21} is contained in $\mathbb{AG}(\mathcal{L})$ and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $|\bigcup_{i\neq 1} U_{ij}| \leq 2$. Further, if $|U_{ij}| = |U_{k\ell}| = 1$ for some $\{i, j\} \cap \{k, \ell\} = \emptyset$, then the graph $(H_4 \cup (u_1, u_2)) - (v_2, v_4)$ is contained in $\mathbb{AG}(\mathcal{L})$ and by [3, Lemma 3.6], we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. That is, $E(\left\langle \bigcup_{i\neq 1} U_{ij} \right\rangle) = \emptyset$.

Case 3.1. Assume $|\bigcup_{i\neq 1} U_{ij}| = 2$. Let $|U_{ij}| = 2$ for some $i \neq 1$. If $I \in (\bigcup_{k\neq i,j} U_{1k}) \cup U_{(ij)^c}$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup U_k \cup U_{k'} \cup I$, where $k' \in \{2, 3, 4, 5\} \setminus \{i, j, k\}$ form $K_{4,5}$ in $\mathbb{AG}(\mathcal{L})$, a contradiction. Therefore, $\bigcup_{k\neq i,j} U_{1k} = U_{(ij)^c} = \emptyset$.

If U_{1i} or U_{1j} has two elements, say $|U_{1i}| \geq 2$, then the subgraph $G_{22} = \mathbb{AG}(\mathcal{L}) - \{I_{1i}, I'_{1i}, (I_k, I_i), (I_k, I_j), (I_k, I_{ij}), (I_k, I'_{ij})\}$ contains $K_{5,3}$ with partite sets $X = U_i \cup U_j \cup U_k \cup U_{ij}$ and $Y = U_1 \cup U_{k'}$, where $k \in \{2, 3, 4, 5\} \setminus \{i, j\}$ and $k' \in \{2, 3, 4, 5\} \setminus \{i, j, k\}$. Note that any N_2-embedding of $K_{5,3}$ has one hexagonal, six rectangular faces and out of which three faces contains the vertex I_k because $deg_{K_{5,3}}(I_k) = 3$. Since I_{1i} and I'_{1i} are adjacent to $I_j, I_k, I_{k'}$ in $\mathbb{AG}(\mathcal{L})$, it requires two distinct faces that contains I_k to embed the vertices I_{1i} and I'_{1i}. So, after embedding I_{1i} and I'_{1i} in any N_2-embedding of $K_{5,3}$, it may adopt at most three distinct edges with one end in I_k and another end in one of the vertices of X. But, I_k is adjacent to $\{I_i, I_j, I_{ij}, I'_{ij}\} \subset X$, a contradiction. Thus, $|U_{1i}|, |U_{1j}| \leq 1$.

Suppose $|U_{1i}|, |U_{1j}| = 1$ and $\bigcup_{p\neq 1} U_{pqr} \neq \emptyset$. Then, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Suppose $|U_{1i}| = 1$ and $U_{1j} = \emptyset$. If $U_{(1i)^c} \neq \emptyset$, then the sets $X = U_i \cup U_j \cup U_{ij} \cup \{[I_{1i}, I_{(1i)^c}]\}$ and $Y = U_1 \cup U_m \cup U_n$, where $m, n \in \{1, \ldots, 5\} \setminus \{1, i, j\}$ form $K_{5,4}$ in $\mathbb{AG}(\mathcal{L})$, a contradiction. Also, if $I, J \in \bigcup_{p\neq 1} U_{pqr}$, then it is not difficult to verify that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if $\bigcup_{k\notin\{i,j\}} U_{1k} = \emptyset, |\bigcup_{k\in\{i,j\},p\neq 1} (U_{1k} \cup U_{pqr})| \leq 2$ with $|U_{1k}|, |U_{pqr}| \leq 1$ and $U_{(mn)^c} = \emptyset$ if $U_{mn} \neq \emptyset$ for all $1 \leq m, n \leq 5$.

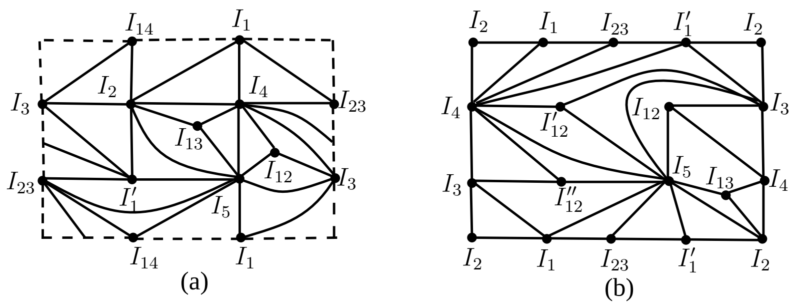
Moreover, it is not difficult to verify that the same argument is also valid for $|U_{ij}| = |U_{mn}| = 1$ for some $i, m \neq 1$. Since $E(\left\langle \bigcup_{i\neq 1} U_{ij} \right\rangle) = \emptyset$, let $\{i, j\} \cap \{m, n\} = \ell$. So $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $\bigcup_{k\neq \ell} U_{1k} = \emptyset$, $|U_{1\ell}| \leq 1$, $|U_{1\ell} \cup \bigcup_{p\neq 1} U_{pqr}| \leq 2$ and $U_{(mn)^c} = \emptyset$ if $U_{mn} \neq \emptyset$ for all $1 \leq m, n \leq 5$.

Case 3.2. Suppose $|U_{ij}| = 1$ for some unique $i \neq 1$. If $I, J \in \bigcup_{k \neq i,j} U_{1k} \cup U_{(ij)^c}$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_1 \cup I \cup J \cup \bigcup_{k \neq i,j} U_k$ form $K_{3,6} \cup (K_4 - e)$ in $\mathbb{AG}(\mathcal{L})$ and so by Proposition 1.1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $|\bigcup_{k \neq i,j} U_{1k} \cup U_{(ij)^c}| \leq 1$.

Case 3.2.1. Suppose $I \in \bigcup_{k \neq i,j} U_{1k} \cup U_{(ij)^c}$. If $|U_{1i}| \geq 2$, then the graph induced by sets $X = \{I_1, I'_1, I_i, I_{1i}, I'_{1i}, [I_{ij}, I]\}$ and $Y = \bigcup_{k \neq i,j} U_k$ contains a minor $K_{6,3} \cup (K_4 - e)$ in $\mathbb{AG}(\mathcal{L})$ and so by Proposition 1.1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $|U_{1i}|, |U_{1j}| \leq 1$. Further, if $J \in U_{pqr} \cup U_{pqrs}$ for some $p \neq 1$, then the set $\{I_1, [I'_1, J], I_2, I_3, I_4, I_5, [I_{ij}, I]\}$ form K_7 in $\mathbb{AG}(\mathcal{L})$, a contradiction. For the remaining cases, by using Figure 3(a), we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.

Case 3.2.2. Suppose $\bigcup_{k \neq i,j} U_{1k} \cup U_{(ij)^c} = \emptyset$. Let $\max\{|U_{1i}|, |U_{1j}|\} = |U_{1i}|$ and $\ell, m \in \{2, 3, 4, 5\} \setminus \{i, j\}$. Clearly $|U_{1i}| \leq 3$, otherwise, the sets $X = U_1 \cup U_i \cup U_{1i}$ and $Y = U_j \cup U_\ell \cup U_m$ form $K_{7,3}$ in $\mathbb{AG}(\mathcal{L})$. If $|U_{1i}| \geq 2$ and $|U_{1j}| \geq 2$, then $\mathbb{AG}(\mathcal{L}) - \{I_{1j}, I'_{1j}, I_{ij}, (I_j, I_\ell), (I_j, I'_\ell)\}$ contains $K_{5,3}$ with $X = U_1 \cup U_i \cup U_{1i}$ and $Y = U_j \cup U_\ell \cup U_m$ which is similar to the graph G_{15} (refer Case 4.2.2 of [3, Theorem 5.2]) so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

Let $|U_{1i}| = 3$. If $I \in \bigcup_{p \neq 1} U_{pqr} \cup U_{pqrs}$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{ij}, I, (I_1, I_i), (I'_1, I_i)\}$ contains $K_{6,3}$ with $X = U_1 \cup U_i \cup U_{1i}$ and $Y = U_j \cup U_\ell \cup U_m$. Note that any N_2-embedding of $K_{6,3}$ has only rectangular faces. Further, in $\mathbb{AG}(\mathcal{L})$, I_{ij} is adjacent to I_1, I'_1, I_ℓ, I_m and I is adjacent to I_1, I'_1. So, to embed the vertices I_{ij} and I, it requires two distinct rectangular faces that contain both I_1 and I'_1. Next, to embed the edges (I_1, I_i) and (I'_1, I_i), it requires two more distinct rectangular faces with diagonals I_1, I_i and I'_1, I_i. In such a case, one cannot construct the remaining five distinct rectangular faces by using the existing vertices and edges, a contradiction. Therefore, $\bigcup_{p \neq 1} U_{pqr} \cup U_{pqrs} = \emptyset$. In this case, that is $|U_{1i}| = 3$, $|U_{1j}| \leq 1$ with $\bigcup_{p \neq 1} U_{pqr} \cup U_{pqrs} = \emptyset$, and by the help of Figure 3(b), we get $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$.



(a) $|\bigcup_{k \neq i,j} U_{1k} \cup U_{(ij)^c}| = 1$, $|U_{1i}|, |U_{1j}| \leq 1$ and $\bigcup_{p \neq 1} U_{pqr} \cup U_{pqrs} = \emptyset$.

(b) $\bigcup_{k \neq i,j} U_{1k} \cup U_{(ij)^c} = \emptyset$, $|U_{1i}| = 3$, $|U_{1j}| \leq 1$ and $\bigcup_{p \neq 1} U_{pqr} \cup U_{pqrs} = \emptyset$.

Figure 3. $|\bigcup_{n=1}^{5} U_n| = 6$ with $|U_{ij}| = 1$ for $i \neq 1$.

Let $|U_{1i}| = 2$. Then, $U_{(1i)^c} = \emptyset$; otherwise, the minor subgraph induced by the set $\{I_1, [I'_1, I_{ij}], I_2, I_3, I_4, I_5, [I_{1i}, I_{(1i)^c}]\}$ form K_7 in $\mathbb{AG}(\mathcal{L})$. Suppose $|U_{1j}| = 1$. If $I \in \bigcup_{p \neq 1} U_{pqr} \cup U_{pqrs}$, then $\mathbb{AG}(\mathcal{L}) - \{I_{1j}, I_{ij}, I, (I_1, I_i), (I'_1, I_i)\}$ contains $K_{5,3}$ with $X = U_1 \cup U_i \cup U_{1i}$ and $Y = U_j \cup U_\ell \cup U_m$.

By using the structure of the graph G_{15} (refer Case 4.2.2 of [3, Theorem 5.2]), we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Suppose $U_{1j} = \emptyset$. If $|\bigcup_{p\neq 1; pqr\neq (1i)^c} U_{pqr}| \geq 2$ or $\bigcup_{p\neq 1; pqr\neq (1i)^c} U_{pqr}, U_{2345} \neq \emptyset$, then the reader can verify that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $|U_{1j}| = 1$ with $\bigcup_{p\neq 1} U_{pqr} \cup U_{pqrs} = \emptyset$ or $U_{1j} = \emptyset$ with $|\bigcup_{p\neq 1; pqr\neq (1i)^c} U_{pqr}| \leq 1$ and $U_{2345} = \emptyset$ while $|\bigcup_{p\neq 1; pqr\neq (1i)^c} U_{pqr}| = 1$.

Let $|U_{1i}|, |U_{1j}| \leq 1$. Then, $U_{(1i)^c}, U_{(1j)^c} = \emptyset$ whenever $U_{1i}, U_{1j} \neq \emptyset$. Note that by Theorem 2.8 [10], $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 1$ if $\bigcup_{p\neq 1} U_{pqr} = \emptyset$.

Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if $|U_{1i} \cup U_{1j}| = 2$ with $|\bigcup_{p\neq 1; pqr\neq (1i)^c,(1j)^c} U_{pqr}| = 1$ (or) $|U_{1i} \cup U_{1j}| = 1$ with $|\bigcup_{p\neq 1; pqr\neq (1i)^c,(1j)^c} U_{pqr}| = 2$, $U_{2345} = \emptyset$ or $|\bigcup_{p\neq 1; pqr\neq (1i)^c,(1j)^c} U_{pqr}| = 1$ (or) $|U_{1i} \cup U_{1j}| = \emptyset$ with $1 \leq |\bigcup_{p\neq 1} U_{pqr}| \leq 2$.

Case 3.3. Suppose $U_{ij} = \emptyset$ for all $i \neq 1$. Then, the subgraph induced by the neighborhood set of each vertex in U_{1mn} for all $2 \leq m, n \leq 5$ is an edge and so it does not play any role in determining the value of the crosscap. If $|\bigcup_{2\leq k\leq 5} U_{1k}| \geq 5$ or $|U_{1k}| \geq 4$ or $|U_{pqr}| \geq 4$ for $p \neq 1$, then $K_{3,x}$ where $x \geq 7$ is a subgraph of $\mathbb{AG}(\mathcal{L})$ and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

Case 3.3.1. Let $|U_{1k}| \in \{2, 3\}$. Clearly, $U_{(1m)^c} = \emptyset$ whenever $U_{1m} \neq \emptyset$ for all $2 \leq m \leq 5$; otherwise, $K_{3,6} \cup (K_4 - e)$ is a subgraph of $\mathbb{AG}(\mathcal{L})$. If $|U_{pqr}| = 2$ for $p \neq 1$, then $pqr \neq (1k)^c$ and so $\{p, q, r\} \cap \{k\} \neq \emptyset$. Therefore, we assume that $p = k$. Now the subgraph $G_{23} = \mathbb{AG}(\mathcal{L}) - \{(I_p, I_q), (I_p, I_r), (I_p, I_\ell)\}$ contains $K_{6,4} - 4e$ with partite set $X = U_p \cup U_q \cup U_r \cup U_\ell \cup U_{pqr}$ and $Y = U_1 \cup U_{1k}$, where $\ell \in \{2, 3, 4, 5\} \backslash \{p, q, r\}$. Clearly, every face in any N_2-embedding of $K_{6,4} - 4e$ is rectangular. So to embed the edges $(I_p, I_q), (I_p, I_r)$ and (I_p, I_ℓ), it requires three rectangular faces which contains I_p, a contradiction to $deg_{K_{6,4}-4e}(I_p) = 2$.

Suppose $|U_{1k}| = 3$ for some $2 \leq k \leq 5$. Then $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ provided $|\bigcup_{\ell\neq k} U_{1\ell}| \leq 1$ with $U_{(1k)^c}, U_{(1\ell)^c} = \emptyset$ and $|\bigcup_{p\neq 1, pqr\neq (1k)^c,(1\ell)^c} U_{pqr}| \leq 1$.

Suppose $|U_{1k}| = 2$ for some $2 \leq k \leq 5$. If $|U_{1\ell}| = 2$ for some $\ell \in \{2, 3, 4, 5\} \setminus k$, then $|\bigcup_{p\neq 1, pqr\neq (1k)^c,(1\ell)^c} U_{pqr} \cup U_{2345}| = \emptyset$, otherwise, $\mathbb{AG}(\mathcal{L})$ contains G_{23}. Therefore, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ provided $|\bigcup_{\ell\neq k} U_{1\ell}| \leq 2$ with $U_{(1k)^c}, U_{(1\ell)^c} = \emptyset$. Moreover, if $|U_{1\ell}| = 2$, then $|\bigcup_{p\neq 1, pqr\neq (1k)^c,(1\ell)^c} U_{pqr} \cup U_{2345}| = \emptyset$ and if $|U_{1\ell}| \leq 1$, then $|U_{pqr}| \leq 1$ for all $p \neq 1$ and $pqr \neq (1k)^c, (1\ell)^c$.

Case 3.3.2. Suppose $|U_{1k}| \leq 1$ for all $k \in \{2, 3, 4, 5\}$.

Suppose $|\bigcup_{U_{1k}\neq\emptyset} U_{(1k)^c}| \geq 2$, say $I, J \in \bigcup_{U_{1k}\neq\emptyset} U_{(1k)^c}$. Then, the contraction of $\mathbb{AG}(\mathcal{L})$ induced by $\{I_1\} \cup \{[I'_1, I'_{(1k^c)}]\} \cup \{[I_{1k}, I_{(1k)^c}]\} \cup \bigcup_{\ell\neq 1} U_\ell$ in $\mathbb{AG}(\mathcal{L})$ form K_7, a contradiction. Thus $|\bigcup_{U_{1k}\neq\emptyset} U_{(1k)^c}| \leq 1$.

Claim A: If $|\bigcup_{2\leq k\leq 5} U_{1k}| = 4$, then $\bigcup_{p\neq 1} U_{pqr} = \emptyset$; equivalently, $\bigcup_{2\leq k\leq 5} U_{(1k)^c} = \emptyset$.

Let $|\bigcup_{2\leq k\leq 5} U_{1k}| = 4$. Assume on the contrary that $U_{pqr} \neq \emptyset$ for some $p \neq 1$. Then $pqr = (1m)^c$ where $m \in \{2, 3, 4, 5\} \setminus \{p, q, r\}$. Now the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{1q}, I_{1r}, I_{1m}, (I_p, I_1), (I_p, I'_1)\}$ contains $K_{4,4}$ with partite sets $X = U_1 \cup U_p \cup U_{1p}$ and $Y = U_q \cup U_r \cup U_m \cup U_{(1m)^c}$. Note that each face of any N_2-embedding of $K_{4,4}$ is rectangular, the vertices I_{1q}, I_{1r}, I_{1m} are adjacent to $I_p \in X$ and two vertices of Y, and I_p is adjacent to $I_1, I'_1 \in X$. So to embed the remaining three vertices and two edges, it requires five rectangular faces which contains I_p but $deg_{K_{4,4}}(I_p) = 4$, a contradiction. Therefore, the claim holds.

Claim B: If $|\bigcup_{2\le k\le 5} U_{1k}| \le 3$ and $|\bigcup_{U_{1k}\neq\emptyset} U_{(1k)^c}| = 1$, then $U_{pqr} = \emptyset$ for all $p \neq 1, pqr \neq (1k)^c$.

If $U_{(1k)^c} \neq \emptyset$ for $2 \le k \le 5$ and $U_{pqr} \neq \emptyset$ for $p \neq 1$ and $pqr \neq (1k)^c$, then the contraction of $\mathbb{AG}(\mathcal{L})$ induced by $\left\langle \{I_1\} \cup \{[I'_1, I_{pqr}]\} \cup \{[I_{1k}, I_{(1k)^c}]\} \cup \bigcup_{\ell\neq 1} U_\ell \right\rangle$ contains K_7, a contradiction.

Claim C: In case of $|\bigcup_{2\le k\le 5} U_{1k}| \le 3$ and $\bigcup_{U_{1k}\neq\emptyset} U_{(1k)^c} = \emptyset$.

Claim C1: If $|\bigcup_{2\le k\le 5} U_{1k}| = 3$, then $|U_{pqr}| \le 2$ for $p \neq 1, pqr \neq (1k)^c$.

Suppose $\bigcup_{U_{1k}\neq\emptyset} U_{(1k)^c} = \emptyset$ and $|U_{pqr}| \ge 3$ for some $p \neq 1, pqr \neq (1k)^c$. This implies that $|U_{1p}| = |U_{1q}| = |U_{1r}| = 1$. Now consider the graph $\mathbb{AG}(\mathcal{L}) - \{I_r, I_{1p}, I_{1q}, I_{1r}\}$ which contains $K_{5,3}$ with partite sets $X = U_p \cup U_q \cup U_{pqr}$ and $Y = U_1 \cup U_\ell$ where $\ell \notin \{1, p, q, r\}$. Notice that any N_2-embedding of $K_{5,3}$ has one hexagonal face and six rectangular faces. Label the hexagonal face as F_1. Now, try to embed the left-out vertices of $\mathbb{AG}(\mathcal{L})$ into a N_2-embedding of $K_{5,3}$. In $\mathbb{AG}(\mathcal{L})$, I_r is adjacent to I_1, I'_1, I_p, I_q and I_ℓ so that I_r should be embedded into the face F_1. Since I_{1p} and I_{1q} are adjacent to both I_r and I_ℓ, we have both I_{1p} and I_{1q} embedded together with I_r in F_1 and further the face F_1 should have the path $I_p - I_\ell - I_q$. The point to remember here is the other neighbors of I_p and I_q in F_1 are I_1 and I'_1. Also, I_{1r} is adjacent to I_ℓ, I_p and I_q, so to embed I_{1r} it requires a rectangular face, say F_2, that contains the path $I_p - I_\ell - I_q$. The point here is the fourth vertex of F_2 must be either I_1 or I'_1. At last, since $deg_{K_{3,5}}(I_p) = 3$, there must be another rectangular face, say F_3, in any N_2-embedding of $K_{5,3}$ that should have I_p. But, the edge (I_p, I_ℓ) is already used twice for forming the faces F_1 and F_2 so that the two neighbors of I_p in F_3 must be I_1 and I'_1. This contradicts the fact that at least one of the edges (I_p, I_1) or (I_p, I'_1) was used twice in F_1 and F_2. Thus, the claim holds true.

Claim C2: If $|\bigcup_{2\le k\le 5} U_{1k}| \le 2$, then $2 \le |\bigcup_{p\neq 1, pqr\neq(1k)^c} U_{pqr}| \le 4$ with at most one $|U_{pqr}| \in \{2, 3\}$. Further, if $|\bigcup_{p\neq 1, pqr\neq(1k)^c} U_{pqr}| = 2$, then there exists U_{pqr} such that $|U_{pqr}| = 2$.

First recall that if $|\bigcup_{2\le k\le 5} U_{1k}| \in \{1, 2\}$, $\bigcup_{U_{1k}\neq\emptyset} U_{(1k)^c} = \emptyset$ and $|\bigcup_{p\neq 1, pqr\neq(1k)^c} U_{pqr}| \le 2$ with $|U_{pqr}| \le 1$, then by Theorem 2.8 of [10], $\mathbb{AG}(\mathcal{L})$ is projective.

- If $|U_{pqr}| \ge 4$ for some $p \neq 1$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{7,3}$ with partite sets $U_p \cup U_q \cup U_r \cup U_{pqr}$ and $U_1 \cup U_m$ where $m \in \{2, \dots, 5\} \setminus \{p, q, r\}$.
- Suppose there exist two sets $U_{p_1q_1r_1}$ and $U_{p_2q_2r_2}$ from the collection $\{U_{pqr} : p \neq 1,\ pqr \neq (1k)^c\}$ each having more than two elements. Clearly $|\{p_1, q_1, r_1\} \cap \{p_2, q_2, r_2\}| = 2$. So let us take $p_1 = p_2 = p$ and $q_1 = q_2 = q$. Now, consider the graph $\mathbb{AG}(\mathcal{L}) - \{I_{pqr_2}, I'_{pqr_2}, (I_p, I_q), (I_p, I_{r_1}), (I_q, I_{r_1}), (I_1, I_{r_2}), (I'_1, I_{r_2})\}$ which is isomorphic to $K_{5,3}$ with partite sets $X = U_p \cup U_q \cup U_{r_1} \cup U_{pqr_1}$ and $Y = U_1 \cup U_{r_2}$. Any N_2-embedding of $K_{5,3}$ has one hexagonal face and six rectangular faces. Note that I_{pqr_2} and I'_{pqr_2} are adjacent to I_1, I'_1 and I_{r_1}. To embed the vertices I_{pqr_2} and I'_{pqr_2} into a N_2-embedding of $K_{5,3}$, we have two possibilities; (i) both I_{pqr_2} and I'_{pqr_2} together with its edges are embedded in two rectangular faces, or (ii) I_{pqr_2} and I'_{pqr_2} together with its edges are embedded in hexagonal and rectangular faces respectively.

  (i) In this case, both rectangular faces must have I_1, I'_1 and I_{r_1}. Now, embedding of the edges (I_{r_1}, I_p) and (I_{r_1}, I_q) together with the fact that $deg_{K_{5,3}}(I_{r_1}) = 3$ implies that the hexagonal face must contain I_{r_1}. So, the edges either (I_{r_1}, I_1) or (I_{r_1}, I_1) belong to the hexagonal face, a contradiction because it is used twice in two rectangular faces.

  (ii) In this case, to embed the edges $(I_p, I_q), (I_p, I_{r_1})$ and (I_q, I_{r_1}), at least two rectangular faces are required. Finally, to embed the edges (I_1, I_{r_2}) and (I'_1, I_{r_2}), it requires two more rectangular faces in which the diagonals are I_1, I_{r_2} and I'_1, I_{r_2} respectively. But, such a case does not exist and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

- Suppose $|\bigcup_{p\neq 1, pqr\neq(1k)^c} U_{pqr}| \geq 5$. Then, the possibilities from the collection $\{U_{pqr} : p \neq 1,\ pqr \neq (1k)^c\}$ are (i) one set with three elements and two singleton sets, and (ii) one set with two elements and three singleton sets. For case (i), the graph $\mathbb{AG}(\mathcal{L}) - \{I_4, I_{245}, I_{345}, (I_2, I_3)\}$ has $K_{5,3}$ with partite sets $X = U_2 \cup U_3 \cup U_{234}$ and $Y = U_1 \cup U_5$ which behave as a similar structure of the graph given in Claim C1. So $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. We leave it to the reader to prove $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$ for case (ii).

Thus, the claim holds true.

Case 4. Let $|\bigcup_{n=1}^5 U_n| = 5$ and let $\max_{1\leq p<q\leq 5} |U_{pq}| = |U_{ij}|$. Clearly $\mathbb{AG}(\mathcal{L})$ is projective when $U_{ij} = \emptyset$. If $|U_{ij}| \geq 5$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = \bigcup_{k\neq i,j} U_k$ form $K_{3,7}$ in $\mathbb{AG}(\mathcal{L})$, a contradiction.

Case 4.1. Assume that $|U_{ij}| \in \{3, 4\}$. If $U_{\ell m} \neq \emptyset$ or $U_{\ell mn} \neq \emptyset$ for some $\ell, m, n \notin \{i, j\}$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_\ell \cup U_m \cup U_n \cup U_{\ell m} \cup U_{\ell mn}$ form $K_{5,4}$ in $\mathbb{AG}(\mathcal{L})$, a contradiction. So, any non-empty two index sets U_{pq} and three index sets U_{pqr} must have either i or j as one of their indices. Therefore, every vertex in U_{ijk}, for any k, is adjacent to exactly two vertices in $\mathbb{AG}(\mathcal{L})$, hence, these vertices do not play any role in finding the crosscap value. Thus, we avoid the sets U_{ijk} for all k from $V(\mathbb{AG}(\mathcal{L}))$. Also, if $U_{ik}, U_{i\ell}, U_{im} \neq \emptyset$ for $k, \ell, m \notin \{i, j\}$, then $G_{24} = \mathbb{AG}(\mathcal{L}) - \{I_{ik}, I_{i\ell}, I_{im}, (I_i, I_j)\}$ contains $K_{5,3}$ with partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_k \cup U_\ell \cup U_m$. Notice that any N_2-embedding of $K_{5,3}$ has one hexagonal and six rectangular faces. Also, in $\mathbb{AG}(\mathcal{L})$, I_{ik} is adjacent to I_j, I_ℓ, I_m; $I_{i\ell}$ is adjacent to I_j, I_k, I_m and I_{im} is adjacent to I_j, I_k, I_ℓ. So to embed the vertices $I_{ik}, I_{i\ell}, I_{im}$ in a N_2-embedding of $K_{5,3}$, it requires either one hexagon with a rectangular face or three rectangular faces which contains I_j. If $I_{ik}, I_{i\ell}, I_{im}$ are embedded in three rectangular faces, then since $deg_{K_{5,3}}(I_j) = 3$, no other face contains I_j so the edge (I_i, I_j) cannot be drawn without crossing. If not, two vertices must be inserted in the hexagonal face and the other vertex should be inserted in a rectangle face. In such cases, the third face which contains I_j does not exist because each edge occurs in exactly two faces. So $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore one of the sets U_{ik} or $U_{i\ell}$ or U_{im} must be empty for $k, \ell, m \notin \{i, j\}$. A slight modification of the proof would show that one of the sets U_{jk} or $U_{j\ell}$ or U_{jm} must be empty for $k, \ell, m \notin \{i, j\}$.

Case 4.1.1. Suppose $|U_{ij}| = 4$. If $|U_{pq}| \geq 2$ for $p \in \{i, j\}$ and $q \notin \{i, j\}$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{pq}, I'_{pq}, (I_i, I_j)\}$ has a similar structure to the graph G_{16} (refer Case 5.1 [3, Theorem 5.2]) so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, at most two sets from $\{U_{ik}, U_{i\ell}, U_{im}\}$ and two sets from $\{U_{jk}, U_{j\ell}, U_{jm}\}$, where $k, \ell, m \notin \{i, j\}$, may have an element.

Further, if $U_{pq} \neq \emptyset$ for $p \in \{i, j\}$ and $q \notin \{i, j\}$, then we claim that the set $U_{p'q'} \cup U_{(pq)^c} = \emptyset$, where $p' \in \{i, j\} \setminus \{p\}$ and $q \neq q' \notin \{i, j\}$. Suppose not, $I \in U_{p'q'} \cup U_{(pq)^c}$, then the graph $\mathbb{AG}(\mathcal{L}) - \{[I_{pq}, I]\}$ contains $K_{6,3}$ with partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = \bigcup_{k\neq i,j} U_k$. Since the merged vertex $[I_{pq}, I]$ is adjacent to all the five vertices of $\bigcup_{n=1}^5 U_n$, it requires a face of length at least five in an N_2-embedding of $K_{6,3}$. A contradiction to the fact that every face in any N_2-embedding of $K_{6,3}$ is a rectangle.

Thus, in the case of $|U_{ij}| = 4$, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ provided $\bigcup_{\ell,m\notin\{i,j\}} U_{\ell m} \cup U_{(ij)^c} = \emptyset$, $|\bigcup_{p\in\{i,j\}; q\notin\{i,j\}} U_{pq}| \leq 2$, in which $|U_{pq}| \leq 1$ and $U_{p'q'}, U_{(pq)^c} = \emptyset$ when $|U_{pq}| = 1$, where $p', q' \notin \{p, q\}$.

Case 4.1.2. Suppose $|U_{ij}| = 3$. In every part of the case, let $k, \ell, m \in \{1, \ldots, 5\} \setminus \{i, j\}$. If $|U_{pq}| = 3$ for $p \in \{i, j\}$ and $q \in \{k, \ell, m\}$, then the graph $\mathbb{AG}(\mathcal{L}) - \{I_{pq}, I'_{pq}, I''_{pq}, (I_{p'}, I_{q'}), (I_{p'}, I_{q''})\}$, where $p' \in \{i, j\} \setminus \{p\}$ and distinct $q', q'' \in \{k, \ell, m\} \setminus \{q\}$, contains a similar structure of the graph G'_{15} (refer Case 5.2 [3, Theorem 5.2]) so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Also, if $|U_{pq}| = |U_{p'q'}| = 2$ for distinct $p, p' \in \{i, j\}$ and distinct

$q, q' \in \{k, l, m\}$, then $\mathbb{AG}(\mathcal{L}) - \{I_{pq}, I'_{pq}, I_{p'q'}, I'_{p'q'}, (I_p, I_{p'})\}$ contains a similar structure of the graph G_{15} (refer to Case 4.2.2 [3, Theorem 5.2]) so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, at most one set from the collection $\{U_{ik}, U_{i\ell}, U_{im}, U_{jk}, U_{j\ell}, U_{jm}\}$ has two elements.

(i) Without loss of generality, let us take $|U_{ik}| = 2$. Then, by Case 4.1, we have $|U_{i\ell} \cup U_{im}| \leq 1$ and at most two sets from the collection $\{U_{jk}, U_{j\ell}, U_{jm}\}$ have an element. But, our next claims are:

Claim A: $|U_{i\ell} \cup U_{im} \cup U_{jk}| \leq 1$ and $U_{j\ell} \cup U_{jm} \cup U_{(ik)^c} = \emptyset$.

Suppose $|U_{i\ell} \cup U_{im} \cup U_{jk}| = 2$. Since $|U_{i\ell} \cup U_{im}| \leq 1$ and $|U_{jk}| \leq 1$, we choose $I \in U_{i\ell} \cup U_{im}$ and $J \in U_{jk}$. Now the graph $\mathbb{AG}(\mathcal{L}) - \{I_{ik}, I'_{ik}, (I_i, I_j), (I_j, [I, J])\}$ contains $K_{6,3}$ with partite sets $X = U_i \cup U_j \cup U_{ij} \cup \{[I, J]\}$ and $Y = U_k \cup U_\ell \cup U_m$. Note that all the faces in any N_2-embedding of $K_{6,3}$ are rectangular. Since I_{ik} and I'_{ik} are adjacent to I_j, I_ℓ and I_m, to embed the vertices I_{ik} and I'_{ik}, it requires two distinct rectangular faces that should have the vertex I_j in its boundary. Since $deg_{K_{6,3}}(I_j) = 3$, there is exactly one more rectangular face containing I_j, in which the two edges (I_j, I_i) and $(I_j, [I, J])$ cannot be embedded so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

Suppose $I \in U_{j\ell} \cup U_{jm} \cup U_{(ik)^c}$. Then, $K_{5,3}$ is a subgraph of the graph $\mathbb{AG}(\mathcal{L}) - \{I_{ik}, I'_{ik}, I\}$ with partite sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_k \cup U_\ell \cup U_m$. Note that $\tilde{\gamma}(K_{5,3}) = 2$. Since $I_{ik} - I - I'_{ik}$ is a path in $\mathbb{AG}(\mathcal{L})$, these three vertices should be embedded into a single face of a N_2-embedding of $K_{5,3}$. First, embed the path $I_{ik} - I - I'_{ik}$ together with the edges $(I_{ik}, I_j), (I_{ik}, I_\ell), (I_{ik}, I_m), (I'_{ik}, I_j), (I'_{ik}, I_\ell)$ and (I'_{ik}, I_m) into a face. Then, clearly the middle vertex I of the path cannot adopt any edge incident with I, so the edges (I, I_i) and (I, I_k) cannot be embedded. Therefore, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Thus, the claim holds true.

Claim B: If $|U_{i\ell} \cup U_{im} \cup U_{jk}| = 1$, say $|U_{i\ell}| = 1$, then $U_{(i\ell)^c} = \emptyset$.

If $|U_{i\ell}| = 1$ with $U_{(i\ell)^c} \neq \emptyset$, then just replace I by $I_{i\ell}$ and J by $I_{(i\ell)^c}$ in the proof of the case $|U_{i\ell} \cup U_{im} \cup U_{jk}| = 2$, and we get $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

(ii) Let $|U_{ik}| = 1$. Here, our claim is:

Claim C: $|U_{i\ell} \cup U_{im} \cup U_{jk} \cup U_{j\ell} \cup U_{jm} \cup U_{(ik)^c}| \leq 2$, $|U_{j\ell} \cup U_{jm} \cup U_{(ik)^c}| \leq 1$ and $|\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c}| \leq 1$. Further, if $|\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c}| = 1$, then $U_{j\ell} \cup U_{jm} = \emptyset$ and $|U_{i\ell} \cup U_{im} \cup U_{jk}| \leq 1$.

A slight modification of the proof of Claim B would show that $|U_{j\ell} \cup U_{jm} \cup U_{(ik)^c}| \leq 1$ and $|\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c}| \leq 1$.

Suppose $|U_{i\ell} \cup U_{im} \cup U_{jk} \cup U_{j\ell} \cup U_{jm} \cup U_{(ik)^c}| = 3$. Since $|U_{i\ell} \cup U_{im}| \leq 1$ and $|U_{j\ell} \cup U_{jm} \cup U_{(ik)^c}| \leq 1$, let $I \in U_{i\ell} \cup U_{im}$, $J \in U_{jk}$ and $K \in U_{j\ell} \cup U_{jm} \cup U_{(ik)^c}$. Then the sets $X = U_i \cup U_j \cup U_{ij} \cup \{[I, J]\} \cup \{K, I_{ik}\}$ and $Y = U_k \cup U_\ell \cup U_m$ form $K_{7,3}$.

Let $|\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c}| = 1$. If $U_{(ik)^c} \neq \emptyset$, then clearly $U_{j\ell} \cup U_{jm} = \emptyset$ and $|U_{i\ell} \cup U_{im} \cup U_{jk}| \leq 1$. If $U_{pq}, U_{(pq)^c} \neq \emptyset$ for $pq \neq ik$, then we have to show that $pq \neq j\ell, jm$. If not, then $X = U_i \cup U_j \cup U_{ij}$ and $Y = U_k \cup U_\ell \cup U_m$ form $K_{5,3}$ in the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{ik}, [I_{pq}, I_{(pq)^c}]\}$. Note that $\tilde{\gamma}(K_{5,3}) = 2$. Since the vertex I_{ik} is adjacent to the vertex $[I_{pq}, I_{(pq)^c}]$, the vertices I_{ik} and $[I_{pq}, I_{(pq)^c}]$ have to be embedded in a single face of any N_2-embedding of $K_{5,3}$. Here, the vertex $[I_{pq}, I_{(pq)^c}]$ is adjacent to all of the five vertices of $\bigcup_{n=1}^{5} U_n$ and the vertex I_{ik} is adjacent to exactly three vertices of $\bigcup_{n=1}^{5} U_n$. Clearly, a single face cannot embed such eight edges onto it, a contradiction. Therefore, the claim holds true.

Thus, in the case of $|U_{ij}| = 3$, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ provided $\bigcup_{\ell, m \notin \{i,j\}} U_{\ell m} \cup U_{(ij)^c} = \emptyset$, $|\bigcup_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq}| \leq 3$, where the choice i or j for p is placed at most two times in the union, in which at most one of the sets U_{pq} has two elements and $|\bigcup_{U_{pq} \neq \emptyset} U_{(pq)^c}| \leq 1$. Further, if $|U_{pq}| = 2$ for some

$p \in \{i, j\}; q \notin \{i, j\}$, then $\bigcup\limits_{p',q' \notin \{p,q\}} U_{p'q'} \cup \bigcup\limits_{U_{pq} \neq \emptyset} U_{(pq)^c} = \emptyset$, and if $|\bigcup\limits_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq}| = 3$ with $|U_{pq}| \leq 1$, then the three choices for q are not distinct. Moreover, if $|\bigcup\limits_{U_{pq} \neq \emptyset} U_{(pq)^c}| = 1$, then $|\bigcup\limits_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq}| \leq 2$ with the choice for two pairs of p, q's are not mutually disjoint.

Case 4.2. Assume $|U_{ij}| = 2$. If $|\bigcup\limits_{\ell,m \notin \{i,j\}} U_{\ell m} \cup U_{(ij)^c}| \geq 2$, then the sets $X = U_i \cup U_j \cup U_{ij}$ and $Y = V(\mathbb{AG}(\mathcal{L})) \setminus X$ form $K_{4,5}$ in $\mathbb{AG}(\mathcal{L})$, a contradiction.

Further, if four non-empty sets exist other than U_{ij} in such a way that two sets of the form U_{ik} and two other sets of the form U_{jk} for $k \neq i, j$ are non-empty, say $U_{i\ell}, U_{im} \neq \emptyset$ and $U_{j\ell}, U_{jn} \neq \emptyset$ for distinct $\ell, m, n \notin \{i, j\}$. Then, the sets $X = \{I_i, I_j, I_{ij}, [I_{i\ell}, I_{jn}]\}$ and $Y = \{I_\ell, I_m, I_n, [I_{im}, I_{j\ell}]\}$ form $(H_3 \cup (u_2, u_3)) - (u_2, v_1)$. Therefore, by [3, Lemma 3.6], we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

Case 4.2.1. Suppose $|\bigcup\limits_{\ell,m \notin \{i,j\}} U_{\ell m} \cup U_{(ij)^c}| = 1$ and let $I \in \bigcup\limits_{\ell,m \notin \{i,j\}} U_{\ell m} \cup U_{(ij)^c}$. If $|U_{pq}| \geq 2$ for some $p \in \{i, j\}$ and $q \notin \{i, j\}$, then clearly $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Let $|U_{pq}| = 1$ for some $p \in \{i, j\}$ and $q \notin \{i, j\}$. If $J \in U_{p'q'} \cup U_{p'q''}$ for $p' \in \{i, j\} \setminus \{p\}$ and $q', q'' \notin \{i, j, q\}$, then the sets $X = \{I_i, I_j, I_{ij}, I'_{ij}, [I_{pq}, J]\}$ and $Y = \{I_q, I_{q'}, I_{q''}, I\}$ form $K_{5,4} - e$ in $\mathbb{AG}(\mathcal{L})$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Further, similar verification gives us $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$ whenever $U_{(pq)^c} \neq \emptyset$ or there exist $p \in \{i, j\}$ such that $U_{pq} \neq \emptyset$ for all $q \notin \{i, j\}$.

Therefore, in the case of $|U_{ij}| = 2$ with $|\bigcup\limits_{\ell,m \notin \{i,j\}} U_{\ell m} \cup U_{(ij)^c}| = 1$, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ provided $|\bigcup\limits_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq}| \leq 2$ with $|U_{pq}| \leq 1$ and $U_{p'q'}, U_{(pq)^c} = \emptyset$ when $|U_{pq}| = 1$ where $p', q' \notin \{p, q\}$.

Case 4.2.2. Suppose $\bigcup\limits_{\ell,m \notin \{i,j\}} U_{\ell m} \cup U_{(ij)^c} = \emptyset$. If there exist $p, p_1 \in \{i, j\}$ and $q, q_1 \notin \{i, j\}$ with $pq \neq p_1 q_1$ such that $|U_{pq}|, |U_{p_1 q_1}| = 2$, then $\mathbb{AG}(\mathcal{L})$ contains a structure of $K_{3,8} - 4e$ and it can be verified that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. So assume $|U_{pq}| = 2$ for unique $p \in \{i, j\}$ and $q \notin \{i, j\}$. If $|U_{p'q'} \cup U_{p'q''}| > 1$ for $p' \in \{i, j\} \setminus \{p\}$ and $q', q'' \in \{1, \ldots, 5\} \setminus \{i, j, q\}$, then the sets $X = U_p \cup U_q \cup U_{pq}$ and $Y = U_{p'} \cup U_{q'} \cup U_{q''} \cup U_{p'q'} \cup U_{p'q''}$ form $K_{4,5}$ in $\mathbb{AG}(\mathcal{L})$, a contradiction. If $|U_{p'q'} \cup U_{p'q''}| = 1$, then $|U_{pq'} \cup U_{pq''} \cup U_{p'q}| \leq 1$ with $U_{pq'} = \emptyset$ if $|U_{p'q''}| = 1$ and $U_{pq''} = \emptyset$ if $|U_{p'q'}| = 1$ because of the facts that no two sets of the form U_{ik} and no two sets of the form U_{jk} are non-empty, and the edges $(I_{p'q'}, I_{pq''})$ and $(I_{p'q''}, I_{pq'})$ are in $\mathbb{AG}(\mathcal{L})$. Similarly, if $U_{p'q'}, U_{p'q''} = \emptyset$, then $|U_{pq'} \cup U_{pq''}| \leq 1$ and $|U_{p'q}| \leq 1$. Moreover, $U_{(pq)^c} = \emptyset$ whenever $U_{pq} \neq \emptyset$.

Finally, assume $|U_{pq}| \leq 1$ for all $p \in \{i, j\}$ and $q \notin \{i, j\}$. Since no two sets from U_{ik} and no two sets from U_{jk} are non-empty, $|\bigcup\limits_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq}| \leq 4$, where the choice i or j for p is placed at most once in the union. If the subgraph induced by $\left\langle \bigcup\limits_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq} \right\rangle$ in $\mathbb{AG}(\mathcal{L})$ has an edge, then $U_{(pq)^c} = \emptyset$ for all $U_{pq} \neq \emptyset$. If not, $|\bigcup\limits_{p \in \{i,j\}; q \notin \{i,j\}} U_{(pq)^c}| \leq 1$, where the union is taken over all non-empty U_{pq}. Also, by Theorem 2.1, $\mathbb{AG}(\mathcal{L})$ is projective whenever $|\bigcup\limits_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq}| \leq 2$ with $|U_{pq}| \leq 1$ and $\bigcup\limits_{U_{pq} \neq \emptyset} U_{(pq)^c} = \emptyset$. Further, if two non-empty sets $U_{p_1 q_1}$ and $U_{p_2 q_2}$ exists in the collection $\{U_{pq} : p \in \{i, j\}; q \notin \{i, j\}\}$, then $\{p_1, q_1\} \cap \{p_2, q_2\} \neq \emptyset$.

Therefore, in the case of $|U_{ij}| = 2$ with $\bigcup\limits_{\ell,m \notin \{i,j\}} U_{\ell m} \cup U_{(ij)^c} = \emptyset$, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ provided $2 \leq |\bigcup\limits_{p \in \{i,j\}; q \notin \{i,j\}} U_{pq}| \leq 4$ in which at most one of the sets U_{pq} has two elements, where the choice i or j for p is placed at most once in the union. Moreover, one of the following is satisfied:

(a) If $|U_{p_f q_f}| = 2$ for some $p_f \in \{i, j\}, q_f \notin \{i, j\}$, then $\bigcup\limits_{U_{pq} \neq \emptyset} U_{(pq)^c} = \emptyset$. Further, at most one of the sets

$U_{p_g q_g}$ is non-empty with the property that $\{p_f, q_f\} \cap \{p_g, q_g\} = \emptyset$.

(b) If $|U_{pq}| \leq 1$ for all $p \in \{i, j\}, q \notin \{i, j\}$, then $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \leq 1$. Further, if $|U_{(pq)^c}| = 1$ for some $|U_{pq}| = 1$, then every non-empty set U_{p_f, q_f} should have the property that $\{p_f, q_f\} \cap \{p, q\} \neq \emptyset$.

Case 4.3. Suppose $|U_{ij}| = 1$. If $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| \geq 6$, then $K_8 - 3e$ is a minor of $\mathbb{AG}(\mathcal{L})$ and so by Proposition 1.1, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| \leq 5$.

(i) Let $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| \in \{4, 5\}$. Suppose non-empty sets $U_{p_1q_1}, U_{p_2q_2}, U_{p_3q_3}, U_{p_4q_4}$ exists which satisfies the conditions $\{p_1, q_1\} \cap \{p_2, q_2\} = \emptyset$ and $\{p_3, q_3\} \cap \{p_4, q_4\} = \emptyset$. If $\{p_1, q_1\} \cap \{p_3, q_3\} = \emptyset$ or $\{p_2, q_2\} \cap \{p_4, q_4\} = \emptyset$, then the vertex subset $\{\bigcup_{n=1}^{5} U_n \cup [I_{p_1q_1}, I_{p_2q_2}] \cup [I_{p_3q_3}, I_{p_4q_4}]\}$ form K_7. So, let us take $p_1 \in \{p_1, q_1\} \cap \{p_3, q_3\}$ and $p_2 \in \{p_2, q_2\} \cap \{p_4, q_4\}$. Then, the sets $X = U_{p_1} \cup U_{q_1} \cup U_t \cup U_{p_1q_1} \cup U_{p_3q_3}$ and $Y = U_{p_2} \cup U_{q_2} \cup U_{p_2q_2} \cup U_{p_4q_4}$, where $t \in \{1, \ldots, 5\} \setminus \{p_1, p_2, q_1, q_2\}$, form $K_{5,4} - 4e$ in $\mathbb{AG}(\mathcal{L})$. Note that $\tilde{\gamma}(K_{4,5} - 4e) = 2$. Now it is not hard to demonstrate that the edges $(I_{p_1}, I_{q_1}), (I_{q_1}, I_{p_3q_3}), (I_{p_1}, I_t), (I_{q_1}, I_t), (I_{p_1q_1}, I_t), (I_{p_3q_3}, I_t), (I_{p_2}, I_{q_2})$ and $(I_{q_2}, I_{p_4q_4})$ of $\mathbb{AG}(\mathcal{L})$ cannot be embedded into any N_2-embedding of $K_{4,5} - 4e$.

Also, notice that the set $\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$ when $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| = 5$. Otherwise, either K_7 or a graph similar to the structure of the graph G_{24}, given in Case 4.1 of Theorem 2.2, will be a subgraph of $\mathbb{AG}(\mathcal{L})$.

Claim A: $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \leq 1$ when $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| = 4$. Moreover, if $\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c} = \emptyset$, then the subgraph induced by the set $\bigcup_{1\leq p\neq q\leq 5} U_{pq}$ has more than one edge and if $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| = 1$, say $|U_{(rs)^c}| = 1$, then the vertex in U_{rs} is adjacent to at most two vertices of $\bigcup_{U_{pq}\neq\emptyset; pq\neq rs} U_{pq}$. Further, if there is an adjacency between the vertex of U_{rs} and a vertex of $\bigcup_{U_{pq}\neq\emptyset; pq\neq rs} U_{pq}$, then the subgraph induced by the set $\bigcup_{U_{pq}\neq\emptyset; pq\neq rs} U_{pq}$ is an empty graph.

Assume that $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| = 4$. Since $|U_{pq}| \leq 1$ for all $1 \leq p \neq q \leq 5$, let us take $U_{p_1q_1}, U_{p_2q_2}, U_{p_3q_3}, U_{p_4q_4} \neq \emptyset$.

Let $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \geq 2$. If $|U_{(p_1q_1)^c}| \geq 2$, then the graph $\mathbb{AG}(\mathcal{L}) - \{\bigcup_{pq\neq p_1q_1} U_{pq}, (p_1, q_1)\}$ is similar to the graph G_{24} (Case 4.1 of Theorem 2.2) with partite sets $X = U_{p_1} \cup U_{q_1} \cup U_{p_1q_1}$ and $Y = \bigcup_{r\neq p_1, q_1} U_r \cup U_{(p_1q_1)^c}$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Suppose $|U_{(p_1q_1)^c}|, |U_{(p_2q_2)^c}| \geq 1$. If $\{p_3, q_3\} \cap \{p_4, q_4\} = \emptyset$, then the graph induced by the set $\{\bigcup_{n=1}^{5} U_n \cup [I_{p_1q_1}, I_{(p_1q_1)^c}] \cup [I_{p_2q_2}, I_{(p_2q_2)^c}] \cup [I_{p_3q_3}, I_{p_4q_4}]\}$ contains $K_8 - 3e$ so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Also, if $\{p_3, q_3\} \cap \{p_4, q_4\} \neq \emptyset$, say $p_3 \in \{p_3, q_3\} \cap \{p_4, q_4\}$, then the sets $X = U_{p_3} \cup U_{p_3q_3} \cup U_{p_4q_4} \cup \{[I_{p_1q_1}, I_{(p_1q_1)^c}]\} \cup \{[I_{p_2q_2}, I_{(p_2q_2)^c}]\}$ and $Y = \bigcup_{r\neq p_3} U_r$ contains $K_{5,4} - 2e$ in $\mathbb{AG}(\mathcal{L})$. Note that $\tilde{\gamma}(K_{5,4} - 2e) = 2$ and every face in any N_2-embedding of $K_{5,4} - 2e$ is rectangular. Since $\langle Y \rangle = K_4$ and K_4 cannot be embedded in N_2 along with rectangular embedding, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

Let $|U_{(p_1q_1)^c}| = 1$. If $I_{p_1q_1}$ is adjacent to all vertices of $U_{p_2q_2} \cup U_{p_3q_3} \cup U_{p_4q_4}$, then the sets $X = U_{p_1} \cup U_{q_1} \cup U_{p_1q_1}$ and $Y = \bigcup_{r\neq p_1, q_1} U_r \cup U_{p_2q_2} \cup U_{p_3q_3} \cup U_{p_4q_4} \cup U_{(p_1q_1)^c}$ form $K_{3,7}$ in $\mathbb{AG}(\mathcal{L})$. Suppose $I_{p_1q_1}$ is adjacent to $I_{p_2q_2}$. As mentioned earlier, $I_{p_3q_3}$ is not adjacent to $I_{p_4q_4}$. If $I_{p_2q_2}$ is adjacent to either $I_{p_3q_3}$ or $I_{p_4q_4}$, say $I_{p_3q_3}$, then the set $\{\bigcup_{n=1}^{5} U_n \cup [I_{p_1q_1}, I_{(p_1q_1)^c}] \cup [I_{p_2q_2}, I_{p_3q_3}]\}$ form K_7. Therefore, $E\left(\left\langle U_{p_2q_2} \cup U_{p_3q_3} \cup U_{p_4q_4} \right\rangle\right) = \emptyset$. Thus, the claim holds true.

Claim B: $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \leq 3$ when $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| \leq 3$.

Assume that $|\bigcup_{1\leq p\neq q\leq 5} U_{pq}| \leq 3$. If $|U_{(pq)^c}| \geq 4$ for $1 \leq p \neq q \leq 5$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,7}$ with partite sets $X = U_p \cup U_q \cup U_{pq}$ and $Y = \bigcup_{k\neq p,q} U_k \cup U_{(pq)^c}$. If three sets $U_{(p_1q_1)^c}, U_{(p_2q_2)^c}, U_{(p_3q_3)^c}$ are non-empty, then the three merged vertices $[I_{p_fq_f}, I_{(p_fq_f)^c}]$, for all $1 \leq f \leq 3$, together with the vertices in $\bigcup_{n=1}^{5} U_n$ form $K_8 - 3e$. If $|U_{(p_1q_1)^c}|$, $|U_{(p_2q_2)^c}| \geq 2$, then the sets $X = U_{p_1} \cup U_{q_1} \cup U_{p_1q_1}$ and $Y = \bigcup_{r\neq p_1,q_1} U_r \cup U_{(p_1q_1)^c}$ form $K_{3,5}$ which has crosscap 2. Clearly, the path $I_{(p_2q_2)^c} - I_{p_2q_2} - I'_{(p_2q_2)^c}$ together with its edges, could not be embedded into any N_2-embedding of $K_{3,5}$.

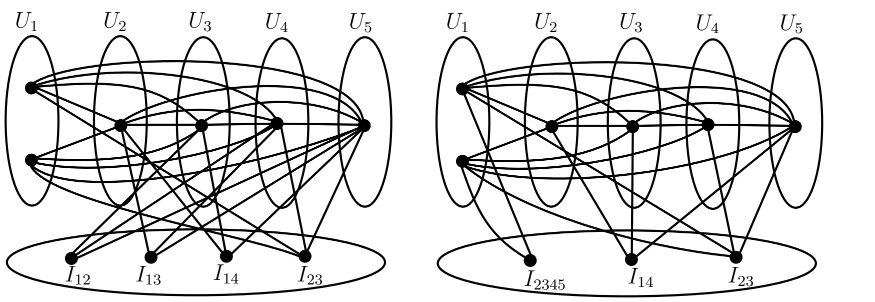
Claim C: [C1] If $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| = 3$, then $|U_{(pq)^c}| = 3$ for $1 \leq p \neq q \leq 5$ and no non-empty set U_{rs} exists with $\{r, s\} \cap \{p, q\} = \emptyset$.

[C2] Let $|\bigcup_{U_{pq}\neq\emptyset} U_{(pq)^c}| \in \{2, 1\}$. If there exists a unique set $U_{(pq)^c} \neq \emptyset$ for $1 \leq p \neq q \leq 5$, then at most one non-empty set U_{rs} exist with $\{r, s\} \cap \{p, q\} = \emptyset$. If two sets $U_{(p_1q_1)^c}, U_{(p_2q_2)^c} \neq \emptyset$ for $1 \leq p_1, q_1, p_2, q_2 \leq 5$ and $\{p_1, q_1\}\cap\{p_2, q_2\} \neq \emptyset$, then no non-empty set U_{rs} exist with $\{r, s\}\cap\{p_f, q_f\} = \emptyset$ for $1 \leq f \leq 2$.

The proofs of claims [C1] and [C2] are merely verifications that can be done by the reader.

Now, to determine conditions for $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ (given in the statement), we have to eliminate projective conditions of $\mathbb{AG}(\mathcal{L})$ which was given in Theorem 2.1. □

Example 2.1. As an illustration, we consider the case (iii)[c] in Theorem 2.2. Let $|U_1| = 2, |U_i| = 1$ for $i = 2, 3, 4, 5$, and $|U_{14}| = |U_{23}| = 1$. If $|U_{12}| = |U_{13}| = 1$, then the corresponding 6-partite graph, given in Figure 4(a), is a crosscap two graph. Also, if $|U_{2345}| = 1$, then the crosscap of corresponding 6-partite graph, given in Figure 4(b), is not equal to two. Moreover, the 6-partite graph G in Figure 4(b) is minimal with respect to $\tilde{\gamma}(G) \neq 2$.



(a) A crosscap two 6-partite graph

(b) A minimal 6-partite graph with crosscap $\neq 2$

Figure 4. undefined.

3. The case when $|A(\mathcal{L})| = 6$

Finally we look into the lattice with 6 atoms. We close the paper by presenting its statement and proof.

Theorem 3.1. *Let $\mathcal{L}$ be a lattice with $|A(\mathcal{L})| = 6$. Then, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ if and only if one of the following conditions hold:*

(i) $|\bigcup_{n=1}^{6} U_n| = 7$, *there is U_i such that $|U_i| = 2$ with* $\bigcup_{1\le p\ne q\le 6} U_{pq} = \emptyset$, $\bigcup_{j,k,\ell,m,n\ne i} (U_{jk\ell} \cup U_{jk\ell m} \cup U_{jk\ell mn}) = \emptyset$ *and at most three sets U_{ipq}'s has exactly one element.*

(ii) $|\bigcup_{n=1}^{6} U_n| = 6$, $|\bigcup_{1\le i\ne j\le 6} U_{ij}| \le 2$, $\bigcup_{U_{ij}, U_{ijk}\ne\emptyset} \left(U_{(ij)^c} \cup U_{(ijk)^c}\right) = \emptyset$ *and one of the following cases is satisfied:*

[a] *In case of* $\bigcup_{1\le i\ne j\le 6} U_{ij} = \emptyset$:

[a1] *If there is a unique $|U_{ijk}| = 3$ for $1 \le i \ne j \ne k \le 6$ or $|U_{ijk}| = |U_{\ell mn}| = 2$ for some $ijk \ne \ell mn$ and $1 \le i, j, k, \ell, m, n \le 6$, then there exist at most eight distinct non-empty U_{pqr}'s (including $U_{ijk}, U_{\ell mn}$) in which at most two distinct U_{pqr}'s are non-empty such that the intersection of all the sets at their indices has exactly two elements, where $1 \le p \ne q \ne r \le 6$.*

[a2] *If there is a unique $|U_{ijk}| = 2$ for $1 \le i \ne j \ne k \le 6$, then there exist at most nine distinct non-empty U_{pqr}'s (including U_{ijk}) in which at most three distinct U_{pqr}'s are non-empty such that the intersection of all the sets at their indices has exactly two elements, where $1 \le p \ne q \ne r \le 6$.*

[a3] *If $|U_{ijk}| \le 1$ for all $1 \le i \ne j \ne k \le 6$, then there exist at most ten distinct non-empty U_{ijk}'s in which exactly three distinct U_{ijk}'s are non-empty such that the intersection of all the sets at their indices has exactly two elements, where $1 \le i \ne j \ne k \le 6$.*

[b] *If $|U_{ij}| = 1$ for some unique $1 \le i \ne j \le 6$, then* $\bigcup_{\{\ell,m,n\}\cap\{i,j\}=\emptyset} U_{\ell mn} = \emptyset$, *and* $\left|\bigcup_{\{\ell,m,n\}\cap\{i,j\}\ne\emptyset} U_{\ell mn}\right| \le 6$ *with at most two distinct non-empty $U_{\ell mn}$'s in which at most one set $U_{\ell mn}$ has two elements such that the intersection of all the sets at their indices has exactly two elements.*

[c] *If $|U_{ij}| = 2$ for some unique $1 \le i \ne j \le 6$, then* $\bigcup_{\{\ell,m,n\}\cap\{i,j\}=\emptyset} U_{\ell mn} = \bigcup_{1\le k\le 6} U_{ijk} = \emptyset$, *and* $\left|\bigcup_{|\{\ell,m,n\}\cap\{i,j\}|=1} U_{\ell mn}\right| \le 4$ *with $|U_{\ell mn}| \le 1$ in which at most two distinct $U_{\ell mn}$'s are non-empty such that the intersection of all the sets at their indices has exactly two elements.*

[d] *If $|U_{ij}| = |U_{ik}| = 1$ for some $1 \le i \ne j \ne k \le 6$, then* $\bigcup_{\{i\} or \{j,k\}\not\subseteq\{\ell,m,n\}\cap\{i,j,k\}} U_{\ell mn} = \emptyset$, *and* $\left|\bigcup_{\{i\} or \{j,k\}\subseteq\{\ell,m,n\}\cap\{i,j,k\}} U_{\ell mn}\right| \le 5$ *with $|U_{\ell mn}| \le 1$ in which at most two distinct $U_{\ell mn}$'s are non-empty such that the intersection of all the sets at their indices has exactly two elements.*

Proof. Suppose that $|\bigcup_{n=1}^{6} U_n| \ge 8$. Then $\mathbb{AG}(\mathcal{L})$ contains $K_{2,2,2,2}$ or $K_8 - 3e$ as a subgraph so that by Proposition 1.1, we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Thus, $|\bigcup_{n=1}^{6} U_n| \le 7$.

Case 1. Let $|\bigcup_{n=1}^{6} U_n| = 7$. Then, $|U_1| = 2$ and $|U_2| = \ldots = |U_6| = 1$. Clearly, $K_7 - e$ is a subgraph of $\mathbb{AG}(\mathcal{L})$. If $I \in \bigcup_{i\ne 1} (U_{ij} \cup U_{ijk} \cup U_{ijk\ell} \cup U_{ijk\ell m})$, then merge the vertices I and I_1 so that K_7 is a subgraph of $\mathbb{AG}(\mathcal{L})$, a contradiction. If $U_{1j} \ne \emptyset$ for some $j = 2, \ldots, 6$, then $X = U_1 \cup U_j \cup U_{1j}$ and $Y = \bigcup_{j'\ne 1,j} U_{j'}$ form a subgraph H_3 (refer [3, Lemma 3.6]) in $\mathbb{AG}(\mathcal{L})$ and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \ge 3$. Therefore, $\bigcup_{i,j} U_{ij} = \emptyset$.

If $|U_{1jk}| \geq 2$ for some $2 \leq j, k \leq 6$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,6} \cup (K_4 - e)$ as a minor, so by Proposition 1.1 we have $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. So, $|U_{1jk}| \leq 1$ for all $2 \leq j, k \leq 6$.

Suppose $|\bigcup U_{1jk}| \geq 4$. Then, it is not hard to verify that $K_{8,3} - 4e$ is a subgraph of $\mathbb{AG}(\mathcal{L})$ with the partition set $X \supset \{\bigcup U_{1jk} \cup U_1\}$. Further, assume that a vertex in X is adjacent to at least six vertices of X or, two vertices in X is adjacent to at least five vertices of X. Note that any N_2-embedding of $K_{8,3} - 4e$ has two hexagonal and seven rectangular faces. Since the maximum degree of vertices of X in $K_{8,3} - 4e$ is 3, at most one vertex of X may adopt 5 distinct edges from $\langle X \rangle$. But, $\langle X \rangle$ contains either a vertex of degree 6 or two vertices of degree 5, a contradiction.

Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $\bigcup_{k \neq 1}(U_{ij} \cup U_{k\ell m} \cup U_{k\ell mn} \cup U_{k\ell mnp}) = \emptyset$ with at most three $U_{1\ell m}$'s having one element.

Case 2. Let $|\bigcup_{n=1}^{6} U_n| = 6$. Then $|U_1| = \ldots = |U_6| = 1$. If the subgraph induced by $V(\mathbb{AG}(\mathcal{L})) \setminus \{\bigcup_{n=1}^{6} U_n\}$ has an edge (I, J), then merge the vertices I and J so that the resulting vertex is adjacent to all the vertices of $\bigcup_{n=1}^{6} U_n$. Therefore, K_7 is a minor of $\mathbb{AG}(\mathcal{L})$, a contradiction. Thus, $\left\langle V(\mathbb{AG}(\mathcal{L})) \setminus \{\bigcup_{n=1}^{6} U_n\} \right\rangle$ is an empty graph.

If $|\bigcup_{i,j} U_{ij}| \geq 3$, then the structure given for G_{21} is a subgraph of $\mathbb{AG}(\mathcal{L})$, so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Therefore, $|\bigcup_{i,j} U_{ij}| \leq 2$.

Case 2.1. Assume $U_{ij} = \emptyset$ for all $1 \leq i, j \leq 6$. As mentioned earlier, we have $U_{(ijk)^c} = \emptyset$ when $U_{ijk} \neq \emptyset$.

If $|U_{ijk} \cup U_{ij\ell} \cup U_{ijm} \cup U_{ijn}| \geq 4$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{(I_i, I_k), (I_j, I_k), (I_k, I_{ij\ell}), (I_k, I_{ijm}), (I_k, I_{ijn})\}$ contains $K_{3,7} - 3e$ with partite sets $X = U_i \cup U_j \cup U_k \cup U_{ijk} \cup U_{ij\ell} \cup U_{ijm} \cup U_{ijn}$ and $Y = U_\ell \cup U_m \cup U_n$ (take $\ell, m, n \in \{1, \ldots, 6\} \setminus \{i, j, k\}$ in case of ℓ, m, n does not exist in the union of the assumption). Note that any N_2-embedding of $K_{3,7} - 3e$ have two hexagonal and six rectangular faces. The edges $(I_i, I_k), (I_j, I_k), (I_k, I_{ij\ell}), (I_k, I_{ijm})$ can be inserted without crossing in two hexagonal faces. In such a case one cannot find a rectangular face with diagonals I_k and I_{ijn} and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

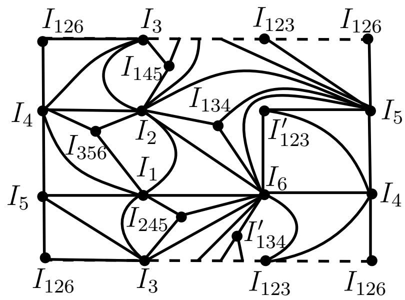
Also, if $|U_{ijk}| \geq 4$ for some $1 \leq i \neq j \neq k \leq 6$, then the sets $X = U_i \cup U_j \cup U_k \cup U_{ijk}$ and $Y = \bigcup_{\ell \neq i,j,k} U_\ell$ form $K_{7,3}$ in $\mathbb{AG}(\mathcal{L})$.

(i) Assume $|U_{ijk}| = 3$ for some $1 \leq i \neq j \neq k \leq 6$. If $|U_{\ell mn}| \geq 2$ for some $\ell mn \neq ijk$, then the the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{\ell mn}, I'_{\ell mn}, (I_i, I_j), (I_i, I_k), (I_j, I_k)\}$ is similar to the structure of G_{16} (refer Case 5.1 of [3, Theorem 5.2]) so that $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. If $|U_{ij\ell}| = |U_{ijm}| = 1$, then the graph $\mathbb{AG}(\mathcal{L}) - \{I_{ij\ell}, I_{ijm}, (I_k, I_i), (I_k, I_j)\}$ contains $K_{6,3}$ with partite sets $X = U_i \cup U_j \cup U_k \cup U_{ijk}$ and $Y = U_\ell \cup U_m \cup U_n$. Note that every face in any N_2-embedding of $K_{6,3}$ is rectangular, $I_{ij\ell}$ is adjacent to I_k, I_m, I_n and I_{ijm} is adjacent to I_k, I_ℓ, I_n. So, to embed the vertices $I_{ij\ell}$ and I_{ijm}, it requires two rectangular faces that contains I_k. Now the edges (I_k, I_i) and (I_k, I_j) cannot be embedded into a single rectangular face which has I_k, a contradiction. Similarly, we get $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$ whenever $|U_{pqr}| = |U_{pqs}| = |U_{pqt}| = 1$ for $1 \leq p \neq q \neq r \neq s \neq t \leq 6$ with $pq \neq ij$.

(ii) Assume $|U_{ijk}| = 2$ for some $1 \leq i \neq j \neq k \leq 6$. If $|U_{\ell mn}| = |U_{pqr}| = 2$ for $\ell mn, pqr \neq ijk$, then clearly $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. If $|U_{ij\ell}| = 2$ with $U_{ijm} \neq \emptyset$ for $1 \leq \ell \neq m \leq 6$, then the graph $\mathbb{AG}(\mathcal{L}) - \{(I_k, I_i), (I_k, I_j), (I_k, I_{ij\ell}), (I_k, I'_{ij\ell}), (I_k, I_{ijm})\}$ contains $K_{8,3} - 3e$ with partite sets $X = U_i \cup U_j \cup U_k \cup U_{ijk} \cup U_{ij\ell} \cup U_{ijm}$ and $Y = \bigcup_{n \neq i,j,k} U_n$. Note that $\tilde{\gamma}(K_{8,3} - 3e) = 2$, and any N_2-embedding of $K_{8,3} - 3e$ have one hexagonal and nine rectangular faces. Now, to recover a N_2-embedding of $\mathbb{AG}(\mathcal{L})$ from a N_2-embedding of $K_{8,3} - 3e$, we have to embed five edges in which one end is at I_k and the another end at a vertex of X. So it is required that I_k has to be part of at least four faces of a N_2-embedding of $K_{8,3} - 3e$, a contradiction to $deg_{K_{8,3}-3e}(I_k) = 3$.

Suppose $|U_{pqr}| = 2$ for $pq \neq ij$. If $|U_{ij\ell}| = |U_{ijm}| = 1$ for $1 \leq \ell \neq m \leq 6$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{pqr}, I'_{pqr}, (I_k, I_i), (I_k, I_j), (I_k, I_{ij\ell}), (I_k, I_{ijm}), (I_i, I_j)\}$ contains $K_{3,7} - 2e$ with partite sets $X = U_i \cup U_j \cup U_k \cup U_{ijk} \cup U_{ij\ell} \cup U_{ijm}$ and $Y = U_\ell \cup U_m \cup U_n$. Note that any N_2-embedding of $K_{3,7} - 2e$ have one hexagonal and 8 rectangular faces. Clearly, each rectangular face can adopt at most one new edge and so to embed the edges $(I_k, I_i), (I_k, I_j), (I_k, I_{ij\ell}), (I_k, I_{ijm})$ it requires one hexagonal and two rectangular faces which contains I_k. In addition, we have to embed the vertices I_{pqr} and I'_{pqr}. Here, the vertices I_{pqr} and I'_{pqr} are adjacent to at least one of I_i or I_j or I_k. If it is I_k, then clearly the edge (I_k, I_{pqr}) cannot be embedded. So, let I_{pqr}, I'_{pqr} be adjacent to I_i. But, I_i is used in embedding the first five edges. So, the remaining two rectangular faces are required to embed the edges (I_i, I_{pqr}) and (I_i, I'_{pqr}). In such a case, the edge (I_i, I_j) could not be embedded in N_2, a contradiction. For all the remaining cases, one can retrieve a N_2-embedding of $\mathbb{AG}(\mathcal{L})$ from Figure 5.

If $|U_{ijk}| \leq 1$ for all $1 \leq i \neq j \neq k \leq 6$, then by eliminating the projective cases, we will get the result as in the statement.



$|U_{ijk}| = |U_{\ell mn}| = 2$ for some $ijk \neq \ell mn$

Figure 5. $|\bigcup_{n=1}^{6} U_n| = 6$ with $U_{ij} = \emptyset$ for all $1 \leq i \neq j \leq 6$.

Case 2.2. Assume $1 \leq |\bigcup\limits_{i,j} U_{ij}| \leq 2$. Let $U_{ij} \neq \emptyset$ for some $1 \leq i \neq j \leq 6$. Clearly, $U_{(ij)^c} = \emptyset$ and $U_{pqr} = \emptyset$ for all $\{p, q, r\} \cap \{i, j\} = \emptyset$.

Choose p and q such that $\{p, q\} \cap \{i, j\} \neq \emptyset$. If $U_{pqr_1}, U_{pqr_2}, U_{pqr_3} \neq \emptyset$ with $r_1, r_2, r_3 \notin \{i, j\}$, then, the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{pqr_2}, I_{pqr_3}, (I_{r_1}, I_p), (I_{r_1}, I_q), (I_{r_1}, I_{ij})\}$ contains $K_{5,3}$ with partite sets $X = U_p \cup U_q \cup U_{r_1} \cup U_{ij} \cup U_{pqr_1}$ and $Y = U_r \cup U_{r_2} \cup U_{r_3}$, where $r \notin \{p, q, r_1, r_2, r_3\}$. Any N_2-embedding of $K_{5,3}$ have one hexagonal and six rectangular faces. Note that I_{pqr_2}, I_{pqr_3} are adjacent to I_{r_1} and two vertices of Y. So, to embed the vertices I_{pqr_2}, I_{pqr_3} together with edges $(I_{r_1}, I_p), (I_{r_1}, I_q)$ and (I_{r_1}, I_{ij}), it requires a hexagon with three rectangular faces or five rectangular faces which contains I_{r_1}, a contradiction to $deg_{K_{5,3}}(I_{r_1}) = 3$.

(i) Assume $|U_{ij}| = 1$ for some unique $1 \leq i \neq j \leq 6$. Choose ℓ, m and n in such a way that $\{\ell, m, n\} \cap \{i, j\} \neq \emptyset$. If $|U_{\ell mn}| \geq 3$, then $\mathbb{AG}(\mathcal{L})$ contains $K_{3,7}$ as a subgraph, a contradiction. Also, if $|U_{\ell mn}| = |U_{\ell_1 m_1 n_1}| = 2$, then the graph $\mathbb{AG}(\mathcal{L}) - \{I_{\ell_1 m_1 n_1}, I'_{\ell_1 m_1 n_1}, (I_i, I_j)\}$ contains a subgraph similar to the structure of G_{16} (refer Case 5.1 of [3, Theorem 5.2]) and so $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$. Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ whenever $\bigcup\limits_{\{\ell,m,n\}\cap\{i,j\}=\emptyset} U_{\ell mn} \cup U_{(ij)^c} = \emptyset$ and, $\left|\bigcup\limits_{\{\ell,m,n\}\cap\{i,j\}\neq\emptyset} U_{\ell mn}\right| \leq 6$ in which at most one set $U_{\ell mn}$ has two elements with at most two distinct sets $U_{\ell mn}$'s non-empty such that the intersection of all the sets at their indices has exactly two elements.

(ii) Assume $|\bigcup_{i,j} U_{ij}| = 2$.

Suppose $|U_{ij}| = 2$ for $1 \leq i \neq j \leq 6$. If $U_{ijk} \neq \emptyset$ for some $1 \leq k \leq 6$, then $K_{6,3}$ with partite sets $X = U_i \cup U_j \cup U_k \cup U_{ij} \cup U_{ijk}$ and $Y = U_p \cup U_q \cup U_r$, where $p, q, r \in \{1, \ldots, 6\} \setminus \{i, j, k\}$. Every face in any N_2-embedding of $K_{6,3}$ is rectangular. So, to embed the edges $(I_k, I_i), (I_k, I_j), (I_k, I_{ij}), (I_k, I'_{ij})$, it requires four rectangular faces all of which contains I_k but $deg_{K_{3,6}}(I_k) = 3$, a contradiction. Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ when $\bigcup_{\{\ell,m,n\}\cap\{i,j\}=\emptyset} U_{\ell mn} \cup \bigcup_k U_{ijk} \cup U_{(ij)^c} = \emptyset$, and $\left|\bigcup_{|\{\ell,m,n\}\cap\{i,j\}|=1} U_{\ell mn}\right| \leq 4$ in which $|U_{\ell mn}| \leq 1$ with at most two distinct sets $U_{\ell mn}$'s non-empty such that the intersection of all the sets at their indices has exactly two elements.

Suppose distinct $U_{ij}, U_{pq} \neq \emptyset$ for $1 \leq i, j, p, q \leq 6$. Then, $\{i, j\} \cap \{p, q\} \neq \emptyset$. So let $|U_{ij}| = |U_{ik}| = 1$ for some $1 \leq i \neq j \neq k \leq 6$. Let us take ℓ, m and n such that $\{\ell, m, n\} \cap \{i, j, k\}$ contains either $\{i\}$ or $\{j, k\}$. If $|U_{\ell mn}| \geq 2$, then the subgraph $\mathbb{AG}(\mathcal{L}) - \{I_{\ell mn}, I'_{\ell mn}, (I_i, I_j), (I_i, I_k), (I_j, I_{ik}), (I_k, I_{ij})\}$ contains $K_{5,3}$ with partite sets $X = U_i \cup U_j \cup U_k \cup U_{ij} \cup U_{ik}$ and $Y = U_p \cup U_q \cup U_r$ where $p, q, r \in \{1, \ldots, 6\} \setminus \{i, j, k\}$. Note that $I_{\ell mn}, I'_{\ell mn}$ are adjacent to three vertices of $K_{5,3}$ and so to embed the vertices $I_{\ell mn}, I'_{\ell mn}$ together with edges $(I_i, I_j), (I_i, I_k)$, it requires one hexagonal and one rectangular face. Thereafter one cannot find two rectangular faces with diagonal vertices I_j, I_{ik} and I_k, I_{ij} respectively. Therefore, either (I_j, I_{ik}) or (I_k, I_{ij}) cannot be embedded without crossing. Therefore, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) \geq 3$.

Thus, $\tilde{\gamma}(\mathbb{AG}(\mathcal{L})) = 2$ when $\bigcup_{\{i\} or \{j,k\} \not\subseteq \{\ell,m,n\}\cap\{i,j,k\}} U_{\ell mn} \cup U_{(ij)^c} \cup U_{(ik)^c} = \emptyset$, and $\left|\bigcup_{\{i\} or \{j,k\} \subseteq \{\ell,m,n\}\cap\{i,j,k\}} U_{\ell mn}\right| \leq 5$ in which $|U_{\ell mn}| \leq 1$ with at most two distinct sets $U_{\ell mn}$'s non-empty such that the intersection of all the sets at their indices has exactly two elements. □

Example 3.1. As an illustration, we consider the case (ii)[a1] in Theorem 3.1. Let $|U_i| = 1$ for $i = 1, \ldots, 6$, $|U_{123}| = |U_{134}| = 2$ and $|U_{126}| = |U_{145}| = |U_{245}| = 1$. If $|U_{356}| = 1$, then the corresponding crosscap two 8-partite graph is given in Figure 6. Also, if $|U_{125}| = 1$, then the crosscap of corresponding 8-partite graph, as in Figure 7, is not equal to two. Moreover, the 8-partite graph G in Figure 7 is minimal with respect to $\tilde{\gamma}(G) \neq 2$.

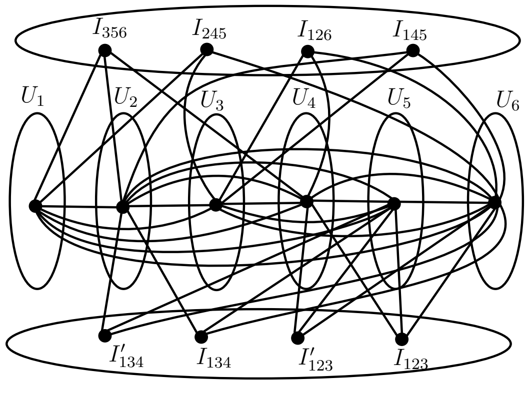


Figure 6. A crosscap two 8-partite graph.

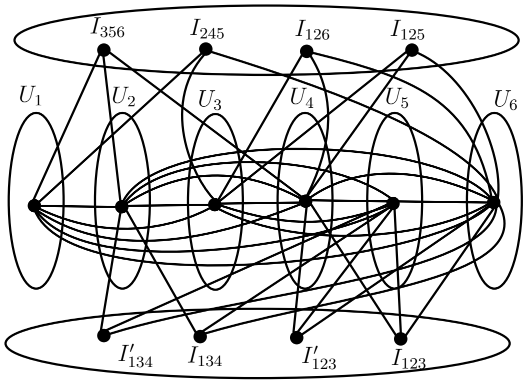


Figure 7. A minimal 8-partite graph with crosscap ≠ 2.

4. Conclusions

Note that no complete set of forbidden subgraphs for the two-crosscap surface (Klein bottle) is known yet. In this regard, an open problem will be to determine a family of graphs that has crosscap number two. This series of papers provides a class of r-partite graphs, where $3 \leq r \leq 8$, that (1) can be embedded or (2) cannot be embedded in the two-crosscap surface. This was done by using the complete classification of all lattices whose annihilating-ideal graph has crosscap number two.

For the future work, one can determine all forbidden r-partite, $r \geq 4$, subgraphs for the crosscap two surface. Also, it would be interesting to classify all lattices whose annihilating-ideal graph can be embedded in the non-orientable surfaces of crosscap three.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding the publishing of this paper.

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