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Research article

The convergence rate for the laws of logarithms under sub-linear expectations

Qunying Wu*

College of Science, Guilin University of Technology, Guilin 541004, China

* **Correspondence:** Email: wqy666@glut.edu.cn.

Abstract: Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables in a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. The necessary and sufficient conditions for the convergence rate on the laws of the logarithms and the law of the iterated logarithm are obtained.

Keywords: sub-linear expectation; convergence rate; laws of logarithms; law of the iterated logarithm
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1. Introduction

Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables. Complete convergence first established by Hsu and Robbins [1] (for the sufficiency) and Erdős [2, 3] (for the necessity) proceeds as follows:

$$\sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) < \infty, \quad \text{for any} \quad \epsilon > 0,$$

if and only if $EX = 0$ and $EX^2 < \infty$. Baum and Katz [4] extended the above result and obtained the following theorem:

$$\sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| \geq \epsilon n^{1/p}) < \infty, \quad \text{for} \quad 0 < p < 2, \quad r \geq p, \quad \text{any} \quad \epsilon > 0, \tag{1.1}$$

if and only if $E|X|^r < \infty$, and when $r \geq 1$, $EX = 0$.

There are several extensions of the research on complete convergence. One of them is the study of the convergence rate of complete convergence. The first work was the convergence rate, achieved by Heyde [5]. He got the result of $\lim\limits_{\epsilon \to 0} \epsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) = EX^2$ under the conditions $EX = 0$ and

$EX^2 < \infty$. For more results on the convergence rate, see Chen [6], Spătaru [7], Gut and Spătaru [8], Spătarut and Gut [9], Gut and Steinebach [10], He and Xie [11], Kong and Dai [12], etc.

But (1.1) does not hold for $p = 2$. However, by replacing $n^{1/p}$ by $\sqrt{n \ln n}$ and $\sqrt{n \ln \ln n}$, Gut and Spătaru [8] and Spătarut and Gut [9] established the following results called the convergence rate of the law of the (iterated) logarithm. Supposing that $\{X, X_n; n \geq 1\}$ is a sequence of i.i.d. random variables with $EX = 0$ and $EX^2 = \sigma^2 < \infty$, Gut and Spătaru [8] and Spătarut and Gut [9] obtained the following results respectively:

$$\lim_{\epsilon \to 0} \epsilon^{2+2\delta} \sum_{n=1}^{\infty} \frac{\ln^{\delta} n}{n} P(|S_n| \geq \epsilon \sqrt{n \ln n}) = \frac{E|\mathcal{N}|^{2+2\delta} \sigma^{2+2\delta}}{\delta + 1}, \quad 0 \leq \delta \leq 1, \tag{1.2}$$

where $\mathcal{N}$ is the standard normal distribution, and

$$\lim_{\epsilon \to 0} \epsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \ln n} P(|S_n| \geq \epsilon \sqrt{n \ln \ln n}) = \sigma^2. \tag{1.3}$$

Motivated by the above results, the purpose of this paper is to extend (1.2) and (1.3) to sub-linear expectation space (to be introduced in Section 2), which was introduced by Peng [13, 14], and to study the necessary conditions of (1.2).

Under the theoretical framework of the traditional probability space, in order to infer the model, all statistical models must assume that the error (and thus the response variable) is subject to a certain uniquely determined probability distribution, that is, the distribution of the model is deterministic. Classical statistical modeling and statistical inference are based on such distribution certainty or model certainty. "Distribution certainty" modeling has yielded a set of mature theories and methods. However, the real complex data in economic, financial and other fields often have essential and non negligible probability and distribution uncertainty. The probability distribution of the response variable to be studied is uncertain and does not meet the assumptions of classical statistical modeling. Therefore, classical probability statistical modeling methods cannot be used for this type of data modeling. Driven by uncertainty issues, Peng [14, 15] established a theoretical framework for sub-linear expectation spaces from the perspective of expectations. Sub-linear expectation has a wide range of application backgrounds and prospects. In recent years, a series of research achievements on limit theory in sub-linear expectation spaces has been established. See Peng [14, 15], Zhang [16–18], Hu [19], Wu and Jiang [20, 21], Wu et al. [22], Wu and Lu [23], etc. Wu [24], Liu and Zhang [25], Ding [26] and Liu and Zhang [27] obtained the convergence rate for complete moment convergence. However, the convergence rate results for the (iterative) logarithmic law have not been reported yet. The main difficulty in studying it is that the sub-linear expectation and capacity are not additive, which makes many traditional probability space tools and methods no longer effective; thus, it is much more complex and difficult to study it.

In Section 2, we will provide the relevant definitions of sub-linear expectation space, the basic properties and the lemmas that need to be used in this paper.

2. Preliminaries

Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathcal{H}$ be a linear space of random variables on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the set of local Lipschitz functions on $\mathbb{R}_n$. In this case, for $X \in \mathcal{H}$, X is called a random variable.

Definition 2.1. A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function: $\mathcal{H} \to \mathbb{R}$ satisfying the following for all $X, Y \in \mathcal{H}$:
(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$;
(b) Constant preservation: $\hat{\mathbb{E}}c = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda\hat{\mathbb{E}}X, \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space. The conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ is defined by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X), \ \ \forall X \in \mathcal{H}.$$

Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \ \ V(\Omega) = 1 \ \text{ and } \ V(A) \leq V(B) \ \text{ for } \ \forall A \subseteq B, A, B \in \mathcal{G}.$$

The upper and lower capacities $(\mathbb{V}, v)$ corresponding to $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ are respectively defined as

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}\xi; I(A) \leq \xi, \xi \in \mathcal{H}\}, \ \ v(A) := 1 - \mathbb{V}(A^c), \ \forall A \in \mathcal{F}, \ A^c := \Omega - A.$$

The Choquet integrals is defined by

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^0 (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

From all of the definitions above, it is easy to obtain the following Proposition 2.1.

Proposition 2.1. *Let $X, Y \in \mathcal{H}$ and $A, B \in \mathcal{F}$.*
(i) $\hat{\varepsilon}X \leq \hat{\mathbb{E}}X, \ \hat{\mathbb{E}}(X + a) = \hat{\mathbb{E}}X + a, \ \forall a \in \mathbb{R}$;
(ii) $|\hat{\mathbb{E}}(X - Y)| \leq \hat{\mathbb{E}}|X - Y|, \ \hat{\mathbb{E}}(X - Y) \geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y$;
(iii) $v(A) \leq \mathbb{V}(A), \ \mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B), \ v(A \cup B) \leq v(A) + \mathbb{V}(B)$;
(iv) *If $f \leq I(A) \leq g, f, g \in \mathcal{H}$, then*

$$\hat{\mathbb{E}}f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g, \hat{\varepsilon}f \leq v(A) \leq \hat{\varepsilon}g. \tag{2.1}$$

(v) *(Lemma 4.5 (iii) in Zhang [16]) For any $c > 0$,*

$$\hat{\mathbb{E}}\left(|X| \wedge c\right) \leq \int_0^c \mathbb{V}(|X| > x)\mathrm{d}x \leq C_{\mathbb{V}}(|X|), \tag{2.2}$$

where, here and hereafter, $a \wedge b := \min(a, b)$, and $a \vee b := \max(a, b)$ for any $a, b \in \mathbb{R}$.
(vi) *Markov inequality:* $\mathbb{V}(|X| \geq x) \leq \hat{\mathbb{E}}(|X|^p)/x^p, \ \forall \ x > 0, p > 0$;
Jensen inequality: $\left(\hat{\mathbb{E}}(|X|^r)\right)^{1/r} \leq \left(\hat{\mathbb{E}}(|X|^s)\right)^{1/s}$ for $0 < r \leq s$.

Definition 2.2. (Peng [14, 15])
(i) (Identical distribution) Let X_1 and X_2 be two random variables on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\hat{\mathbb{E}}(\varphi(X_1)) = \hat{\mathbb{E}}(\varphi(X_2)), \ \text{ for all } \ \varphi \in C_{l,Lip}(\mathbb{R}_n).$$

A sequence $\{X_n; n \geq 1\}$ of random variables is said to be identically distributed if for each $i \geq 1$, $X_i \overset{d}{=} X_1$.
(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent of another random vector $\mathbf{X} = (X_1, \ldots, X_m), X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}$ if for each $\varphi \in C_{l,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$, there is $\hat{\mathbb{E}}(\varphi(\mathbf{X}, \mathbf{Y})) = \hat{\mathbb{E}}[\hat{\mathbb{E}}(\varphi(\mathbf{x}, \mathbf{Y}))|_{\mathbf{x}=\mathbf{X}}]$.
(iii) (Independent and identically distributed) A sequence $\{X_n; n \geq 1\}$ of random variables is said to be i.i.d., if X_{i+1} is independent of $(X_1, \ldots, X_i)$ and $X_i \overset{d}{=} X_1$ for each $i \geq 1$.

From Definition 2.2 (ii), it can be verified that if Y is independent of X, and $X \geq 0, \hat{\mathbb{E}}Y \geq 0$, then $\hat{\mathbb{E}}(XY) = \hat{\mathbb{E}}(X)\hat{\mathbb{E}}(Y)$. Further, if Y is independent of X and $X, Y \geq 0$, then

$$\hat{\mathbb{E}}(XY) = \hat{\mathbb{E}}(X)\hat{\mathbb{E}}(Y), \quad \hat{\varepsilon}(XY) = \hat{\varepsilon}(X)\hat{\varepsilon}(Y). \tag{2.3}$$

For convenience, in all subsequent parts of this article, let $\{X, X_n; n \geq 1\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $S_n = \sum_{i=1}^n X_i$. For any $X \in \mathcal{H}$ and $c > 0$, set $X^{(c)} := (-c) \vee X \wedge c$. The symbol c represents a positive constant that does not depend on n. Let $a_x \sim b_x$ denote $\lim_{x\to\infty} a_x/b_x = 1$, $a_x \ll b_x$ denote that there exists a constant $c > 0$ such that $a_x \leq cb_x$ for sufficiently large x, $[x]$ denote the largest integer not exceeding x, and $I(\cdot)$ denote an indicator function.

To prove the main results of this article, the following three lemmas are required.

Lemma 2.1. *(Theorem 3.1 (a) and Corollary 3.2 (b) in Zhang [16]) Let $\{X_k; k \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.*
(i) If $\hat{\mathbb{E}}X_k \leq 0$, then for any $x, y > 0$,

$$\mathbb{V}(S_n \geq x) \leq \mathbb{V}\left(\max_{1\leq k\leq n} X_k > y\right) + \exp\left(-\frac{x^2}{2(xy + B_n)}\left\{1 + \frac{2}{3}\ln\left(1 + \frac{xy}{B_n}\right)\right\}\right);$$

(ii) If $\hat{\varepsilon}X_k \leq 0$, then there exists a constant $c > 0$ such that for any $x > 0$,

$$\nu(S_n \geq x) \leq c\frac{B_n}{x^2},$$

where $B_n = \sum_{k=1}^n \hat{\mathbb{E}}X_k^2$.

Here we give the notations of a G-normal distribution which was introduced by Peng [14].

Definition 2.3. (G-normal random variable) For $0 \leq \underline{\sigma}^2 \leq \bar{\sigma}^2 < \infty$, a random variable ξ in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a G-normal $\mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ distributed random variable (write $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ under $\hat{\mathbb{E}}$), if for any $\varphi \in C_{l,Lip}(\mathbb{R})$, the function $u(x, t) = \hat{\mathbb{E}}\left(\varphi(x + \sqrt{t}\xi)\right)$ $(x \in \mathbb{R}, t \geq 0)$ is the unique viscosity solution of the following heat equation:

$$\partial_t u - G(\partial^2_{xx} u) = 0, \quad u(0, x) = \varphi(x),$$

where $G(\alpha) = (\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)/2$.

From Peng [14], if $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ under $\hat{\mathbb{E}}$, then for each convex function φ,

$$\hat{\mathbb{E}}(\varphi(\xi)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(\bar{\sigma}x)\mathrm{e}^{-x^2/2}\mathrm{d}x. \tag{2.4}$$

If $\sigma = \bar{\sigma} = \underline{\sigma}$, then $\mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2]) = \mathcal{N}(0, \sigma^2)$ which is a classical normal distribution.

In particular, notice that $\varphi(x) = |x|^p, p \geq 1$ is a convex function, taking $\varphi(x) = |x|^p, p \geq 1$ in (2.4), we get

$$\hat{\mathbb{E}}(|\xi|^p) = \frac{2\bar{\sigma}^p}{\sqrt{2\pi}} \int_0^\infty x^p \mathrm{e}^{-x^2/2} \mathrm{d}x < \infty. \tag{2.5}$$

Equation (2.5) implies that

$$C_{\mathbb{V}}(|\xi|^p) = \int_0^\infty \mathbb{V}(|\xi|^p > x)\mathrm{d}x \leq 1 + \int_1^\infty \frac{\hat{\mathbb{E}}(|\xi|^{2p})}{x^2}\mathrm{d}x < \infty, \text{ for any } p \geq 1/2.$$

Lemma 2.2. *(Theorem 4.2 in Zhang [17], Corollary 2.1 in Zhang [18]) Let* $\{X, X_n; n \geq 1\}$ *be a sequence of i.i.d. random variables in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*. Suppose that*
(i) $\lim_{c\to\infty} \hat{\mathbb{E}}(X^2 \wedge c)$ *is finite;*
(ii) $x^2\mathbb{V}(|X| \geq x) \to 0$ *as* $x \to \infty$*;*
(iii) $\lim_{c\to\infty} \hat{\mathbb{E}}(X^{(c)}) = \lim_{c\to\infty} \hat{\mathbb{E}}((-X)^{(c)}) = 0.$
Then for any bounded continuous function φ*,*

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left(\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right) = \hat{\mathbb{E}}\left(\varphi(\xi)\right),$$

and if $F(x) := \mathbb{V}(|\xi| \geq x)$*, then*

$$\lim_{n\to\infty} \mathbb{V}(|S_n| > x\sqrt{n}) = F(x), \text{ if } x \text{ is a continuous point of } F, \tag{2.6}$$

where $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ *under* $\hat{\mathbb{E}}$*,* $\bar{\sigma}^2 = \lim_{c\to\infty} \hat{\mathbb{E}}(X^2 \wedge c)$ *and* $\underline{\sigma}^2 = \lim_{c\to\infty} \hat{\varepsilon}(X^2 \wedge c)$*.*

Lemma 2.3. *(Lemma 2.1 in Zhang [17]) Let* $\{X_n; n \geq 1\}$ *be a sequence of independent random variables in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*, and* $0 < \alpha < 1$ *be a real number. If there exist real constants* $\beta_{n,k}$ *such that*

$$\mathbb{V}(|S_n - S_k| \geq \beta_{n,k} + \epsilon) \leq \alpha, \text{ for all } \epsilon > 0 \ k \leq n,$$

then

$$(1-\alpha)\mathbb{V}\left(\max_{k\leq n}(|S_k| - \beta_{n,k}) > x + \epsilon\right) \leq \mathbb{V}(|S_n| > x), \text{ for all } x > 0, \ \epsilon > 0.$$

3. The main results and the proofs

The results of this article are as follows.

Theorem 3.1. *Let* $\{X, X_n; n \geq 1\}$ *be a sequence of i.i.d. random variables in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*. Suppose that*

$$C_{\mathbb{V}}(X^2) < \infty, \ \lim_{c\to\infty} \hat{\mathbb{E}}(X^{(c)}) = \lim_{c\to\infty} \hat{\mathbb{E}}((-X)^{(c)}) = 0. \tag{3.1}$$

Then for $0 \leq \delta \leq 1$*,*

$$\lim_{\epsilon\to0} \epsilon^{2+2\delta} \sum_{n=2}^{\infty} \frac{\ln^\delta n}{n} \mathbb{V}(|S_n| \geq \epsilon\sqrt{n\ln n}) = \frac{C_{\mathbb{V}}(|\xi|^{2\delta+2})}{\delta+1}, \tag{3.2}$$

where, here and hereafter, $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ *under* $\hat{\mathbb{E}}$*,* $\bar{\sigma}^2 = \lim_{c\to\infty} \hat{\mathbb{E}}(X^2 \wedge c)$ *and* $\underline{\sigma}^2 = \lim_{c\to\infty} \hat{\varepsilon}(X^2 \wedge c)$*.*

Conversely, if (3.2) holds for $\delta = 1$, then (3.1) holds.

Theorem 3.2. *Under the conditions of Theorem 3.1,*

$$\lim_{\epsilon\to 0}\epsilon^2\sum_{n=3}^{\infty}\frac{1}{n\ln n}\mathbb{V}(|S_n|\geq\epsilon\sqrt{n\ln\ln n}) = C_{\mathbb{V}}(\xi^2). \tag{3.3}$$

Remark 3.1. *Theorems 3.1 and 3.2 not only extend Theorem 3 in [8] and Theorem 2 in [9], respectively, from the probability space to sub-linear expectation space, but they also study and obtain necessary conditions for Theorem 3.1.*

Remark 3.2. *Under the condition* $\lim_{c\to\infty}\hat{\mathbb{E}}(|X|-c)^+ = 0$ *(* $\lim_{c\to\infty}\hat{\mathbb{E}}(X^2-c)^+ = 0 \Rightarrow \lim_{c\to\infty}\hat{\mathbb{E}}(|X|-c)^+ = 0$*), it is easy to verify that* $\hat{\mathbb{E}}(\pm X) = \lim_{c\to\infty}\hat{\mathbb{E}}((\pm X)^{(c)})$*. So, Corollary 3.9 in Ding [26] has two more conditions than Theorem 3.2:* $\hat{\mathbb{E}}$ *is continuous and* $\lim_{c\to\infty}\hat{\mathbb{E}}(X^2-c)^+ = 0$*. Therefore, Corollary 3.9 in Ding [26] and Theorem 3.2 cannot be inferred from each other.*

Proof of the direct part of Theorem 3.1. Note that

$$\begin{aligned}
&\epsilon^{2+2\delta}\sum_{n=2}^{\infty}\frac{\ln^{\delta}n}{n}\mathbb{V}(|S_n|\geq\epsilon\sqrt{n\ln n})\\
&=\epsilon^{2+2\delta}\sum_{n=2}^{\infty}\frac{\ln^{\delta}n}{n}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln n})+\epsilon^{2+2\delta}\sum_{n=2}^{\infty}\frac{\ln^{\delta}n}{n}\left(\mathbb{V}(|S_n|\geq\epsilon\sqrt{n\ln n})-\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln n})\right)\\
&:= I_1(\epsilon)+I_2(\epsilon).
\end{aligned}$$

Hence, in order to establish (3.2), it suffices to prove that

$$\lim_{\epsilon\to 0}I_1(\epsilon) = \frac{C_{\mathbb{V}}(|\xi|^{2\delta+2})}{\delta+1} \tag{3.4}$$

and

$$\lim_{\epsilon\to 0}I_2(\epsilon) = 0. \tag{3.5}$$

Given that $\frac{\ln^{\delta}n}{n}$ and $\mathbb{V}(|\xi|\geq\varepsilon\sqrt{\ln n})$ is monotonically decreasing with respect to n, it holds that

$$\begin{aligned}
I_1(\epsilon) &= \epsilon^{2+2\delta}\sum_{n=2}^{\infty}\frac{\ln^{\delta}n}{n}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln n})\\
&= \epsilon^{2+2\delta}\frac{\ln^{\delta}2}{2}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln 2})+\epsilon^{2+2\delta}\sum_{n=3}^{\infty}\int_{n-1}^{n}\frac{\ln^{\delta}n}{n}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln n})\mathrm{d}x\\
&\leq \epsilon^{2+2\delta}\frac{\ln^{\delta}2}{2}+\epsilon^{2+2\delta}\sum_{n=3}^{\infty}\int_{n-1}^{n}\frac{\ln^{\delta}x}{x}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln x})\mathrm{d}x\\
&= \epsilon^{2+2\delta}\frac{\ln^{\delta}2}{2}+\epsilon^{2+2\delta}\int_{2}^{\infty}\frac{\ln^{\delta}x}{x}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln x})\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
I_1(\epsilon) &= \epsilon^{2+2\delta}\sum_{n=2}^{\infty}\frac{\ln^\delta n}{n}\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln n})\\
&= \epsilon^{2+2\delta}\sum_{n=2}^{\infty}\int_n^{n+1}\frac{\ln^\delta n}{n}\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln n})\mathrm{d}x\\
&\ge \epsilon^{2+2\delta}\sum_{n=2}^{\infty}\int_n^{n+1}\frac{\ln^\delta x}{x}\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln x})\mathrm{d}x\\
&= \epsilon^{2+2\delta}\int_2^{\infty}\frac{\ln^\delta x}{x}\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln x})\mathrm{d}x.
\end{aligned}$$

Therefore, (3.4) follows from

$$\begin{aligned}
\lim_{\epsilon\to0}I_1(\epsilon) &= \lim_{\epsilon\to0}\epsilon^{2+2\delta}\int_2^{\infty}\frac{\ln^\delta x}{x}\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln x})\mathrm{d}x\\
&= \lim_{\epsilon\to0}\int_{\varepsilon\sqrt{\ln 2}}^{\infty}2y^{2\delta+1}\mathbb{V}(|\xi|\ge y)\mathrm{d}y \qquad (\text{let } y=\varepsilon\sqrt{\ln x})\\
&= \int_0^{\infty}2y^{2\delta+1}\mathbb{V}(|\xi|\ge y)\mathrm{d}y=\frac{C_{\mathbb{V}}(|\xi|^{2+2\delta})}{\delta+1}.
\end{aligned}$$

Let $M\ge 40$; write $A_{M,\epsilon}:=\exp(M\epsilon^{-2})$.

$$\begin{aligned}
|I_2(\epsilon)| &\le \epsilon^{2+2\delta}\sum_{2\le n\le[A_{M,\epsilon}]}\frac{\ln^\delta n}{n}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}}\ge\epsilon\sqrt{\ln n}\right)-\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln n})\right|\\
&\quad+\epsilon^{2+2\delta}\sum_{n>[A_{M,\epsilon}]}\frac{\ln^\delta n}{n}\mathbb{V}(|S_n|\ge\epsilon\sqrt{n\ln n})+\epsilon^{2+2\delta}\sum_{n>[A_{M,\epsilon}]}\frac{\ln^\delta n}{n}\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln n})\\
&:= I_{21}(\epsilon)+I_{22}(\epsilon)+I_{23}(\epsilon). \qquad (3.6)
\end{aligned}$$

Let us first estimate $I_{21}(\epsilon)$. For any $\beta>\epsilon^2$,

$$\begin{aligned}
I_{21}(\epsilon) &\sim \epsilon^{2+2\delta}\int_2^{A_{M,\epsilon}}\frac{\ln^\delta x}{x}\left|\mathbb{V}\left(\frac{|S_{[x]}|}{\sqrt{[x]}}\ge\epsilon\sqrt{\ln x}\right)-\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln x})\right|\mathrm{d}x\\
&\le \epsilon^{2+2\delta}\int_2^{A_{\beta,\epsilon}}\frac{2\ln^\delta x}{x}\mathrm{d}x\\
&\quad+\epsilon^{2+2\delta}\int_{A_{\beta,\epsilon}}^{A_{M,\epsilon}}\frac{\ln^\delta x}{x}\sup_{n\ge A_{\beta,\epsilon}}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}}\ge\epsilon\sqrt{\ln x}\right)-\mathbb{V}(|\xi|\ge\epsilon\sqrt{\ln x})\right|\mathrm{d}x\\
&\le 2\beta^{1+\delta}+\int_0^{\sqrt{M}}2y^{1+2\delta}\sup_{n\ge A_{\beta,\epsilon}}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}}\ge y\right)-F(y)\right|\mathrm{d}y. \qquad (3.7)
\end{aligned}$$

By (2.2), $\hat{\mathbb{E}}(X^2\wedge c)\le\int_0^c\mathbb{V}(X^2\ge x)\mathrm{d}x$; also, notice that $\mathbb{V}(X^2\ge x)$ is a decreasing function of x. So, $C_{\mathbb{V}}(X^2)=\int_0^\infty\mathbb{V}(X^2\ge x)\mathrm{d}x<\infty$ implies that $\lim_{c\to\infty}\hat{\mathbb{E}}(X^2\wedge c)$ is finite and $\lim_{x\to\infty}x^2\mathbb{V}(|X|\ge x)=$

$\lim_{x\to\infty} x\mathbb{V}(X^2 \geq x) = 0$. Therefore, (3.1) implies the conditions of Lemma 2.2. From (2.6),

$$\lim_{\epsilon\to 0}\sup_{n\geq A_{\beta,\epsilon}}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}}\geq y\right)-F(y)\right| = 0, \text{ if } y \text{ is a continuous point of } F. \tag{3.8}$$

Note that $F(y)$ is a monotonically decreasing function, so its discontinuous points are countable. Hence (3.8) holds for each y, except on a set with the null Lebesgue measure. Combining $y^{2\delta+1}\sup_{n\geq A_{\beta,\epsilon}}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}}\geq y\right)-F(y)\right| \leq 2M^{\delta+1/2}$ for any $0\leq y\leq\sqrt{M}$, by the Lebesgue bounded convergence theorem, (3.8) leads to the following:

$$\lim_{\epsilon\to 0}\int_0^{\sqrt{M}} y^{2\delta+1}\sup_{n\geq A_{\beta,\epsilon}}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}}\geq y\right)-F(y)\right|\mathrm{d}y = 0. \tag{3.9}$$

Let $\epsilon\to 0$ first, then let $\beta\to 0$; from (3.7) and (3.9), we get

$$\lim_{\epsilon\to 0} I_{21}(\epsilon) = 0. \tag{3.10}$$

Next, we estimate that $I_{22}(\epsilon)$. For $0<\mu<1$, let $\varphi_\mu(x)\in C_{l,Lip}(\mathbb{R})$ be an even function such that $0\leq\varphi_\mu(x)\leq 1$ for all x and $\varphi_\mu(x)=0$ if $|x|\leq\mu$ and $\varphi_\mu(x)=1$ if $|x|>1$. Then

$$I(|x|\geq 1)\leq\varphi_\mu(x)\leq I(|x|\geq\mu). \tag{3.11}$$

Given (2.1) and (3.11), and that X, X_i are identically distributed, for any $x>0$ and $0<\mu<1$, we get

$$\mathbb{V}(|X_i|\geq x)\leq\hat{\mathbb{E}}\left[\varphi_\mu\left(\frac{X_i}{x}\right)\right]=\hat{\mathbb{E}}\left[\varphi_\mu\left(\frac{X}{x}\right)\right]\leq\mathbb{V}(|X|\geq\mu x). \tag{3.12}$$

Without loss of generality, we assume that $\bar{\sigma}=1$. For $n\geq\exp(M\epsilon^{-2})\geq\exp(40\epsilon^{-2})$, set $b_n := \epsilon\sqrt{n\ln n}/20$; from Proposition 2.1 (ii) and the condition that $\lim_{c\to\infty}\hat{\mathbb{E}}(X^{(c)})=0$,

$$\begin{aligned}\sum_{i=1}^n\left|\hat{\mathbb{E}}X_i^{(b_n)}\right| &= n\left|\lim_{c\to\infty}\hat{\mathbb{E}}(X^{(c)})-\hat{\mathbb{E}}X^{(b_n)}\right|\leq n\lim_{c\to\infty}\hat{\mathbb{E}}\left|X^{(c)}-X^{(b_n)}\right|\\ &= n\lim_{c\to\infty}\hat{\mathbb{E}}\left(|X|\wedge c-b_n\right)^+\leq n\lim_{c\to\infty}\frac{\hat{\mathbb{E}}\left(|X|\wedge c\right)^2}{b_n}=\frac{n\bar{\sigma}^2}{b_n}=\frac{20\sqrt{n}}{\epsilon\sqrt{\ln n}}\\ &\leq \frac{\epsilon}{2}\sqrt{n\ln n},\quad\text{for } M\geq 40,\ n\geq\exp(M\epsilon^{-2}).\end{aligned}$$

Using Lemma 2.1 for $\{X_i^{(b_n)}-\hat{\mathbb{E}}X_i^{(b_n)}; 1\leq i\leq n\}$, and taking $x=\epsilon\sqrt{n\ln n}/2$ and $y=2b_n=\epsilon\sqrt{n\ln n}/10$ in Lemma 2.1 (i), by Proposition 2.1 (i), $\hat{\mathbb{E}}(X_i^{(b_n)}-\hat{\mathbb{E}}X_i^{(b_n)})=0$, and noting that $|X_i^{(b_n)}-\hat{\mathbb{E}}X_i^{(b_n)}|\leq y$, $B_n=\sum_{i=1}^n\hat{\mathbb{E}}(X_i^{(b_n)}-\hat{\mathbb{E}}X_i^{(b_n)})^2\leq 4n\hat{\mathbb{E}}(X_i^{(b_n)})^2\leq 4n$; combining this with (3.12) we get

$$\mathbb{V}(S_n\geq\epsilon\sqrt{n\ln n}) \leq \mathbb{V}\left(\sum_{i=1}^n(X_i^{(b_n)}-\hat{\mathbb{E}}X_i^{(b_n)})\geq\epsilon\sqrt{n\ln n}/2\right)+\sum_{i=1}^n\mathbb{V}(|X_i|\geq b_n)$$

$$\begin{aligned}
&\le \exp\left(-\frac{\epsilon^2 n\ln n}{4(\epsilon^2 n\ln n/20+4n)}\left\{1+\frac{2}{3}\ln\frac{\epsilon^2 n\ln n}{80n}\right\}\right)+n\mathbb{V}(|X|\ge \mu b_n)\\
&\le c(\epsilon^2\ln n)^{-3}+n\mathbb{V}(|X|\ge \mu\epsilon\sqrt{n\ln n}/20)
\end{aligned}$$

from $\frac{\epsilon^2 n\ln n}{4(\epsilon^2 n\ln n/20+4n)}\left\{1+\frac{2}{3}\ln\left(1+\frac{\epsilon^2\ln n}{80}\right)\right\}\ge 3\ln\left(\frac{\epsilon^2\ln n}{80}\right)$.

Since $\{-X,-X_i\}$ also satisfies the (3.1), we can replace the $\{X,X_i\}$ with $\{-X,-X_i\}$ in the upper form

$$\mathbb{V}(-S_n\ge\epsilon\sqrt{n\ln n})\le c(\epsilon^2\ln n)^{-3}+n\mathbb{V}(|X|\ge\mu\epsilon\sqrt{n\ln n}/20).$$

Therefore

$$\mathbb{V}(|S_n|\ge\epsilon\sqrt{n\ln n})\ll(\epsilon^2\ln n)^{-3}+n\mathbb{V}(|X|\ge c\epsilon\sqrt{n\ln n}).$$

This implies the following from Markov's inequality and (2.5),

$$\begin{aligned}
I_{22}(\epsilon)+I_{23}(\epsilon) &\ll \epsilon^{2+2\delta}\sum_{n\ge A_{M,\epsilon}}\frac{\ln^\delta n}{n}\left(n\mathbb{V}(|X|\ge c\epsilon\sqrt{n\ln n})+\frac{1}{\epsilon^6\ln^3 n}+\frac{\hat{\mathbb{E}}|\xi|^6}{\epsilon^6\ln^3 n}\right)\\
&\sim \epsilon^{2+2\delta}\int_{A_{M,\epsilon}}^\infty \ln^\delta x\mathbb{V}(|X|\ge c\epsilon\sqrt{x\ln x})\mathrm{d}x+c\epsilon^{-4+2\delta}\int_{A_{M,\epsilon}}^\infty\frac{\mathrm{d}x}{x\ln^{3-\delta}x}\\
&\le \epsilon^{2+2\delta}\int_{M\epsilon^{-1}}^\infty\frac{2^\delta y}{\ln^{1-\delta}y}\mathbb{V}(|X|\ge c\epsilon y)\mathrm{d}y+cM^{-2+\delta}\\
&\ll \epsilon^{2+2\delta}\int_{M\epsilon^{-1}}^\infty y\mathbb{V}(|X|\ge\epsilon y)\mathrm{d}y+M^{-2+\delta}\\
&\le \epsilon^{2\delta}\int_0^\infty z\mathbb{V}(|X|\ge z)\mathrm{d}z+M^{-2+\delta}\\
&= \epsilon^{2\delta}C_{\mathbb{V}}(X^2)/2+M^{-2+\delta}.
\end{aligned}$$

Let $\epsilon\to0$ first, then let $M\to\infty$; we get

$$\lim_{\epsilon\to0}(I_{22}(\epsilon)+I_{23}(\epsilon))=0.$$

Combining this with (3.10) and (3.6), (3.5) is established. □

Proof of the converse part of Theorem 3.1. If (3.2) holds for $\delta=1$, then

$$\sum_{n=2}^\infty\frac{\ln n}{n}\mathbb{V}(|S_n|\ge\epsilon\sqrt{n\ln n})<\infty\ \text{ for any }\ \epsilon>0. \tag{3.13}$$

Take ξ as defined by Lemma 2.2 ($\hat{\mathbb{E}}|\xi|<\infty$ from (2.5)) and the bounded continuous function ψ such that $I(x>q\hat{\mathbb{E}}|\xi|+1)\le\psi(x)\le I(x>q\hat{\mathbb{E}}|\xi|)$ for any fixed $q>0$. Therefore, for any $\epsilon>0, q>0$ and $n\ge\exp\left(\frac{q\hat{\mathbb{E}}|\xi|+1}{\epsilon}\right)^2$, according to (2.1), Lemma 2.2 and the Markov inequality, one has

$$\begin{aligned}
\mathbb{V}(|S_n|\ge\epsilon\sqrt{n\ln n}) &\le \mathbb{V}(|S_n|\ge(q\hat{\mathbb{E}}|\xi|+1)\sqrt{n})\le\hat{\mathbb{E}}\left(\psi\left(\frac{|S_n|}{\sqrt{n}}\right)\right)\\
&\to \hat{\mathbb{E}}(\psi(|\xi|))\le\mathbb{V}(|\xi|>q\hat{\mathbb{E}}|\xi|)\le\frac{\hat{\mathbb{E}}|\xi|}{q\hat{\mathbb{E}}|\xi|}\\
&= \frac{1}{q}.
\end{aligned}$$

From the arbitrariness of q, letting $q \to \infty$, we get the following for any $\epsilon > 0$,

$$\mathbb{V}(|S_n| \geq \epsilon\sqrt{n \ln n}) \to 0, \quad n \to \infty. \tag{3.14}$$

So, there is an n_0 such that $\mathbb{V}(|S_n| \geq \epsilon\sqrt{n \ln n}) < 1/4$ for $n \geq n_0$. Now for $n \geq 2n_0$, if $k \leq n/2$, then $n - k \geq n/2 \geq n_0$, and, combining this with (2.1), (3.11) and (3.12), we get that,

$$\begin{aligned}\mathbb{V}(|S_n - S_k| \geq 2\epsilon\sqrt{n \ln n}) &\leq \hat{\mathbb{E}}\left(\varphi_{1/2}\left(\frac{|S_n - S_k|}{2\epsilon\sqrt{n \ln n}}\right)\right) = \hat{\mathbb{E}}\left(\varphi_{1/2}\left(\frac{|S_{n-k}|}{2\epsilon\sqrt{n \ln n}}\right)\right) \\ &\leq \mathbb{V}(|S_{n-k}| \geq \epsilon\sqrt{(n-k)\ln(n-k)}) < 1/2.\end{aligned}$$

Also, if $n/2 < k \leq n$, then $n, k \geq n/2 \geq n_0$; thus,

$$\mathbb{V}(|S_n - S_k| \geq 2\epsilon\sqrt{n \ln n}) \leq \mathbb{V}(|S_n| \geq \epsilon\sqrt{n \ln n}) + \mathbb{V}(|S_k| \geq \epsilon\sqrt{n \ln n} \geq \epsilon\sqrt{k \ln k}) < 1/2.$$

Taking $\alpha = 1/2, \beta_{n,k} = 0$ in Lemma 2.3, for $n \geq 2n_0$,

$$\mathbb{V}\left(\max_{k \leq n} |S_k| \geq 4\epsilon\sqrt{n \ln n}\right) \leq \mathbb{V}(|S_n| \geq 2\epsilon\sqrt{n \ln n}).$$

Since $\max_{k \leq n} |X_k| \leq 2 \max_{k \leq n} |S_k|$, it follows that for $n \geq 2n_0$

$$\mathbb{V}\left(\max_{k \leq n} |X_k| \geq 8\epsilon\sqrt{n \ln n}\right) \leq \mathbb{V}(|S_n| \geq 2\epsilon\sqrt{n \ln n}). \tag{3.15}$$

Let $Y_k = \varphi_{8/9}\left(\frac{X_k}{9\epsilon\sqrt{n \ln n}}\right)$. Then,

$$\begin{aligned}I\left(\max_{k \leq n} |X_k| \geq 8\epsilon\sqrt{n \ln n}\right) &= 1 - I\left(\max_{k \leq n} |X_k| < 8\epsilon\sqrt{n \ln n}\right) \\ &= 1 - \prod_{k=1}^{n} I(|X_k| < 8\epsilon\sqrt{n \ln n}) \\ &\geq 1 - \prod_{k=1}^{n}(1 - Y_k).\end{aligned}$$

Since $\{X_k; k \geq 1\}$ is a sequence of i.i.d. random variables, $\{1 - Y_k; k \geq 1\}$ is also a sequence of i.i.d. random variables, and $1 - Y_k \geq 0$; given (2.1), (2.3) and $\hat{\mathbb{E}}(-X) = -\hat{\varepsilon}(X)$, it can be concluded that,

$$\begin{aligned}\mathbb{V}\left(\max_{k \leq n} |X_k| \geq 8\epsilon\sqrt{n \ln n}\right) &\geq \hat{\mathbb{E}}\left(1 - \prod_{k=1}^{n}(1 - Y_k)\right) = 1 - \hat{\varepsilon}\left(\prod_{k=1}^{n}(1 - Y_k)\right) \\ &= 1 - \prod_{k=1}^{n}\hat{\varepsilon}(1 - Y_k) = 1 - \prod_{k=1}^{n}(1 - \hat{\mathbb{E}}Y_k) \\ &\geq 1 - \prod_{k=1}^{n} \mathrm{e}^{-\hat{\mathbb{E}}Y_k} = 1 - \mathrm{e}^{-n\hat{\mathbb{E}}Y} \geq 1 - \mathrm{e}^{-n\mathbb{V}(|X| \geq 9\epsilon\sqrt{n \ln n})} \\ &\sim n\mathbb{V}(|X| \geq 9\epsilon\sqrt{n \ln n}).\end{aligned}$$

Hence, by (3.15) and (3.13)

$$\sum_{n=2}^{\infty} \ln n \mathbb{V}(|X| \geq \sqrt{n \ln n}) < \infty.$$

On the other hand,

$$\sum_{n=2}^{\infty} \ln n \mathbb{V}(|X| \geq \sqrt{n \ln n}) \sim \int_2^{\infty} \ln x \mathbb{V}(|X| \geq \sqrt{x \ln x}) \mathrm{d}x \sim \int_{\sqrt{2 \ln 2}}^{\infty} 2y \mathbb{V}(|X| \geq y) \mathrm{d}y \sim C_{\mathbb{V}}(X^2).$$

Hence,

$$C_{\mathbb{V}}(X^2) < \infty. \tag{3.16}$$

Next, we prove that $\lim\limits_{c\to\infty} \hat{\mathbb{E}}(X^{(c)}) = \lim\limits_{c\to\infty} \hat{\mathbb{E}}((-X)^{(c)}) = 0$. For $c_1 > c_2 > 0$, by (2.2) and (3.16),

$$\begin{aligned} \left|\hat{\mathbb{E}}(\pm X)^{(c_1)} - \hat{\mathbb{E}}(\pm X)^{(c_2)}\right| &\leq \hat{\mathbb{E}}\left|(\pm X)^{(c_1)} - (\pm X)^{(c_2)}\right| = \hat{\mathbb{E}}\left(|X| \wedge c_1 - c_2\right)^+ \\ &\leq \frac{\hat{\mathbb{E}}(|X| \wedge c_1)^2}{c_2} \leq \frac{C_{\mathbb{V}}(X^2)}{c_2} \ll \frac{1}{c_2}. \end{aligned}$$

This implies that

$$\lim_{c_1 > c_2 \to \infty} \left|\hat{\mathbb{E}}(\pm X)^{(c_1)} - \hat{\mathbb{E}}(\pm X)^{(c_2)}\right| = 0.$$

By the Cauchy criterion, $\lim\limits_{c\to\infty} \hat{\mathbb{E}}(X^{(c)})$ and $\lim\limits_{c\to\infty} \hat{\mathbb{E}}((-X)^{(c)})$ exist and are finite. It follows that $\lim\limits_{c\to\infty} \hat{\mathbb{E}}(X^{(c)}) = \lim\limits_{n\to\infty} \hat{\mathbb{E}}(X^{(n)}) := a$. So, for any $\epsilon > 0$, when n is large enough, $|\hat{\mathbb{E}}(X^{(n)}) - a| < \epsilon$; by Proposition 2.1 (iii), Lemma 2.1 (ii), $\hat{\mathbb{E}}(-X_k^{(n)} + \hat{\mathbb{E}}X_k^{(n)})^2 \leq 4\hat{\mathbb{E}}(X_k^{(n)})^2 \leq 4C_{\mathbb{V}}(X^2)$ and (3.16),

$$\begin{aligned} v\left(\frac{S_n}{n} < a - 2\epsilon\right) &\leq v\left(\left(\frac{S_n}{n} < a - 2\epsilon, \forall 1 \leq k \leq n, |X_k| \leq n\right) \cup (\exists 1 \leq k \leq n, |X_k| > n)\right) \\ &\leq v\left(\sum_{k=1}^{n} X_k^{(n)} < (a - 2\epsilon)n\right) + \sum_{k=1}^{n} \mathbb{V}(|X_k| > n) \\ &= v\left(\sum_{k=1}^{n}(-X_k^{(n)} + \hat{\mathbb{E}}X_k^{(n)}) > (2\epsilon - a)n + n\hat{\mathbb{E}}X^{(n)}\right) + \sum_{k=1}^{n} \mathbb{V}(|X_k| > n) \\ &\leq v\left(\sum_{k=1}^{n}(-X_k^{(n)} + \hat{\mathbb{E}}X_k^{(n)}) > \epsilon n\right) + \sum_{k=1}^{n} \mathbb{V}(|X_k| > n) \\ &\ll \frac{\sum_{k=1}^{n} \hat{\mathbb{E}}(-X_k^{(n)} + \hat{\mathbb{E}}X_k^{(n)})^2}{n^2} + \sum_{k=1}^{n} \frac{\hat{\mathbb{E}}(|X_k| \wedge n)^2}{n^2} \\ &\ll \frac{1}{n} \to 0, \quad n \to \infty. \end{aligned}$$

It is concluded that,

$$\lim_{n\to\infty} \mathbb{V}\left(\frac{S_n}{n} \geq a - 2\epsilon\right) = 1 \text{ for any } \epsilon > 0.$$

If $a > 0$, taking $\epsilon < a/2$, then $\epsilon_1 := a - 2\epsilon > 0$, and

$$\lim_{n\to\infty} \mathbb{V}\left(\frac{|S_n|}{n} \geq \epsilon_1\right) \geq \lim_{n\to\infty} \mathbb{V}\left(\frac{S_n}{n} \geq \epsilon_1\right) = 1. \tag{3.17}$$

On the other hand, by (3.14),

$$\lim_{n\to\infty}\mathbb{V}\left(\frac{|S_n|}{n}\geq\epsilon_1\right)\leq\lim_{n\to\infty}\mathbb{V}\left(|S_n|\geq\epsilon_1\sqrt{n\ln n}\right)=0,$$

which contradicts (3.17). It follows that $a\leq 0$. Similarly, we can prove that $b:=\lim_{c\to\infty}\hat{\mathbb{E}}((-X)^{(c)})\leq 0$. From $(-X)^{(c)}=-X^{(c)}$ and

$$0\geq a+b=\lim_{c\to\infty}\left(\hat{\mathbb{E}}(X^{(c)})+\hat{\mathbb{E}}(-X^{(c)})\right)\geq\lim_{c\to\infty}\hat{\mathbb{E}}\left(X^{(c)}-X^{(c)}\right)=0,$$

we conclude that $a=b=0$, i.e., $\lim_{c\to\infty}\hat{\mathbb{E}}(X^{(c)})=\lim_{c\to\infty}\hat{\mathbb{E}}((-X)^{(c)})=0$. This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Note that

$$\begin{aligned}
&\epsilon^2\sum_{n=3}^{\infty}\frac{1}{n\ln n}\mathbb{V}(|S_n|\geq\epsilon\sqrt{n\ln\ln n})\\
=&\ \epsilon^2\sum_{n=3}^{\infty}\frac{1}{n\ln n}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln\ln n})+\epsilon^2\sum_{n=3}^{\infty}\frac{1}{n\ln n}\left(\mathbb{V}(|S_n|\geq\epsilon\sqrt{n\ln\ln n})-\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln\ln n})\right)\\
:=&\ J_1(\epsilon)+J_2(\epsilon).
\end{aligned}$$

Hence, in order to establish (3.3), it suffices to prove that

$$\lim_{\epsilon\to 0}J_1(\epsilon)=C_{\mathbb{V}}(\xi^2)\tag{3.18}$$

and

$$\lim_{\epsilon\to 0}J_2(\epsilon)=0.\tag{3.19}$$

Obviously, (3.18) follows from

$$\begin{aligned}
\lim_{\epsilon\to 0}J_1(\epsilon)&=\lim_{\epsilon\to 0}\epsilon^2\int_3^{\infty}\frac{1}{x\ln x}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln\ln x})\mathrm{d}x\\
&=\lim_{\epsilon\to 0}\int_{\varepsilon\sqrt{\ln\ln 3}}^{\infty}2y\mathbb{V}(|\xi|\geq y)\mathrm{d}y\qquad(\text{let }\ y=\varepsilon\sqrt{\ln\ln x})\\
&=\int_0^{\infty}2y\mathbb{V}(|\xi|\geq y)\mathrm{d}y=C_{\mathbb{V}}(\xi^2).
\end{aligned}$$

Let $M\geq 32$; write $B_{M,\epsilon}:=\exp(\exp(M\epsilon^{-2}))$.

$$\begin{aligned}
|J_2(\epsilon)|\ &\leq\ \epsilon^2\sum_{3\leq n\leq[B_{M,\epsilon}]}\frac{1}{n\ln n}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}}\geq\epsilon\sqrt{\ln\ln n}\right)-\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln\ln n})\right|\\
&\quad+\epsilon^2\sum_{n>[B_{M,\epsilon}]}\frac{1}{n\ln n}\mathbb{V}(|S_n|\geq\epsilon\sqrt{n\ln\ln n})+\epsilon^2\sum_{n>[B_{M,\epsilon}]}\frac{1}{n\ln n}\mathbb{V}(|\xi|\geq\epsilon\sqrt{\ln\ln n})\\
&:=\ J_{21}(\epsilon)+J_{22}(\epsilon)+J_{23}(\epsilon).
\end{aligned}\tag{3.20}$$

Let us first estimate $J_{21}(\epsilon)$. For any $\beta > \epsilon^2$,

$$
\begin{aligned}
I_{21}(\epsilon) &\sim \epsilon^2 \int_3^{B_{M,\epsilon}} \frac{1}{x\ln x}\left|\mathbb{V}\left(\frac{|S_{[x]}|}{\sqrt{[x]}} \geq \epsilon\sqrt{\ln\ln x}\right) - \mathbb{V}(|\xi| \geq \epsilon\sqrt{\ln\ln x})\right| \mathrm{d}x \\
&\leq \epsilon^2 \int_3^{B_{\beta,\epsilon}} \frac{2}{x\ln x}\mathrm{d}x \\
&\quad +\epsilon^2 \int_{B_{\beta,\epsilon}}^{B_{M,\epsilon}} \frac{1}{x\ln x} \sup_{n\geq B_{\beta,\epsilon}}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}} \geq \epsilon\sqrt{\ln\ln x}\right) - \mathbb{V}(|\xi| \geq \epsilon\sqrt{\ln\ln x})\right| \mathrm{d}x \\
&\leq 2\beta + \int_0^{\sqrt{M}} 2y \sup_{n\geq B_{\beta,\epsilon}}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}} \geq y\right) - F(y)\right| \mathrm{d}y.
\end{aligned}
$$

Similar to (3.9) we have

$$
\lim_{\epsilon\to 0} \int_0^{\sqrt{M}} y \sup_{n\geq B_{\beta,\epsilon}}\left|\mathbb{V}\left(\frac{|S_n|}{\sqrt{n}} \geq y\right) - F(y)\right| \mathrm{d}y = 0.
$$

Therefore, let $\epsilon \to 0$ first, then let $\beta \to 0$; we get

$$
\lim_{\epsilon\to 0} J_{21}(\epsilon) = 0. \tag{3.21}
$$

Next, we estimate that $J_{22}(\epsilon)$. Without loss of generality, we still assume that $\bar{\sigma} = 1$. For $n \geq \exp(\exp(M\epsilon^{-2})) \geq \exp(\exp(32\epsilon^{-2}))$, set $a_n := \epsilon\sqrt{n\ln\ln n}/16$; from Proposition 2.1 (ii) and the condition that $\lim_{c\to\infty} \hat{\mathbb{E}}(X^{(c)}) = 0$,

$$
\begin{aligned}
\sum_{i=1}^{n}\left|\hat{\mathbb{E}}X_i^{(a_n)}\right| &= n\left|\lim_{c\to\infty}\hat{\mathbb{E}}(X^{(c)}) - \hat{\mathbb{E}}X^{(a_n)}\right| \leq n\lim_{c\to\infty}\hat{\mathbb{E}}\left|X^{(c)} - X^{(a_n)}\right| \\
&= n\lim_{c\to\infty}\hat{\mathbb{E}}\,(|X|\wedge c - a_n)^+ \leq n\lim_{c\to\infty}\frac{\hat{\mathbb{E}}\,(|X|\wedge c)^2}{a_n} = \frac{n\bar{\sigma}^2}{a_n} \\
&= \frac{16\sqrt{n}}{\epsilon\sqrt{\ln\ln n}} \leq \frac{\epsilon}{2}\sqrt{n\ln\ln n}.
\end{aligned}
$$

Using Lemma 2.1 for $\{X_i^{(a_n)} - \hat{\mathbb{E}}X_i^{(a_n)}; 1 \leq i \leq n\}$, and taking $x = \epsilon\sqrt{n\ln\ln n}/2$ and $y = 2a_n = \epsilon\sqrt{n\ln\ln n}/8$ in Lemma 2.1 (i), if we note that $|X_i^{(a_n)} - \hat{\mathbb{E}}X_i^{(a_n)}| \leq y$, and $B_n \leq 4n$, combined with (3.12) we get

$$
\begin{aligned}
\mathbb{V}(S_n \geq \epsilon\sqrt{n\ln\ln n}) &\leq \mathbb{V}\left(\sum_{i=1}^{n}(X_i^{(a_n)} - \hat{\mathbb{E}}X_i^{(a_n)}) \geq \epsilon\sqrt{n\ln\ln n}/2\right) + \sum_{i=1}^{n}\mathbb{V}(|X_i| \geq a_n) \\
&\leq \exp\left(-\frac{\epsilon^2 n\ln\ln n}{4(\epsilon^2 n\ln\ln n/16 + 4n)}\left\{1 + \frac{2}{3}\ln\frac{\epsilon^2 n\ln\ln n}{64n}\right\}\right) \\
&\quad +n\mathbb{V}(|X| \geq \mu a_n) \\
&\leq c(\epsilon^2\ln\ln n)^{-2} + n\mathbb{V}(|X| \geq \mu\epsilon\sqrt{n\ln\ln n}/16)
\end{aligned}
$$

from $\frac{\epsilon^2 n\ln\ln n}{4(\epsilon^2 n\ln\ln n/16+4n)}\left\{1 + \frac{2}{3}\ln\left(1 + \frac{\epsilon^2\ln\ln n}{64}\right)\right\} \geq 2\ln\left(\frac{\epsilon^2\ln\ln n}{64}\right)$.

Since $\{-X, -X_i\}$ also satisfies (3.1), we can replace the $\{X, X_i\}$ with $\{-X, -X_i\}$ in the upper form

$$\mathbb{V}(-S_n \geq \epsilon \sqrt{n \ln \ln n}) \leq c(\epsilon^2 \ln \ln n)^{-2} + n\mathbb{V}(|X| \geq \mu\epsilon \sqrt{n \ln \ln n}/16).$$

Therefore

$$\mathbb{V}(|S_n| \geq \epsilon \sqrt{n \ln \ln n}) \ll (\epsilon^2 \ln \ln n)^{-2} + n\mathbb{V}(|X| \geq c\epsilon \sqrt{n \ln \ln n}).$$

This implies the following from Markov's inequality and (2.5):

$$\begin{aligned}
J_{22}(\epsilon) + J_{23}(\epsilon) &\ll \epsilon^2 \sum_{n \geq B_{M,\epsilon}} \frac{1}{n \ln n} \left(n\mathbb{V}(|X| \geq c\epsilon \sqrt{n \ln \ln n}) + \frac{1}{\epsilon^4 (\ln \ln n)^2} + \frac{\hat{\mathbb{E}}|\xi|^4}{\epsilon^4 (\ln \ln n)^2} \right) \\
&\sim \epsilon^2 \int_{B_{M,\epsilon}}^{\infty} \frac{\mathbb{V}(|X| \geq c\epsilon \sqrt{x \ln \ln x})}{\ln x} \mathrm{d}x + c\epsilon^{-2} \int_{B_{M,\epsilon}}^{\infty} \frac{\mathrm{d}x}{x \ln x (\ln \ln x)^2} \\
&\leq \epsilon^2 \int_{\sqrt{M}\epsilon^{-1}}^{\infty} \frac{y}{\ln y \ln \ln y} \mathbb{V}(|X| \geq c\epsilon y) \mathrm{d}y + cM^{-1} \\
&\leq \int_{\sqrt{M}}^{\infty} z\mathbb{V}(|X| \geq z) \mathrm{d}z + cM^{-1} \\
&\to 0, \quad M \to \infty.
\end{aligned}$$

Hence

$$\lim_{\epsilon \to 0} (J_{22}(\epsilon) + J_{23}(\epsilon)) = 0.$$

Combining this with (3.20) and (3.21), (3.19) is established.

□

4. Conclusions

Statistical modeling is one of the key and basic topics in statistical theory research and practical application research. Under the theoretical framework of traditional probability space, in order to infer the model, all statistical models must assume that the error (and therefore the response variable) follows a unique and deterministic probability distribution, that is, the distribution of the model is deterministic. However, complex data in the fields of economics, finance, and other fields often have inherent and non negligible probability and distribution uncertainties. The probability distribution of the response variables that need to be studied is uncertain and does not meet the assumptions of classical statistical modeling. Therefore, classical probability statistical modeling methods cannot be used to model these types of data. How to analyze and model uncertain random data has been an unresolved and challenging issue that has long plagued statisticians. Driven by uncertainty issues, Peng [13] established a theoretical framework for the sub-linear expectation space from the perspective of expectations, providing a powerful tool for analyzing uncertainty problems. The sub-linear expectation has a wide range of potential applications. In recent years, the limit theory for sub-linear expectation spaces has attracted much attention from statisticians, and a series of research results have been achieved. This article overcomes the problem of many traditional probability space tools and methods no longer being effective due to the non additivity of sub-linear expectations and capacity; it also demonstrates the development of sufficient and necessary conditions for the rate convergence of logarithmic laws in sub-linear expectation spaces.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

Regarding this article, the author claims no conflict of interest.

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