AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 24740–24750.
DOI: 10.3934/math.20231261
Received: 24 March 2023
Revised: 23 May 2023
Accepted: 07 August 2023
Published: 23 August 2023

Research article

On the positive solutions for IBVP of conformable differential equations

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Abstract: A problem with integral boundary conditions (IBVP) involving conformable fractional derivatives is considered in this article. The upper and lower solutions technique is used to discuss the existence and uniqueness of positive solutions. The fixed point Theorem of Schauder proves the existence of positive solutions, and the fixed point Theorem of Banach proves the uniqueness of solutions. Our results are illustrated by an example.

Keywords: IBVP; positive solutions; fixed point; conformable differential equations
Mathematics Subject Classification: 34A08, 34A12, 34B18, 47H10

1. Introduction

Since the inception of the fractional derivatives by the scientist Leibniz in the year 1695, this type of derivative has known development in all branches of mathematics and even included applications in engineering and science (see [3, 10, 16, 18–21]).

Recently, a new definition has drawn much interest from many researchers, namely conformable fractional derivative introduced in [14] by Khalil et al. Since that time, several equations and applications have been studied and several articles have been published regarding this type of derivative (see [1, 4–7, 9, 12, 17, 23–25]). Many researchers study existence and positivity problems using the upper and lower solutions technique due to its effectiveness and good results (see [2, 8, 11, 13, 15, 22]). The upper and lower solutions method is associated with the use of fixed point theory to prove the existence and uniqueness of the solution.

Xu and Sun [22] proved the existence of positive solutions for the IBVP of the fractional differential equations

$$\begin{cases} D^{s}y(r)+p(r,y(r))=D^{s-1}q(r,y(r)), & r\in(0,1),\\ y(0)=0, \quad y(1)=\int_0^1 q(r,y(r))\,dr, \end{cases} \tag{1.1}$$

where D^s is the standard Riemann-Liouville derivative and $s \in (1, 2]$.

Zhong and Wang [25] used the fixed point theorem in a cone to show the existence of positive solutions of the BVP

$$\begin{cases} T_s y(r) + p(r, y(r)) = 0, & r \in [0, 1], \\ y(0) = 0, \quad y(1) = \lambda \int_0^1 y(r)\, dr, \end{cases} \tag{1.2}$$

where $s \in (1, 2]$, T_s denotes the conformable derivative of order s and λ is positive number.

The purpose of this paper is to examine an integral boundary value problem of conformable differential equations defined as follows

$$\begin{cases} T_s y(r) + p(r, y(r)) = T_{s-1} q(r, y(r)), & r \in (0, 1), \\ y(0) = 0, \quad y(1) = \int_0^1 q(r, y(r))\, dr, \end{cases} \tag{1.3}$$

where $s \in (1, 2]$, the functions $p, q : [0, 1] \times [0, \infty) \to [0, \infty)$ are continuous such that $q(r, y)$ is non-decreasing on y.

In this context, the main contributions of this paper is to apply the conformable derivative introduced in [14] to an integral boundary value problem which is the generalization of the problem (1.2). So, our study is organized as follow. After recalling some definitions and results of the conformable derivative in Section 2, and we give, in Section 3, the proof of our results concerning the existence and uniqueness of positive solutions. In Section 4, we write the conclusion in which we explain the contribution of this research.

2. Preliminaries

The purpose of this section is to provide the most important materials and preliminaries results for understanding conformable derivatives, (See [1, 14, 25]).

Definition 2.1. *[1, 14] Let $s \in (1, 2]$. The conformable fractional derivative of a function $p : [0, \infty) \to \mathbb{R}$ of order s is defined by*

$$T_s p(r) = \lim_{\varepsilon \to 0} \frac{p\left(r + \varepsilon r^{1-s}\right) - p(r)}{\varepsilon}.$$

If $T_s p(r)$ exists on $(0, \infty)$, then $T_s p(0) = \lim_{r \to 0^+} T_s p(r)$.

Definition 2.2. *[1, 14] Let $s \in (m, m+1]$, $m \in \mathbb{N}_0$ and function $p : [0, \infty) \to \mathbb{R}$.*

(a) The conformable fractional derivative of a function p of order s is defined by

$$T_s p(r) = T_\beta p^{(m)}(r) \text{ with } \beta = s - m.$$

(b) The fractional integral of a function p of order s is defined by

$$I_s p(r) = \frac{1}{m!} \int_0^r (r - \zeta)^m \zeta^{s-m-1} p(\zeta)\, d\zeta.$$

Lemma 2.1. *[1, 25] Let s be in $(m, m+1]$.*

(a) If p is a continuous function on $[0, \infty)$, then, for all $r > 0$, $T_s I_s p(r) = p(r)$.

(b) $T_s r^k = 0$ for r in $[0, 1]$ and $k = 0, 1, 2, ..., m$.

(c) If $T_s p(r)$ *is continuous on* $[0,\infty)$*, then*

$$I_s T_s p(r) = p(r) + c_0 + c_1 r + c_2 r^2 + ... + c_m r^m,$$

for some real numbers $c_k, k = 0, 1, 2, ..., m$.

Lemma 2.2. *Let* $y \in C([0,1], \mathbb{R})$*. Then* y *is a solution of (1.3) if and only if*

$$y(r) = \int_0^r q(\zeta, y(\zeta))\, d\zeta + \int_0^1 Q(r,\zeta)\, p(\zeta, y(\zeta))\, d\zeta,$$

where

$$Q(r,\zeta) = \begin{cases} (1-r)\,\zeta^{s-1}, & 0 \le \zeta \le r \le 1, \\ r(1-\zeta)\,\zeta^{s-2}, & 0 < r \le \zeta \le 1. \end{cases} \tag{2.1}$$

Proof. Note that

$$T_s \int_0^r q(\zeta, y(\zeta))\, d\zeta = T_{s-1} q(r, y(r)).$$

So

$$T_s\left(y(r) - \int_0^r q(\zeta, y(\zeta))\, d\zeta\right) + p(r, y(r)) = 0, \quad r \in (0,1).$$

Then, by Lemma 2.1 we have

$$y(r) - \int_0^r q(\zeta, y(\zeta))\, d\zeta + c_0 + c_1 r = -I_s p(r, y(r)),$$

the boundary conditions $y(0) = 0$, implies $c_0 = 0$ and

$$y(1) - \int_0^1 q(\zeta, y(\zeta))\, d\zeta + c_1 = -\int_0^1 (1-\zeta)\,\zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta,$$

$$c_1 = -\int_0^1 (1-\zeta)\,\zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta.$$

Hence

$$\begin{aligned} y(r) &= \int_0^r q(r, y(r))\, dr + \int_0^1 r(1-\zeta)\,\zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta \\ &\quad - \int_0^r (r-\zeta)\,\zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta \\ &= \int_0^r q(r, y(r))\, dr + \int_0^r r(1-\zeta)\,\zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta \\ &\quad + \int_r^1 r(1-\zeta)\,\zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta - \int_0^r (r-\zeta)\,\zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta \\ &= \int_0^r q(r, y(r))\, dr + \int_0^r \zeta^{s-1} p(\zeta, y(\zeta))\, d\zeta - \int_0^r r\zeta^{s-1} p(\zeta, y(\zeta))\, d\zeta \\ &\quad + \int_r^1 r(1-\zeta)\,\zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta \end{aligned}$$

$$= \int_0^r q(r, y(r))\, dr + \int_0^r (1-r)\, \zeta^{s-1} p(\zeta, y(\zeta))\, d\zeta$$
$$+ \int_r^1 r(1-\zeta)\, \zeta^{s-2} p(\zeta, y(\zeta))\, d\zeta.$$

So,

$$y(r) = \int_0^r q(\zeta, y(\zeta))\, d\zeta + \int_0^1 Q(r, \zeta)\, p(\zeta, y(\zeta))\, d\zeta.$$

Lemma 2.3 ([25])**.** *For any* (r, ζ) *in* $(0,1] \times (0,1]$*,*

$$0 \leq \omega(r)\, Q(\zeta, \zeta) \leq Q(r, \zeta) \leq Q(\zeta, \zeta),$$

where $\omega(r) = r(1-r)$*.*

3. Main results

Let the norm

$$\|y\| = \max_{r \in [0,1]} |y(r)|,$$

of the Banach space $X = C([0,1])$. Denote $\Omega := \{y \in X : y(r) \geq 0, r \in [0,1]\}$.

Let $a, b \in \mathbb{R}^+$ with $a < b$. For any $y \in [a, b]$, we define the upper control function with

$$U(r, y) = \sup\{p(r, \lambda) : a \leq \lambda \leq y\},$$

and the lower control function with

$$L(r, y) = \inf\{p(r, \lambda) : y \leq \lambda \leq b\}.$$

Clearly, $U(r, y)$ and $L(r, y)$ are monotonous non-decreasing on y and

$$L(r, y) \leq p(r, y) \leq U(r, y).$$

We need the following hypothesis:

(A) Let $\underline{y}(r), \overline{y}(r) \in \Omega$ with $a \leq \underline{y}(r) \leq \overline{y}(r) \leq b$ and

$$\underline{y}(r) \leq \int_0^r q\left(\zeta, \underline{y}(\zeta)\right) d\zeta + \int_0^1 Q(r, \zeta)\, L\left(\zeta, \underline{y}(\zeta)\right) d\zeta,$$

$$\overline{y}(r) \geq \int_0^r q(\zeta, \overline{y}(\zeta))\, d\zeta + \int_0^1 Q(r, \zeta)\, U(\zeta, \overline{y}(\zeta))\, d\zeta,$$

for all $r \in [0,1]$.

The function $\underline{y}(r)$ called the lower solution and the function $\overline{y}(r)$ is the upper solution of (1.3).

We need the following Lemma in the proof of the Theorem below

Lemma 3.1. *For each $r_1, r_2 \in [0,1]$, $r_1 < r_2$, the function Q defined by (2.1) satisfies*

$$\int_0^1 |Q(r_1,\zeta) - Q(r_2,\zeta)|\, d\zeta \le \max_{s\in(1,2]} \left\{ \frac{1}{s}, \frac{1}{s(s-1)}, \frac{2s-1}{s(s-1)} \right\} |r_1 - r_2| .$$

Proof. Let $\zeta \in [0,1]$, for each $r_1, r_2 \in [0,1]$, such that $r_1 < r_2$ we have three cases:

Case 1. For $0 \le \zeta \le r_1 < r_2 \le 1$, we have

$$\begin{aligned}\int_0^1 |Q(r_1,\zeta) - Q(r_2,\zeta)|\, d\zeta &= \int_0^1 \left|(1-r_1)\zeta^{s-1} - (1-r_2)\zeta^{s-1}\right| d\zeta \\ &= |r_1 - r_2| \int_0^1 \zeta^{s-1} d\zeta = \frac{1}{s}|r_1 - r_2| .\end{aligned}$$

Case 2. For $0 \le r_1 < r_2 \le \zeta \le 1$, we have

$$\begin{aligned}\int_0^1 |Q(r_1,\zeta) - Q(r_2,\zeta)|\, d\zeta &= \int_0^1 \left|r_1(1-\zeta)\zeta^{s-2} - r_2(1-\zeta)\zeta^{s-2}\right| d\zeta \\ &= |r_1 - r_2| \int_0^1 (1-\zeta)\zeta^{s-2} d\zeta \\ &= \frac{1}{s(s-1)}|r_1 - r_2| .\end{aligned}$$

Case 3. For $0 \le r_1 < \zeta \le r_2 \le 1$

$$\begin{aligned}\int_0^1 |Q(r_1,\zeta) - Q(r_2,\zeta)|\, d\zeta &= \int_0^1 \left|r_1(1-\zeta)\zeta^{s-2} - (1-r_2)\zeta^{s-1}\right| d\zeta \\ &= \int_0^1 \left|r_1\zeta^{s-2} - r_1\zeta^{s-1} - \zeta^{s-1} + r_2\zeta^{s-1}\right| d\zeta \\ &\le \int_0^1 \left(\zeta^{s-1}\left|\frac{r_1}{\zeta} - 1\right| + \zeta^{s-1}|r_2 - r_1|\right) d\zeta \\ &\le \int_0^1 \left(\zeta^{s-1}\left|\frac{r_1}{\zeta} - \frac{r_2}{\zeta}\right| + \zeta^{s-1}|r_2 - r_1|\right) d\zeta \\ &= |r_2 - r_1| \int_0^1 \left(\zeta^{s-2} + \zeta^{s-1}\right) d\zeta = \frac{2s-1}{s(s-1)}|r_2 - r_1| .\end{aligned}$$

Theorem 3.1. *The problem (1.3) has at least one positive solution $y \in \Omega$ if (A) holds. Furthermore,*

$$\underline{y}(r) \le y(r) \le \overline{y}(r), \text{ for all } r \in [0,1] .$$

Proof. Let

$$\Sigma := \left\{ y \in \Omega : \underline{y}(r) \le y(r) \le \overline{y}(r), r \in [0,1] \right\} .$$

It is easy to see that $\|y\| \le b$, so $\Sigma \subset X$ is closed, convex and bounded. If $y \in \Sigma$, $\exists R_p, R_q > 0$ two constants such that

$$\max_{r\in[0,1]} p(r, y(r)) < R_p,$$

and

$$\max_{r\in[0,1]} q(r, y(r)) < R_q.$$

From Lemma 2.2 we define the operator F as

$$Fy(r) = \int_0^r q(\zeta, y(\zeta))\, d\zeta + \int_0^1 Q(r,\zeta)\, p(\zeta, y(\zeta))\, d\zeta.$$

The continuity of p and q give the continuity of the operator F on Σ. Then, for $y \in \Sigma$ we have

$$\begin{aligned} Fy(r) &= \int_0^r q(\zeta, y(\zeta))\, d\zeta + \int_0^1 Q(r,\zeta)\, p(\zeta, y(\zeta))\, d\zeta \\ &\leq R_q + R_p \int_0^1 Q(\zeta,\zeta)\, d\zeta \\ &= R_q + \frac{R_p}{s(s+1)}. \end{aligned}$$

Hence $F(\Sigma)$ is uniformly bounded.

Now, for each $y \in \Sigma$, $r_1, r_2 \in [0,1]$, $r_1 < r_2$, we get

$$\begin{aligned} |(Fy)(r_1) - (Fy)(r_2)| &\leq \left| \int_0^{r_1} q(\zeta, y(\zeta))\, d\zeta - \int_0^{r_2} q(\zeta, y(\zeta))\, d\zeta \right| \\ &\quad + \int_0^1 |Q(r_1,\zeta) - Q(r_2,\zeta)|\, p(\zeta, y(\zeta))\, d\zeta \\ &= R_q |r_1 - r_2| + \int_0^1 |Q(r_1,\zeta) - Q(r_2,\zeta)|\, p(\zeta, y(\zeta))\, d\zeta \\ &\leq \left(R_q + R_p \max_{s\in(1,2]} \left\{ \frac{1}{s}, \frac{1}{s(s-1)}, \frac{2s-1}{s(s-1)} \right\} \right) |r_1 - r_2|. \end{aligned}$$

Therefore, $F(\Sigma)$ is equicontinuous. By Ascoli-Arzele Theorem, $F : \Sigma \to X$ is compact.

Next we will show that $F(\Sigma) \subset \Sigma$. Let $y \in \Sigma$, then from the hypothesis (A) we get

$$\begin{aligned} (Fy)(r) &= \int_0^r q(\zeta, y(\zeta))\, d\zeta + \int_0^1 Q(r,\zeta)\, p(\zeta, y(\zeta))\, d\zeta \\ &\leq \int_0^r q(\zeta, y(\zeta))\, d\zeta + \int_0^1 Q(r,\zeta)\, U(\zeta, y(\zeta))\, d\zeta \\ &\leq \int_0^r q(\zeta, \overline{y}(\zeta))\, d\zeta + \int_0^1 Q(r,\zeta)\, U(\zeta, \overline{y}(\zeta))\, d\zeta \\ &\leq \overline{y}(r). \end{aligned}$$

Similarly, $(Fy)(r) \geq \underline{y}(r)$.

As a conclusion, by the Schauder fixed point theorem, F has at least one fixed point, $y \in \Sigma$. So, the Eq (1.3) has at least one positive solution for all $y \in X$ and $\underline{y}(r) \leq y(r) \leq \overline{y}(r)$, for all $r \in [0,1]$.

Corollary 3.1. *Assume that there exist continuous functions h_1, h_2, h_3 and h_4 such that*

$$0 \leq h_1(r) \leq p(r,z) \leq h_2(r) < \infty, (r,z) \in [0,1] \times [0,+\infty), \tag{3.1}$$

$$0 \leq h_3(r) \leq q(r,z) \leq h_4(r) < \infty, (r,z) \in [0,1] \times [0,+\infty), \tag{3.2}$$

and at least one of $h_1(r)$ and $h_3(r)$ is not identically equal to 0*. Then (1.3) has at least one positive solution $y \in X$ and*

$$\int_0^r h_3(\zeta)\,d\zeta + \int_0^1 Q(r,\zeta)\,h_1(\zeta)\,d\zeta \leq y(r) \leq \int_0^r h_4(\zeta)\,d\zeta + \int_0^1 Q(r,\zeta)\,h_2(\zeta)\,d\zeta. \tag{3.3}$$

Proof. Consider the problem

$$\begin{cases} T_s x(r) + h_2(r) = T_{s-1} h_4(r), & r \in (0,1), \\ x(0) = 0, & x(1) = \int_0^1 h_4(r)\,dr, \end{cases} \tag{3.4}$$

which is equivalent to

$$x(r) = \int_0^r h_4(\zeta)\,d\zeta + \int_0^1 Q(r,\zeta)\,h_2(\zeta)\,d\zeta. \tag{3.5}$$

By the definitions of control function, we have

$$h_1(r) \leq L(r,x) \leq U(r,x) \leq h_2(r), \quad (r,x) \in [0,1] \times [a,b],$$

where a, b are minimal and maximal of $x(r)$ on $[0,1]$. Therefore we have

$$x(r) \geq \int_0^r q(\zeta, x(\zeta))\,d\zeta + \int_0^1 Q(r,\zeta)\,U(\zeta, x(\zeta))\,d\zeta.$$

Thus (3.5) is an upper solution of (1.3). On the other hand, we can prove

$$\int_0^r h_1(\zeta)\,d\zeta + \int_0^1 Q(r,\zeta)\,h_3(\zeta)\,d\zeta,$$

is a lower solution of (1.3). According to Theorem 3.1, (1.3) has at least one positive solution $y \in X$ and we obtain (3.3).

Corollary 3.2. *Suppose that*

(i) (3.2) and $0 \leq h_1(r) \leq p(r,z), r \in [0,1]$ *hold,*

(ii) $p(r,z)$ uniformly converges to $h(r)$ on $[0,1]$ *as $z \to \infty$,*

(iii) at least one of $h_1(r)$ and $h_3(r)$ is not identically equal to 0.

Then (1.3) has at least one positive solution $y \in X$.

Proof. From (ii), there exist η, $K > 0$ such that

$$|p(r,z) - h(r)| < \eta, (r,z) \in [0,1] \times [K,+\infty),$$

hence

$$p(r,z) < h(r) + \eta, (r,z) \in [0,1] \times [K,+\infty).$$

Let $v = \max_{(r,z)\in[0,1]\times[0,K]} p(r,z)$, hence

$$h_1(r) \le p(r,z) \le h(r) + \eta + v, (r,z) \in [0,1] \times [0,\infty).$$

From Corollary 3.1, the IBVP (1.3) has at least one solution, $y \in X$, satisfies

$$\begin{aligned}\int_0^r h_3(\zeta)\,d\zeta + \int_0^1 Q(r,\zeta)\,h_1(\zeta)\,d\zeta &\le y(r)\\ &\le \int_0^r h_4(\zeta)\,d\zeta + \int_0^1 Q(r,\zeta)\,h(\zeta)\,d\zeta + \frac{\eta+v}{s(s+1)}.\end{aligned}$$

Theorem 3.2. *Let (A) holds and assume that for any $r \in [0,1]$, $z, z^* \in \Sigma$,*

$$|p(r,z) - p(r,z^*)| \le L_p \|z - z^*\|,$$

$$|q(r,z) - q(r,z^*)| \le L_q \|z - z^*\|,$$

where $L_p, L_q > 0$ are constants satisfie

$$L_q + \frac{L_p}{s(s+1)} < 1. \tag{3.6}$$

Then the IBVP (1.3) has a unique positive solution on Σ.

Proof. We show in Theorem 3.1 that $F : \Sigma \to \Sigma$. So, for any $r \in [0,1]$, $z, z^* \in \Sigma$, we have

$$\begin{aligned}|(Fz)(r) - (Fz^*)(r)| &\le \int_0^r |q(\zeta, z(\zeta)) - q(\zeta, z^*(\zeta))|\,d\zeta\\ &\quad + \int_0^1 Q(r,\zeta)\,|p(\zeta, z(\zeta)) - p(\zeta, z^*(\zeta))|\,d\zeta\\ &\le L_q \|z - z^*\| + \frac{L_p}{s(s+1)} \|z - z^*\|\\ &= \left(L_q + \frac{L_p}{s(s+1)}\right) \|z - z^*\|.\end{aligned}$$

Since (3.6) is hold, then F is a contraction mapping that has unique fixed point $y \in \Sigma$. Therefore, the IBVP (1.3) has a unique positive solution on Σ.

In order to illustrate our results, we provide an example.

Example 3.1. *Consider the IBVP*

$$\begin{cases} T_{\frac{7}{4}} y(r) + r^3 + \frac{ry(r)}{4+y(r)} = T_{\frac{3}{4}} \left(\frac{\pi}{2} + r + \frac{1}{2}\tan^{-1} y(r)\right), & r \in (0,1),\\ y(0) = 0, \quad y(1) = \int_0^1 \left(\frac{\pi}{2} + \zeta + \frac{1}{2}\tan^{-1} y(\zeta)\right) d\zeta, \end{cases} \tag{3.7}$$

where $q(r,y) = \frac{\pi}{2} + r + \tan^{-1} y(r)$, $p(r,y) = r^3 + \frac{ry(r)}{4+y(r)}$. We can find q is non-decreasing on y, and

$$\frac{\pi}{2} + r \le q(r,y) \le \frac{3\pi}{4} + r,$$

$$r^3 \leq p(r,y) \leq r^3 + 1 \leq 2,$$

for $(r,y) \in [0,1] \times [0,\infty)$.

So, the IBVP (3.7) has at least one solution according to the above Corollaries. In addition, we have

$$|p(r,z) - p(r,z^*)| \leq \frac{1}{4}\|z - z^*\|,$$

$$|q(r,z) - q(r,z^*)| \leq \frac{1}{2}\|z - z^*\|,$$

this implies $L_q + \frac{L_p}{s(s+1)} = \frac{1}{2} + \frac{\frac{1}{4}}{\frac{7}{4}\left(\frac{7}{4}+1\right)} < 1$, *so the IBVP(3.7) has a unique positive solution due to Theorem 3.2.*

4. Conclusions

In this paper, we study an integral boundary problem with a conformable fractional derivative, such that our problem is more general than the problem studied in [25], so if $q(r, y(r)) \equiv 0$ or constant then we obtain results for the problem (1.2). Especially, the uniqueness has not been studied in the work [25].

The method of upper and lower solutions is more applicable and easily used for more general problems. Also, the fixed point theorems play an important role to show the existence and the uniquenes.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests

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