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Research article

A study of mixed generalized quasi-Einstein spacetimes with applications in general relativity

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Abstract: In the present paper we study Ricci pseudo-symmetry, Z-Ricci pseudo-symmetry and concircularly pseudo-symmetry conditions on a mixed generalized quasi-Einstein spacetime $MG(QE)_4$. Also, it is proven that if $d \neq \Lambda$, then $MG(QE)_4$ spacetime does not admit heat flux, where d and Λ are the function and the cosmological constant, respectively. In the end of this paper we construct a non-trivial example of $MG(QE)_4$ to prove its existence.

Keywords: quasi-Einstein manifolds; mixed generalized quasi-Einstein manifolds; Einstein's field equation; heat flux
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1. Introduction

A Riemannian (or semi-Riemannian) manifold (M^n, g), $(n \geq 3)$ is named an Einstein manifold if the Ricci tensor $Ric(\neq 0)$ of type $(0,2)$ satisfies: $Ric = \frac{r}{n}g$, where r represents the scalar curvature of (M^n, g). Einstein manifolds form a natural subclass of several classes of (M^n, g) determined by a curvature restriction imposed on their Ricci tensor [1]. Also, Einstein manifolds play a key role in Riemannian geometry, the general theory of relativity as well as in mathematical physics.

Approximately two decades ago, the idea of quasi-Einstein manifolds was proposed and studied by Chaki and Maity [2]. An (M^n, g), $(n > 2)$ is said to be a quasi-Einstein manifold $(QE)_n$ if its

$Ric(\neq 0)$ satisfies

$$Ric(\zeta_1, \zeta_2) = ag(\zeta_1, \zeta_2) + bU(\zeta_1)U(\zeta_2), \tag{1.1}$$

where $a, b(\neq 0) \in \mathbb{R}$ and U is a non-zero 1-form such that

$$g(\zeta_1, \varrho) = U(\zeta_1), \quad g(\varrho, \varrho) = U(\varrho) = 1, \tag{1.2}$$

for all vector fields ζ_1 and a unit vector field ϱ called the generator of $(QE)_n$. Also, the 1-form U is named the associated 1-form. From (1.1) it is clear that for $b = 0$, $(QE)_n$ reduces to an Einstein manifold. The notion of $(QE)_n$ came into existence during the study of exact solutions of Einstein's field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For example, the Robertson-Walker spacetimes are $(QE)_4$. Also, $(QE)_4$ can be taken as a model of the perfect fluid spacetime in general relativity [3–5].

An $(M^n, g), (n \geq 3)$ is said to be a generalized quasi-Einstein manifold $G(QE)_n$ [6–8] if its $Ric(\neq 0)$ satisfies

$$Ric(\zeta_1, \zeta_2) = ag(\zeta_1, \zeta_2) + bU(\zeta_1)U(\zeta_2) + cV(\zeta_1)V(\zeta_2), \tag{1.3}$$

where $a, b(\neq 0), c(\neq 0) \in \mathbb{R}$ and $U(\neq 0)$, $V(\neq 0)$ are 1-forms such that

$$g(\zeta_1, \varrho) = U(\zeta_1), \quad g(\zeta_1, \sigma) = V(\zeta_1), \quad g(\varrho, \varrho) = 1, \quad g(\sigma, \sigma) = 1, \tag{1.4}$$

where σ and ϱ are mutually orthogonal unit vector fields, i.e., $g(\varrho, \sigma) = 0$ and are known as generators of $G(QE)_n$.

In 2007, Bhattacharya et al. [9] introduced the notion of mixed generalized quasi-Einstein manifolds. An $(M^n, g), (n \geq 3)$ is said to be a mixed generalized quasi-Einstein manifold $MG(QE)_n$ if its $Ric(\neq 0)$ satisfies

$$\begin{aligned} Ric(\zeta_1, \zeta_2) = {} & ag(\zeta_1, \zeta_2) + bU(\zeta_1)U(\zeta_2) + cV(\zeta_1)V(\zeta_2) \\ & + d[U(\zeta_1)V(\zeta_2) + U(\zeta_2)V(\zeta_1)], \end{aligned} \tag{1.5}$$

where $a, b(\neq 0), c(\neq 0), d(\neq 0) \in \mathbb{R}$ and $U(\neq 0)$, $V(\neq 0)$ are 1-forms defined in (1.4).

$MG(QE)_n$ has wide applications in cosmology and the general theory of relativity and is studied by several authors, such as [10–13] and many others.

Putting $\zeta_1 = \zeta_2 = e_j$ in (1.5), where $\{e_j\}$ is an orthonormal basis of the tangent space at each point of $MG(QE)_n$ and taking summation over i $(1 \leq j \leq n)$, we get

$$r = na + b + c, \tag{1.6}$$

where r is the scalar curvature of $MG(QE)_n$.

Let $\mathcal{K}$ be the Riemannian curvature tensor of an (M^n, g). The k-nullity distribution $\mathcal{N}(k)$ of an (M^n, g) is defined by [14, 15]

$$\mathcal{N}(k) : p \to \mathcal{N}_p(k) = \{\zeta_3 \in T_pM^n : \mathcal{K}[\zeta_1, \zeta_2)\zeta_3 = k(g(\zeta_2, \zeta_3)\zeta_1 - g(\zeta_1, \zeta_3)\zeta_2]\}, \tag{1.7}$$

for all $\zeta_1, \zeta_2 \in \mathcal{T}_pM^n$, where k is some smooth function.

In a similar manner, the k-nullity distribution $\mathcal{N}(k)$ of a Lorentzian manifold can be defined. In a $(QE)_n$, if the generator ϱ belongs to some k-nullity distribution $\mathcal{N}(k)$, then (M^n, g) is called an $\mathcal{N}(k)$-$(QE)_n$ [16]. In 2007, Özgür and Triphati [17] proved that in an $\mathcal{N}(k)$-$(QE)_n$, k is not arbitrary, that is equal to $\frac{a+b}{n-1}$.

A spacetime is a time oriented (M^4, g) with Lorentz metric of signature $(+,+,+,-)$. A 4-dimensional Lorentzian manifold is said to be $MG(QE)_4$ with the generator ϱ if its $Ric(\neq 0)$ satisfies (1.5). Here $U(\neq 0)$ and $V(\neq 0)$ being1-forms such that σ is the heat flux vector field perpendicular to the velocity vector field ϱ. Therefore, for any ζ_1, we have

$$g(\zeta_1, \varrho) = U(\zeta_1), \;\; g(\zeta_1, \sigma) = V(\zeta_1), \;\; g(\varrho, \varrho) = -1, \;\; g(\sigma, \sigma) = 1. \tag{1.8}$$

From (1.5) and (1.8) we have

$$\begin{gathered} Ric(\zeta_1, \varrho) = (a-b)U(\zeta_1) - dV(\zeta_1), Ric(\zeta_1, \sigma) = (a+c)V(\zeta_1) + dU(\zeta_1), \\ r = 4a - b + c. \end{gathered} \tag{1.9}$$

In [18], a generalized $(0, 2)$ type symmetric Z tensor was introduced by Mantica and Molinari and defined as follows

$$Z(\zeta_1, \zeta_2) = Ric(\zeta_1, \zeta_2) + \phi g(\zeta_1, \zeta_2), \tag{1.10}$$

where ϕ is an arbitrary scalar function. The properties of the Z tensor in several ways to a different extent have been studied in [19, 20]. If the Z tensor at each point of the spacetime vanishes, then the spacetime is said to be Z flat.

Einstein's field equation (without cosmological constant) is given by

$$Ric(\zeta_1, \zeta_2) - \frac{r}{2} g(\zeta_1, \zeta_2) = \varLambda T(\zeta_1, \zeta_2), \tag{1.11}$$

where T and $\varLambda$ represent the energy-momentum tensor and the Einstein gravitational constant, respectively.

The idea of perfect fluid spacetime came into existence while discussing the structure of the universe. In general relativity the matter content of the spacetime is described by T. The matter content is supposed to be a fluid having pressure and density and possessing kinematical and dynamical quantities like acceleration, velocity, vorticity, shear and expansion. In a perfect fluid spacetime, the energy-momentum tensor T is given through the relation

$$T(\zeta_1, \zeta_2) = \mu g(\zeta_1, \zeta_2) + (\mu + \psi) U(\zeta_1) U(\zeta_2), \tag{1.12}$$

where ψ and μ stand for the energy density and isotropic pressure, respectively. ϱ is the unit timelike velocity vector field such that $g(\zeta_1, \varrho) = U(\zeta_1)$ for all ζ_1. In case of fluid matter distribution, the energy momentum tensor is given by Ellis [21] as

$$\begin{aligned} T(\zeta_1, \zeta_2) = \mu g(\zeta_1, \zeta_2) &+ (\mu + \psi) U(\zeta_1) U(\zeta_2) + V(\zeta_1) V(\zeta_2) \\ &+ U(\zeta_1) V(\zeta_2) + U(\zeta_2) V(\zeta_1), \end{aligned} \tag{1.13}$$

where $g(\zeta_1, \varrho) = U(\zeta_1)$, $g(\zeta_1, \sigma) = V(\zeta_1)$, $A = U(\varrho) = -1$, $V(\sigma) > 0$, $g(\varrho, \sigma) = 0$. σ is the heat conduction vector field perpendicular to the velocity vector field ϱ.

Definition 1.1. *An (M^n, g) is called Ricci-pseudosymmetric [22], if the tensors $\mathcal{K} \cdot Ric$ and $Q(g, Ric)$ are linearly dependent, where*

$$(\mathcal{K}(\zeta_1, \zeta_2) \cdot Ric)(\zeta_3, \zeta_4) = -Ric(\mathcal{K}(\zeta_1, \zeta_2)\zeta_3, \zeta_4) - Ric(\zeta_3, \mathcal{K}(\zeta_1, \zeta_2)\zeta_4), \tag{1.14}$$

$$Q(g, Ric)(\zeta_3, \zeta_4; \zeta_1, \zeta_2) = Ric((\zeta_1 \wedge_g \zeta_2)\zeta_3, \zeta_4) - Ric(\zeta_3, (\zeta_1 \wedge_g \zeta_2)\zeta_4), \tag{1.15}$$

and

$$(\zeta_1 \wedge_g \zeta_2)\zeta_3 = g(\zeta_2, \zeta_3)\zeta_1 - g(\zeta_1, \zeta_3)\zeta_2, \tag{1.16}$$

for all ζ_1, ζ_2, ζ_3, ζ_4 on M^n and $\mathcal{K}$ denotes the curvature tensor of $\mathcal{M}^n$. Then $(\mathcal{M}^n, g)$ is Ricci-pseudosymmetric if and only if

$$(\mathcal{K}(\zeta_1, \zeta_2) \cdot Ric)(\zeta_3, \zeta_4) = L_s Q(g, Ric)(\zeta_3, \zeta_4; \zeta_1, \zeta_2) \tag{1.17}$$

holds on G_s, where $G_s = \{\zeta_1 \in M^n : Ric \neq \frac{r}{n} g \text{ at } \zeta_1\}$, where L_s is a certain function on G_s.

The concircular curvature tensor N of type $(1, 3)$ on an (M^n, g) $(n \geq 3)$ is defined by [23]

$$N(\zeta_1, \zeta_2)\zeta_3 = \mathcal{K}(\zeta_1, \zeta_2)\zeta_3 - \frac{r}{n(n-1)}[g(\zeta_2, \zeta_3)\zeta_1 - g(\zeta_1, \zeta_3)\zeta_2], \tag{1.18}$$

where r is the scalar curvature of the manifold.

In view of (1.18), it follows that

$$\overline{N}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \overline{\mathcal{K}}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \frac{r}{n(n-1)}[g(\zeta_2, \zeta_3)g(\zeta_1, \zeta_4) - g(\zeta_1, \zeta_3)g(\zeta_2, \zeta_4)], \tag{1.19}$$

where $\overline{N}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = g(N(\zeta_1, \zeta_2)\zeta_3, \zeta_4)$, $\overline{\mathcal{K}}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = g(\mathcal{K}(\zeta_1, \zeta_2)\zeta_3, \zeta_4)$ and ζ_1, ζ_2, ζ_3, ζ_4 $\in \chi(M^n)$. Here, $\overline{K}$ is the curvature tensor of type $(0, 4)$ and $\overline{N}(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is the concircular curvature tensor of type (0,4) which satisfies the following properties: $\overline{N}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = -\overline{N}(\zeta_2, \zeta_1, \zeta_3, \zeta_4)$ and $\overline{N}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = -\overline{N}(\zeta_1, \zeta_2, \zeta_4, \zeta_3)$.

From (1.19), we have

$$\sum_{j=1}^{n} \overline{N}(e_j, \zeta_2, e_j, \varrho) = -Ric(\zeta_2, \varrho) + \frac{r}{n} g(\zeta_2, \varrho). \tag{1.20}$$

2. Ricci pseudo-symmetric $MG(QE)_4$ spacetimes

In this section, we consider Ricci-pseudosymmetric $MG(QE)_4$ spacetime. Therefore, from (1.5) and (1.14)–(1.17) we have

$$\begin{aligned}
& a[g(\mathcal{K}(\zeta_1, \zeta_2)\zeta_3, \zeta_4) + g(\zeta_3, \mathcal{K}(\zeta_1, \zeta_2)\zeta_4)] + b[U(\mathcal{K}(\zeta_1, \zeta_2)\zeta_3)U(\zeta_4) + U(\zeta_3)A(\mathcal{K}(\zeta_1, \zeta_2)\zeta_4)] \\
& + c[V(\mathcal{K}(\zeta_1, \zeta_2)\zeta_3)V(\zeta_4) + V(\zeta_3)V(\mathcal{K}(\zeta_1, \zeta_2)\zeta_4)] + d[U(\mathcal{K}(\zeta_1, \zeta_2)\zeta_3)V(\zeta_4) \\
& + U(\zeta_4)V(\mathcal{K}(\zeta_1, \zeta_2)\zeta_3) + U(\zeta_3)V(\mathcal{K}(\zeta_1, \zeta_2)\zeta_4) + V(\zeta_3)U(\mathcal{K}(\zeta_1, \zeta_2)\zeta_4)] \\
& = L_s\Big(b[g(\zeta_2, \zeta_3)U(\zeta_1)U(\zeta_4) - g(\zeta_1, \zeta_3)U(\zeta_2)U(\zeta_4) + g(\zeta_2, \zeta_4)U(\zeta_1)A(\zeta_3) \\
& - g(\zeta_1, \zeta_4)U(\zeta_2)A(\zeta_3)] + c[g(\zeta_2, \zeta_3)V(\zeta_1)V(\zeta_4) - g(\zeta_1, \zeta_3)V(\zeta_2)V(\zeta_4) \\
& + g(\zeta_2, \zeta_4)V(\zeta_1)V(\zeta_3) - g(\zeta_1, \zeta_4)V(\zeta_2)V(\zeta_3)] \\
& + d[g(\zeta_2, \zeta_3)[U(\zeta_1)V(\zeta_4) + U(\zeta_4)V(\zeta_1)] - g(\zeta_1, \zeta_3)[U(\zeta_2)V(\zeta_4) + U(\zeta_4)V(\zeta_2)] \\
& + g(\zeta_2, \zeta_4)[U(\zeta_1)V(\zeta_3) + U(\zeta_3)V(\zeta_1)] - g(\zeta_1, \zeta_4)[U(\zeta_2)V(\zeta_3) + U(\zeta_3)V(\zeta_2)]\Big).
\end{aligned} \tag{2.1}$$

Now, contracting the foregoing equation over ζ_1 and ζ_4, we obtain

$$\begin{aligned} &b[U(\mathcal{K}(\varrho,\zeta_2)\zeta_3)-U(\zeta_3)Ric(\zeta_2,\varrho)]+c[V(\mathcal{K}(\sigma,\zeta_2)\zeta_3)-V(\zeta_3)Ric(\zeta_2,\sigma)]\\ &+d[U(\mathcal{K}(\sigma,\zeta_2)\zeta_3)+V(\mathcal{K}(\varrho,\zeta_2)\zeta_3)-U(\zeta_3)Ric(\zeta_2,\sigma)-V(\zeta_3)Ric(\zeta_2,\sigma)]\\ &=L_s[b\{-g(\zeta_2,\zeta_3)-4U(\zeta_2)U(\zeta_3)\}+c\{g(\zeta_2,\zeta_3)-4V(\zeta_2)V(\zeta_3)\}\\ &-4d\{U(\zeta_2)V(\zeta_3)+U(\zeta_3)V(\zeta_2)\}]. \end{aligned} \tag{2.2}$$

Putting $\zeta_3=\varrho$ in (2.2), we obtain

$$\begin{aligned} &bRic(\zeta_2,\varrho)+c\overline{\mathcal{K}}(\sigma,\zeta_2,\varrho,\sigma)+d\{\overline{\mathcal{K}}(\varrho,\zeta_2,\varrho,\sigma)+Ric(\zeta_2,\sigma)\}\\ &=L_s[(3b+c)U(\zeta_2)+4dV(\zeta_2)], \end{aligned} \tag{2.3}$$

where $\overline{K}$ is the curvature tensor of type $(0,4)$.

By putting $\zeta_3=\varrho$ and $\zeta_4=\sigma$ in (2.1) and using (1.8), we obtain

$$\overline{\mathcal{K}}(\zeta_1,\zeta_2,\varrho,\sigma)=L_s[U(\zeta_2)V(\zeta_1)-U(\zeta_1)V(\zeta_2)],\quad b+c\neq 0. \tag{2.4}$$

In view of (1.8), (2.3) and (2.4), we arrive at

$$U(\zeta_2)[ab-b^2+d^2-3bL_s]+V(\zeta_2)[-bd+ad+cd-3dL_s]=0. \tag{2.5}$$

Putting $\zeta_2=\varrho$ in (2.5), we get

$$L_s=\frac{ab-b^2+d^2}{3b}. \tag{2.6}$$

Again, putting $\zeta_2=\sigma$ in (2.5), we get

$$\text{either } d=0 \quad or \quad L_s=\frac{a-b+c}{3}. \tag{2.7}$$

If $d=0$, then from (2.6) we find $L_s=\frac{a-b}{3}$, as $b\neq 0$. If $d\neq 0$, then $L_s=\frac{a-b+c}{3}$. Comparing this with (2.6), it follows that $c=d=0$ and thus $MG(QE)_4$ spacetime reduces to a quasi Einstein spacetime. Therefore, from (2.4) we have

$$\mathcal{K}(\zeta_1,\zeta_2)\varrho=\frac{(a-b)}{3}[U(\zeta_2)\zeta_1-U(\zeta_1)\zeta_2], \tag{2.8}$$

which means that the generator ϱ belongs to the $\frac{(a-b)}{3}$-nullity distribution. Thus, the manifold turns into $\mathcal{N}\left(\frac{a-b}{3}\right)$ quasi-Einstein spacetime. Therefore, we can state the following result:

Theorem 2.1. *Every Ricci-pseudosymmetric $MG(QE)_4$ spacetime is a $\mathcal{N}\left(\frac{a-b}{3}\right)$ quasi-Einstein spacetime, for some certain function $L_s=\frac{a-b}{3}$, where $b+c\neq 0$.*

3. Z-Ricci pseudo-symmetric $MG(QE)_4$ spacetime

An (M^n,g), $(n\geq 3)$ is called Z-Ricci pseudo-symmetric if and only if the following relation

$$Z\cdot\mathcal{Q}=f_{\mathcal{Q}}P(g,\mathcal{Q}), \tag{3.1}$$

holds on the set $G_Q = \{\zeta_1 \in M^n : P(g,Q) \neq 0 \quad at \quad \zeta_1\}$, where Q is the Ricci operator defined by $Ric(\zeta_1,\zeta_2) = g(Q\zeta_1,\zeta_2)$ and f_Q is a smooth function on G_Q. The tensor $P(g,Q)$ is defined by

$$P(g,Q)(\zeta_4;\zeta_1,\zeta_2) = Q((\zeta_1 \wedge_g \zeta_2)\zeta_4), \tag{3.2}$$

for all vector fields ζ_1, ζ_2, ζ_4.

Now, if $MG(QE)_4$ is a Z-Ricci pseudosymmetric, then from (3.1) we get

$$(Z(\zeta_1,\zeta_2)\cdot Q)\zeta_4 = f_Q P(g,Q)(\zeta_4;\zeta_1,\zeta_2). \tag{3.3}$$

If $f_Q = 0$, then (M^n, g) reduces to a Z-Ricci semisymmetric manifold. Now considering

$$\begin{aligned}(Z(\zeta_1,\zeta_2)\cdot Q)\zeta_4 &= ((\zeta_1 \wedge_z \zeta_2)\cdot Q)\zeta_4\\ &= (\zeta_1 \wedge_z \zeta_2)Q\zeta_4 - Q((\zeta_1 \wedge_z \zeta_2)\zeta_4)\\ &= Z(\zeta_2,Q\zeta_4)\zeta_1 - Z(\zeta_1,Q\zeta_4)\zeta_2 - Z(\zeta_2,\zeta_4)Q\zeta_1 + Z(\zeta_1,\zeta_4)Q\zeta_2.\end{aligned} \tag{3.4}$$

Also,

$$\begin{aligned}P(g,Q)(\zeta_4;\zeta_1,\zeta_2) &= Q((\zeta_1 \wedge_g \zeta_2)\zeta_4)\\ &= Q(g(\zeta_2,\zeta_4)\zeta_1 - g(\zeta_1,\zeta_4)\zeta_2)\\ &= g(\zeta_2,\zeta_4)Q\zeta_1 - g(\zeta_1,\zeta_4)Q\zeta_2.\end{aligned} \tag{3.5}$$

By virtue of (3.4) and (3.5), (3.3) turns to

$$\begin{aligned}&Z(\zeta_2,Q\zeta_4)\zeta_1 - Z(\zeta_1,Q\zeta_4)\zeta_2 - Z(\zeta_2,\zeta_4)Q\zeta_1 + Z(\zeta_1,\zeta_4)Q\zeta_2\\ &= f_Q\{g(\zeta_2,\zeta_4)Q\zeta_1 - g(\zeta_1,\zeta_4)Q\zeta_2\}.\end{aligned} \tag{3.6}$$

From (1.5), it follows that

$$Q\zeta_1 = a\zeta_1 + bU(\zeta_1)\varrho + cV(\zeta_1)\sigma + d[U(\zeta_1)\sigma + V(\zeta_1)\varrho]. \tag{3.7}$$

By virtue of (3.7), (3.6) becomes

$$\begin{aligned}&Z(\zeta_2,a\zeta_4)\zeta_1 + bU(\zeta_4)Z(\zeta_2,\varrho)\zeta_1 + cV(\zeta_4)Z(\zeta_2,\sigma)\zeta_1 + dU(\zeta_4)Z(\zeta_2,\sigma)\zeta_1\\ &+ dV(\zeta_4)Z(\zeta_2,\varrho)\zeta_1 - [Z(\zeta_1,a\zeta_4)\zeta_2 + bU(\zeta_4)Z(\zeta_1,\varrho)\zeta_2 + cV(\zeta_4)Z(\zeta_1,\sigma)\zeta_2\\ &+ dU(\zeta_4)Z(\zeta_1,\sigma)\zeta_2 + dV(\zeta_4)Z(\zeta_1,\varrho)\zeta_2]\\ &= \{f_Q g(\zeta_2,\zeta_4) + Z(\zeta_2,\zeta_4)\}Q\zeta_1 - \{f_Q g(\zeta_1,\zeta_4) + Z(\zeta_1,\zeta_4)\}Q\zeta_2.\end{aligned} \tag{3.8}$$

In view of (1.5) and (1.10), (3.8) takes the form

$$\begin{aligned}&aZ(\zeta_2,\zeta_4)\zeta_1 + bU(\zeta_4)\{(a-b+\phi)U(\zeta_2) - dV(\zeta_2)\}\zeta_1 + cV(\zeta_4)\{(a+c+\phi)V(\zeta_2) + dU(\zeta_2)\}\zeta_1\\ &+ dU(\zeta_4)\{(a+c+\phi)V(\zeta_2) + dU(\zeta_2)\}\zeta_1 + dV(\zeta_4)\{(a-b+\phi)U(\zeta_2) - dV(\zeta_2)\}\zeta_1\\ &- aZ(\zeta_1,\zeta_4)\zeta_2 - bU(\zeta_4)\{(a-b+\phi)U(\zeta_1) - dV(\zeta_1)\}\zeta_2 - cV(\zeta_4)\{(a+c+\phi)V(\zeta_1) + dU(\zeta_1)\}\zeta_2\\ &- dU(\zeta_4)\{(a+c+\phi)V(\zeta_1) + dU(\zeta_1)\}\zeta_2 - dV(\zeta_4)\{(a-b+\phi)U(\zeta_1) - dV(\zeta_1)\}\zeta_2\\ &= \{f_Q g(\zeta_2,\zeta_4) + Ric(\zeta_2,\zeta_4) + \phi g(\zeta_2,\zeta_4)\}Q\zeta_1 - \{f_Q g(\zeta_1,\zeta_4) + Ric(\zeta_1,\zeta_4) + \phi g(\zeta_1,\zeta_4)\}Q\zeta_2,\end{aligned}$$

which by putting $\zeta_1 = \varrho$, $\zeta_2 = \sigma$ yields

$$\begin{aligned}
&a\{(a+c+\phi)V(\zeta_4) + dU(\zeta_4)\}\varrho - bdU(\zeta_4)\varrho + c(a+c+\phi)V(\zeta_4)\varrho + d(a+c+\phi)U(\zeta_4)\varrho \\
&- d^2V(\zeta_4)\varrho - a\{(a-b+\phi)U(\zeta_4) - dV(\zeta_4)\}\sigma + b(a-b+\phi)U(\zeta_4)\sigma + cdV(\zeta_4)\sigma \\
&+ d^2U(\zeta_4)\sigma + d(a-b+\phi)V(\zeta_4)\sigma = \{f_Q V(\zeta_4) + (a+c)V(\zeta_4) + dU(\zeta_4) + \phi V(\zeta_4)\}Q\varrho \\
&- \{f_Q U(\zeta_4) + (a-b)U(\zeta_4) - dV(\zeta_4) + \phi U(\zeta_4)\}Q\sigma.
\end{aligned}$$

Taking the inner product of the foregoing equation with ϱ, we lead to

$$d(f_Q + b + c)U(\zeta_4) + \{(b+c)(a+c+\phi) - f_Q(a-b)\}V(\zeta_4) = 0. \tag{3.9}$$

Now by putting $\zeta_4 = \varrho$ in (3.9), we obtain $d(f_Q+b+c) = 0$. Thus, we have either $d = 0$ or $f_Q = -(b+c)$. For the first case $d = 0$, $MG(QE)_4$ spacetime reduces to a $G(QE)_4$ spacetime. Hence, we can state the following theorem:

Theorem 3.1. *A Z-Ricci pseudo-symmetric $MG(QE)_4$ spacetime is a $G(QE)_4$ spacetime.*

4. Concircularly pseudo-symmetric $MG(QE)_4$ spacetime

An (M^n, g), $(n \geq 3)$ is said to be concircularly pseudo-symmetric if and only if the following relation

$$(N(\zeta_1, \zeta_2) \cdot Ric)(\zeta_3, \zeta_4) = L_S Q(g, Ric)(\zeta_3, \zeta_4; \zeta_1, \zeta_2) \tag{4.1}$$

holds on the set G_s, where $G_s = \{\zeta_1 \in M^n : Ric \neq \frac{r}{n} g \ at \ \zeta_1\}$ and L_s is a certain function on G_s. In view of (1.14)–(1.16), (4.1) turns to

$$\begin{aligned}
&Ric(N(\zeta_1,\zeta_2)\zeta_3)\zeta_4) + Ric(\zeta_3, N(\zeta_1,\zeta_2)\zeta_4) \\
&= L_s[g(\zeta_2,\zeta_3)Ric(\zeta_1,\zeta_4) - g(\zeta_1,\zeta_3)Ric(\zeta_2,\zeta_4) \\
&+ g(\zeta_2,\zeta_4)Ric(\zeta_1,\zeta_3) - g(\zeta_1,\zeta_4)Ric(\zeta_2,\zeta_3)].
\end{aligned} \tag{4.2}$$

By using (1.5) in (4.2) it follows that

$$\begin{aligned}
&a[g(N(\zeta_1,\zeta_2)\zeta_3,\zeta_4) + g(\zeta_3, N(\zeta_1,\zeta_2)\zeta_4)] + b[U(N(\zeta_1,\zeta_2)\zeta_3)U(\zeta_4) + U(\zeta_3)U(N(\zeta_1,\zeta_2)\zeta_4)] \\
&+ c[V(N(\zeta_1,\zeta_2)\zeta_3)V(\zeta_4) + V(\zeta_3)V(N(\zeta_1,\zeta_2)\zeta_4)] + d[U(N(\zeta_1,\zeta_2)\zeta_3)V(\zeta_4) + U(\zeta_4)V(N(\zeta_1,\zeta_2)\zeta_3) \\
&+ U(\zeta_3)V(N(\zeta_1,\zeta_2)\zeta_4) + V(\zeta_3)U(N(\zeta_1,\zeta_2)\zeta_4)] \\
&= L_s\Big(b[g(\zeta_2,\zeta_3)U(\zeta_1)U(\zeta_4) - g(\zeta_1,\zeta_3)U(\zeta_2)U(\zeta_4) + g(\zeta_2,\zeta_4)U(\zeta_1)U(\zeta_3) - g(\zeta_1,\zeta_4)U(\zeta_2)U(\zeta_3)] \\
&+ c[g(\zeta_2,\zeta_3)V(\zeta_1)V(\zeta_4) - g(\zeta_1,\zeta_3)V(\zeta_2)V(\zeta_4) + g(\zeta_2,\zeta_4)V(\zeta_1)V(\zeta_3) - g(\zeta_1,\zeta_4)V(\zeta_2)V(\zeta_3)] \\
&+ d[g(\zeta_2,\zeta_3)[U(\zeta_1)V(\zeta_4) + U(\zeta_4)V(\zeta_1)] - g(\zeta_1,\zeta_3)[U(\zeta_2)V(\zeta_4) + U(\zeta_4)V(\zeta_2)] \\
&+ g(\zeta_2,\zeta_4)[U(\zeta_1)V(\zeta_3) + U(\zeta_3)V(\zeta_1)] - g(\zeta_1,\zeta_4)[U(\zeta_2)V(\zeta_3) + U(\zeta_3)V(\zeta_2)]\Big).
\end{aligned} \tag{4.3}$$

Now, contracting the foregoing equation over ζ_1 and ζ_4, we have

$$b[U(N(\varrho,\zeta_2)\zeta_3) + U(\zeta_3)\{-Ric(\zeta_2,\varrho) + \frac{r}{4}g(\zeta_2,\sigma)\}] + c[V(N(\sigma,\zeta_2)\zeta_3)$$

$$
\begin{aligned}
&+V(\zeta_3)\{-Ric(\zeta_2,\sigma)+\frac{r}{4}g(\zeta_2,\sigma)\}]+d[U(N(\sigma,\zeta_2)\zeta_3)+V(N(\varrho,\zeta_2)\zeta_3)\\
&+U(\zeta_3)\{-Ric(\zeta_2,\sigma)+\frac{r}{4}g(\zeta_2,\sigma)\}+V(\zeta_3)\{-Ric(\zeta_2,\varrho)+\frac{r}{4}g(\zeta_2,\varrho)\}]\\
&=L_s[b\{-g(\zeta_2,\zeta_3)-4U(\zeta_2)U(\zeta_3)\}+c\{g(\zeta_2,\zeta_3)-4V(\zeta_2)V(\zeta_3)\}\\
&-4d\{U(\zeta_2)V(\zeta_3)+U(\zeta_3)V(\zeta_2)\}].
\end{aligned}
\tag{4.4}
$$

Putting $\zeta_3=\varrho$ in (4.4), we get

$$
\begin{aligned}
&b[Ric(\zeta_2,\varrho)-\frac{r}{4}U(\zeta_2)]+c\overline{N}(\sigma,\zeta_2,\varrho,\sigma)\\
&+d[\overline{N}(\varrho,\zeta_2,\varrho,\sigma)+Ric(\zeta_2,\sigma)-\frac{r}{4}V(\zeta_2)]\\
&=L_s[(3b+c)U(\zeta_2)+4dV(\zeta_2)].
\end{aligned}
\tag{4.5}
$$

Putting $\zeta_3=\varrho$ and $\zeta_4=\sigma$ in (4.3) and using (1.8), we can easily find

$$
\begin{aligned}
&b[-g(N(\zeta_1,\zeta_2)\sigma,\varrho)]+c[g(N(\zeta_1,\zeta_2)\varrho,\sigma)]+d[g(N(\zeta_1,\zeta_2)\varrho,\varrho)-g(N(\zeta_1,\zeta_2)\sigma,\sigma)]\\
&=L_s\Big(b[-A(\zeta_1)B(\zeta_2)+A(\zeta_2)B(\zeta_1)]+c[A(\zeta_2)B(\zeta_1)-A(\zeta_1)B(\zeta_2)]\\
&+d[A(\zeta_2)A(\zeta_1)-A(\zeta_1)A(\zeta_2)+B(\zeta_2)(-B(\zeta_1))+B(\zeta_1)B(\zeta_2)]\Big).
\end{aligned}
$$

On simplification, we obtain

$$
\overline{N}(\zeta_1,\zeta_2,\varrho,\sigma)=L_s[U(\zeta_2)V(\zeta_1)-U(\zeta_1)V(\zeta_2)],\ \text{where}\ \ b+c\neq 0. \tag{4.6}
$$

From (1.18) and (4.6), we obtain

$$
\overline{\mathcal{K}}(\zeta_1,\zeta_2,\varrho,\sigma)=\Big(\frac{r}{12}+L_s\Big)[U(\zeta_2)V(\zeta_1)-U(\zeta_1)V(\zeta_2)]. \tag{4.7}
$$

In view of (1.9) and (4.6), from (4.5) it follows that

$$
U(\zeta_2)[-3b^2-bc+4d^2-12bL_s]+V(\zeta_2)[-5b+3c-12L_s]d=0. \tag{4.8}
$$

Putting $\zeta_2=\varrho$ in (4.8) gives

$$
L_s=\frac{-3b^2-bc+4d^2}{12b}. \tag{4.9}
$$

Again, putting $\zeta_2=\sigma$ in (4.8), we get

$$
d=0 \quad or \quad L_s=\frac{-b+c}{4}. \tag{4.10}
$$

If $d=0$, then from (4.9) we find $L_s=\frac{-3b-c}{12}$, as $b\neq 0$. If $d\neq 0$, then $L_s=\frac{-b+c}{4}$. Comparing this with (4.9), it follows that $c=d=0$ and thus $MG(QE)_4$ spacetime reduces to a $(QE)_4$ spacetime. Therefore, from (4.7) we have

$$
\mathcal{K}(\zeta_1,\zeta_2)\varrho=\frac{(a-b)}{3}[U(\zeta_2)\zeta_1-U(\zeta_1)\zeta_2], \tag{4.11}
$$

which means that the generator ϱ belongs to the $\frac{(a-b)}{3}$-nullity distribution. Thus, the manifold turns into $\mathcal{N}\Big(\frac{a-b}{3}\Big)$ quasi-Einstein spacetime. Therefore, we have the following result:

Theorem 4.1. *Every concircularly pseudo-symmetric $MG(QE)_4$ spacetime is a $\mathcal{N}\Big(\frac{a-b}{3}\Big)$ quasi-Einstein spacetime, for some certain function $L_s=\frac{a-b}{3}$, where $b+c\neq 0$.*

5. Heat flux in $MG(QE)_4$ spacetimes

Consider an $MG(QE)_4$ spacetime satisfying Einstein's field equation without cosmological constant whose matter content is viscous fluid. Then, by (1.11) and (1.13), the Ric is of the form

$$\begin{aligned} Ric(\zeta_1,\zeta_2) &= (\Lambda\mu+\frac{r}{2})g(\zeta_1,\zeta_2)+\Lambda(\mu+\psi)U(\zeta_1)U(\zeta_2) \\ &\quad +\Lambda V(\zeta_1)V(\zeta_2)+\Lambda[U(\zeta_1)V(\zeta_2)+U(\zeta_2)V(\zeta_1)]. \end{aligned} \tag{5.1}$$

From (1.5) and (5.1), we have

$$\begin{aligned} & ag(\zeta_1,\zeta_2)+bU(\zeta_1)U(\zeta_2)+cV(\zeta_1)V(\zeta_2) \\ & +d[U(\zeta_1)V(\zeta_2)+U(\zeta_2)V(\zeta_1)] \\ & =(\Lambda\mu+\frac{r}{2})g(\zeta_1,\zeta_2)+\Lambda(\mu+\psi)U(\zeta_1)U(\zeta_2) \\ & +\Lambda V(\zeta_1)V(\zeta_2)+\Lambda[U(\zeta_1)V(\zeta_2)+U(\zeta_2)V(\zeta_1)]. \end{aligned} \tag{5.2}$$

Putting $\zeta_2=\varrho$ in (5.2), it follows that

$$(a-b-\frac{r}{2}+\Lambda\psi)U(\zeta_1)=(d-\Lambda)V(\zeta_1), \tag{5.3}$$

for all ζ_1. Removing ζ_1 from the above equation we have

$$(a-b-\frac{r}{2}+\Lambda\psi)\varrho=(d-\Lambda)\sigma. \tag{5.4}$$

Taking the inner product in (5.3) by ϱ yields

$$a-b-\frac{r}{2}+\Lambda\psi=0. \tag{5.5}$$

Using (5.5) in (5.3) we get $B=0$ (which is inadmissible), provided $d\neq\Lambda$. Thus we have the following result:

Theorem 5.1. *An $MG(QE)_4$ spacetime can not admit heat flux if the smooth function d is not equal to the cosmological constant Λ.*

6. Example of $MG(QE)_4$ spacetime

In this section, we construct a non-trivial example to prove the existence of an $MG(QE)_4$ spacetime. We assume a Lorentzian manifold $(\mathcal{M}^4,g)$ endowed with the Lorentzian metric g given by

$$ds^2=g_{ij}d\zeta_1^i d\zeta_1^j=-\frac{\omega}{r}(d\zeta_1^2)+\frac{1}{\frac{c}{r}-4}(dr^2)+r^2(d\theta^2)+(r\sin\theta)^2(d\psi)^2, \tag{6.1}$$

where i, $j=1,2,3,4$ and ω, c are constants. Then the covariant and contravariant components of the metric are respectively given by

$$g_{11}=-\frac{\omega}{r},\quad g_{22}=\frac{1}{\frac{c}{r}-4},\quad g_{33}=r^2,\quad g_{44}=(r\sin\theta)^2,\quad g_{ij}=0\quad \text{for}\ \ 1\leq i\neq j\leq 4, \tag{6.2}$$

and

$$g^{11} = -\frac{r}{\omega}, \;\; g^{22} = \frac{c}{r} - 4, \;\; g^{33} = \frac{1}{r^2}, \;\; g^{44} = \frac{1}{(rsin\theta)^2}, \;\; g^{ij} = 0 \quad \text{for } \; 1 \le i \ne j \le 4. \tag{6.3}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{aligned}
&\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = 4r - c, \; \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = -\frac{1}{2r}, \; \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = \frac{c}{2r(c-4r)}, \; \begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 42 \end{Bmatrix} = \frac{1}{r}, \\
&\begin{Bmatrix} 4 \\ 43 \end{Bmatrix} = cot(\theta), \; \begin{Bmatrix} 2 \\ 44 \end{Bmatrix} = (4r - c)(sin\theta)^2, \; \begin{Bmatrix} 3 \\ 44 \end{Bmatrix} = \frac{-sin(2\theta)}{2}.
\end{aligned} \tag{6.4}$$

The non-zero derivatives of (6.4) are

$$\begin{aligned}
&\frac{\partial}{\partial r}\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = 4, \; \frac{\partial}{\partial r}\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = \frac{1}{2r^2}, \; \frac{\partial}{\partial r}\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = -\frac{c(c-8r)}{2r^2(c-4r)^2}, \; \frac{\partial}{\partial r}\begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = \frac{\partial}{\partial r}\begin{Bmatrix} 4 \\ 42 \end{Bmatrix} = -\frac{1}{r^2}, \\
&\frac{\partial}{\partial \theta}\begin{Bmatrix} 4 \\ 43 \end{Bmatrix} = cosec^2(\theta), \; \frac{\partial}{\partial r}\begin{Bmatrix} 2 \\ 44 \end{Bmatrix} = 4(sin\theta)^2, \; \frac{\partial}{\partial \theta}\begin{Bmatrix} 2 \\ 44 \end{Bmatrix} = (4r - c)(sin(2\theta)), \; \frac{\partial}{\partial \theta}\begin{Bmatrix} 3 \\ 44 \end{Bmatrix} = -cos(2\theta).
\end{aligned}$$

For the Riemannian curvature tensor,

$$\mathcal{K}^l_{ijK} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial \zeta_1^j} & \frac{\partial}{\partial \zeta_1^K} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ iK \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ iK \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mK \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II}.$$

The non-zero components of (I) are:

$$\begin{aligned}
\mathcal{K}^1_{221} &= \frac{\partial}{\partial r}\begin{Bmatrix} 1 \\ 21 \end{Bmatrix} = \frac{1}{2r^2}, \\
\mathcal{K}^2_{332} &= -\frac{\partial}{\partial r}\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = -4, \\
\mathcal{K}^2_{442} &= -\frac{\partial}{\partial r}\begin{Bmatrix} 2 \\ 44 \end{Bmatrix} = -4(sin\theta)^2, \\
\mathcal{K}^3_{443} &= -\frac{\partial}{\partial \theta}\begin{Bmatrix} 3 \\ 44 \end{Bmatrix} = cos(2\theta),
\end{aligned}$$

and the non-zero components of (II) are:

$$\begin{aligned}
\mathcal{K}^1_{221} &= \begin{Bmatrix} m \\ 21 \end{Bmatrix}\begin{Bmatrix} 1 \\ m2 \end{Bmatrix} - \begin{Bmatrix} m \\ 22 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} \\
&= \begin{Bmatrix} 1 \\ 21 \end{Bmatrix}\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 22 \end{Bmatrix}\begin{Bmatrix} 1 \\ 21 \end{Bmatrix} \\
&= \frac{2c - 4r}{4r^2(c-4r)},
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}^{1}_{313} &= \left\{ {m \atop 31} \right\} \left\{ {1 \atop m3} \right\} - \left\{ {m \atop 33} \right\} \left\{ {1 \atop m1} \right\} \\
&= -\left\{ {2 \atop 33} \right\} \left\{ {1 \atop 21} \right\} \\
&= \frac{4r-c}{2r},
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}^{1}_{441} &= \left\{ {m \atop 41} \right\} \left\{ {1 \atop m4} \right\} - \left\{ {m \atop 44} \right\} \left\{ {1 \atop m1} \right\} \\
&= -\left\{ {2 \atop 44} \right\} \left\{ {1 \atop 21} \right\} \\
&= \frac{(4r-c)(sin\theta)^2}{2r},
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}^{2}_{332} &= \left\{ {m \atop 22} \right\} \left\{ {2 \atop m3} \right\} - \left\{ {m \atop 33} \right\} \left\{ {2 \atop m2} \right\} \\
&= -\left\{ {2 \atop 32} \right\} \left\{ {2 \atop 33} \right\} \\
&= \frac{4r-c}{r},
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}^{2}_{442} &= \left\{ {m \atop 42} \right\} \left\{ {2 \atop m4} \right\} - \left\{ {m \atop 44} \right\} \left\{ {2 \atop m2} \right\} \\
&= \left\{ {4 \atop 42} \right\} \left\{ {2 \atop 44} \right\} - \left\{ {2 \atop 44} \right\} \left\{ {2 \atop 22} \right\} \\
&= -\frac{(sin\theta)^2(c-8r)}{2r},
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}^{3}_{443} &= \left\{ {m \atop 43} \right\} \left\{ {3 \atop m4} \right\} - \left\{ {m \atop 44} \right\} \left\{ {3 \atop m3} \right\} \\
&= \left\{ {4 \atop 43} \right\} \left\{ {3 \atop 44} \right\} - \left\{ {2 \atop 44} \right\} \left\{ {2 \atop 23} \right\} \\
&= -cos(\theta)^2 - \frac{(4r-c)(sin\theta)^2}{r}.
\end{aligned}$$

Adding components corresponding to (I) and (II), we have

$$\mathcal{K}^{1}_{221} = \frac{c-3r}{r^2(c-4r)},\ \mathcal{K}^{1}_{313} = \frac{4r-c}{2r},\ \mathcal{K}^{1}_{414} = \frac{(4r-c)(sin\theta)^2}{2r},$$

$$\mathcal{K}^{2}_{332} = -\frac{c}{2r},\ \mathcal{K}^{2}_{442} = -\frac{c(sin\theta)^2}{2r},\ \mathcal{K}^{3}_{443} = -\frac{(5r-c)(sin\theta)^2}{r}.$$

Thus, the non-zero components of the curvature tensor, up to symmetry are

$$\mathcal{K}_{1221} = -\frac{\omega(c-3r)}{r^3(c-4r)},\ \mathcal{K}_{1331} = \frac{\omega(c-4r)}{2r^2},\ \mathcal{K}_{1441} = \frac{\omega(c-4r)(sin\theta)^2}{2r^2},$$

$$\mathcal{K}_{2332} = \frac{c}{2(4r-c)},\ \mathcal{K}_{2442} = \frac{c(sin\theta)^2}{2(4r-c)},\ \mathcal{K}_{3443} = r(c-5r)(sin\theta)^2$$

and the non-zero components of the Ricci tensor are

$$Ric_{11} = g^{jh}\mathcal{K}_{1j1h} = g^{22}\mathcal{K}_{1212} + g^{33}\mathcal{K}_{1313} + g^{44}\mathcal{K}_{1414} = -\frac{\omega}{r^3},$$

$$Ric_{22} = g^{jh}\mathcal{K}_{2j2h} = g^{11}\mathcal{K}_{2121} + g^{33}\mathcal{K}_{2323} + g^{44}\mathcal{K}_{2424} = -\frac{3}{r(c-4r)},$$

$$Ric_{33} = g^{jh}\mathcal{K}_{3j3h} = g^{11}\mathcal{K}_{3131} + g^{22}\mathcal{K}_{3232} + g^{44}R_{3434} = -3,$$

$$Ric_{44} = g^{jh}\mathcal{K}_{4j4h} = g^{11}\mathcal{K}_{4141} + g^{22}\mathcal{K}_{4242} + g^{33}\mathcal{K}_{4343} = -3(sin\theta)^2.$$

The scalar curvature *scal* is given by

$$scal = g^{11}Ric_{11} + g^{22}Ric_{22} + g^{33}Ric_{33} + g^{44}Ric_{44} = -\frac{8}{r^2}.$$

Let us consider the associated scalars a, b, c, d defined by $a = -\frac{3}{r^2},\ b = -\frac{4}{r^2},\ c = \frac{2}{r^2},\ d = \frac{3}{r^2}$, and the 1-forms are defined by

$$A_i(\zeta_1) = \begin{cases} \sqrt{\frac{\omega}{r}}, & \text{if i=1} \\ 0, & \text{if 1=2,3,4} \end{cases} \quad and \quad B_i(\zeta_1) = \begin{cases} \frac{1}{2r^2}, & \text{if i=4} \\ 0, & \text{if i=1,2,3} \end{cases}$$

where generators are unit vector fields, then from (1.5), we have

$$Ric_{11} = ag_{11} + bU_1U_1 + cV_1V_1 + d(U_1V_1 + U_1V_1), \tag{6.5}$$

$$Ric_{22} = ag_{22} + bU_2U_2 + cV_2V_2 + d(U_2V_2 + U_2V_2), \tag{6.6}$$

$$Ric_{33} = ag_{33} + bU_3U_3 + cV_3V_3 + d(U_3V_3 + U_3V_3), \tag{6.7}$$

$$Ric_{44} = ag_{44} + bU_4U_4 + cV_4V_4 + d(U_4V_4 + U_4V_4). \tag{6.8}$$

Now,

$$\begin{aligned} R.H.S.\ of\ (6.5) &= ag_{11} + bU_1U_1 + cV_1V_1 + d(U_1V_1 + U_1V_1) \\ &= -\frac{3}{r^2} \times -\frac{\omega}{r} - \frac{4}{r^2} \times \frac{\omega}{r} + 0 + 0 \\ &= \frac{3\omega}{r^3} - \frac{4\omega}{r^3} - \\ &= -\frac{\omega}{r^3} \\ &= R_{11} \\ &= L.H.S.\ of\ (6.5). \end{aligned} \tag{6.9}$$

By a similar argument it can be shown easily that (6.6), (6.7) and (6.8) are also true. Hence, $(\mathbb{R}^4,g)$ is an $MG(QE)_4$.

7. Conclusions

The most modern approaches to mathematical general relativity begin with the concept of a manifold. After Riemannian manifolds, the structure of Lorentzian manifold is the most significant subclass of pseudo-Riemannian manifolds. The theory of general relativity is mainly studied on a semi-Riemannian manifold which sometimes is not an Einstein spacetime. Thus, it was always necessary to expand the concept of Einstein manifolds to quasi-Einstein, generalized quasi-Einstein and mixed generalized quasi-Einstein manifolds. Mixed generalized quasi-Einstein manifolds play a key role in the general relativity and cosmology and has wide applications in general relativistic viscous fluid spacetime admitting heat flux and stress. In the present work, we investigate some geometric and physical properties of mixed generalized quasi-Einstein spacetimes in general relativity and cosmology satisfying certain conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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