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Research article

Analysis of fractional global differential equations with power law

Abdon Atangana[1,2] **and Muhammad Altaf Khan**[1,*]

[1] Faculty of Natural and Agricultural Sciences, University of the Free State, South Africa
[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

* **Correspondence:** Email: altafdir@gmail.com.

Abstract: We have considered a special class of ordinary differential equations in which the differential operators are those of the Caputo fractional global derivative. These equations are generalizations of the well-known differential equations with the Caputo fractional derivative. Due to the various possible applications of these equations to model real-world problems we have first introduced some new inequalities that will be used in all fields of science, technology and engineering where these equations could be applied. We used Nagumo's principles to establish the existence and uniqueness of the solution for this class of equations with additional conditions. We have applied the midpoint principle to obtain a numerical scheme that will be used to solve these equations numerically. Some illustrative examples are presented with excellent results.

Keywords: numerical scheme; power law kernel; global derivative; Nagumo's principles
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1. Introduction

The generalization of differential and integral operators has gained popularity in recent years since it has been discovered that these operators are useful for capturing phenomena with complex dynamics [1–5]. Particularly in the context of fractional calculus, the concept of differentiation and integration of a function with another function has attracted a lot of attention. The most recent extensions are based on the Riemann-Stieltjes integral concept, which can be seen as a generalization of the fractional Riemann, Caputo-Fabrizio and Atangana-Baleanu integrals [6–10]. Therefore, these operators can be viewed as fractional derivatives and integrals of a function with respect to another with respect to a function say $g(t)$. When $g(t) = t$ we recover the classical fractional differential and integral operators. It is therefore required that, the function $g(t)$ should be nonzero, continuously differentiable and increasing, it is also possible to have $g(t)$ decreasing but never constant. A unique selection of the $g(t)$ function leads to a special class of fractional differential and integral equations. For example, choosing $g(t) = t^{\lambda}$ will lead to fractal-fractional differential and integral operators. We will then point out that new classes of differential and integral operators will result in new classes of differential and integral equations, which will require fresh research to grasp. Numerical methods are required to solve these equations due to the potential applications of the fractional global derivative with the power-law kernel, certain significant inequalities, existence and uniqueness [11–15]. Since these differential and integral operators will be used for modeling in many fields of science, technology and engineering, the first goal of this work is to establish some significant inequalities using existing theory. If precise solutions are not possible, the second goal is to identify specific circumstances in which these equations admit a singular solution. The final goal is to develop a numerical method to solve these equations based on current theory; in this study, the well-known midpoint approximation will be applied [16]. The above-mentioned processes will subsequently be followed by the structure of this work.

This work is organized as follows. In Section 2, we give details of the fractional global differential equations with the power law and some useful theorems regarding the proposed study. In Section 3, we give the existence and uniqueness of the nonlinear equations with global derivatives by using the power law. In Section 4, we use the generalized Caratheodory principle to give results for a general global fractional model to obtain the unique solution; further, some new results in the form of theorems are shown. In Sections 5 and 6, we use the numerical approach to solve the model and then provide some illustrative examples with details respectively. Finally, in Section 7, we summarize the results.

2. A fractional global differential equation with the power law kernel

The concept of the global fractional differential equation was first introduced in [17]. In this section, we consider the following general nonlinear equation:

$$\left\{ {}_{t_0}^{RL}D_g^{\alpha} y(t) = f(t, y(t)), \text{ if } t > 0, \; y(t_0) = t_0, \text{ if } t = t_0. \right. \tag{2.1}$$

Here, the operator ${}_{t_0}^{RL}D_g^{\alpha} y(t)$ is defined as follows:

$${}_{t_0}^{RL}D_g^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} D_g \int_{t_0}^{t} y(\tau)(t-\tau)^{-\alpha} d\tau, \tag{2.2}$$

where from the Riemann-Stieltjes integral,

$${}_{t_0}J_g f(t) = \int_{t_0}^{t} f(\tau) dg(t); \tag{2.3}$$

if the function $g(t)$ is differentiable with $g'(t) \neq 0$ for all $t \in [t_0, T]$, we have

$${}_{t_0}J_g f(t) = \int_{t_0}^{t} f(\tau) g'(\tau) d\tau. \tag{2.4}$$

With the fundamental theorem of calculus, we yield the corresponding differential operator:

$$D_g f(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{g(t+h) - g(t)}.$$

The above formula has been independently obtained and studied by several authors from different scholarly backgrounds. Some work related to this can be found in [9, 10].

We start our analysis by providing some useful inequalities under some conditions of the function f.

We assume that $t \in [t_0, T]$. We assume that the function $g(t)$ is differentiable that $g'(t) > 0$ and that it is continuous and bounded. That is to say $\forall t \in [t_0, T]$, and there exists $M'_g > 0$ such that

$$||g'(t)||_\infty = \sup_{t\in[t_0,T]} |g'(t)| < M'_g.$$

Theorem 1. *Assume that $\forall t \in [t_0, T]$; the function $f(t, y(t))$ satisfies*

$$|f(t, y(t))| < C(1 + |y(t)|),$$

where C is a constant; then,

$$|y(t)| < \frac{||g'||_\infty C(t-t_0)^\alpha}{\Gamma(\alpha+1)} \exp\Big[\frac{||g'||_\infty C(t-t_0)^\alpha}{\Gamma(\alpha+1)}\Big].$$

Proof. Since $g'(t)$ exists and is positive, bounded and continuous, then we convert the differential equation into an integro differential equation as follows:

$$\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_0}^t g'(\tau)f(\tau, y(\tau))(t-\tau)^{\alpha-1}d\tau, \\
|y(t)| &= \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^t g'(\tau)f(\tau, y(\tau))(t-\tau)^{\alpha-1}d\tau\Big| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^t |g'(\tau)|f(\tau, y(\tau))(t-\tau)^{\alpha-1}d\tau| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^t \sup_{l\in[t_0,\tau]} |g'(l)||f(\tau, y(\tau))|(t-\tau)^{\alpha-1}d\tau \\
&\leq \frac{||g||_\infty}{\Gamma(\alpha)}\int_{t_0}^t |f(\tau, y(\tau))|(t-\tau)^{\alpha-1}d\tau.
\end{aligned}$$

Using the hypothesis, we get

$$\begin{aligned}
|y(t)| &\leq \frac{||g||_\infty}{\Gamma(\alpha)}\int_{t_0}^t |C + C|y||(t-\tau)^{\alpha-1}d\tau \\
&\leq \frac{||g'||_\infty}{\Gamma(\alpha)}\int_{t_0}^t C(t-\tau)^{\alpha-1}d\tau + \frac{C||g'||_\infty}{\Gamma(\alpha)}\int_{t_0}^t |y|(t-\tau)^{\alpha-1}d\tau \\
&\leq \frac{||g'||_\infty}{\Gamma(\alpha)}C(t-t_0)^\alpha + \frac{||g'||_\infty C}{\Gamma(\alpha)}\int_{t_0}^t |y|(t-\tau)^{\alpha-1}d\tau.
\end{aligned}$$

We can put for the sake of clarity $|y(t)| = w(t)$; then, we have

$$w(t) \leq \frac{||g'||_\infty}{\Gamma(\alpha+1)}C(t-t_0)^\alpha + \frac{||g'||_\infty C}{\Gamma(\alpha)}\int_{t_0}^t w(\tau)(t-\tau)^{\alpha-1}d\tau.$$

By virtue of the Gronwall inequality, we have

$$w(t) < \frac{\|g'\|_\infty}{\Gamma(\alpha+1)} C(t-t_0)^\alpha \exp\Big[\frac{\|g'\|_\infty C(t-t_0)^\alpha}{\Gamma(\alpha+1)}\Big].$$

Therefore,

$$|y(t)| < \frac{\|g'\|_\infty}{\Gamma(\alpha+1)} C(t-t_0)^\alpha \exp\Big[\frac{\|g'\|_\infty C(t-t_0)^\alpha}{\Gamma(\alpha+1)}\Big],$$

which completes the proof. □

Theorem 2. *Assume that for all $t \in [t_0, T]$, there exists a $k > 0$ such that*

$$|f(\tau, y(\tau))|^2 < k(1 + |y|^2),$$

and if $\alpha > 1/2$, then

$$|y(t)|^2 < \frac{\overline{M}}{\Gamma^2(\alpha)} \frac{(t-t_0)^{2\alpha-1}}{(2\alpha-1)} k \exp\Big[\frac{\overline{M}k}{\Gamma^2(\alpha)(2\alpha-1)}(t-t_0)^{2\alpha-1}\Big].$$

Proof. If $g'(t)$ is continuous and bounded, then

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t g'(\tau)(t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \\ |y(t)|^2 &= \frac{1}{\Gamma^2(\alpha)} \Big| \int_{t_0}^t g'(\tau)(t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \Big|^2. \end{aligned}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} |y(t)|^2 &\leq \frac{1}{\Gamma^2(\alpha)} \int_{t_0}^t |g'(\tau)|^2 d\tau \int_{t_0}^t |t-\tau|^{2\alpha-2} |f(\tau, y(\tau))|^2 d\tau \\ &= \frac{1}{\Gamma^2(\alpha)} \int_{t_0}^t (g'(\tau))^2 d\tau \int_{t_0}^t (t-\tau)^{2\alpha-2} |f(\tau, y(\tau))|^2 d\tau. \end{aligned}$$

By integration by parts, we have

$$|y(t)|^2 \leq \frac{1}{\Gamma^2(\alpha)} \Big\{ g'(t)g(t) - g'(t_0)g(t_0) + \frac{g^2(t_0)}{2} - \frac{g^2(t)}{2} \Big\} \times \int_{t_0}^t (t-\tau)^{2\alpha-2} |f(\tau, y(\tau))|^2 d\tau.$$

By hypothesis, we get

$$\begin{aligned} |y(t)|^2 &\leq \frac{1}{\Gamma^2(\alpha)} \Big\{ \|g'\|_\infty \|g\|_\infty + \frac{\|g\|_\infty^2}{2} + \frac{|g(t_0)|^2}{2} + |g'(t_0)||g(t_0)| \Big\} \\ &\quad \times \Big\{ \int_{t_0}^t (t-\tau)^{2\alpha-2} k d\tau + \int_{t_0}^t k|y|^2 (t-\tau)^{2\alpha-2} d\tau \Big\} \\ &< \frac{\overline{M}}{\Gamma^2(\alpha)} \Big\{ \frac{(t-t_0)^{2\alpha-1}}{2\alpha-1} k + \int_{t_0}^t k|y(\tau)|^2 (t-\tau)^{2\alpha-2} d\tau \Big\}. \end{aligned}$$

By virtue of the Gronwall inequality, we have

$$|y(t)|^2 < \frac{\overline{M}}{\Gamma^2(\alpha)} \frac{(t-t_0)^{2\alpha-1}}{2\alpha-1} k \exp\Big[\frac{\overline{M}k}{\Gamma^2(\alpha)(2\alpha-1)}(t-t_0)^{2\alpha-1}\Big],$$

with the condition that $2\alpha - 1 > 0$ implies that $\alpha > 1/2$, which completes the proof. □

Theorem 3. *Assume that for all $t \in [t_0, T]$, there exists $C > 0$ such that $\forall t \in [t_0, T]$*

$$|f(t, y(t))| < C(1 + |y(t)|).$$

Then

$$|y(t)| < C\ {}_{t_0}J_t^{\alpha} g'(t) + \int_{t_0}^{t} g'(\tau)_{t_0} J_\tau^{\alpha} g'(\tau) \exp\Big[\int_{\tau}^{t} g'(q)(t-q)^{\alpha-1} dq d\tau\Big].$$

Proof. We have that for all $t \in [t_0, T]$,

$$\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \\
|y(t)| &= \Big|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau\Big|, \\
|y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} |g'(\tau)|(t-\tau)^{\alpha-1} \Big|f(\tau, y(\tau))\Big| d\tau, \\
|y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} |g'(\tau)|(t-\tau)^{\alpha-1} c(1 + |y(\tau)|) d\tau \\
&< \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} C d\tau + \frac{C}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} |y(\tau)| d\tau.
\end{aligned}$$

We let

$$v(t) = \frac{C}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} d\tau, \quad w(t) = |y(t)|.$$

We have

$$w(t) = v(t) + \frac{C}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} w(\tau) d\tau.$$

For $t > t_0$, $v(t)$ is continuous and $g'(\tau)(t-\tau)^{\alpha-1}$; therefore,

$$\begin{aligned}
w(t) &< v(t) + \int_{t_0}^{t} g'(\tau) v(\tau) \exp\Big[\int_{\tau}^{t} g'(q)(t-q)^{\alpha-1}\Big] dq d\tau, \\
|y(t)| &\leq C t_0 J_t^{\alpha} g'(t) + \int_{t_0}^{t} g'(\tau) t_0 J_\tau^{\alpha} g'(q) \exp\Big[\int_{t}^{\tau} g'(q)(t-q)^{\alpha-1} dq\Big] d\tau.
\end{aligned}$$

Note that

$$\frac{C}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(\tau-q)^{\alpha-1} d\tau = {}_{t_0}J_t^{\alpha} g'(t)\, C,$$

which completes the proof. □

Theorem 4. *Assume that for all $t \in [t_0, T]$, there exists $C > 0$, such that for all $t \in [t_0, T]$*

$$|f(t, y(t))| < C(1 + |y(t)|).$$

If in addition $g'(t) \in L^p[t_0, T]$, $p > 1$, then we have

$$|y(t)| < \frac{C}{\Gamma(\alpha)} ||g'||_p \frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1} \exp\Big[\frac{C}{\Gamma(\alpha)} ||g'||_p \frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1}\Big],$$

$\alpha > \frac{q-1}{q}$ *and*

$$\frac{1}{q} + \frac{1}{p} = 1.$$

Proof. For all $t \in [t_0, T]$, we have

$$\begin{aligned}
|y(t)| &< \frac{C}{\Gamma(\alpha)} \int_{t_0}^t g'(\tau)(t-\tau)^{\alpha-1} d\tau + \frac{C}{\Gamma(\alpha)} \int_{t_0}^t g'(\tau)(t-\tau)^{\alpha-1} |y(\tau)| d\tau, \\
|y(t)| &< \frac{C}{\Gamma(\alpha)} \Big(\int_{t_0}^t (g'(\tau))^p\Big)^{1/p} \Big(\int_{t_0}^t (t-\tau)^{\alpha q - q} d\tau\Big)^{1/q} + \frac{C}{\Gamma(\alpha)} \int_{t_0}^t g'(\tau)(t-\tau)^{\alpha-1} |y(\tau)| d\tau \\
&< \frac{C}{\Gamma(\alpha)} ||g'||_p \frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1} + \frac{C}{\Gamma(\alpha)} \int_{t_0}^t g'(\tau)(t-\tau)^{\alpha-1} |y(\tau)| d\tau.
\end{aligned}$$

For $\alpha > \frac{q-1}{q}$ using the Gronwall inequality, we have

$$\begin{aligned}
|y(t)| &< \frac{C}{\Gamma(\alpha)} ||g'||_p \frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1} \exp\Big[\frac{C}{\Gamma(\alpha)} \int_{t_0}^t g'(\tau)(t-\tau)^{\alpha-1} d\tau\Big] \\
&< \frac{C}{\Gamma(\alpha)} ||g'||_p \frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1} \exp\Big[\frac{C}{\Gamma(\alpha)} ||g'||_p \frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1} + 1\Big].
\end{aligned}$$

Therefore,

$$|y(t)| < \frac{C}{\Gamma(\alpha)} ||g'||_p \frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1} \exp\Big[\frac{c||g'||_p}{\Gamma(\alpha)} \frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1}\Big],$$

which completes the proof. □

Theorem 5. *Assume that for all $t \in [t_0, T]$, we can find a positive non-null function $M(t)$ such that*

$$|f(t, y(t))| < m(t),$$

then

$$|y(t)| < \frac{||g'||_\infty}{\Gamma(\alpha)} {}_{t_0}^{RL}J_t^\alpha m(t).$$

Proof. Let us start with the following:

$$\begin{aligned}
|y(t)| &= \frac{1}{\Gamma(\alpha)} \Big| \int_{t_0}^t g'(\tau) f(\tau, y(\tau))(t-\tau)^{\alpha-1} d\tau \Big|, \\
|y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^t |g'(\tau)||f(\tau, y(\tau))|(t-\tau)^{\alpha-1} d\tau, \\
|y(t)| &\leq \frac{||g'||_\infty}{\Gamma(\alpha)} \int_{t_0}^t m(t)(t-\tau)^{\alpha-1} d\tau \leq ||g'||_\infty {}_{t_0}^{RL}J_t^\alpha m(t),
\end{aligned}$$

which completes the proof; however, if the function $M(t) \in L^p[t_0, T]$, then we have

$$|y(t)| \quad < \quad \frac{\|g'\|_\infty}{\Gamma(\alpha)}\|m\|_p^p \frac{(t-t_0)^{\alpha p - p + 1}}{\alpha p - p + 1},$$

if $p > 1/2$ and $\alpha > \frac{p-1}{p}$. If $1 < p < 2$, we have

$$|y(t)| \quad < \quad \frac{\|g'\|_\infty}{\Gamma(\alpha)}\|m\|_p \Big(\frac{(t-t_0)^{\alpha q - q + 1}}{\alpha q - q + 1}\Big)^{1/q},$$

with

$$\frac{1}{q} + \frac{1}{p} = 1$$

and $\alpha > \frac{q-1}{q}$. □

Theorem 6. *Assume that for all $t \in [t_0, T]$, $g(t)$ is twice differentiable continuous* $([t_0, T], \Sigma, \mu)$. *Assume that*

$$f(t, y(t)), \overline{f}(t, y(t)) : [t_0, T] \to \mathbb{R}$$

is continuous and belonged to $L^p[t_0, T]$; then,

$$|y(t)| < \|g\|^{1/2}\|g''\|^{1/2}\frac{(t-t_0)^{\alpha p - p + 1}}{\alpha p - p + 1}\Big(\|f(.,y(.))\|_{L^p}^p\Big) + \Big(\|\overline{f}(.,y(.))\|_{L^p}^p\Big),$$

if $\alpha > \frac{p-1}{p}$ and $2 < p < \infty$.

Proof. We begin the proof by starting with the following:

$$\begin{aligned}
|y(t)| &= \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^t g'(\tau) f(\tau, y(\tau))(t-\tau)^{\alpha-1} d\tau\Big|, \\
|y(t)| &\leq \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^t \sup_{l \in [t_0, \tau]} (g'(l)) f(\tau, y(\tau))(t-\tau)^{\alpha-1} d\tau\Big| \\
&\leq \frac{\|g'\|_\infty}{\Gamma(\alpha)}\Big|\int_{t_0}^t f(\tau, y(\tau))(t-\tau)^{\alpha-1} d\tau\Big|.
\end{aligned}$$

By the Landau-Kolmogorov inequality, we have

$$\begin{aligned}
|y(t)| &\leq \frac{2\|g\|^{1/2}\|g''\|^{1/2}}{\Gamma(\alpha)}\Big(\int_{t_0}^t |f(\tau, y(\tau))|^p d\tau\Big)\int_{t_0}^t \Big((t-\tau)^{p\alpha - p} d\tau\Big) \\
&\leq \frac{2\|g\|^{1/2}\|g''\|^{1/2}}{\Gamma(\alpha)}\frac{(t-t_0)^{p\alpha - p + 1}}{p\alpha - p + 1}\int_{t_0}^t \Big|\frac{f(\tau, y(\tau)) + \overline{f}(\tau, y(\tau))}{2} + \frac{f(\tau, y(\tau)) - \overline{f}(\tau, y(\tau))}{2}\Big|^p d\tau \\
&\leq \frac{2\|g\|^{1/2}\|g''\|^{1/2}}{\Gamma(\alpha)}\frac{(t-t_0)^{p\alpha - p + 1}}{p\alpha - p + 1}\Big(\Big\|\frac{f(.,y(.)) + \overline{f}(.,y(.))}{2}\Big\|_{L^p}^p + \Big\|\frac{f(.,y(.)) - \overline{f}(.,y(.))}{2}\Big\|_{L^p}^p\Big).
\end{aligned}$$

Using the Clarkson's inequality, we have

$$|y(t)| \quad \leq \quad \frac{2\|g\|^{1/2}\|g''\|^{1/2}}{\Gamma(\alpha)}\frac{(t-t_0)^{p\alpha - p + 1}}{p\alpha - p + 1}\frac{1}{2}\Big(\Big\|f(.,y(.))\|_{L^p}^p + \|\overline{f}(.,y(.))\|_{L^p}^p\Big),$$

if $2 < p < \infty$ and $\alpha > \frac{p-1}{p}$, which completes the proof. However, if $1 < p < 2$, we shall have

$$|y(t)| \leq \frac{\|g\|^{1/2}\|g''\|^{1/2}}{\Gamma(\alpha)}\Big(\frac{(t-t_0)^{q\alpha-q+1}}{q\alpha-q+1}\Big)^{1/q}\Big(\frac{1}{2}\Big\|f(.,y(.))\|_{L^p}^p + \|\overline{f}(.,y(.))\|_{L^p}^p\Big)^{q/p},$$

$\alpha > \frac{q-1}{q}$ and

$$\frac{1}{q} + \frac{1}{p} = 1.$$

□

The above inequalities are of great importance because they appear in several proofs, such as the existence and uniqueness of nonlinear equations. In the next sections, we shall present the existence and uniqueness of the nonlinear equations.

3. Existence and uniqueness

In this section, we shall use different hypotheses to establish the existence and uniqueness of nonlinear equations with global derivatives based on the power law. First, we give results for the existence; then, in the next subsection, we show results for the uniqueness.

3.1. Uniqueness

Euler's method: Let f be a real continuous function on a domain S in the (t, y) phase. We note that by the definitions, ξ-approximate solutions of our equations on $I = [t_0, T]$ constitute a function $\overline{y}(t) \in C(I)$ satisfying the following:

- $(t, \overline{y}(t)) \in D$, $t \in I$.
- $\overline{y} \in C'[I]$, except for a finite set, and $\overline{I} \subset I$, where $\overline{y}'(t)$ may be discontinuous.
- $|{}^{RL}_{t_0}D^{\alpha}_{g}\overline{y} - f(t, \overline{y}(t))| \leq \xi$, $t \in I\ \overline{I}$.

It is also possible that $\overline{y}(t)$ has a piecewise continuous derivative on I; then, we shall write $\overline{y} \in C'_{pw}(I)$,

$$B = \{(t, y) : |t - t_0| \leq a, |y - y_0| < b\};$$

we have that $a, b > 0$; we assume that f is continuous, and we impose

$$N = \max_{(t,y)\in B} |f(t, y)|.$$

We also define

$$\beta = \min\Big\{a, \Big(\frac{b\Gamma(\alpha)}{\|g'\|_\infty N}\Big)^{\frac{1}{\alpha}}\Big\},$$

where

$$\|g'\|_\infty = \sup_{t\in[t_0,T]} |g'(t)|.$$

Theorem 7. *Let $f \in C(B)$. For all $\xi > 0$ there exists an ξ-approximate solution $\overline{y}$ of our equation on $|t - t_0| < \beta$.*

Proof. Let $\xi > 0$; we shall construct $\overline{y}$ on

$$K = [t_0, t_0 + \beta].$$

But $f \in C(B)$ and B is compact; f is indeed uniformly continuous on B. On the other hand $g'(t)$ is bounded and continuous, as well as positive. Therefore, $g'(t)$ is uniformly continuous; then, $g'(t)f(t, y(t))$ is uniformly continuous on B. Therefore for all $\xi > 0$, there exists $\delta_\xi > 0$, such that

$$|g'(t)f(t,y) - g'(\bar{t})f(\bar{t}, y(\bar{t}))| \leq \frac{\xi\Gamma(\alpha+1)}{\|g'\|_\infty T}, \ \forall (t,y), (\bar{t}, y(\bar{t})) \in B,$$

where

$$|t - \widetilde{t}| \leq \delta_\xi, \ \ |y - \widetilde{y}| \leq \xi.$$

We shall subdivide $t_0, t_0 + \beta$ into m subintervals with the end point $t_n = t_0 + nl, \ n = 0, 1, \cdots, m, \ l = \frac{\beta}{m}$, where

$$l \leq \min\Big\{\delta_\xi, \Big(\frac{\delta_\xi \Gamma(\alpha)}{M}\Big)^{\frac{1}{\alpha}}\Big\}.$$

From t_0 until $t = t_1$, we shall have

$$y(t) = g'(t_0)f(t_0, \overline{y}(t_0))\Big\{\frac{(t-t_0)^\alpha}{\alpha} - \frac{(t-t_1)^\alpha}{\alpha}\Big\}f(t, \overline{y}(t_1)),$$

we have

$$\overline{y}(t) = g'(t_0)f(t_0, \overline{y}(t_0))\Big\{\frac{(t-t_0)^\alpha}{\alpha} - \frac{(t-t_1)^\alpha}{\alpha}\Big\} + g'(t_1)f(t_1, \overline{y}(t_1))\Big\{\frac{(t-t_1)^\alpha}{\alpha} - \frac{(t-t_2)^\alpha}{\alpha}\Big\}f(t_2, \overline{y}'(t_2)),$$

we have

$$\begin{aligned}\overline{y}(t) &= g'(t_0)f(t_0, \overline{y}(t_0))\Big\{\frac{(t-t_0)^\alpha}{\alpha} - \frac{(t-t_1)^\alpha}{\alpha}\Big\} + g'(t_1)f(t_1, \overline{y}(t_1))\Big\{\frac{(t-t_1)^\alpha}{\alpha} - \frac{(t-t_2)^\alpha}{\alpha}\Big\} \\ &\quad + g'(t_2)f(t_2, \overline{y}(t_2))\Big\{\frac{(t-t_1)^\alpha}{\alpha} - \frac{(t-t_2)^\alpha}{\alpha}\Big\}.\end{aligned}$$

Repeating thus until $t_{n+1} = T$, we have

$$\begin{aligned}\overline{y}(t) &= \sum_{j=0}^{n} g'(t_j)f(t_j, \overline{y}(t_j))\Big\{\frac{(t-t_j)^\alpha - (t-t_{j+1})^\alpha}{\Gamma(\alpha+1)}\Big\} \\ &= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} g'(t_j)f(t_j, \overline{y}(t_j))(t-\tau)^{\alpha-1}d\tau.\end{aligned}$$

For all $t, \bar{t}$ and K, we have that

$$\begin{aligned}|\overline{y}(t) - \overline{y}(\bar{t})| &= \Big|\sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} g'(t_0)f(t_j, \overline{y}(t_j))\Big\{\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau - \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} g'(t_j)f(t_j, \overline{y}(t_j))\frac{(\bar{t}-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\Big\} \\ &\quad + \sum_{j=n+1}^{m} \int_{t_j}^{t_{j+1}} g'(t_j)f(t_j, \overline{y}(t_j))\frac{(\bar{t}-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\Big|,\end{aligned}$$

if we assume without loss of generality that $t > \bar{t}$

$$
\begin{aligned}
|\overline{y}(t) - \overline{y}(\bar{t})| &\leq \sum_{j=0}^{n} |g'(t_j)||f(t_j, \overline{y}(t_j))| \int_{t_j}^{t_{j+1}} \Big\{ \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(\bar{t}-\tau)^{\alpha-1}}{\Gamma(\alpha)} \Big\} d\tau \\
&\quad + \sum_{j=n+1}^{m} |g'(t_j)| \frac{|f(t_j, \overline{y}(t_j))|}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t-\tau)^{\alpha-1} d\tau \\
&\leq \sum_{j=0}^{n} \frac{|g'(t_j)|}{\Gamma(\alpha)} |f(t_j, \overline{y}(t_j))| \Big\{ \frac{(t-t_j)^\alpha}{\alpha} - \frac{(t-t_{j+1})^\alpha}{\alpha} - \frac{(\bar{t}-t_j)^\alpha}{\alpha} \\
&\quad + \frac{(\bar{t}-t_{j+1})^\alpha}{\alpha} + \sum_{j=n+1}^{m} |g'(t_j)||f(t_j, \overline{y}(t_j))| \Big\} \Big\{ \frac{(t-t_j)^\alpha}{\alpha} - \frac{(t-t_{j+1})^\alpha}{\alpha} \Big\} \\
&\leq \frac{|g'|_\infty N}{\Gamma(\alpha)} \Big(\sum_{j=0}^{n} \Big\{ \frac{(t-t_j)^\alpha}{\alpha} - \frac{(t-t_{j+1})^\alpha}{\alpha} - \frac{(\bar{t}-t_j)^\alpha}{\alpha} + \frac{(\bar{t}-t_{j+1})^\alpha}{\alpha} \Big\} \Big) \\
&\quad + \frac{|g'|\infty N}{\Gamma(\alpha)} \sum_{j=n+1}^{m} \Big\{ \frac{(t-t_j)^\alpha}{\alpha} - \frac{(t-t_{j+1})^\alpha}{\alpha} \Big\}.
\end{aligned}
$$

We should evaluate each term separately:

$$
\begin{aligned}
&\sum_{j=0}^{n} \Big\{ \frac{(t-t_j)^\alpha}{\alpha} - \frac{(t-t_{j+1})^\alpha}{\alpha} - \frac{(\bar{t}-t_j)^\alpha}{\alpha} + \frac{(\bar{t}-t_{j+1})^\alpha}{\alpha} \Big\} \\
&= \frac{(t-t_0)^\alpha}{\alpha} - \frac{(t-t_1)^\alpha}{\alpha} - \frac{(\bar{t}-t_0)^\alpha}{\alpha} + \frac{(\bar{t}-t_1)^\alpha}{\alpha} \\
&\quad + \frac{(t-t_1)^\alpha}{\alpha} - \frac{(t-t_2)^\alpha}{\alpha} - \frac{(\bar{t}-t_1)^\alpha}{\alpha} + \frac{(\bar{t}-t_2)^\alpha}{\alpha} \\
&\quad - \frac{(t-t_n)^\alpha}{\alpha} - \frac{(t-t_{n+1})^\alpha}{\alpha} - \frac{(\bar{t}-t_n)^\alpha}{\alpha} + \frac{(\bar{t}-t_{n+1})^\alpha}{\alpha} \\
&= \frac{(t-t_0)^\alpha}{\alpha} - \frac{(\bar{t}-t_0)^\alpha}{\alpha} - \frac{(t-t_{n+1})^\alpha}{\alpha} + \frac{(\bar{t}-t_{n+1})^\alpha}{\alpha}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\sum_{j=n+1}^{m} \Big\{ \frac{(t-t_j)^\alpha}{\alpha} - \frac{(t-t_{j+1})^\alpha}{\alpha} \Big\} &= \frac{(t-t_{n+1})^\alpha}{\alpha} - \frac{(t-t_{n+2})^\alpha}{\alpha} + \frac{(t-t_{n+2})^\alpha}{\alpha} - \frac{(t-t_{n+3})^\alpha}{\alpha} \\
&\quad + \cdots + \frac{(t-t_{m+1})^\alpha}{\alpha} - \frac{(t-t_m)^\alpha}{\alpha} = \frac{(t-t_{n+1})^\alpha}{\alpha} - \frac{(t-t_m)^\alpha}{\alpha}.
\end{aligned}
$$

But $t = t_n$, therefore,

$$
\sum_{j=n+1}^{m} \Big\{ \frac{(t-t_j)^\alpha}{\alpha} - \frac{(t-t_{j+1})^\alpha}{\alpha} \Big\} = \frac{(t-t_{n+1})^\alpha}{\alpha} = \frac{(t-\bar{t})^\alpha}{\alpha}.
$$

Putting everything together, we get

$$
|\overline{y}(t) - \overline{y}(\bar{t})| \leq \frac{\|g'\|_\infty N}{\Gamma(\alpha+1)} \Big\{ (t-t_0)^\alpha - (\bar{t}-t_0)^\alpha - (t-t_{n+1})^\alpha + (\bar{t}-t_{n+1})^\alpha + (t-\bar{t})^\alpha \Big\}
$$

$$\leq \frac{\|g'\|_\infty N}{\Gamma(\alpha+1)}\Big\{(t-t_0)^\alpha-(\bar{t}-t_0)^\alpha-(t-\bar{t})^\alpha+(\bar{t}-\bar{t})^\alpha+(t-\bar{t})^\alpha\Big\}$$
$$\leq \frac{\|g'\|_\infty N}{\Gamma(\alpha+1)}\{(t-t_0)^\alpha-(\bar{t}-t_0)^\alpha\}.$$

The function $(p-t_0)^\alpha$ is differentiable in $[\bar{t}-t_0, t-t_0]$ by the mean value theorem there exists $p \in [\bar{t}-t_1, t-t_0]$

$$\alpha(p-t_0)^{\alpha-1}(t-\bar{t}) = (t-t_0)^\alpha-(\bar{t}-\bar{t}_0)^\alpha.$$

Therefore, replacing yields

$$|\bar{y}(t)-\bar{y}(\bar{t})| \leq \frac{\|g'\|_\infty N}{\Gamma(\alpha+1)}\alpha(p-t_0)^{\alpha-1}(t-\bar{t}) \leq \Omega(t-\bar{t}), \quad t,\bar{t}\in K.$$

This obtained fact and the previous result imply that

$$|\bar{y}(t)-\bar{y}(\bar{t})| \leq \xi, \quad t\in[t_0,t_{n-1}].$$

Now,

$$\begin{aligned}
\Big|\bar{y}(t) &- \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau\Big| \\
&= \Big|\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} g'(t_j)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(t_j,y(t_j))d\tau-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1}f(\tau,(\tau))d\tau\Big| \\
&= \Big|\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} g'(t_j)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(t_j,y(t_j))d\tau-\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} g'(\tau)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,y(\tau))d\tau\Big| \\
&\leq \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\Big|g'(t)(t-\tau)^{\alpha-1}(f(\tau,y(\tau))-f(t_j,y(t_j)))\Big|d\tau \\
&\leq \frac{\|g'\|_\infty}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t-\tau)^{\alpha-1}\Big|(f(\tau,y(\tau))-f(t_j,y(t_j)))\Big|d\tau \\
&\leq \frac{\|g'\|_\infty}{\Gamma(\alpha)}\xi\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t-\tau)^{\alpha-1}d\tau.\frac{\Gamma(\alpha+1)}{T\|g'\|_\infty}\Big| \\
&\leq \frac{\|g'\|_\infty}{\Gamma(\alpha+1)}\xi\sum_{j=0}^{n}[(t-t_j)^\alpha-(t-t_{j+1})^\alpha]\frac{\Gamma(\alpha+1)}{T\|g'\|_\infty}\Big|.
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{j=0}^{n}[(t-t_j)^\alpha-(t-t_{j+1})^\alpha] &= (t-t_0)^\alpha-(t-t_1)^\alpha+(t-t_1)^\alpha-(t-t_2)^\alpha-\cdots(t-t_n)^\alpha-(t-t_{n+1})^\alpha \\
&= (t-t_0)^\alpha.
\end{aligned}$$

Since $(t - t_{n+1}) = 0$, because $t = t_{n+1}$,

$$\begin{aligned}\left|\overline{y}(t) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} g'(\tau)(t-\tau)^{\tau-1} f(\tau, y(\tau))\right| &< \frac{\|g'\|_\infty}{\Gamma(\alpha+1)}\xi(t-t_0)^\alpha \\ &< \frac{\|g'\|_\infty}{\Gamma(\alpha+1)}\xi T\frac{\Gamma(\alpha+1)}{T\|g'\|_\infty} \\ &= \xi.\end{aligned}$$

Therefore, $\overline{y}$ is an ξ-approximate solution, indeed when $\xi \to 0$. We have that

$$\left|\overline{y}(t) - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} f(\tau, y(\tau))d\tau\right| = 0,$$

that is to say

$$\overline{y}(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} f(\tau, y(\tau))d\tau$$

is the solution. □

We shall note that the required conditions can be imposed on the function $f(t, y(t))$ to derive the existence of a solution. In the next subsection, some conditions will be imposed to demonstrate the uniqueness of the existing solution.

3.2. Uniqueness of existing solution

In this section, we shall use existing theories to establish conditions under which the considered linear equation admits a unique solution.

Nagumo's conditions: By definition, a function satisfies the Nagumo conditions in the domain $\overline{D}$ if

$$|f(t, y) - f(t, \overline{y})| \leq k\frac{|y - \overline{y}|}{t - t_0}, \;\; t \neq t_0, \; k \leq 1,$$

for all $(t, y), (t, \overline{y}) \in \overline{D}$. The following important lemma should be stated.

Lemma 1. *Let $\overline{\delta}(t_0) = 0$ be a non-negative continuous function in $|t - t_0| \leq T$ and $\overline{\delta}(t_0) = 0$, and let $\overline{\delta}(t)$ be differentiable at $t = t_0$, with $\overline{\delta}'(t_0) = 0$. Then the inequality*

$$\overline{\delta}(t) \leq \left|\int_{t_0}^{t} \frac{\overline{\delta}(\tau)}{\tau - t_0} d\tau\right|$$

leads to $\overline{\delta}(t) = 0$, if $|t - t_0| < a$.

Theorem 8. *(Extension of Nagumo's uniqueness theorem) Assume that $f(t, y(y))$ is continuous and meets the requirements stated by Nagumo. Assume in addition to Nagumo's condition that $g'(t)$ is continuous and bounded in $[t_0, T]$, with*

$$M_{g'} = \max_{t\in[t_0,T]} |g'(t)|$$

and that $\frac{M_{g'}}{\lambda} < 1$, *where*

$$\lambda = \min_{t \in [t_0, T]} \Big\{ \min_{\tau t \in [t_0, t]} (t-\tau)^{\alpha-1} \Big\}.$$

Then the equations

$$\begin{aligned} {}_{t_0}^{RL}D^{\alpha}_{g(t)} y(t) &= f(t, y(t)), \ \ if\ t > t_0, \\ y(t_0) &= y_0, \ \ t = t_0, \end{aligned}$$

have a unique solution in $[t_0, T]$.

Proof. Let $(t, y), (t, \overline{y}) \in \overline{D}$, then,

$$|y(t) - \overline{y}(t)| = \frac{1}{\Gamma(\alpha)} \Big| \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} |f(\tau, y(\tau)) - f(\tau, \overline{y}(\tau))| d\tau \Big|.$$

Using the Nagumo criteria yields

$$|y(t) - \overline{y}(t)| \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} |g'|(t-\tau)^{\alpha-1} ||\tau - t_0|^{-1} |y(\tau) - \overline{y}(\tau)| d\tau.$$

We put

$$\overline{\delta}(t) = |y(t) - \overline{y}(t)|,$$

then

$$\begin{aligned} \overline{\delta}(t) &\leq \Big| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} |g'|(t-\tau)^{\alpha-1} (\tau - t_0)^{-1} \overline{\delta}(\tau) d\tau \Big| \\ &\leq \Big| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \max_{l \in [t_0, \tau]} |g'(l)| \frac{1}{(t-\tau)^{1-\alpha}} \frac{1}{(\tau - t_0)} \overline{\delta}(\tau) d\tau \Big| \\ &\leq \Big| \frac{M_{g'}}{\Gamma(\alpha)} \int_{t_0}^{t} \min_{l \in [t_0, \tau]} \{(t-l)^{1-\alpha}\} \frac{\overline{\delta}(\tau)}{\tau - t_0} d\tau \Big| \\ &\leq \frac{M_{g'}}{\Gamma(\alpha)\lambda} \Big| \int_{t_0}^{t} \frac{\overline{\delta}(\tau)}{\tau - t_0} d\tau \Big|. \end{aligned}$$

Using $\frac{M_{g'}}{\Gamma(\alpha)\lambda} \leq 1$, we get

$$\overline{\delta}(\tau) \leq \int_{t_0}^{t} \frac{\overline{\delta}(\tau)}{\tau - t_0} d\tau,$$

where $\overline{\delta}(\tau)$ is a non-negative continuous function in $|t - t_0| < T$ satisfying Nagumo's condition; therefore, $\overline{\delta}(\tau) = 0$ for all $t \in [t_0, T]$, that is to say $y(t) = \overline{y}(t)$ for all $t \in [t_0, T]$, which completes the proof. □

It is worth noting that this condition suggested by Nagumo is only a sufficient condition for the uniqueness of the solutions to initial value problems.

Theorem 9. *(Lipschitz uniqueness) Let $f(t, y(t))$ satisfy the Lipschitz condition; assume that $g'(t)$ is bounded. Then the initial value problem has a unique solution.*

Proof. Let $y(t), \overline{y}(t) \in \overline{D}$ and

$$\begin{aligned} |y(t) - \overline{y}(t)| &\leq \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1}(f(\tau, y(\tau)) - f(\tau, \overline{y}(\tau)))\Big|d\tau \\ &\leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} |g'(\tau)||(t-\tau)^{\alpha-1}|\Big|(f(\tau, y(\tau)) - f(\tau, \overline{y}(\tau)))\Big|d\tau \\ &\leq M_{g'}\int_{t_0}^{t} (t-\tau)^{\alpha-1} L|y(\tau) - \overline{y}(\tau)|d\tau. \end{aligned}$$

We let $\phi(t) = |y(t) - \overline{y}(t)|$, we have

$$\phi(t) \leq M_{g'}\int_{t_0}^{t} (t-\tau)^{\alpha-1} L\phi(\tau)d\tau.$$

By Gronwall's inequality, we get

$$\begin{aligned} \phi(t) &\leq q\exp\Big(\frac{-LM_{g'}}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}d\tau\Big) \\ &\leq q\exp\Big(-LM_{g'}\frac{(t-\tau)^{\alpha}}{\Gamma(\alpha+1)}\Big), \end{aligned}$$

whereas $q = 0$, then for all $t \in [t_0, T]$, $\phi(t) = 0$ implies that $y(t) = \overline{y}(t)$, which completes the proof. □

We shall use other conditions to establish the comprehensive existence and uniqueness.

4. The generalized Caratheodory principle

We shall use some existing theories based on the Caratheodory approach to show that

$$\begin{aligned} {}^{RL}_{t_0}D^{\alpha}_{g}(t)y(t) &= f(t, y(t)), \quad t > 0, \\ y(0) &= y_0, \end{aligned}$$

have a unique solution. In this work, we shall let [18]

$$y_n \to y$$

in $C([t_0, T], \mathbb{R})$; the following topology's uniform convergence on $C([t_0, T], \mathbb{R})$ is closed. We let

$$d_n = \sup_{t_0 \leq t \leq T} |y_n(t) - y(t)|.$$

We define

$$y^n = y - d_n sign(y),$$

which is of course positive for all $t \in [t_0, T]$.

Theorem 10. *Consider the closed interval* $[0,1]$. *The function*

$$[0,1]\times\mathbb{R}\rightarrow f(t,y)\in\mathbb{R}$$

is such that $t\rightarrow f(t,y(t))$ *holds for each* $y\in\mathbb{R}$ *and* $y\rightarrow f(t,y)$ *is continuous for all* $t\in[0,1]$. *If there exists* $M\in([0,1]\times\mathbb{R})$ *such that*

$$|f(t,y(t))|\leq M(x)(1+|y|),(t,y)\in[0,1]\times\mathbb{R},$$

then there exists an absolutely continuous function $\overline{y}(t)$ *such that*

$$\overline{y}(t)=\int_{t_0}^{t}f(s,y(s))ds,\quad\forall t\in[0,1].$$

The proof can be found here [18]. From the above theorem, we have following corollary:

Corollary 1. *Assume that, for all* $(t,y)\in[0,1]\times\mathbb{R}$, $f(t,y(t))$ *satisfies the following:*

$$|f(t,y(t))|<M(t)(1+|y(t)|)$$

for all $t\in[0,1]\times\mathbb{R}$. *Then, for a fixed* $0<\alpha\leq1$, *there exists an absolutely continuous function* $y(t)$ *such that*

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}g'(\tau)(t-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau.$$

Proof. Since $f(t,y(t))$ verified the Caratheodory principle in $[0,1]\times\mathbb{R}$ then for a fixed $\alpha\in[0,1]$

$$\begin{aligned}y(t)\leq|y(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_{0}^{t}g'(\tau)M(\tau)(t-\tau)^{\alpha-1}d\tau+\frac{1}{\Gamma(\alpha)}\int_{0}^{t}g'(\tau)M(\tau)|y(\tau)|(t-\tau)^{\alpha-1}d\tau\\ &\leq \int_{0}^{t}\Omega(\tau)(t-\tau)^{\alpha-1}d\tau+\int_{0}^{t}\Omega(\tau)|y(\tau)|(t-\tau)^{\alpha-1}d\tau\\ &\leq V(t)+\int_{0}^{t}\Omega(\tau)(t-\tau)^{\alpha-1}|y(\tau)|d\tau.\end{aligned}$$

If we put $\phi(t)=|y(\tau)|$, then

$$y(t)\leq\phi(t)\leq V(t)+\int_{0}^{t}\Omega(\tau)(t-\tau)^{\alpha-1}\phi(\tau)d\tau.$$

We see that $\phi(t)$ satisfies the condition of Gronwall, which therefore leads us to

$$\phi(t)\leq V(t)\exp\Big[\int_{0}^{t}\Omega(\tau)(t-\tau)^{\alpha-1}d\tau\Big].$$

On the other hand, however,

$$|f(t,y(t))|\leq M(t)(1+|y(t)|)$$

implies that if $|y(t)|$ is continuous and $\phi(t)$ is also absolutely continuous, as with $1+\phi(t)$, then put

$$\overline{M}(t) = M(t)(1+|y(t)|).$$

$\overline{M}(t)$ is continuous, absolutely; therefore, if D is an open set of $\mathbb{R}$, since $f(t, y(t))$ satisfies the Caratheodory principle D, then for all $(t_0, y_0) \in D$, we assume $\overline{\alpha}$ and $\overline{\beta}$ positive members such that the rectangle

$$\overline{B}(\overline{\alpha},\overline{\beta}) = \{(t,y) : |t-t_0| \le \overline{\alpha}, |y-y_0| \le \overline{\beta}\} \subset D.$$

We now let

$$\overline{I}_a = \{t : |t-t_0| \le \overline{\alpha}\},$$

we choose

$$\overline{m} = \overline{M}(t) \in \overline{B}(\overline{\alpha},\overline{\beta})$$

and set

$$\overline{M}_1(t) = \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g'(\tau)\overline{m}(\tau)d\tau.$$

We chose $\overline{\alpha}_1$ and $\overline{\beta}_1$ such that we have

$$0 < \overline{\alpha}_1 \le \alpha_1 \text{ and } 0 < \overline{\beta}_1 \le \beta_1,$$

indeed

$$|\overline{M}_1(t)| \le \overline{\beta}, \quad t \in \overline{I}_{\overline{\alpha}_1}.$$

We let Λ be the set of functions $\overline{y} \in C[\overline{I}_{\overline{\alpha}_1}, \mathbb{R}]$ satisfying

$$\overline{y}(t_0) = y_0, \quad |\overline{y}(t)| \le \overline{\beta}$$

for all $t \in I_{\overline{\alpha}_1}$. Clearly Λ is a bounded, closed and convex subset of $C[\overline{I}_{\overline{\alpha}_1}, \mathbb{R}]$. Thus for all $\overline{y} \in \Lambda$, we can define the following mapping $\Gamma\overline{y}$:

$$\Gamma\overline{y}(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau) f(\tau, \overline{y}(\tau))(t-\tau)^{\alpha-1} d\tau, \quad t \in \overline{I}_{\overline{\alpha}_1}.$$

Note that the fixed points of the above mapping, if they exist, are considered as a solution of our equation in Λ. The Schwarz theorem for fractional case can be used to show the existence of the fixed points of Γ, n, Λ. We shall now show that Γ is well defined providing that $f(t, \overline{y}(t))$ is integrable for any $\overline{y}(t) \in \Lambda$. $\Gamma y(t)$ is continuous for all $t \in \overline{I}_{\overline{\alpha}}$ since

$$\begin{aligned} |\overline{y}(t_1) - \overline{y}(\bar{t}_1)| &= \frac{1}{\Gamma(\alpha)} \Big| \int_{t_0}^{t_1} (t_1-\tau)^{\alpha-1} f(\tau, \overline{y}(\tau))\overline{q}(\tau)d\tau \\ &\quad - \int_{t_0}^{\bar{t}_1} (\bar{t}_1-\tau)^{\alpha-1} f(\tau, \overline{y}(\tau))\overline{q}(\tau)d\tau \Big|. \end{aligned}$$

Assuming that $t_1 > \bar{t}_1$, we have

$$\begin{aligned}
|\overline{y}(t_1) - \overline{y}(\bar{t}_1)| &= \frac{1}{\Gamma(\alpha)}\Big| \int_{t_0}^{\bar{t}_2} (t_1-\tau)^{\alpha-1} f(\tau, \overline{y}(\tau)) g'(\tau) d\tau + \int_{\bar{t}_1}^{t_1} (t_1-\tau)^{\alpha-1} f(\tau, y(\tau)) g'(\tau) d\tau \\
&\quad - \int_{t_0}^{\bar{t}_1} (\bar{t}_1-\tau)^{\alpha-1} f(\tau, y(\tau)) g'(\tau) d\tau \Big| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{\bar{t}_1} g'(\tau) |f(\tau, y(\tau))| \{(t_1-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1}\} g'(\tau) d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\bar{t}_1}^{t_1} (t_1-\tau)^{\alpha-1} g'(\tau) f(\tau, y(\tau)) d\tau \\
&\leq \frac{M}{\Gamma(\alpha)} \Big[\int_{t_0}^{\bar{t}_1} ((t_1-\tau)^{\alpha-1} - (\bar{t}_1-\tau)^{\alpha-1}) d\tau + \int_{\bar{t}_1}^{t} (t_1-\tau)^{\alpha-1} d\tau \Big] \\
&\leq \frac{M}{\Gamma(\alpha+1)} \Big\{ [(t_1-t_0)^{\alpha} - (t_1-\bar{t}_1)^{\alpha} - (\bar{t}_1-t_0)^{\alpha} + (\bar{t}_1-\bar{t}_1)^{\alpha} + (t_1-\bar{t}_1)^{\alpha}] \Big\} \\
&= \frac{M}{\Gamma(\alpha+1)} \{(t_1-t_0)^{\alpha} - (\bar{t}_1-t_0)^{\alpha}\}.
\end{aligned}$$

By using the mean value theorem, we have

$$|y(t_1) - y(\bar{t}_1)| \leq \frac{M}{\Gamma(\alpha+1)} (l-t_0)^{\alpha} (t_1-\bar{t}_1).$$

Therefore, for all $\xi > 0$, there exists $\delta > 0$ such that

$$|y(t_1) - y(\bar{t}_1)| < \xi$$

implies that $|t_1 - \bar{t}_1| < \delta$; in this case, we have

$$\delta < \frac{\Gamma(\alpha+1)\xi}{M(l-t_0)^{\alpha}},$$

which completes the proof; therefore, Γ is continuous

$$\begin{aligned}
|\Gamma\overline{y}(t)| &\leq \Big| \int_{t_0}^{t} |g'(\tau) f(\tau, \overline{y}(\tau))| (t-\tau)^{\alpha-1} d\tau \Big| \\
&\leq \Big| \int_{t_0}^{t} |g'(\tau)| \overline{M}(\tau) (t-\tau)^{\alpha-1} d\tau \Big| \\
&\leq |\overline{M}_1(t)| \leq \overline{\beta}.
\end{aligned}$$

In fact, if $\overline{y}_n \in \Omega$ and $\overline{y}_n \to \overline{y}$ in Λ, then using the continuity of $f(t, y)$ in y for all fixed t, leads to

$$f(t, \overline{y}_n(t)) \to f(t, \overline{y}(t))$$

as $n \to \infty$ for all $t \in \overline{I}_{\overline{\alpha}_1}$; then, also

$$g'(\tau)(t-\tau)^{\alpha-1} f(\tau, \overline{y}_n(\tau)) \to g'(\tau)(t-\tau)^{\alpha-1} + (t, \overline{y}(t))$$

for all $t \in \overline{I}_{\overline{\alpha}_1}$. Additionally, since

$$|f(t, \overline{y}_n(t))| \leq M(t),$$

the Lebesgue dominated convergence theorem leads to

$$\lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} f(\tau, \overline{y}_n(\tau))d\tau \to \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} f(\tau, y(\tau))d\tau.$$

We have shown Γ to be uniformly continuous, and that $\Lambda\Omega$ is an equilibrium set of $C[I_{\overline{\alpha}_1}, \mathbb{R}]$. Of course Λ is also uniformly bounded which leads to the fact that $\Lambda\Omega$ is relatively compact; therefore, $\Lambda\Omega$ is completely continuous. The Schauder fixed-point theory helps complete the proof. □

We have shown that if the function meets the requirements of the theorem then for all $t \in [t_0, T]$,

$$\lim_{n\to\infty} f(t, y_n(t)) \to f(t, y(t))$$

pointwise, we have to show that

$$f(t_, y_n(t^n)) \to f(t, y(t))$$

in measure in $[t_0, T]$. The continuity of $f'(., y(.))$ ensures that

$$f(t^n, y_n(t^n)) \to f(t, y(t)).$$

However, an additional requirement is that the sequence $(y_n(t))_n$ satisfies the Lipschitz condition, of course with

$$K = \max_{n=0} |k_n|,$$

where k_n is the Lipschitz constant of $(y_n t(t))$. Therefore, with this condition, the continuity of $g'(t)$ and $\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ for every fixed $\alpha \in [0, T]$ and $t_0 \leq s \leq t^n$, $t_0 \leq s < t$:

$$g'(t^n)\frac{(t^n-s)^{\alpha-1}}{\Gamma(\alpha)} f(t^{\cdot} y_n(t^n)) \to g'(t)\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t)).$$

However, f should meet an extra condition for all

$$(y_n(t))_n \in V([t_0, T], \mathbb{R}) \subset C([t_0, \mathbb{R}], \mathbb{R}),$$

where V is a convex set but also compact. In the classical case the result was achieved with some extra definitions which we adapt here in the case of our problem.

Definition 1. *We assume that $a > 0$, $I_a = [0, a]$ and $P', \cdots, P^m,\ q', \cdots, q^m \in L_g^{\alpha,1}[I_a, \mathbb{R}]$, where*

$$L_g^{\alpha,1}[I_a, \mathbb{R}] = \{h : \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)d\tau\}$$

exists.

$$G_g^{\alpha,1}(q, p) = \{h : h \in L_g^{\alpha,1}[I_a, \mathbb{R}],\ p^k(t) \leq h^k(t) \leq q^{(k)}(t),\ \forall t \in I_a, 1 \leq k \leq m\}.$$

For simplicity $G_g^{\alpha,1}$ notation is adopted instead of $G_g^{\alpha,1}(q,p)$. One can easily see that for all $h, \overline{h} \in G_g^{\alpha,1}$, we have that

$$(1-t)h + t\overline{h} \in G_g^{\alpha,1},$$

when $t \in [0,1]$.

Therefore, for any fixed α and g $G_g^{\alpha,1}$ is a convex set.

Definition 2. *Let*

$$G_g^{\alpha,1} = G_g^{\alpha,1}[p,q]$$

and

$$f : I_a \times \mathbb{R} \to \mathbb{R}$$

be such that

$$f\Big(t, \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)(t-\tau)^{\alpha-1} h(\tau) d\tau\Big) \in G_g^{\alpha,1}$$

for all $h \in G_g^{\alpha}$ for every fixed $0 \le \alpha \le 1$; and, $g'(t)$ is continuous. Then f is said to be G_g^{α} integrable on I_a.

Definition 3. *Let $f \in G_g^{\alpha}$ on I_a and let $h \in G_g^{\alpha}$. Let the sequence*

$$y_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t g'(\tau)(t-\tau)^{\alpha-1} h_n(\tau) d\tau,$$

$h_n \in G_g^{\alpha}$ such that $y_n(t) \to y(t)$ uniformly on I_a. Let $t^n = d - d_n$, where d_n is defined as before. For any sequence

$$g'(t^n)\frac{(t^n - s)^{\alpha-1}}{\Gamma(\alpha)} f(t^n, y_n(t^n)) \to g'(t)\frac{(t-s)^{\alpha} - 1}{\Gamma(\alpha)} f(t, y(t)),$$

then f is $G_g^{\alpha,1}$ regular on I. Let

$$G_g^{\alpha,1} = G_g^{\alpha,1}[p,q]$$

and $M \in L_g^{\alpha,1}[\mathbb{R}]$ be such that $|p(t)| \le M(t)$ and $|q(t)| \le M(t)$ for all $t \in I_a$; then, $G_g^{\alpha,1}$ is said to be dominated by M.

The Caratheodory principle can now be extended.

Theorem 11. *Let $a > 0$ and $I_a = [0,a]$. Let f be $G_g^{\alpha,1}$-regular on I. Then we can find at least one absolutely continuous function y satisfying the first theorem under this section.*

Proof. We let

$$\overline{\Omega} = \{z : \xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g'(\tau) z(\tau) dz, \;\; z \in G_g^{\alpha,1}\},$$

we let

$$\Gamma z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g'(\tau) f(t, z(\tau)) d\tau, \;\; z \in \overline{\Omega}.$$

It is also clear from [18] that $\Gamma y \in \overline{\Omega}$, because $f \in G_g^{\alpha,1}$ and Ω is compact. Let $z_n \to z$ for all $z_n \in \overline{\Omega}$; we have to show that $\Gamma z_n \to \Gamma z$. Let $t^n = t - d_n$,

$$\begin{aligned}
d_n &= \sup_{t\in I_a} |z_n(t) - z(t)|,\\
|\Gamma_{z_n} - \Gamma_z| &= |\frac{1}{\Gamma(\alpha)} \int_0^t g'(\tau^n)(t-\tau^n)^{\alpha-1} f^j(\tau^n, z_n(\tau^n))d\tau\\
&\quad -\frac{1}{\Gamma(\alpha)} \int_0^t g'(\tau)(t-\tau)^{\alpha-1} f^j(\tau, y(\tau))d\tau|,\ 1 \le j \le n,\\
\lim_{n\to\infty} |\Gamma_{z_n} - \Gamma_z| &= \lim_{n\to\infty} \Big|\frac{1}{\Gamma(\alpha)} \int_0^t g'(\tau^n)(t-\tau^n)^{\alpha-1} f^j(\tau^n, z_n(\tau^n))d\tau\\
&\quad -\frac{1}{\Gamma(\alpha)} \int_0^t g'(\tau)(t-\tau)^{\alpha-1} f^j(\tau, y(\tau))d\tau\Big|.
\end{aligned}$$

The continuity of $g'(t)$ and $(t-\tau)^{\alpha-1}$ for all $t_0 \le \tau \le t$ allows us to have

$$\lim_{n\to\infty} |\Gamma_{z_n} - \Gamma_z| = \lim_{n\to\infty} \Big|\frac{1}{\Gamma(\alpha)} \int_0^t g'(\tau)(t-\tau)^{\alpha-1} f^j(\tau^n, z^n(\tau^n))d\tau - f^s(\tau, y(\tau))d\tau\Big|.$$

From [18] and by Fatou's lemma,

$$\int_0^t \lim_{n\to\infty}(M(t^n) - f^j(\tau^n, y_n(t^n)))d\tau \le \lim_{n\to\infty} \int_0^t (M(\tau^n) - f^j(\tau^n, y_n(\tau^n)))d\tau,\ \ 1 \le j \le n, t > 0,$$

by letting

$$z_n^1(t) = f^j(t^n, z_n(t^n)),$$

it follows by [18] that

$$\lim_{n\to\infty} \int_0^t z_n^1(\tau)d\tau = \int_0^t z^1(\tau)d\tau.$$

Therefore, we also have

$$\lim_{n\to\infty} \int_0^t g'(\tau)(t-\tau)^{\alpha-1} z_n^1(\tau)d\tau = \int_0^t g'(\tau)(t-\tau)^{\alpha-1} z^1(\tau)d\tau.$$

Then Ω is equicontinuous because M is dominating $G_g^{\alpha,1}$. This implies that $\Gamma_{z_n} \to \Gamma_z$ uniformly. T is continuous. Thus, with Schauder fixed-point theorem the proof is completed. □

5. Numerical scheme

In this section we shall derive a numerical solution to the nonlinear equation. The used method is the extension of Heun's approach. To start, we shall give some important hypotheses:

$$\begin{cases} {}_{t_0}^{RL}D_t^{\alpha} y(t) = f(t, y(t)), & if\ t > t_0,\\ y(t_0) = y_0, & if\ t = t_0. \end{cases} \tag{5.1}$$

(i) $g'(t)$ is continuous and positive.

(ii) $f(t,(t))$ is twice continuously differentiable.

(iii) $g'(t)$ is bounded.

(iv) $f(t, y(t))$ is bounded for all $t \in [t_0, T]$.

(v) $f(t,(t))$ is Lipschitz with respect to y.

Equation (5.1) is converted to the integral equation as follows:

$$\begin{cases} {}^{RL}_{t_0}D^{\alpha}_{g(t)}y(t) = g'(t)f(t, y(t)), & if\ t > t_0, \\ y(t_0 = y_0, & if\ t = t_0. \end{cases} \tag{5.2}$$

$$\begin{cases} y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} g'(\tau)f(\tau, y(\tau))(t-\tau)^{\alpha-1}d\tau, \\ y(t_0 = y_0. \end{cases} \tag{5.3}$$

We put

$$t_{n+1} = (n+1)h,$$

where $h = t_{n+1} - t_n$ in (5.3), and we get

$$\begin{aligned} y(t_{n+1}) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_{n+1}} g'(\tau)f(\tau, y(\tau))(t_{n+1}-\tau)^{\alpha-1}d\tau \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} g'(\tau)f(\tau, y(\tau))(t_{n+1}-\tau)^{\alpha-1}d\tau. \end{aligned} \tag{5.4}$$

Between t_j, t_{j+1} in (5.4), we approximate

$$g'(\tau)f(\tau, y(\tau)) \approx g'(t_{j+1})f\Big(t_j + \frac{h}{2}, \frac{y_j + y_{j+1}}{2}\Big) \tag{5.5}$$

substituting (5.5) into (5.4), we get

$$y_{n+1} = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} g'(t_{j+1})f\Big(t_j + \frac{h}{2}, \frac{y_j + y_{j+1}}{2}\Big)(t_{n+1}-\tau)^{\alpha-1}d\tau. \tag{5.6}$$

Further simplification of (5.6) leads to the below equation:

$$y_{n+1} = \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} g'(t_{j+1})f\Big(t_j + \frac{h}{2}, \frac{y_j + y_{j+1}}{2}\Big) \times [(n-j+1)^{\alpha} - (n-j)^{\alpha}]. \tag{5.7}$$

The method used to get (5.7) is implicit since we have y_{n+1} on the both sides when $j = n$. Therefore,

$$\begin{aligned} y_{n+1} &= \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} g'(t_{j+1})f\Big(t_j + \frac{h}{2}, \frac{y_j + y_{j+1}}{2}\Big) \times [(n-j+1)^{\alpha} - (n-j)^{\alpha}] \\ &\quad + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f\Big(\frac{t_{n+1} + t_n}{2}, \frac{\overline{y}_{n+1} + y_n}{2}\Big) g'(t_{n+1}), \end{aligned} \tag{5.8}$$

where

$$\overline{y}_{n+1} = \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} g'(t_{j+1}) f(t_j, y_j)\{(n-j+1)^{\alpha} - (n-j)^{\alpha}\}. \tag{5.9}$$

Note that

$$g'(t_{n+1}) \quad = \quad \frac{g(t_{n+1}) - g(t_n)}{h}, \tag{5.10}$$

replacing (5.10) in the main Eqs (5.8) and (5.9), we get

$$\begin{aligned} y_{n+1} \quad &= \quad \frac{h^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{j=0}^{n-1} \Big[g\Big(\frac{t_{j+1}+t_j}{2}\Big) - g(t_j)\Big] f\Big(t_j + \frac{h}{2}, \frac{y_j + y_{j+1}}{2}\Big) \times [(n-j+1)^{\alpha} - (n-j)^{\alpha}] \\ &+ \frac{h^{\alpha-1}}{\Gamma(\alpha+1)} [g(t_{n+1}) - g(t_n)] \times [(n-j+1)^{\alpha} - (n-j)^{\alpha}] f\Big(t_n + \frac{h}{2}, \frac{\overline{y}_{n+1} + y_n}{2}\Big), \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} \overline{y}_{n+1} \quad &= \quad \frac{h^{\alpha-1}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \Big[g(t_{j+1}) - g(t_j)\Big] f(t_j, y_j)[(n-j+1)^{\alpha} - (n-j)^{\alpha}], \\ y_1 \quad &= \quad \frac{h^{\alpha-1}}{\Gamma(\alpha+1)} f(t_0, y_0)[g(t_1) - g(t_0)], \; y_0 = y(t_0). \end{aligned} \tag{5.12}$$

6. Illustrative examples

In this section, we shall solve equations and compare their exact solutions with the numerical solutions.

Example 1.

$$ {}_{0}^{RL}D^{\alpha}_{g(t)} y(t) = t^2, \quad y(0) = 0, t \in (0, 1), \tag{6.1}$$

where $g(t) = 2t$. *The exact solution is*

$$y(t) = 2t^{\alpha+2} \frac{\Gamma(3)}{\Gamma(\alpha+3)}.$$

Results for Example 1: Here, we consider the numerical solution of Example 1 by using the scheme given in (5.11). The numerical solution of Example 1 is shown graphically in Figure 1 by comparing it with the exact solution. It can be seen from Figure 1 that the numerical scheme provides accurate results for the exact solution. Further, we consider various values of the fractional order α and provide the error and estimated order of convergence (EOC) for different subintervals. As a result of increasing the number of subintervals N and fractional order α the error is minimizing; see Table 1. This shows that the present method is useful and reliable and can be considered for other scientific and engineering problems.

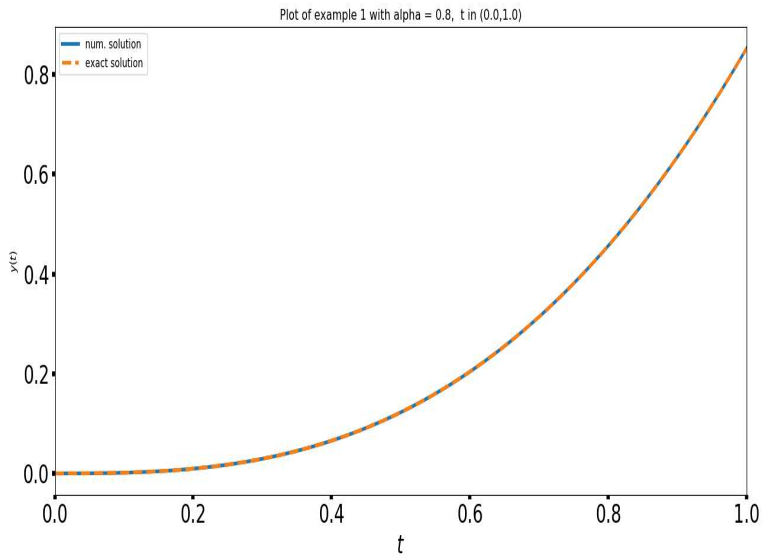

Figure 1. Plot of exact and numerical solutions for Example 1.

Table 1. Results for Example 1 obtained using $t \in (0, 1)$; N = number of subintervals.

N	Error	EOC	N	Error	EOC
Results for $\alpha = 0.6$			*Results for* $\alpha = 0.8$		
80	0.00068825	-	80	0.00017232	-
160	0.0002273	1.598	160	5.001e-05	1.785
320	7.522e-05	1.595	320	1.455e-05	1.782
640	2.49e-05	1.595	640	4.23e-06	1.781
1280	8.24e-06	1.595	1280	1.23e-06	1.783
Results for $\alpha = 0.9$			*Results for* $\alpha = 0.99$		
80	7.658e-05	-	80	3.033e-05	-
160	2.078e-05	1.882	160	7.59e-06	1.999
320	5.66e-06	1.877	320	1.91e-06	1.99
640	1.54e-06	1.875	640	4.8e-07	1.986
1280	4.2e-07	1.876	1280	1.2e-07	1.984

Example 2.

$$ {}_0^{RL}D_{g(t)}^{\alpha} y(t) = t^2, \quad y(0) = 0, t \in (0, 1), \tag{6.2} $$

where $g(t) = t^{\beta}$. *The exact solution is*

$$ y(t) = \beta\Gamma(\beta + 2)t^{\alpha+\beta+1} \frac{1}{\Gamma(\alpha + \beta + 2)}. $$

Results for Example 2: We solve Example 2 by using the numerical approach shown in (5.11), and we have obtained the results both graphically as well as in tabular form. The numerical solution of

Example 2 is given in Figure 2 by comparing it with the exact solution. We observe that the numerical solution obtained through the present scheme is matched well to an exact solution as given in Figure 2. We also consider various values of the fractional order α and provide the error and EOC for different subintervals; see Table 2. It can be observed from Table 2 that by increasing the number of subintervals N and the fractional order α, the error is minimized; see Table 2.

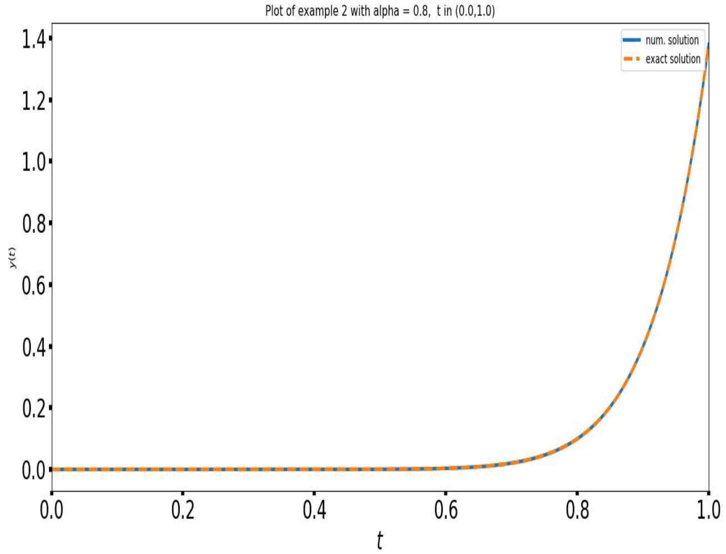

Figure 2. Plot of exact and numerical solutions for Example 2.

Table 2. Results for Example 2 obtained using $t \in (0, 1)$; N = number of subintervals.

N	Error	EOC	N	Error	EOC
Results for $\alpha = 0.6$			Results for $\alpha = 0.8$		
80	0.01643295	-	80	0.0035491	-
160	0.00565334	1.539	160	0.00108545	1.709
320	0.00192324	1.556	320	0.00032859	1.724
640	0.00064911	1.567	640	9.864e-05	1.736
1280	0.00021782	1.575	1280	2.941e-05	1.746
Results for $\alpha = 0.9$			Results for $\alpha = 0.99$		
80	0.00132247	-	80	0.0003269	-
160	0.00037914	1.802	160	8.352e-05	1.969
320	0.00010799	1.812	320	2.144e-05	1.962
640	3.057e-05	1.821	640	5.51e-06	1.959
1280	8.6e-06	1.829	1280	1.42e-06	1.958

Example 3.

$$ {}_0^{RL}D_{g(t)}^{\alpha} = \frac{(4-\alpha)\Gamma(4)}{\Gamma(5-\alpha)} t^{3-\alpha}, \quad y(0) = 0, t \in (0, 8), \tag{6.3} $$

with $g(t) = t$. *The exact solution is* $y(t) = t^3$.

Results for Example 3: We solve Example 3 by using the numerical approach in (5.11) and we have obtained the results both graphically as well as in tabular form. The numerical solution of Example 3 is given in Figure 3 by comparing it with the exact solution. We observe that the scheme matches well for this problem and the exact solution in Figure 3. Further, we provide the results for Example 3 in Table 3 for various values of the fractional order α, and we provide the error and EOC for different subintervals N; see Table 3. Table 3 shows that the error is minimized by increasing the number of subintervals N and the fractional order α.

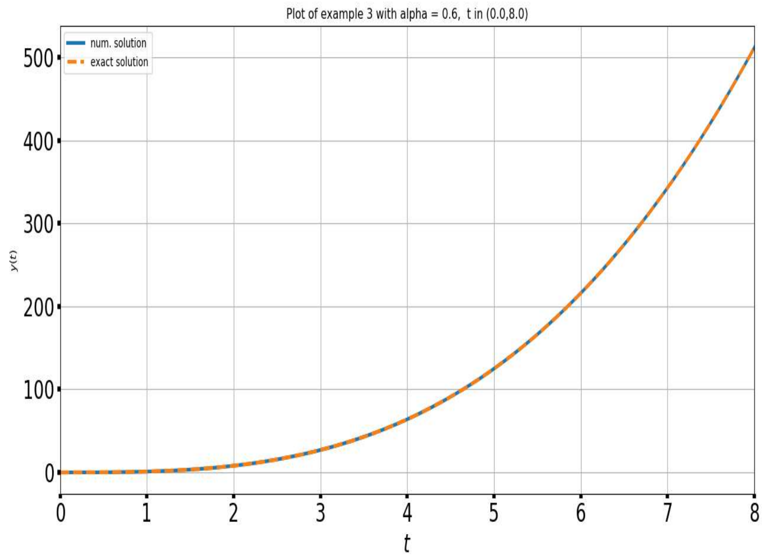

Figure 3. Plot of exact and numerical solutions for Example 3.

Table 3. Results for Example 3 obtained using $t \in (0, 1)$; N = number of subintervals.

N	Error	EOC	N	Error	EOC
Results for $\alpha = 0.6$			*Results for* $\alpha = 0.8$		
80	0.00082668	-	80	0.00023354	-
160	0.00027342	1.596	160	6.783e-05	1.784
320	9.058e-05	1.594	320	1.974e-05	1.781
640	3.001e-05	1.594	640	5.75e-06	1.781
1280	9.94e-06	1.595	1280	1.67e-06	1.782
Results for $\alpha = 0.9$			Results for $\alpha = 0.99$		
80	0.00010951	-	80	4.53e-05	-
160	2.972e-05	1.882	160	1.133e-05	1.999
320	8.09e-06	1.876	320	2.85e-06	1.99
640	2.21e-06	1.875	640	7.2e-07	1.986
1280	6e-07	1.876	1280	1.8e-07	1.984

7. Conclusions

Beyond fractional ordinary nonlinear differential equations with the power law kernel, exponential decay kernel and the generalized Mittag-Leffler kernel, there exists a class of differential equations in which the mentioned fractional differential operators are their generalization. They are fractional differential operators of a given function with respect to another function. This gives them the flexibility to capture complex processes that cannot be captured by using classical fractional differential operators. In this work, we have established very useful inequalities similar to the Gronwall inequality that will be employed for theoretical and applied purposes. Using Nagumo's conditions for existence, we have derived conditions under which the equations admit unique solutions. We have also suggested a methodology that could be used to solve these equations numerically. We considered the detail steeping to solve the fractional system numerically and provided examples and their exact solution, as well as presented their comparison, graphically and in tabulated form. The results indicate that the provided scheme is accurate and one can use this for the solution of other nonlinear systems arising in scientific and engineering areas. We shall extend this work to other fractional operators based on exponential decay and the Mittag-Leffler kernel to obtain the theoretical and numerical results.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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