AIMS Mathematics, 8(10): 24663–24680.
DOI: 10.3934/math.20231257
Received: 10 May 2023
Revised: 14 July 2023
Accepted: 07 August 2023
Published: 21 August 2023

Mathematics

http://www.aimspress.com/journal/Math

Research article

The moment exponential stability of infinite-dimensional linear stochastic switched systems

Guojie Zheng[1,*] **and Taige Wang**[2]

[1] College of Digital Technology and Engineering, Ningbo University of Finance & Economics, Ningbo 315175, China

[2] Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, USA

* **Correspondence:** Email: guojiezheng@yeah.net.

Abstract: This paper studies the 2nd-moment exponential stability of a class of infinite-dimensional linear stochastic switched systems comprising two unstable subsystems. We first construct an algebraic sufficient condition on the existence of multiple Lyapunov functions. Then, two switching strategies are designed to stabilize infinite-dimensional linear stochastic switched systems in terms of the multiple Lyapunov function method. Moreover, the system possesses good robust stability of the switching time with our switching strategies.

Keywords: infinite-dimensional stochastic switched systems; stability analysis; multiple Lyapunov function; Itô formula
Mathematics Subject Classification: 60H30, 93E15

1. Introduction

From the application point of view, many practical systems are often subjected to stochastic disturbances. A natural and widely acceptable way of describing stochastic factors uses white noise, and such a class of induced systems is called stochastic systems (see [1, 2]).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete filtered probability space, on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined. We denote by ω a sample point of Ω, and by $\mathbb{E}(\cdot)$ the expectation with respect to the probability measure $\mathcal{P}$. Let $\mathbb{H}$ be a separable (infinite-dimensional) Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Further, for each $T > 0$, we denote by $L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{H}))$ the Banach space consisting of all $\mathbb{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$ adapted continuous processes $X(\cdot)$ such that $\mathbb{E}\|X(\cdot)\|^2_{C([0,T];\mathbb{H})} < +\infty$, endowed with canonical norms.

Let $A_i : D(A_i) \subset \mathbb{H} \mapsto \mathbb{H}$ $(i = 1, 2)$ be a linear closed densely defined operator in $\mathbb{H}$, which generates a C_0 semi-group $\{S_i(t)\}_{t\geq 0}$. Suppose that the domain of the operator A_i $(i = 1, 2)$ satisfies

$D(A_1) = D(A_2)$. Let $f_i(\cdot)$ be a real-valued $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process in $L^\infty_{\mathcal{F}}(0, +\infty; L^\infty(\Omega))$ $(i = 1, 2)$. Here and in what follows, we omit the variable $\omega(\in \Omega)$ in the defined functions if there is no risk of causing any confusion.

In this paper, we mainly study the following infinite-dimensional linear stochastic switched system driven by multiplicative noise:

$$\begin{cases} dY(t) = A_{\sigma(t)}Y(t)dt + f_{\sigma(t)}(t)Y(t)dW(t), & \text{for } t \in (0, +\infty), \\ Y(0) = Y_0(\omega), \end{cases} \tag{1.1}$$

where the initial data Y_0 belongs to $L^2(\Omega, \mathcal{F}_0, \mathcal{P}; \mathbb{H})$, and the switching signal $\sigma(\cdot)$ is a right-continuous step function taking values in $\{1, 2\}$.

Remark 1.1. *Several notes on system (1.1) are given in order.*

(i) Let $\{t_k\}_{k=1}^\infty$ ($t_k < t_{k+1}$ for each $k \in \mathbb{N}^+$) be the set of switching times of (1.1). The switching signal $\sigma(\cdot)$ is said to be well-defined if it involves a finite number of switches in any finite time interval; the switching signal is said to be with dwell time τ if $t_{k+1} - t_k \geq \tau > 0$ for any two consecutive switching times t_{k+1} and t_k. Clearly, a switching signal $\sigma(\cdot)$ is well-defined if it has a dwell time τ.

(ii) We suppose that the state of (1.1) should be continuous at each switching time, i.e., for each $k \in \mathbb{N}^+$,

$$Y(t_k) = \lim_{t\to t_k-} Y(t), \ \ a.s.$$

(iii) In general, we call (1.1) a linear stochastic switched system in infinite dimensions. When $\sigma(t) \equiv i$ $(i = 1, 2)$ in (1.1), this equation is viewed as the ith subsystem of (1.1). Such an abstract system can formulate lots of important stochastic PDEs, including the stochastic heat equation and the stochastic Schrödinger equation.

(iv) Additive noise and multiplicative noise are two different types of mathematical noise models. Among them, multiplicative noise is common in the financial and economic fields. From a mathematical point of view, multiplicative noise is more difficult to deal with.

Since the early 1990s, the study of stochastic switched systems has become prominent in control theory. There are many studies on this topic, and we mention [3–8] for the related works. In particular, Basak et al. discussed the stability of a semi-linear stochastic differential equation with Markovian switching in [3]. In [8], Zhu et al. investigated the asymptotic stability of nonlinear Markov switched systems. In addition, we refer to [9] for the infinite-dimensional deterministic case, which studies the problem of stabilizability of nonlinear switched systems with a countably infinite number of subsystems in some Banach space. However, the stability of switched stochastic systems in infinite dimensions has so far little been studied. We mention [10, 11] in this field.

In statistical mechanics, the evolution of the probability density function of a nonlinear SODE satisfies the Fokker-Planck equation, which represents an infinite-dimensional system (see [12]). Generally speaking, positions of SODEs and PDEs are equivalent in mathematics. Based on this point, we attempt to study infinite-dimensional stochastic switching systems, which should provide a new perspective to understand finite-dimensional stochastic switching systems. One of the most critical problems on stochastic switched systems is how to design a switching strategy to stabilize the switched systems consisting of unstable subsystems. This paper aims to investigate the 2nd moment stability of infinite-dimensional linear stochastic switched systems (1.1) with two unstable subsystems. It should be pointed out that there exist some articles about the moment stability of stochastic infinite-dimensional equations without Markov switching (see [13, 14]).

From the point of view of spectral theory, (infinite-dimensional) linear systems usually contain stable and unstable subspaces. Applying the advantage of the stable subspace to overcome the drawbacks of the unstable subspace, some effective switching strategies were proposed by [15] to stabilize a specific class of deterministic infinite-dimensional linear switched systems.

This article represents a further extension of the work done by [15], focusing on infinite-dimensional linear stochastic switched systems. In the previous paper [15], the semi-group formulation was invoked for the switched deterministic case, which is an elaborate generalization of finite-dimensional systems constituting the classic framework of the control theory. However, this technique fails to deal with the problem of moment exponential stability for the random model (1.1). This paper tries to solve this problem using the multiple Lyapunov functions method.

It is well known that the Lyapunov functions approach plays an important role in the studies of control theory (see [16]). Multiple Lyapunov functions are a very effective analysis tool in studies of the stability of (stochastic) switched systems (see [17–19]). We should point out that even though there exist Lyapunov functions for each subsystem individually, we need to impose restrictions on switching to guarantee stability. In this article, we start with the definition of the moment stability of stochastic switched systems. Then, we construct an algebraic sufficient condition on the existence of multiple Lyapunov functions in quadratic forms. Inspired by the ideas in [15], two switching strategies are designed to stabilize the infinite-dimensional linear stochastic switched system (1.1).

The contributions of this paper can be summarized as follows:

(i) The pursuit of 2nd-moment exponential stability is carried out using the theory of stochastic differential equations (SDEs) in infinite dimensions, even in cases where the operators A_1 and A_2 are noncommutative.
(ii) We investigate the robust stability of the switching time, i.e., if there exists a small error at each switching time t_k ($k \in \mathbb{N}^+$), the system (1.1) with our switching strategies is still stable.
(iii) Although we have primarily focused on systems with two subsystems, we believe that further discussions involving multiple distinct modes can be explored using a similar approach.

The switching strategies discussed in this article are open-loop switching strategies. In closed-loop switched systems, the behavior of switching signals depends on the system's current state over time. From our perspective, the challenge in studying closed-loop switched systems lies in establishing well-posedness within appropriate function spaces.

This article is organized as follows. In Section 2, we give out some hypotheses and preliminaries. Section 3 is devoted to the main results of this paper. Section 4 demonstrates a nontrivial example and the numerical simulation results.

2. Certain hypotheses and preliminary results

We first provide several useful definitions in this section.

Definition 2.1. Let $\sigma(\cdot)$ be the switching signal with a dwell time, and let $\{t_j\}_{j=1}^{\infty}$ ($t_j < t_{j+1}$ for each $j \in \mathbb{N}^+$) be a set of switching times of (1.1). A $\mathbb{H}$-valued process $Y(\cdot)$ is called a (mild) solution to (1.1)

if it satisfies:
(i) For each $T > 0$, $Y(\cdot) \in L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{H}))$;
(ii) When $t \in [t_j, t_{j+1})$ ($j \in \mathbb{N}$),

$$Y(t) = S_{\sigma(t)}(t - t_j)Y(t_j) + \int_{t_j}^{t} S_{\sigma(t)}(t - t_j - s)f_{\sigma(s)}(s)Y(s)dW(s), \ \ a.s.$$

Moreover, the (mild) solution $Y(\cdot)$ to (1.1) is called a strong solution if it satisfies:
(iii) For each $T > 0$, $Y(t) \in D(A_1) = D(A_2)$ for a.e $(t, \omega) \in [0, T] \times \Omega$, and $A_{\sigma(\cdot)}Y(\cdot) \in L^1(0, T; \mathbb{H})$ a.s. (similar definitions can be found at Section 3.2 of [1]).

Definition 2.2. The system (1.1) is said to be 2nd-moment exponentially stable (or exponentially stable in the mean square) if there exist positive numbers C and ϖ such that for each initial data $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; \mathbb{H})$, the solution to (1.1) satisfies

$$\mathbb{E}\|Y(t)\|^2 \leq Ce^{-\varpi t}\mathbb{E}\|Y_0\|^2, \ \ \text{for } t \geq 0.$$

Remark 2.1. From the perspective of mathematics, the 2nd moment exponential stability and almost sure exponential stability do not imply each other. However, under a restrictive condition, the 2nd moment exponential stability implies almost sure exponential stability (see p.175 of [2]). Hence, plenty of literature investigates the 2nd moment stabilization, and we follow the same fashion for our stabilization design.

2.1. Certain hypotheses

In view of the stabilization of this problem, we impose some assumptions on the Hilbert space $\mathbb{H}$ and the principle operator A_i ($i = 1, 2$) as follows.

Assumption 2.1. There exist subspaces $\mathbb{V}_i$ and $\mathbb{W}_i$ of $\mathbb{H}$ ($i = 1, 2$) such that

$$\mathbb{H} = \mathbb{V}_i \oplus \mathbb{W}_i \text{ and } \mathbb{V}_i \perp \mathbb{W}_i \ \ (i = 1, 2).$$

Moreover, there exist bounded positive operators P_i (i=1,2), and positive numbers α_i, β_i, μ and ν such that for each $i = 1, 2$,

$$\begin{cases} (i) \ \ \mu\|h\|^2 \leq \langle P_i h, h\rangle \leq \nu\|h\|^2, \ \ \text{for } h \in \mathbb{H}; \\ (ii) \ \ 0 < Q_i(h) + \nu\|b_i h\|^2 < \alpha_i\|h\|^2, \ \ \text{for } h \in \mathbb{W}_i \cap D(A_i) \setminus \{0\}; \\ (iii) \ Q_i(h) + \nu\|b_i h\|^2 < -\beta_i\|h\|^2, \ \ \text{for } h \in \mathbb{V}_i \cap D(A_i) \setminus \{0\}, \end{cases} \tag{2.1}$$

where $b_i = \|f_i(\cdot)\|_{L^\infty_{\mathcal{F}}(0,+\infty;L^\infty(\Omega))}$, and the functionals $Q_i(\cdot)$ are defined as

$$Q_i(h) := \langle P_i h, A_i h\rangle + \langle A_i h, P_i h\rangle \ \ (i = 1, 2). \tag{2.2}$$

Remark 2.2. (i) We define functionals $V_i(\cdot)$ ($i = 1, 2$) as follows:

$$V_i(h) := \langle P_i h, h\rangle, \ \ h \in \mathbb{H} \ \ (i = 1, 2). \tag{2.3}$$

The family $\{V_i(\cdot) \mid i = 1, 2\}$ is viewed as a multiple Lyapunov function of (1.1).
(ii) In this article, $\mathbb{V}_i$ and $\mathbb{W}_i$ are called the stable and unstable subspace of $\mathbb{H}$ corresponding to the *ith* subsystem of (1.1) ($i = 1, 2$), respectively.

If Assumption 2.1 holds, then for each $h \in \mathbb{H}$, there exists a unique decomposition $h = v_i + w_i$, where $v_i \in \mathbb{V}_i$ and $w_i \in \mathbb{W}_i$ $(i = 1, 2)$. Now, we define orthogonal projections $\Pi_i : \mathbb{H} \mapsto \mathbb{V}_i$ and $\Phi_i : \mathbb{H} \mapsto \mathbb{W}_i$ $(i = 1, 2)$ as

$$\Pi_i h = v_i \ \text{ and } \ \Phi_i h = w_i.$$

Assumption 2.2. If $h \in D(A_i)$, then $\Pi_i(h) \in D(A_i)$ $(i = 1, 2)$, and $A_i\Pi_i(h) = \Pi_i A_i(h)$.

Assumption 2.3. The parameters μ, ν, α_i and β_i $(i = 1, 2)$ given in Assumption 2.1 satisfy that

$$\frac{\alpha_1 \cdot \alpha_2}{\mu^2} < \frac{\beta_1 \cdot \beta_2}{\nu^2}.$$

Remark 2.3. (i) Assumption 2.2 is natural for infinite-dimensional linear systems, and we will give a nontrivial example in Section 4 to show it is reasonable.
(ii) Roughly speaking, Assumption 2.3 guarantees that the drawbacks of unstable subspace should be overcome by the advantage of stable subspace.

2.2. *Preliminary results*

Given a positive number δ, we define three subsets of $\mathbb{R}^2$ as follows:

$$G_1(\delta) := \{(\tau_1, \tau_2) \ : \ \tau_1, \tau_2 > 0, \ \text{and} \ \frac{\alpha_1}{\mu}\tau_1 - \frac{\beta_2}{\nu}\tau_2 < -\delta\},$$

$$G_2(\delta) := \{(\tau_1, \tau_2) \ : \ \tau_1, \tau_2 > 0, \ \text{and} \ -\frac{\beta_1}{\nu}\tau_1 + \frac{\alpha_2}{\mu}\tau_2 < -\delta\},$$

and

$$G_3(\delta) := \{(\tau_1, \tau_2) \ : \ \tau_1, \tau_2 > 0, \ \text{and} \ \frac{\alpha_1}{\mu}\tau_1 + \frac{\alpha_2}{\mu}\tau_2 < \delta\}.$$

Lemma 2.1. Suppose Assumption 2.3 holds, then for each $\delta > 0$, $G_1(\delta) \cap G_2(\delta) \neq \emptyset$.

The proof can be obtained by the standard method of analytic geometry in $\mathbb{R}^2$. A similar result can be found in Lemma 2.2 of [15]. Here, we omit the detailed proof.

Remark 2.4. It is obvious that when $\delta > 0$, $G_1(\delta) \cap G_2(\delta)$ is the unbounded set in the first quadrant.

Lemma 2.2. Suppose Assumption 2.1 holds. Then, the following conclusions are valid:
(i) $\mathbb{W}_1 \subseteq \mathbb{V}_2$ if and only if $\mathbb{W}_2 \subseteq \mathbb{V}_1$.
(ii) If $\mathbb{W}_2 \subseteq \mathbb{V}_1$, then for each $h \in \mathbb{H}$, $||\Phi_2 h|| \leq ||\Pi_1 h||$.
(iii) For each $h \in \mathbb{H}$, $||h||^2 = ||\Phi_i h||^2 + ||\Pi_i h||^2$ $(i = 1, 2)$.
Via the theory of functional analysis (see Section 3.1 of [20]), we can easily prove these results. Here, we omit the detailed proof.

2.3. *Analysis of the subsystem of (1.1)*

Let $Y_i(\cdot)$ $(i = 1, 2)$ be the solution to the *ith* subsystem of (1.1), i.e.,

$$\begin{cases} dY_i(t) = A_i Y_i(t)dt + f_i(t)Y_i(t)dW(t), t \in (0, +\infty), \\ Y_i(0) = Y_0(\omega), \end{cases} \tag{2.4}$$

where the initial data Y_0 belongs to $L^2(\Omega, \mathcal{F}_0, \mathcal{P}; \mathbb{H})$. Then, we have the following lemma.

Lemma 2.3. Equation (2.4) admits a unique (mild) solution $Y_i(\cdot)$ $(i = 1, 2)$, and

$$Y_i(t) = \exp\Big(\int_0^t f_i(s)dW(s) - \frac{1}{2}\int_0^t |f_i(s)|^2 ds\Big) S_i(t) Y_0, \quad t \geq 0. \tag{2.5}$$

If we furthermore assume that $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; D(A_i))$, then (2.5) is a strong solution to Eq (2.4).

The proof can be obtained by the standard method in Section 3.2 of [1] and Theorem 2.1 of [11]. Here, we omit the details.

Remark 2.5. By the same argument, we can easily find that if the switching signal $\sigma(\cdot)$ has a dwell time, then (1.1) admits a unique (mild) solution. Moreover, if $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; D(A_1))$, then, together with (2.5), (ii) of Remark 1.1, and the differential property of C_0 semi-group (see Theorem 1 in Section 7.4 of [21]), we obtain that (1.1) has a unique strong solution under this switching signal.

Next, we present some estimates for the solution to (2.4).

Lemma 2.4. Suppose Assumption 2.1 holds. Then, the (mild) solution $Y_i(\cdot)$ to (2.4) satisfies that for $0 \leq s < t < +\infty$,

$$\mathbb{E}\|Y_i(t)\|^2 \leq \frac{\nu}{\mu} e^{\frac{\alpha_i(t-s)}{\mu}} \mathbb{E}\|Y_i(s)\|^2 \quad (i = 1, 2). \tag{2.6}$$

Proof. Let the initial data $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; D(A_i))$ $(i = 1, 2)$. It follows from Lemma 2.3 that (2.4) admits a unique strong solution $(i = 1, 2)$.

By Itô formula, we obtain that for $t \in (0, +\infty)$,

$$\begin{aligned} d[e^{rt}V_i(Y_i(t))] &= re^{rt}V_i(Y_i(t))dt + e^{rt}(Q_i(Y_i(t)) + V_i(f_i(t)Y_i(t)))dt \\ &\quad +2e^{rt}\langle P_iY_i(t), f_i(t)Y_i(t)\rangle dW(t), \end{aligned}$$

where r is a negative real number which will be determined later, and where $Q_i(\cdot)$ and $V_i(\cdot)$ $(i = 1, 2)$ are functionals defined in (2.2) and (2.3), respectively. Integrating the above equation over $[s, t]$, taking the expectation, and using (i) and (ii) of (2.1), we obtain

$$\begin{aligned} & e^{rt}\mathbb{E}V_i(Y_i(t)) - e^{rs}\mathbb{E}V_i(Y_i(s)) \\ =\ & \mathbb{E}\int_s^t re^{r\tau}V_i(Y_i(\tau))d\tau + \mathbb{E}\int_s^t e^{r\tau}(Q_i(Y_i(\tau)) + |f_i(\tau)|^2 V_i(Y_i(\tau)))d\tau \\ \leq\ & \mathbb{E}\int_s^t r\mu e^{r\tau}\|Y_i(\tau)\|^2 d\tau + \mathbb{E}\int_s^t \alpha_i e^{r\tau}\|Y_i(\tau)\|^2 d\tau. \end{aligned} \tag{2.7}$$

Taking $r = -\frac{\alpha_i}{\mu}$ in (2.7), we have

$$\mathbb{E}V_i(Y_i(t)) \leq e^{\frac{\alpha_i(t-s)}{\mu}} \mathbb{E}V_i(Y_i(s)).$$

It, together with (i) of (2.1), leads to (2.6), when $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; D(A_i))$ $(i = 1, 2)$.

Since A_i $(i = 1, 2)$ is a closed densely defined operator in $\mathbb{H}$, we obtain $\overline{D(A_i)} = \mathbb{H}$ $(i = 1, 2)$. Using the standard density argument, we have that (2.6) still holds for each $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; \mathbb{H})$. This completes the proof. □

3. The proof of the main result

In this section, we will state the main results of this paper.

3.1. The first switching strategy

We arbitrarily fix a positive number ρ satisfying

$$\rho > 2\ln\frac{\sqrt{2}\nu}{\mu}. \tag{3.1}$$

Lemma 2.1 shows that if Assumption 2.3 holds, then $G_1(\rho)\cap G_2(\rho)\neq\emptyset$. We take

$$(s_1, s_2)\in G_1(\rho)\cap G_2(\rho). \tag{3.2}$$

Clearly, $s_1, s_2 > 0$. We propose the first switching strategy as follows:

$$t_0 := 0,\ \ \sigma(0) := 1,\ \ \text{or}\ \sigma(0) := 2, \tag{3.3}$$

$$t_{k+1} := \begin{cases} t_k + s_1, & \text{if } \sigma(t_k) = 1,\\ t_k + s_2, & \text{if } \sigma(t_k) = 2, \end{cases} \tag{3.4}$$

and

$$\sigma(t_{k+1}) := \begin{cases} 2, & \text{if } \sigma(t_k) = 1,\\ 1, & \text{if } \sigma(t_k) = 2, \end{cases} \tag{3.5}$$

where $k\in\mathbb{N}$.

Now, we present the first result as follows.

Theorem 3.1. Suppose that Assumptions 2.1–2.3 hold, and $\mathbb{W}_1\subseteq\mathbb{V}_2$. Then, there exist positive numbers M and γ such that for each $Y_0\in L^2(\Omega,\mathcal{F}_0,\mathcal{P};\mathbb{H})$, the (mild) solution to (1.1) with the switching strategy (3.3)–(3.5) satisfies

$$\mathbb{E}\|Y(t)\|^2 \le Me^{-\frac{\gamma t}{s_1+s_2}}\mathbb{E}\|Y_0\|^2,\ \text{for } t\ge 0. \tag{3.6}$$

Proof. Without loss of generality, we assume $\sigma(0) = 1$. By the switching strategy (3.3)–(3.5), it shows that $t_1 = s_1$ and $t_2 = s_1 + s_2$. Then, we have $\sigma(t) = 1$, as $t\in[0, t_1)$, and $\sigma(t) = 2$, as $t\in[t_1, t_2)$. The following proof will be split into two steps.

Step 1. *We estimate* $\mathbb{E}\|Y(t_2)\|^2$.

We first suppose $Y_0\in L^2(\Omega,\mathcal{F}_0,\mathcal{P}; D(A_1))$. By Remark 2.4 and the fact $D(A_1) = D(A_2)$, (1.1) admits a unique strong solution. Thus, for each $T > 0$, $Y(t)\in D(A_1) = D(A_2)$ for a.e $(t,\omega)\in[0,T]\times\Omega$. By Assumption 2.1, there exists a unique decomposition

$$Y(t) = \Pi_2 Y(t) + \Phi_2 Y(t),\ \ \text{for}\ \ t\in[0,+\infty).$$

By (*iii*) of Lemma 2.2, we obtain

$$\|Y(t)\|^2 = \|\Pi_2 Y(t)\|^2 + \|\Phi_2 Y(t)\|^2.$$

It follows from (i) of (2.1) that for each $t \in [0, +\infty)$,

$$\|Y(t)\|^2 \leq \frac{1}{\mu}[V_2(\Pi_2 Y(t)) + V_2(\Phi_2 Y(t))], \tag{3.7}$$

where the functional $V_2(\cdot)$ is given in (2.3). By Assumption 2.2 and Itô formula, we have

$$\begin{aligned} d[e^{rt}V_2(\Pi_2 Y(t))] &= re^{rt}V_2(\Pi_2 Y(t))dt + e^{rt}[Q_2(\Pi_2 Y(t)) + |f_2(t)|^2 V_2(\Pi_2 Y(t))]dt \\ &\quad +2\langle P_2\Pi_2 Y(t), f_2(t)\Pi_2 Y(t)\rangle dW(t), \end{aligned}$$

where r is a positive real number which will be determined later, and where $Q_2(\cdot)$ is the functional given in (2.2). Integrating the above equation over $[t_1, t_2]$ and taking the expectation, we obtain

$$\begin{aligned} & e^{rt_2}\mathbb{E}V_2(\Pi_2 Y(t_2)) - e^{rt_1}\mathbb{E}V_2(\Pi_2 Y(t_1)) \\ = \ & \mathbb{E}\int_{t_1}^{t_2} re^{rt}V_2(\Pi_2 Y(t))dt + \mathbb{E}\int_{t_1}^{t_2} e^{rt}[Q_2(\Pi_2 Y(t)) + |f_2(t)|^2 V_2(\Pi_2 Y(t))]dt. \end{aligned}$$

It, together with (i) and (ii) of (2.1), indicates

$$e^{rt_2}\mathbb{E}V_2(\Pi_2 Y(t_2)) - e^{rt_1}\mathbb{E}V_2(\Pi_2 Y(t_1)) \leq \mathbb{E}\int_{t_1}^{t_2} r\nu e^{rt}\|\Pi_2 Y(t)\|^2 dt - \mathbb{E}\int_{t_1}^{t_2} \beta_2 e^{rt}\|\Pi_2 Y(t)\|^2 dt. \tag{3.8}$$

Taking $r = \frac{\beta_2}{\nu}$ in (3.8), it shows

$$\mathbb{E}V_2(\Pi_2 Y(t_2)) \leq e^{-\frac{\beta_2 s_2}{\nu}}\mathbb{E}V_2(\Pi_2 Y(t_1)). \tag{3.9}$$

By the same argument in Lemma 2.4, we have

$$\mathbb{E}V_2(\Phi_2 Y(t_2)) \leq e^{\frac{\alpha_2 s_2}{\mu}}\mathbb{E}V_2(\Phi_2 Y(t_1)). \tag{3.10}$$

Combining with (3.7), (3.9), (3.10) and (i) of (2.1) yields

$$\begin{aligned} \mathbb{E}\|Y(t_2)\|^2 &\leq \frac{1}{\mu}[e^{-\frac{\beta_2 s_2}{\nu}}\mathbb{E}V_2(\Pi_2 Y(t_1)) + e^{\frac{\alpha_2 s_2}{\mu}}\mathbb{E}V_2(\Phi_2 Y(t_1))] \\ &\leq \frac{\nu}{\mu}[e^{-\frac{\beta_2 s_2}{\nu}}\mathbb{E}\|\Pi_2 Y(t_1)\|^2 + e^{\frac{\alpha_2 s_2}{\mu}}\mathbb{E}\|\Phi_2 Y(t_1)\|^2]. \end{aligned} \tag{3.11}$$

Next, we estimate $\mathbb{E}\|\Phi_2 Y(t_1)\|^2$. Since $\mathbb{W}_1 \subseteq \mathbb{V}_2$, it follows from (i) of Lemma 2.2, that $\mathbb{W}_2 \subseteq \mathbb{V}_1$. Together with (ii) of Lemma 2.2 and (i) of (2.1), we have

$$\mathbb{E}\|\Phi_2 Y(t_1)\|^2 \leq \mathbb{E}\|\Pi_1 Y(t_1)\|^2 \leq \frac{1}{\mu}\mathbb{E}V_1(\Pi_1 Y(t_1)). \tag{3.12}$$

We note that $\sigma(t) = 1$, as $t \in [0, t_1)$. By the same method in the proof of (3.9), we obtain

$$\mathbb{E}V_1(\Pi_1 Y(t_1)) \leq e^{-\frac{\beta_1 s_1}{\nu}}\mathbb{E}V_1(\Pi_1 Y_0).$$

It, along with (3.12) and (i) of (2.1), yields

$$\mathbb{E}\|\Phi_2 Y(t_1)\|^2 \leq \frac{1}{\mu}e^{-\frac{\beta_1 s_1}{\nu}}\mathbb{E}V_1(\Pi_1 Y_0) \leq \frac{\nu}{\mu}e^{-\frac{\beta_1 s_1}{\nu}}\mathbb{E}\|\Pi_1 Y_0\|^2. \tag{3.13}$$

Then, it follows from Lemma 2.4 that

$$\mathbb{E}\|\Pi_2 Y(t_1)\|^2 \le \mathbb{E}\|Y(t_1)\|^2 \le \frac{\nu}{\mu} e^{\frac{\alpha_1 s_1}{\mu}} \mathbb{E}\|Y_0\|^2.$$

This, along with (3.11) and (3.13), indicates that

$$\mathbb{E}\|Y(t_2)\|^2 \quad \le \quad (\frac{\nu}{\mu})^2 (e^{\frac{\alpha_2 s_2}{\mu} - \frac{\beta_1 s_1}{\nu}} \mathbb{E}\|\Pi_1 Y_0\|^2 + e^{\frac{\alpha_1 s_1}{\mu} - \frac{\beta_2 s_2}{\nu}} \mathbb{E}\|Y_0\|^2).$$

Since $(s_1, s_2) \in G_1(\rho) \cap G_2(\rho)$, we can deduce

$$\mathbb{E}\|Y(t_2)\|^2 \le 2(\frac{\nu}{\mu})^2 e^{-\rho} \mathbb{E}\|Y_0\|^2.$$

By virtue of (3.1), there exists a positive number γ such that

$$\rho - 2 \ln \frac{\sqrt{2}\nu}{\mu} > \gamma. \tag{3.14}$$

Thus,

$$\mathbb{E}\|Y(t_2)\|^2 \le e^{-\rho + 2 \ln \frac{\sqrt{2}\nu}{\mu}} \mathbb{E}\|Y_0\|^2 \le e^{-\gamma} \mathbb{E}\|Y_0\|^2. \tag{3.15}$$

Since A_i $(i = 1, 2)$ is a closed densely defined operator in $\mathbb{H}$, we have $\overline{D(A_i)} = \mathbb{H}$ $(i = 1, 2)$. By the standard density argument, we have that (3.15) still holds for each $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; \mathbb{H})$.

Step 2. *We prove (3.6).*

Using standard iteration arguments, we obtain that

$$\mathbb{E}\|Y(kt_2)\|^2 \le e^{-k\gamma} \mathbb{E}\|Y_0\|^2. \tag{3.16}$$

For each $t \in [0, +\infty)$, we write $t = kt_2 + r$, where $k \in \mathbb{N}$ and $0 \le r < t_2$. There are only two possibilities: (i) If $r \in [0, t_1)$, by (2.6), we get

$$\mathbb{E}\|Y(t)\|^2 \le \frac{\nu}{\mu} e^{\frac{\alpha_1 r}{\mu}} \mathbb{E}\|Y(kt_2)\|^2.$$

(ii) If $r \in [t_1, t_2)$, we can also obtain that

$$\mathbb{E}\|Y(t)\|^2 \le (\frac{\nu}{\mu})^2 e^{\frac{\alpha_1 s_1 + \alpha_2 (r - s_1)}{\mu}} \mathbb{E}\|Y(kt_2)\|^2.$$

These, together with (3.16), yield

$$\mathbb{E}\|Y(t)\|^2 \le M e^{-k\gamma} \mathbb{E}\|Y_0\|^2 \le M e^{-\frac{\gamma t}{s_1 + s_2}} \mathbb{E}\|Y_0\|^2, \tag{3.17}$$

where the positive number M only depends on μ, ν, α_i, and s_i $(i = 1, 2)$. This completes the proof. □

Remark 3.1. The switching signal $\sigma(\cdot)$ defined in (3.3)–(3.5) has a dwell time $\tau = \min\{s_1, s_2\}$. Since $G_1(\rho) \cap G_2(\rho)$ is the unbounded set in the first quadrant, s_1, s_2 can be taken arbitrarily large. Therefore, the switching frequency for system (1.1) can be arbitrarily low under the first switching strategy. This prevents the actuator from fast switching, which will possibly damage the systems.

3.2. The second switching strategy

We will give a further discussion for this problem in this subsection. Next, we provide another hypothesis.

Assumption 3.1. There exist positive numbers δ, $\bar{\delta}$ such that $\delta > 2\ln\frac{\sqrt{6}\nu}{\mu}$, $\bar{\delta} > 2\ln\frac{\sqrt{6}\nu}{\mu}$, and the following conclusions hold.
(*i*) $G_1(\delta)\cap G_2(\delta)\cap G_3(\bar{\delta})\neq\emptyset$;
(*ii*) $\varrho_1\leq e^{-\bar{\delta}}$, or $\varrho_2\leq e^{-\bar{\delta}}$, where $\varrho_1 := \|\Phi_2\Phi_1\|$ and $\varrho_2 := \|\Phi_1\Phi_2\|$.

Remark 3.2. It is obvious that Assumption 3.1 can be directly derived from the conditions of Theorem 3.1. First, taking a positive number $\delta > 2\ln\frac{\sqrt{6}\nu}{\mu}$ and using Assumption 2.2, we have $A_1(\delta)\cap A_2(\delta)\neq\emptyset$. Then, we can choose a positive number $\bar{\delta} > 2\ln\frac{\sqrt{6}\nu}{\mu}$ such that $A_1(\delta)\cap A_2(\delta)\cap A_3(\bar{\delta})\neq\emptyset$. Second, if $\mathbb{W}_1\subseteq\mathbb{V}_2$ (or $\mathbb{W}_2\subseteq\mathbb{V}_1$), then $\varrho_1=\|\Phi_2\Phi_1\|=0$ (or $\varrho_2=\|\Phi_1\Phi_2\|=0$). We observe that (*ii*) of Assumption 3.1 also holds.

Suppose that Assumptions 2.1–2.3 and 3.1 hold. Taking $(\bar{s}_1,\bar{s}_2)\in G_1(\delta)\cap G_2(\delta)\cap G_3(\bar{\delta})$, we give the second switching strategy as follows:

$$\bar{t}_0 := 0,\ \ \sigma(0) := \begin{cases} 1, & \text{if } \varrho_1\leq e^{-\bar{\delta}},\\ 2, & \text{if } \varrho_2\leq e^{-\bar{\delta}},\end{cases} \tag{3.18}$$

$$\bar{t}_{k+1} := \begin{cases} \bar{t}_k+\bar{s}_1, & \text{if } \sigma(\bar{t}_k)=1,\\ \bar{t}_k+\bar{s}_2, & \text{if } \sigma(\bar{t}_k)=2,\end{cases} \tag{3.19}$$

and

$$\sigma(\bar{t}_{k+1}) := \begin{cases} 2, & \text{if } \sigma(\bar{t}_k)=1,\\ 1, & \text{if } \sigma(\bar{t}_k)=2,\end{cases} \tag{3.20}$$

where $k\in\mathbb{N}$.

The second result is presented as follows.

Theorem 3.2. Suppose Assumptions 2.1–2.3 and 3.1 hold. Then, there exist positive numbers M and ϖ such that for each $Y_0\in L^2(\Omega,\mathcal{F}_0,\mathcal{P};\mathbb{H})$, the (mild) solution to (1.1) with the switching strategy (3.18)–(3.20) satisfies

$$\mathbb{E}\|Y(t)\|^2\leq Me^{-\frac{\varpi t}{\bar{s}_1+\bar{s}_2}}\mathbb{E}\|Y_0\|^2,\ \text{for } t\geq 0. \tag{3.21}$$

Proof. Without loss of generality, we assume $\varrho_1=\|\Phi_2\Phi_1\|\leq e^{-\bar{\delta}}$, where $\bar{\delta}$ is the positive number given in Assumption 3.1. It follows from the switching strategy (3.18)–(3.20) that $\bar{t}_1=\bar{s}_1$ and $\bar{t}_2=\bar{s}_1+\bar{s}_2$. Then, we have $\sigma(t)=1$, as $t\in[0,\bar{t}_1)$, and $\sigma(t)=2$, as $t\in[\bar{t}_1,\bar{t}_2)$. The following proof will be split into two steps.

***Step 1**. We first estimate* $\mathbb{E}\|Y(\bar{t}_2)\|^2$.

Suppose that $Y_0\in L^2(\Omega,\mathcal{F}_0,\mathcal{P};D(A_1))$. By the same method in the proof of (3.11), we obtain

$$\mathbb{E}\|Y(\bar{t}_2)\|^2\leq\frac{\nu}{\mu}[e^{-\frac{\beta_2\bar{s}_2}{\nu}}\mathbb{E}\|\Pi_2Y(\bar{t}_1)\|^2+e^{\frac{\alpha_2\bar{s}_2}{\mu}}\mathbb{E}\|\Phi_2Y(\bar{t}_1)\|^2]. \tag{3.22}$$

Then, we estimate $\mathbb{E}\|\Pi_2 Y(\bar{t}_1)\|^2$ in (3.22). Since $\sigma(t) = 1$, as $t \in [0, \bar{t}_1)$, it follows from Lemma 2.4 that

$$\mathbb{E}\|\Pi_2 Y(\bar{t}_1)\|^2 \leq \mathbb{E}\|Y(\bar{t}_1)\|^2 \leq \frac{\nu}{\mu} e^{\frac{\alpha_1 \bar{s}_1}{\mu}} \mathbb{E}\|Y_0\|^2. \tag{3.23}$$

Next, by Assumption 2.1, we have

$$\Phi_2 Y(\bar{t}_1) = \Phi_2 \Pi_1 Y(\bar{t}_1) + \Phi_2 \Phi_1 Y(\bar{t}_1).$$

It is easy to obtain

$$\mathbb{E}\|\Phi_2 Y(\bar{t}_1)\|^2 \leq 2\mathbb{E}\|\Phi_2 \Pi_1 Y(\bar{t}_1)\|^2 + 2\mathbb{E}\|\Phi_2 \Phi_1 Y(\bar{t}_1)\|^2. \tag{3.24}$$

Meanwhile, it follows from (i) of (2.1) that

$$\mathbb{E}\|\Phi_2 \Pi_1 Y(\bar{t}_1)\|^2 \leq \mathbb{E}\|\Pi_1 Y(\bar{t}_1)\|^2 \leq \frac{1}{\mu} \mathbb{E}\|V_1(\Pi_1 Y(\bar{t}_1))\|^2.$$

Using the same argument in the proof of (3.9), we obtain

$$\mathbb{E}\|V_1(\Pi_1 Y(\bar{t}_1))\|^2 \leq e^{-\frac{\beta_1 \bar{s}_1}{\nu}} \mathbb{E}\|V_1(\Pi_1 Y_0)\|^2 \leq \nu e^{-\frac{\beta_1 \bar{s}_1}{\nu}} \mathbb{E}\|\Pi_1 Y_0\|^2.$$

Thus,

$$\mathbb{E}\|\Phi_2 \Pi_1 Y(\bar{t}_1)\|^2 \leq \frac{\nu}{\mu} e^{-\frac{\beta_1 \bar{s}_1}{\nu}} \mathbb{E}\|\Pi_1 Y_0\|^2. \tag{3.25}$$

Then, it follows from Lemma 2.4 and $\|\Phi_2 \Phi_1\| \leq e^{-\bar{\delta}}$ that

$$\mathbb{E}\|\Phi_2 \Phi_1 Y(\bar{t}_1)\|^2 \leq \|\Phi_2 \Phi_1\|^2 \mathbb{E}\|Y(\bar{t}_1)\|^2 \leq \|\Phi_2 \Phi_1\|^2 \frac{\nu}{\mu} e^{\frac{\alpha_1 \bar{s}_1}{\mu}} \mathbb{E}\|Y_0\|^2 \leq \frac{\nu}{\mu} e^{-2\bar{\delta}} e^{\frac{\alpha_1 \bar{s}_1}{\mu}} \mathbb{E}\|Y_0\|^2.$$

It, combining with (3.22)–(3.25), indicates

$$\mathbb{E}\|Y(\bar{t}_2)\|^2 \leq (\frac{\nu}{\mu})^2 (e^{\frac{\alpha_1 \bar{s}_1}{\mu} - \frac{\beta_2 \bar{s}_2}{\nu}} \mathbb{E}\|Y_0\|^2 + 2e^{\frac{\alpha_2 \bar{s}_2}{\mu} - \frac{\beta_1 \bar{s}_1}{\nu}} \mathbb{E}\|Y_0\|^2 + 2e^{-2\bar{\delta}} e^{\frac{\alpha_1 \bar{s}_1 + \alpha_2 \bar{s}_2}{\mu}} \mathbb{E}\|Y_0\|^2).$$

Since $(\bar{s}_1, \bar{s}_2) \in G_1(\delta) \cap G_2(\delta) \cap G_3(\bar{\delta})$, we have

$$\mathbb{E}\|Y(\bar{t}_2)\|^2 \leq (3(\frac{\nu}{\mu})^2 e^{-\delta} + 2(\frac{\nu}{\mu})^2 e^{-\bar{\delta}}) \mathbb{E}\|Y_0\|^2.$$

By Assumption 3.1, there exists a positive number ϖ such that

$$\delta - 2\ln\frac{\sqrt{3}\nu}{\mu} > \varpi + \ln 2, \text{ and } \bar{\delta} - 2\ln\frac{\sqrt{2}\nu}{\mu} > \varpi + \ln 2. \tag{3.26}$$

Therefore,

$$\mathbb{E}\|Y(\bar{t}_2)\|^2 \leq (e^{-\delta + 2\ln\frac{\sqrt{3}\nu}{\mu}} + e^{-\bar{\delta} + 2\ln\frac{\sqrt{2}\nu}{\mu}}) \mathbb{E}\|Y_0\|^2 \leq e^{-\varpi} \mathbb{E}\|Y_0\|^2. \tag{3.27}$$

Then, by the standard density argument, we have that (3.27) still holds for each $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; \mathbb{H})$.

Step 2. *We prove (3.21).*

Using standard iteration arguments, we have that for each $k \in \mathbb{N}$,

$$\mathbb{E}\|Y(k\bar{t}_2)\|^2 \le e^{-k\varpi}\mathbb{E}\|Y_0\|^2.$$

For each $t \in (0, +\infty)$, we write $t = k\bar{t}_2 + r$, where $k \in \mathbb{N}$ and $0 \le r < \bar{t}_2$. Then, by the same method in the proof of (3.17), we obtain

$$\mathbb{E}\|Y(t)\|^2 \le Me^{-\frac{\varpi t}{\bar{s}_1+\bar{s}_2}}\mathbb{E}\|Y_0\|^2,$$

where the positive number M only depends on μ, ν, α_i and s_i $(i = 1, 2)$. This completes the proof. □

Remark 3.3. (i) Similar, $\bar{\tau} = \min\{\bar{s}_1, \bar{s}_2\}$ is a dwell time of the switching signal $\sigma(\cdot)$ given in (3.18)–(3.20).

(ii) It is obvious that Theorem 3.2 is a generalization of Theorem 3.1. However, we can not get arbitrarily large dwell time in Theorem 3.1. The reason is that $(\bar{s}_1, \bar{s}_2) \in G_1(\delta) \cap G_2(\delta) \cap G_3(\bar{\delta})$, which is a bounded subset of $\mathbb{R}^2$ in the first quadrant.

3.3. Robust stability of the switching time

In this subsection, we discuss the robust stability of the switching time under the conditions of Theorem 3.1. Suppose that there exists a small error at each switching time t_k $(k \in \mathbb{N}^+)$, when we carry out the switched system (1.1) with the switching strategy (3.3)–(3.5). Let ϵ be a positive number satisfying

$$\{(r_1, r_2) \in \mathbb{R}^2 : |r_i - s_i| < \epsilon \ (i = 1, 2)\} \subset G_1(\rho) \cap G_2(\rho), \tag{3.28}$$

where (s_1, s_2) is the pair given in (3.2), and ρ is the positive number given in (3.1). Let ϵ_k $(k \in \mathbb{N}^+)$ be the error at each switching time t_k $(k \in \mathbb{N}^+)$, and let $\{\theta_k\}_{k=1}^{\infty}$ be the set of the actual switching times. Then, the practical switching strategy becomes

$$\theta_0 := 0, \ \ \sigma(0) := 1, \ \text{ or } \ \sigma(0) := 2, \tag{3.29}$$

$$\theta_k := t_k + \epsilon_k, \tag{3.30}$$

and

$$\sigma(\theta_k) := \begin{cases} 2, & \text{if } \sigma(\theta_{k-1}) = 1, \\ 1, & \text{if } \sigma(\theta_{k-1}) = 2, \end{cases} \tag{3.31}$$

where $k \in \mathbb{N}^+$ and t_k is defined in (3.4).

Then, we have the following result.

Theorem 3.3. Suppose that Assumptions 2.1–2.3 hold, and $\mathbb{W}_1 \subseteq \mathbb{V}_2$. If the error ϵ_k in (3.30) satisfies

$$|\epsilon_k| < \frac{\epsilon}{2}, \quad k \in \mathbb{N}^+, \tag{3.32}$$

then, there exist positive numbers M and γ such that for each $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; \mathbb{H})$, the (mild) solution to (1.1) with the practical switching strategy (3.29)–(3.31) satisfies

$$\mathbb{E}\|Y(t)\|^2 \le Me^{-\frac{\gamma t}{s_1+s_2}}\mathbb{E}\|Y_0\|^2, \ \text{ for } t \ge 0. \tag{3.33}$$

Proof. Without loss of generality, we assume $\sigma(0) = 1$. By the actual switching strategy (3.29)–(3.31), we have $\sigma(t) = 1$, as $t \in [0, \theta_1)$, and $\sigma(t) = 2$, as $t \in [\theta_1, \theta_2)$. Let

$$\tau_1 = \theta_1 \ \text{ and } \ \tau_2 = \theta_2 - \theta_1.$$

By the definition of t_k in (3.3) and (3.4) (which is the switching strategy in theory), we have

$$s_1 = t_1 \ \text{ and } \ s_2 = t_2 - t_1.$$

It, together with (3.30) and (3.32), indicates that

$$(\tau_1, \tau_2) \in \{(r_1, r_2) \in \mathbb{R}^2 : |r_i - s_i| < \epsilon \ (i = 1, 2)\} \subset G_1(\rho) \cap G_2(\rho).$$

Using the standard density argument, we only need to prove (3.33) under the condition $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P}; D(A_1))$. By the same argument in the proof of (3.15), we have

$$\mathbb{E}\|Y(\theta_2)\|^2 \le e^{-\gamma}\mathbb{E}\|Y_0\|^2,$$

where γ is the positive number given in (3.14). Using iteration arguments, we obtain that for each $k \in \mathbb{N}^+$,

$$\mathbb{E}\|Y(\theta_{2k})\|^2 \le e^{-k\gamma}\mathbb{E}\|Y_0\|^2. \tag{3.34}$$

(i) In the first case that $t \in [0, \theta_2)$, we can apply Lemma 2.4 to get (3.33).
(ii) Next, we consider the second case that $t \in [\theta_2, +\infty)$. Clearly, there exist $k \in \mathbb{N}^+$ such that $\theta_{2k} \le t < \theta_{2(k+1)}$. This, along with (3.30) and (3.32), yields

$$t_{2k} - \frac{\epsilon}{2} \le t < t_{2(k+1)} + \frac{\epsilon}{2}.$$

It, together with the fact that $t_{2(k+1)} = (k + 1)(s_1 + s_2)$, shows

$$-k < -\frac{t - \frac{\epsilon}{2}}{s_1 + s_2} + 1. \tag{3.35}$$

By Lemma 2.4, we obtain

$$\mathbb{E}\|Y(t)\|^2 \le M\mathbb{E}\|Y(\theta_{2k})\|^2,$$

where M only depends on $\epsilon, \mu, \nu, \alpha_i$ and s_i $(i = 1, 2)$. This, along with (3.34) and (3.35), indicates (3.33).
In summary, we conclude that (3.33) holds. Hence, this completes the proof. □

Remark 3.4. In fact, the system (1.1) with the second switching strategy (3.18)–(3.20) also possesses the robust stability of the switching time. The proof is similar to Theorem 3.3. We omit it.

4. Illustrative examples and numerical simulations

In this section, we will give an example of the stochastic switched heat equation. Let $\mathbb{H} \equiv L^2(0, 1)$, equipped with the inner product $\langle \phi, \varphi \rangle = \int_0^1 \phi(x)\varphi(x)dx$, for $\phi, \varphi \in L^2(0, 1)$. Let $\lambda_k = k^2\pi^2$ and $e_k(x) = \sqrt{2}\sin(k\pi x)$ $(k \in \mathbb{N}^+)$. Then, $\{e_k\}_{k=1}^{\infty}$ constitutes an orthonormal basis of $L^2(0, 1)$.

The first stochastic heat equation is presented as follows:

$$\begin{cases} dy(x,t)=[y_{xx}(x,t)+(\lambda_1+1)y(x,t)]dt+f_1(t)y(x,t)dW(t), & x\in(0,1),\ t>0, \\ y(0,t)=y(1,t)=0, & t>0, \end{cases} \tag{4.1}$$

where $f_1(t)=\sin t$. Next, we define a linear bounded operator B on $L^2(0,1)$ as

$$Bg=k_2\langle g,e_2\rangle_{L^2(0,1)}e_2+k_3\langle g,e_3\rangle_{L^2(0,1)}e_3,\quad g\in L^2(0,1),$$

where $k_2=\lambda_3+2$ and $k_3=2\lambda_3-\lambda_2+2$. We introduce the second stochastic heat equation as follows:

$$\begin{cases} dy(x,t)=[y_{xx}(x,t)+By(x,t)]dt+f_2(t)y(x,t)dW(t), & x\in(0,1),t>0, \\ y(0,t)=y(1,t)=0,t>0, \end{cases} \tag{4.2}$$

where $f_2(t)=\cos t$.

Remark 4.1. (*i*) $D(A_1)=D(A_2)=H^2(0,1)\cap H_0^1(0,1)$, and the operator A_i $(i=1,2)$ is a linear closed densely defined operator, which generate a C_0 semi-group $\{S_i(t)\}_{t\geq 0}$ $(i=1,2)$ on $L^2(0,1)$.
(*ii*) Clearly, $b_i=\|f_i(\cdot)\|_{L^\infty_{\mathcal{F}}(0,+\infty;L^\infty(\Omega))}=1$ $(i=1,2)$.

Indeed, both (4.1) and (4.2) are unstable. We conduct numerical simulations to show this point. We discretize these two systems by the finite difference method. The time and the space steps are chosen as $l=0.00005$ and $h=0.01$, respectively. Let the initial data be $y_0(x)=100(x-x^3)$. We performed the numerical simulation 30 times under the MATLAB environment and presented one of them in the following. Figures 1 and 2 are the numerical simulations for (4.1) and (4.2), respectively. These show that (4.1) and (4.2) are unstable.

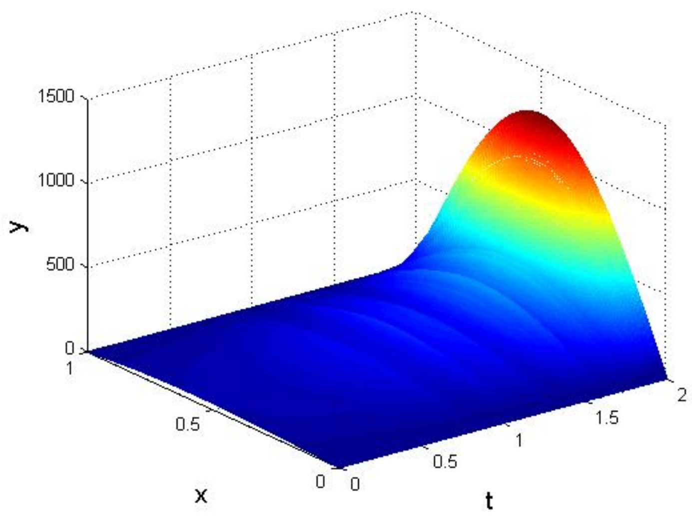

Figure 1. Subsystem (4.1) with the initial data y_0.

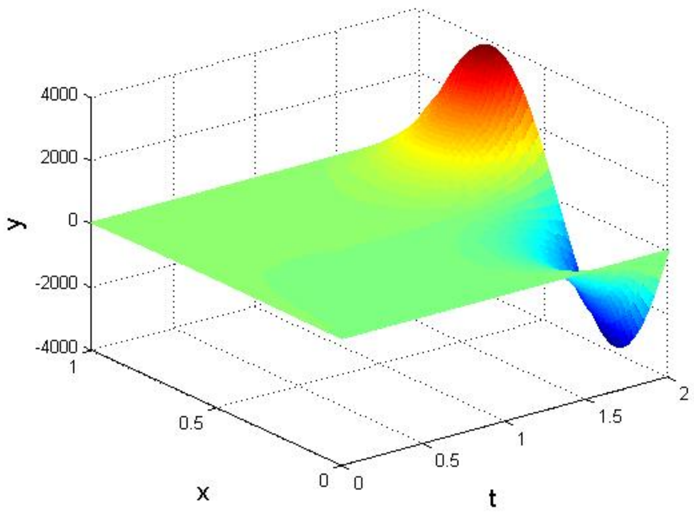

Figure 2. Subsystem (4.2) with the initial data y_0.

Now, we will put them into an abstract framework. Let $A_1 = \frac{\partial^2}{\partial x^2} + (\lambda_1 + 1)I$, where I is the identity operator on $L^2(0,1)$, and $A_2 = \frac{\partial^2}{\partial x^2} + B$. Let $\sigma(\cdot)$ be a right-continuous step function taking values in $\{1,2\}$. The stochastic switched parabolic system generated by the switching signal $\sigma(\cdot)$ from (4.1) and (4.2) can be written as

$$\begin{cases} dy(x,t) = A_{\sigma(t)}y(x,t) + f_{\sigma(t)}(t)y(x,t)dW(t), \quad x \in (0,1), \ t > 0, \\ y(0,t) = y(1,t) = 0, \ t > 0. \end{cases} \tag{4.3}$$

We let $V_i(y) = \langle y, y \rangle_{L^2(0,1)}$ $(i = 1,2)$ be the Lyapunov function of (4.1) and (4.2), respectively. That is, the operators $P_1 = P_2 = I$ and the parameters $\mu = \nu = 1$ in (2.1). Then, we define $\mathbb{W}_1 = \text{span}\{e_1\}$, $\mathbb{V}_1 = \text{span}\{e_2, e_3, e_4, \ldots\}$, $\mathbb{W}_2 = \text{span}\{e_2, e_3\}$, and $\mathbb{V}_2 = \text{span}\{e_1, e_4, e_5, \ldots\}$. It is easy to check that
(i) $L^2(0,1) = \mathbb{W}_i \oplus \mathbb{V}_i$ and $\mathbb{W}_i \perp \mathbb{V}_i$ $(i = 1,2)$.
(ii) $\mathbb{W}_1 \subseteq \mathbb{V}_2$ and $\mathbb{W}_2 \subseteq \mathbb{V}_1$.

Remark 4.2. Indeed, $\{e_k(x)\}_{k=1}^{+\infty}$ is the set of eigenfunctions of minus Laplacian, and $\{\lambda_k(x)\}_{k=1}^{+\infty}$ is the set of corresponding eigenvalues (see Section 6.5 of [21]).

Using Itô formula, we can obtain the parameters $\beta_1 = \lambda_2 - \lambda_1 - 2$, $\beta_2 = \lambda_1 - 1$, $\alpha_1 = 2$, and $\alpha_2 = \lambda_3 - \lambda_2 + 3$ in (2.1) and the Assumptions 2.1–2.3 hold in this case. Applying (3.1), we first take $\rho = 0.7$. Second, using (3.2), we choose the parameters $s_1 = 0.34$ and $s_2 = 0.16$. Then, the switching strategy (3.3)–(3.5) can be determined by s_1 and s_2. Figures 3 and 4 are the simulation of (4.3) under this switching strategy. These confirm that the experiment agrees with the theoretical results in Section 3. We performed it 30 times, and these results all show that our method yields satisfactory performance.

Remark 4.3. According to Section 3.1, the parameters s_1 and s_2 are crucial in switching strategy (3.3)–(3.5), and they can be chosen by (3.2). To this end, we first need to choose ρ by using (3.1).

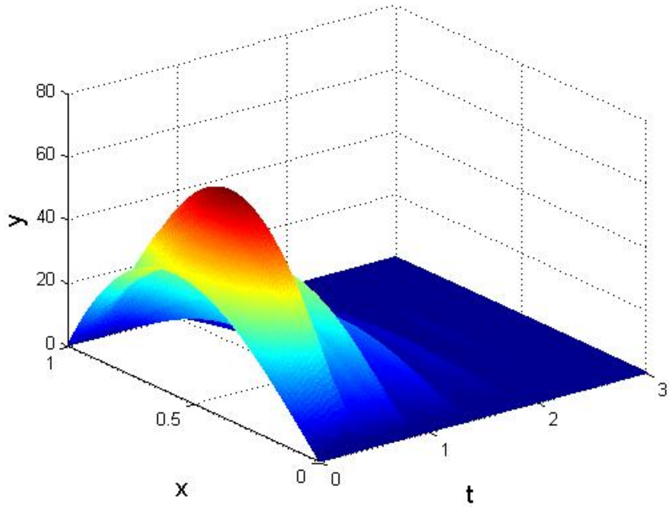

Figure 3. The switched system (4.3) with the initial data y_0.

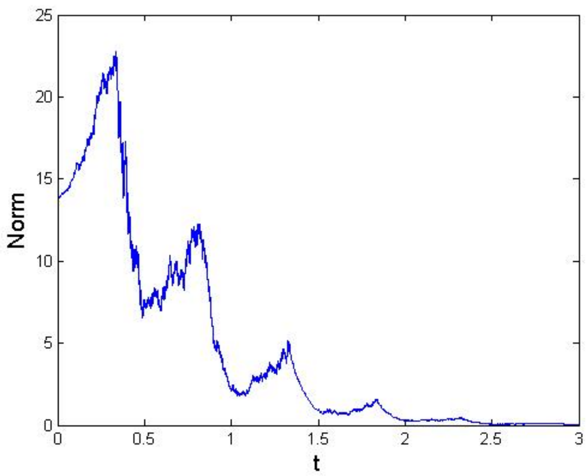

Figure 4. $L^2(0, 1)$ norm of the the switched system (4.3).

5. Conclusions

In this paper, we propose two switching strategies for a class of infinite-dimensional linear stochastic switched systems driven by multiplicative noise. These switching strategies stabilize the systems and prevent the systems from fast switching. In addition, we examine the robust stability of the switching time, meaning that even in the presence of small errors at each switching time t_k ($k \in \mathbb{N}^+$), the system (1.1) remains stable when utilizing our switching strategies. One further study direction is to extend the scheme of this article to switched systems with multiple distinct modes.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their sincere thanks to the referees for their valuable suggestions.

Conflict of interest

The authors declare that they have no competing interests in this paper.

References

1. Q. Lü, X. Zhang, *Mathematical control theory for stochastic partial differential equations*, Cham: Springer, 2021. https://doi.org/10.1007/978-3-030-82331-3
2. X. R. Mao, *Stochastic differential equations and applications*, 2 Eds., Horwood Publishing, 2007.
3. G. K. Basak, A. Bisi, M. K. Ghosh, Stability of a random diffusion with linear drift, *J. Math. Anal. Appl.*, **202** (1996), 604–622. https://doi.org/10.1006/jmaa.1996.0336
4. H. Lin, P. J. Antsaklis, Stability and stabilizability of switched linear systems: A survey of recent results, *IEEE Trans. Automatic Control*, **54** (2009), 308–322. https://doi.org/10.1109/TAC.2008.2012009
5. S. X. Luo, F. Q. Deng, Stabilization of hybrid stochastic systems in the presence of asynchronous switching and input delay, *Nonlinear Anal. Hybrid Syst.*, **32** (2019), 254–266. https://doi.org/10.1016/j.nahs.2018.12.008
6. L. E. Shaikhet, Stability of stochastic hereditary systems with Markov switching, *Theory Stoch. Pro.*, **2** (1996), 180–184.
7. A. R. Teel, A. Subbaraman, A. Sferlazza, Stability analysis for stochastic hybrid systems: A survey, *Automatica*, **50** (2014), 2435–2456. https://doi.org/10.1016/j.automatica.2014.08.006
8. F. B. Zhu, Z. Z. Han, J. F. Zhang, Stability analysis of stochastic differential equations with Markovian switching, *Syst. Control Lett.*, **61** (2012), 1209–1214. https://doi.org/10.1016/j.sysconle.2012.08.013
9. R. Zawiski, Stabilizability of nonlinear infinite dimensional switched systems by measures of noncompactness in the space c_0, *Nonlinear Anal. Hybrid Syst.*, **25** (2017), 79–89. https://doi.org/10.1016/j.nahs.2017.03.004
10. M. J. Anabtawi, Practical stability of nonlinear stochastic hybrid parabolic systems of Itô-type: Vector Lyapunov functions approach, *Nonlinear Anal. Real World Appl.*, **12** (2011), 1386–1400. https://doi.org/10.1016/j.nonrwa.2010.09.029
11. J. H. Bao, X. R. Mao, C. G. Yuan, Lyapunov exponents of hybrid stochastic heat equations, *Syst. Control Lett.*, **61** (2012), 165–172. https://doi.org/10.1016/j.sysconle.2011.10.009

12. L. P. Kadanoff, *Statistical physics: Statics, dynamics and renormalization*, World Scientific, 2000. https://doi.org/ 10.1142/4016
13. A. A. Kwiecinska, Almost sure and moment stability of stochastic partial differential equations, *Probab. Math. Stat.*, **21** (2011), 405–415.
14. B. Xie, The moment and almost surely exponential stability of stochastic heat equations, *Proc. Amer. Math. Soc.*, **136** (2008), 3627–3634. https://doi.org/10.1090/S0002-9939-08-09458-6
15. G. J. Zheng, J. D. Xiong, X. Yu, C. Xu, Stabilization for infinite-dimensional switched linear systems, *IEEE Trans. Automatic Control*, **65** (2020), 5456–5463. https://doi.org/ 10.1109/TAC.2020.2972788
16. Y. C. Liu, Q. D. Zhu, Adaptive neural network asymptotic control design for MIMO nonlinear systems based on event-triggered mechanism, *Inform. Sci.*, **603** (2022), 91–105. https://doi.org/10.1016/j.ins.2022.04.048
17. M. S. Branicky, Multiple Lyapunov functions and other analysis for switched and hybrid systems, *IEEE Trans. Automatic Control*, **43** (1998), 475–482. https://doi.org/ 10.1109/9.664150
18. Z. K. She, B. Xue, Discovering multiple Lyapunov functions for switched hybrid systems, *SIAM J. Control Optim.*, **52** (2014), 3312–3340. https://doi.org/10.1137/130934313
19. R. Shorten, K. S. Narendra, O. Mason, A result on common quadratic Lyapunov functions, *IEEE Trans. Automatic Control,* **48** (2003), 110–113. https://doi.org/10.1109/TAC.2002.806661
20. K. Yosida, *Functional analysis*, 6 Eds., Berlin, Heidelberg: Springer, 1980.
21. L. C. Evans, *Partial differential equations*, American Mathematical Society, 1998.