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Research article

Some variant of Tseng splitting method with accelerated Visco-Cesaro means for monotone inclusion problems

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Abstract: In this paper, we examine the convergence analysis of a variant of Tseng's splitting method for monotone inclusion problem and fixed point problem associated with an infinite family of η-demimetric mappings in Hilbert spaces. The qualitative results of the proposed variant shows strong convergence characteristics under a suitable set of control conditions. We also provide a numerical example to demonstrate the applicability of the variant with some applications.

Keywords: monotone inclusion problem; forward-backward-forward method; viscosity method; Ces*á*ro means method; Nesterov's accelerated method
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1. Introduction

Let the triplet $(\mathcal{H}, < \cdot, \cdot >, \|\cdot\|)$ denote the real Hilbert space with the inner product and induced norm. The classical monotone inclusion problem aims to find

$$s^* \in \mathcal{H} \quad \text{such that } 0 \in As^* + Bs^*, \tag{1.1}$$

where $A \subseteq \mathcal{H} \times \mathcal{H}$ is a multi-valued operator and B is a single-valued operator on $\mathcal{H}$. In the context of monotone operator theory, (1.1) has been largely considered for various problems in signal processing, subgradient algorithms, image recovery problem, variational inequality problem and evolution equations, see [17, 19, 37] and the references cited therein.

In order to study the problem (1.1), one can employ effective iterative algorithms. The elegant forward-backward (FB) iterative algorithm [34, 35] is prominent among various iterative algorithms to solve (1.1). However, the FB iterative algorithm exhibits weak convergence, assuming the stronger conditions for the operators A and B [44]. Recently, Gibali and Thong [23] considered a modified variant of the Tseng's splitting method to obtain strong convergence results in Hilbert spaces.

Fixed point problem (FPP) is another important framework to study a variety of problems arising in various branches of sciences [17, 24, 25]. In 2017, Takahashi [38] proposed and analyzed a new unifying class of nonlinear operators namely the class of η-demimetric operators in Hilbert spaces as follows:

Let C be a nonempty subset of a real Hilbert space $\mathcal{H}$. An operator $W : C \to C$ is said to be η-demimetric [38], where $\eta \in (-\infty, 1)$, if $Fix(W) \neq \emptyset$ such that

$$\langle s - t, (Id - W)s\rangle \geq \frac{1}{2}(1 - \eta)\|(Id - W)s\|^2, \text{ for all } s \in C \text{ and } t \in Fix(W),$$

where Id indicates the identity operator and $Fix(W) = \{t \in C \mid t = Wt\}$ denotes the set of all fixed points of the operator W. Note that

$$\|Ws - t\|^2 \leq \|s - t\|^2 + \eta\|s - Ws\|^2, \text{ for all } s \in C \text{ and } t \in Fix(W),$$

is an equivalent representation of an η-demimetric operator. The class of η-demimetric operators have been studied extensively in various instances of FPP in Hilbert spaces, see [39, 40, 42]. On the other hand, Baillon [13] established the nonlinear ergodic theorem for nonexpansive operator as follows:

Theorem 1.1. *[13] Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $W : C \to C$ be a nonexpansive operator such that $Fix(W) \neq \emptyset$ then for all $s \in C$, the Cesáro means*

$$W_n x = \frac{1}{n+1}\sum_{i=0}^{n} W_i x, \ \forall\, n \in \{0, 1, 2, \cdots, \},$$

weakly converges to a fixed point of W.

Since then, the classical Cesáro means method has been considered for various classes of nonlinear operators, see [18, 28, 29] and the references cited therein. Note that the Cesáro means method fails to converge strongly for the class of nonexpansive operators [22]. In order to establish the strong convergence results, one has to impose additional conditions on the algorithm. In 1967, Halpern [27] introduced and analyzed an iterative algorithm which strongly converges to the nearest fixed point of the nonexpansive operator. It is remarked that the Halpern iterative algorithm coincides with the Cesáro means method for linear operators. In 2000, Moudafi [33] proposed and analyzed the strongly convergent viscosity iterative algorithm by utilizing a strict contraction operator instead of the anchor point in the Halpern iterative algorithm. In order to generalize the classical Cesáro means method for an infinite family of η-demimetric operators, we first consider the following auxiliary operator:

$$\begin{aligned}
Q_{n,n+1} &= Id, \\
Q_{n,n} &= \beta_n T_n^{'} Q_{n,n+1} + (1-\beta_n)Id, \\
Q_{n,n-1} &= \beta_{n-1} T_{n-1}^{'} Q_{n,n} + (1-\beta_{n-1})Id, \\
&\vdots \\
Q_{n,m} &= \beta_m T_m^{'} Q_{n,m+1} + (1-\beta_m)Id, \\
&\vdots \\
Q_{n,2} &= \beta_2 T_2^{'} Q_{n,3} + (1-\beta_2)Id, \\
W_n = Q_{n,1} &= \beta_1 T_1^{'} Q_{n,2} + (1-\beta_1)Id,
\end{aligned}$$

where $0 \leq \beta_m \leq 1$ and $T_m^{'} = \gamma s + (1-\gamma)T_m s$ for all $s \in C$ with T_m being η-demimetric operator and $0 < \gamma < 1-\eta$. It is well-known in the context of operator W_n that each $T_m^{'}$ is nonexpansive and the limit $\lim_{n\to\infty} Q_{n,m}$ exists. Moreover,

$$Ws = \lim_{n\to\infty} W_n s = \lim_{n\to\infty} Q_{n,1} s, \quad \text{for all } \ s \in C.$$

It follows from [41] that

$$Fix(W) = \bigcap_{n=1}^{\infty} Fix(W_n). \tag{1.2}$$

Moreover, to enhance the speed of convergence of the proposed iterative algorithm, we also utilize the inertial extrapolation technique essentially due to Polyak [36], see also [1–11, 31].

The rest of the paper is organized as follows: We present relevant preliminary concepts and results in Section 2. We show the convergence analysis of the proposed iterative algorithm in Section 3 and compute a numerical experiment for the viability of the algorithm in Section 4. Section 5 includes an experiment on image deblurring with applications.

2. Preliminaries

We start this section with the mathematical preliminary concepts required in the sequel. Throughout the paper, we assume the triplet $(\mathcal{H}, < \cdot, \cdot >, \|\cdot\|)$ to be the real Hilbert space with the inner product and induced norm. For a nonempty closed convex subset C of the Hilbert space $\mathcal{H}$, $P_C^{\mathcal{H}}$ denotes the associated metric projection operator which is firmly nonexpansive and satisfies $\langle s - P_C^{\mathcal{H}} s, P_C^{\mathcal{H}} s - t\rangle \geq 0$, for all $s \in \mathcal{H}$ and $t \in C$. Recall that a set-valued operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be monotone, if for all $s, t \in \mathcal{H}$, $u \in Ax$ and $v \in Ay$, we have $\langle s - t, u - v\rangle \geq 0$. Moreover, A is said to be maximal monotone if there is no proper monotone extension of A. For a monotone operator A, the associated resolvent operator J_m^A of index $m > 0$ is defined as

$$J_m^A = (Id + mA)^{-1},$$

where $(\cdot)^{-1}$ indicates the inverse operator. Note that the resolvent operator J_m^A is well defined everywhere on Hilbert space $\mathcal{H}$. Further, J_m^A is single valued and satisfies the firmly nonexpansiveness. Furthermore, $x \in A^{-1}(0)$ if and only if $s \in Fix(J_m^A)$.
The rest of this section is organized with the celebrated results required in the sequel. The following lemma is a special case of Lemma 2.4 in [30].

Lemma 2.1. *Let $\mu, \nu, \bar{\xi} \in \mathcal{H}$. Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma = 1$ then we have*

(i) $\|\mu + \nu\|^2 \leq \|\mu\|^2 + 2\langle \nu, \mu + \nu\rangle$;
(ii) $\|\alpha\mu + \beta\nu + \gamma\bar{\xi}\|^2 = \alpha\|\mu\|^2 + \beta\|\nu\|^2 + \gamma\|\bar{\xi}\|^2 - \alpha\beta\|\mu - \nu\|^2 - \alpha\gamma\|\mu - \bar{\xi}\|^2 - \beta\gamma\|\nu - \bar{\xi}\|^2$.

Lemma 2.2. *[12] Let $W : C \to C$ be an operator defined on a nonempty closed convex subset C of a real Hilbert space $\mathcal{H}$ and let (p_n) be a sequence in C. If $p_n \rightharpoonup p$ and if $(Id - W)p_n \to 0$, then $p \in Fix(W)$*

Lemma 2.3. *[38] Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and let $W : C \to \mathcal{H}$ be an η-demimetric operator with $\eta \in (-\infty, 1)$. Then $Fix(W)$ is closed and convex.*

Lemma 2.4. *[42] Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and let $W : C \to \mathcal{H}$ be an η-demimetric operator with $\eta \in (-\infty, 1)$ and $Fix(W) \neq \emptyset$. Let γ be a real number with $0 < \gamma < 1 - \eta$ and set $T' = (1 - \gamma)Id + \gamma W$, then T' is a quasi-nonexpansive operator of C into $\mathcal{H}$.*

Lemma 2.5. *[15] Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space and $W : C \to C$ be a nonexpansive operator. For each $s \in C$ and the Cesáro means $W_n s = \frac{1}{n+1}\sum_{i=0}^{n} W_i s$, then $\limsup_{n\to\infty} \|W_n s - W(W_n s)\| = 0$.*

Lemma 2.6. *[14] Let $A \subseteq \mathcal{H}\times\mathcal{H}$ be a maximal monotone operator and let B be a Lipschitz continuous and monotone operator on $\mathcal{H}$. Then $A + B$ is a maximal monotone operator.*

Lemma 2.7. *[23] Let $A \subseteq \mathcal{H} \times \mathcal{H}$ be a maximal monotone operator and let B be an operator on $\mathcal{H}$. Define $S_\mu := (Id + \mu A)^{-1}(Id - \mu B), \mu > 0$. Then we have $Fix(S_\mu) = (A + B)^{-1}(0)$, for all $\mu > 0$.*

Lemma 2.8. *[46] Let (b_n) be a sequence of nonnegative real numbers and there exists $n_0 \in \mathbb{N}$ such that*

$$b_{n+1} \leq (1 - \psi_n)b_n + \psi_n c_n + d_n, \ \forall\, n \geq n_0,$$

where $(\psi_n) \subset (0, 1)$ and (c_n), (d_n) with the following conditions hold:

(I) $\sum_{n=1}^{\infty} \psi_n = \infty$;
(II) $\limsup_{n\to\infty} c_n \leq 0$;
(III) $\sum_{n=1}^{\infty} d_n < \infty, \forall\, 0 \leq d_n (0 \leq n)$;

then $\lim_{n\to\infty} b_n = 0$.

Lemma 2.9. *[32] Let (q_n) be a sequence of nonnegative real numbers. Suppose that there is a subsequence (q_{n_j}) of (q_n) such that $q_{n_j} < q_{n_j+1}$ for all $j \in \mathbb{N}$, then there exists a nondecreasing sequence (ε_k) of $\mathbb{N}$ such that $\lim_{k\to\infty} \varepsilon_k = \infty$ and satisfy the following properties such that*

$$q_{\varepsilon_k} \leq q_{\varepsilon_k+1} \text{ and } q_k \leq q_{\varepsilon_k+1},$$

for some large number $k \in \mathbb{N}$. Thus, ε_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $q_n < q_{n+1}$.

3. Main results

In this section, we prove the following strong convergence result.

Theorem 3.1. *Let $A \subseteq \mathcal{H} \times \mathcal{H}$ be a maximal monotone operator and let B be a monotone and ρ-Lipschitz operator for some $\rho > 0$ on a real Hilbert space $\mathcal{H}$. Let W_n be the W-operator and let h be a λ-contraction on $\mathcal{H}$ with $\lambda \in [0, 1)$. Assume that $\Gamma = (A + B)^{-1}(0) \cap Fix(W) \neq \emptyset$, $\mu_1 > 0$, $\sigma \in (0, 1)$, $\{\bar{\xi}_n\} \subset [0, 1)$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0, 1)$. For given $p_0, p_1 \in \mathcal{H}$, let the iterative sequence $\{p_n\}$ be generated by*

$$\begin{cases} u_n = p_n + \bar{\xi}_n(p_n - p_{n-1}); \\ v_n = J^A_{\mu_n}(Id - \mu_n B)u_n; \\ s_n = v_n - \mu_n(Bv_n - Bu_n); \\ p_{n+1} = \alpha_n h(p_n) + (1 - \alpha_n - \beta_n)p_n + \beta_n \frac{1}{n}\sum_{i=0}^{n-1} W_i s_n. \end{cases} \tag{3.1}$$

Assume that the following step size rule

$$\mu_{n+1} = \begin{cases} \min\{\frac{\sigma\|u_n - v_n\|}{\|Bu_n - Bv_n\|}, \mu_n\}, & \text{if } Bu_n - Bv_n \neq 0; \\ \mu_n, & \text{otherwise}, \end{cases}$$

and conditions:

(C1) $\sum_{n=1}^{\infty} \bar{\xi}_n \|p_n - p_{n-1}\| < \infty$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and for each* $n \in \mathbb{N}$, $0 < a^* < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < b^* < 1 - \alpha_n$, *where* a^*, b^* *be positive real numbers,*

hold. Then the sequence $\{p_n\}$ generated by (3.1) converges strongly to an element in Γ.

The following results from [23] are crucial for the analysis of our main result.

Lemma 3.1. *[23] The sequence μ_n generated by (3.1) is a nonincreasing sequence with a lower bound of* $\min\{\mu_1, \frac{\sigma}{\rho}\}$.

Lemma 3.2. *[23] Assume that Conditions (C1) and (C2) hold and let (s_n) be any sequence generated by (3.1), we have*

$$\|s_n - \bar{p}\|^2 \leq \|p_n - \bar{p}\|^2 - (1 - \sigma^2 \frac{\mu_n^2}{\mu_{n+1}^2})\|p_n - v_n\|^2 \tag{3.2}$$

and

$$\|s_n - v_n\| \leq \sigma \frac{\mu_n}{\mu_{n+1}} \|p_n - v_n\|. \tag{3.3}$$

Lemma 3.3. *Assume that Conditions (C1) and (C2) hold and suppose that*

$$\lim_{n\to\infty} \|p_n - u_n\| = \lim_{n\to\infty} \|p_n - v_n\| = \lim_{n\to\infty} \|p_n - s_n\| = \lim_{n\to\infty} \Big\|s_n - \frac{1}{n}\sum_{i=0}^{n-1} W_i s_n\Big\| = 0.$$

Let (p_n) and (u_n) be two sequences generated by (3.1). If a subsequence (p_{n_t}) of p_n converges weakly to some $p^* \in \mathcal{H}$ then $p^* \in \Gamma$.

Proof. Let $p^* \in \mathcal{H}$ such that $p_{n_t} \rightharpoonup p^*$ then $p^* \in (A+B)^{-1}(0)$ follows from [23, Lemma 7]. Since $\lim_{n\to\infty} \|p_n - s_n\| = 0$ and $p_{n_t} \rightharpoonup p^*$ therefore we have $s_{n_t} \rightharpoonup p^*$. Since

$$\lim_{n\to\infty} \|s_n - \frac{1}{n}\sum_{i=0}^{n-1} W_i s_n\| = 0,$$

therefore, utilizing Lemma 2.2, we get $p^* \in Fix(W_i)$ and hence $p^* \in \Gamma$. □

Now we are able to prove the main result of this section.

Proof of Theorem 3.1. For simplicity, the proof is divided into the following three steps:
Step 1. Show that the sequence (p_n) is bounded.
Let $\bar{p} \in \Gamma$, then for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \|u_n - \bar{p}\|^2 &= \|p_n - \bar{p} + \bar{\xi}_n(p_n - p_{n-1})\|^2 \\ &\leq \|p_n - \bar{p}\|^2 + \bar{\xi}_n^2\|p_n - p_{n-1}\|^2 + 2\bar{\xi}_n\langle p_n - \bar{p}, p_n - p_{n-1}\rangle. \end{aligned} \tag{3.4}$$

Set $W_n = \frac{1}{N+1}\sum_{i=0}^{N} W_i$ and utilizing Lemma 2.4 we have

$$\begin{aligned} \|W_n s - W_n t\| &= \Big\|\frac{1}{n}\sum_{i=0}^{n-1} W_i s - \frac{1}{n}\sum_{i=0}^{n-1} W_i t\Big\| \leq \frac{1}{n}\sum_{i=0}^{n-1}\|W_i s - W_i t\| \\ &\leq \frac{1}{n}\sum_{i=0}^{n-1}\|s - t\| = \|s - t\|. \end{aligned}$$

It follows from the above estimate that W_n is a nonexpansive operator. Moreover, for any $\bar{p} \in \Gamma$, we have that $W_n\bar{p} = \frac{1}{n}\sum_{i=0}^{n-1} W_i\bar{p} = \bar{p}$. Since $\lim_{n\to\infty}(1 - \sigma^2\frac{\mu_n^2}{\mu_{n+1}^2}) = 1 - \sigma^2 > 0$, therefore for each $n \geq n_0$ where $n_0 \in \mathbb{N}$, we have that

$$1 - \sigma^2\frac{\mu_n^2}{\mu_{n+1}^2} > 0. \tag{3.5}$$

From (3.2) and (3.5), we obtain

$$\|s_n - \bar{p}\| \leq \|p_n - \bar{p}\|. \tag{3.6}$$

Further, from (C2) and (3.6), we have

$$\begin{aligned} \|p_{n+1} - \bar{p}\| &= \|\alpha_n(h(p_n) - \bar{p}) + (1 - \alpha_n - \beta_n)(p_n - \bar{p}) + \beta_n(W_n s_n - \bar{p})\| \\ &\leq \alpha_n\|h(p_n) - \bar{p}\| + (1 - \alpha_n - \beta_n)\|p_n - \bar{p}\| + \beta_n\|W_n s_n - \bar{p}\| \\ &\leq \alpha_n\lambda\|p_n - \bar{p}\| + \alpha_n\|h(\bar{p}) - \bar{p}\| + (1 - \alpha_n)\|p_n - \bar{p}\| \\ &= [1 - \alpha_n(1 - \lambda)]\|p_n - \bar{p}\| + \alpha_n(1 - \lambda)\frac{\|h(\bar{p}) - \bar{p}\|}{1 - \lambda} \\ &\leq \max\{\|p_n - \bar{p}\|, \frac{\|h(\bar{p}) - \bar{p}\|}{1 - \lambda}\}. \end{aligned}$$

Thus, for all $n \geq n_0$, $\|p_{n+1} - \bar{p}\| \leq \max\{\|p_{n_0} - \bar{p}\|, \frac{\|h(\bar{p}) - \bar{p}\|}{1-\lambda}\}$. This implies that (p_n) is bounded.
Step 2. Compute the following two estimates:

$$(i) : \beta_n(1 - \sigma^2\frac{\mu_n^2}{\mu_{n+1}^2})\|p_n - v_n\|^2 + \beta_n(1 - \alpha_n - \beta_n)\|p_n - W_n s_n\|^2$$

$$\leq \|p_n - \bar{p}\|^2 - \|p_{n+1} - \bar{p}\|^2 + \alpha_n\|h(p_n) - \bar{p}\|^2; \tag{3.7}$$

$$\begin{aligned}(ii) :&\|p_{n+1} - \bar{p}\|^2 \leq [1 - \alpha_n(1 - \lambda)]\|p_n - \bar{p}\|^2 \\ &+ \alpha_n(1 - \lambda)[\frac{2}{1 - \lambda}(\beta_n\|p_n - W_n s_n\|\|p_{n+1} - \bar{p}\| + \langle h(\bar{p}) - \bar{p}, p_{n+1} - \bar{p}\rangle)].\end{aligned} \tag{3.8}$$

Utilizing Lemma 2.1(ii), we obtain

$$\begin{aligned}\|p_{n+1} - \bar{p}\|^2 &= \|\alpha_n(h(p_n) - \bar{p}) + (1 - \alpha_n - \beta_n)(p_n - \bar{p}) + \beta_n(W_n s_n - \bar{p})\|^2 \\ &= \alpha_n\|h(p_n) - \bar{p}\|^2 + (1 - \alpha_n + \beta_n)\|p_n - \bar{p}\|^2 + \beta_n\|(W_n s_n - \bar{p})\|^2 \\ &\quad -\alpha_n(1 - \alpha_n - \beta_n)\|h(p_n) - p_n\|^2 \\ &\quad -\beta_n(1 - \alpha_n - \beta_n)\|p_n - W_n s_n\|^2 - \alpha_n\beta_n\|h(p_n) - W_n s_n\|^2 \\ &\leq \alpha_n\|h(p_n) - \bar{p}\|^2 + (1 - \alpha_n - \beta_n)\|p_n - \bar{p}\|^2 + \beta_n\|s_n - \bar{p}\|^2 \\ &\quad -\beta_n(1 - \alpha_n - \beta_n)\|p_n - W_n s_n\|^2.\end{aligned}$$

Now utilizing (3.2) in the above estimate, we get

$$\begin{aligned}\|p_{n+1} - \bar{p}\|^2 &\leq \alpha_n\|h(p_n) - \bar{p}\|^2 + (1 - \alpha_n)\|p_n - \bar{p}\|^2 - \beta_n(1 - \alpha_n - \beta_n)\|p_n - W_n s_n\|^2 \\ &\quad -\beta_n(1 - \sigma^2\frac{\mu^2}{\mu_{n+1}^2})\|p_n - v_n\|^2 \\ &\leq \alpha_n\|h(p_n) - \bar{p}\|^2 + \|p_n - \bar{p}\|^2 - \beta_n(1 - \alpha_n - \beta_n)\|p_n - W_n s_n\|^2 \\ &\quad -\beta_n(1 - \sigma^2\frac{\mu^2}{\mu_{n+1}^2})\|p_n - v_n\|^2.\end{aligned}$$

Simplifying the above estimate, we have the desired estimate (3.7).
Next, by using (C2) and setting $j_n = (1 - \beta_n)p_n + \beta_n W_n s_n$, we get

$$\|j_n - \bar{p}\| \leq \|p_n - \bar{p}\| \tag{3.9}$$

and

$$\|p_n - j_n\| = \beta_n\|p_n - W_n s_n\|. \tag{3.10}$$

Utilizing (3.9), (3.10), Lemma 2.1(i) and (ii), the desired estimate (3.8) follows from the following calculation:

$$\begin{aligned}&\|p_{n+1} - \bar{p}\|^2 \\ &\quad = \|(1 - \alpha_n)(j_n - \bar{p}) + \alpha_n(h(p_n) - h(\bar{p})) - \alpha_n(p_n - j_n) - \alpha_n(\bar{p} - h(\bar{p}))\|^2 \\ &\quad \leq \|(1 - \alpha_n)(j_n - \bar{p}) + \alpha_n(h(p_n) - h(\bar{p}))\|^2 - 2\alpha_n\langle p_n - j_n + \bar{p} - h(\bar{p}), p_{n+1} - \bar{p}\rangle \\ &\quad \leq (1 - \alpha_n)\|j_n - \bar{p}\|^2 + \alpha_n\|h(p_n) - h(\bar{p})\|^2 - 2\alpha_n\langle p_n - j_n + \bar{p} - h(\bar{p}), p_{n+1} - \bar{p}\rangle \\ &\quad \leq (1 - \alpha_n)\|p_n - \bar{p}\|^2 + \alpha_n\lambda\|p_n - \bar{p}\|^2 + 2\alpha_n\langle p_n - j_n, \bar{p} - p_{n+1}\rangle + 2\alpha_n\langle h(\bar{p}) - \bar{p}, p_{n+1} - \bar{p}\rangle \\ &\quad \leq [1 - \alpha_n(1 - \lambda)]\|p_n - \bar{p}\|^2 + 2\alpha_n\|p_n - j_n\|\|p_{n+1} - \bar{p}\| + 2\alpha_n\langle h(\bar{p}) - \bar{p}, p_{n+1} - \bar{p}\rangle \\ &\quad = [1 - \alpha_n(1 - \lambda)]\|p_n - \bar{p}\|^2 + 2\alpha_n\beta_n\|p_n - W_n s_n\|\|p_{n+1} - \bar{p}\| + 2\alpha_n\langle h(\bar{p}) - \bar{p}, p_{n+1} - \bar{p}\rangle\end{aligned}$$

$$= [1 - \alpha_n(1-\lambda)]\|p_n - \bar{p}\|^2 + \alpha_n(1-\lambda)[\frac{2}{1-\lambda}(\beta_n\|p_n - W_n s_n\|\|p_{n+1} - \bar{p}\| + \langle h(\bar{p}) - \bar{p}, p_{n+1} - \bar{p}\rangle)].$$

Step 3. Show that $\lim_{n\to\infty} \|p_n - \bar{p}\| = 0$.
We consider the two possible cases on the sequence $(\|p_n - \bar{p}\|)$.
Case A. For all $n \geq n_0$, $\|p_{n+1} - \bar{p}\|^2 \leq \|p_n - \bar{p}\|^2$ and $n_0 \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \|p_n - \bar{p}\|$ exists. Since $\lim_{n\to\infty} (1 - \sigma^2 \frac{\mu_n^2}{\mu_{n+1}^2}) = 1 - \sigma^2 > 0$. By using (C2) and (3.7), we have

$$\lim_{n\to\infty} \|p_n - v_n\| = \lim_{n\to\infty} \|p_n - W_n s_n\| = 0. \tag{3.11}$$

From (3.3), we get

$$\lim_{n\to\infty} \|s_n - v_n\| = 0. \tag{3.12}$$

By the definition of (u_n) and (C1), we have

$$\lim_{n\to\infty} \|u_n - p_n\| = \lim_{n\to\infty} \bar{\xi}_n \|p_n - p_{n-1}\| = 0. \tag{3.13}$$

By using the triangle inequality, we obtain the following estimates:

$$\begin{aligned} \|u_n - v_n\| &\leq \|u_n - p_n\| + \|p_n - v_n\| \to 0, \text{ as } n \to \infty; \\ \|u_n - s_n\| &\leq \|u_n - v_n\| + \|v_n - s_n\| \to 0, \text{ as } n \to \infty; \\ \|p_n - s_n\| &\leq \|p_n - v_n\| + \|v_n - s_n\| \to 0, \text{ as } n \to \infty; \\ \|s_n - W_n s_n\| &\leq \|p_n - s_n\| + \|p_n - W_n s_n\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

By using Lemma 2.5, we have

$$\limsup_{n\to\infty} \|W_n s_n - W(W_n s_n)\| = 0. \tag{3.14}$$

Note that for all $n \in \mathbb{N}$, we get

$$\begin{aligned} \|p_{n+1} - p_n\| &\leq \|p_{n+1} - W_n s_n\| + \|p_n - W_n s_n\| \\ &\leq \alpha_n \|h(p_n) - p_n\| + (2 - \beta_n)\|p_n - W_n s_n\|. \end{aligned} \tag{3.15}$$

From (3.11) and (C2), the estimate (3.15) implies that

$$\lim_{n\to\infty} \|p_{n+1} - p_n\| = 0. \tag{3.16}$$

Similarly, from (3.13), (3.16) and the following triangle inequality, we have

$$\|p_{n+1} - u_n\| \leq \|p_{n+1} - p_n\| + \|p_n - u_n\| \to 0, \text{ as } n \to \infty.$$

Since (p_n) is bounded, then there exists a subsequence (p_{n_t}) of (p_n) with $p_{n_t} \rightharpoonup p^* \in \mathcal{H}$. Now utilizing Lemma 3.3 we have $p^* \in \Gamma$.

By making use of the estimate (3.16), we get

$$\limsup_{n\to\infty} \langle h(\bar{p}) - \bar{p}, p_{n+1} - \bar{p}\rangle \leq \limsup_{n\to\infty} \langle h(\bar{p}) - \bar{p}, p_{n+1} - p_n\rangle + \limsup_{n\to\infty} \langle h(\bar{p}) - \bar{p}, p_n - \bar{p}\rangle \leq 0. \tag{3.17}$$

From the estimate (3.17) and Lemma 2.8, we get $\lim_{n\to\infty}\|p_n - \bar{p}\| = 0$.
Case B. There exists a subsequence $(\|p_{n_k} - \bar{p}\|^2)$ of $(\|p_n - \bar{p}\|^2)$ such that $\|p_{n_k} - \bar{p}\| < \|p_{n_k+1} - \bar{p}\|$ for all $k \in \mathbb{N}$.

It follows from Lemma 2.9 that there exists a nondecreasing sequence $(b_m) \in \mathbb{N}$ such that $\lim_{m\to\infty} b_m = \infty$, for all $m \in \mathbb{N}$ with the inequality $\|p_{b_m} - \bar{p}\|^2 \le \|p_{b_m+1} - \bar{p}\|^2$ holds. In the similar fashion from (3.7), we obtain

$$\begin{aligned}
&\beta_{b_m}(1-\sigma^2\frac{\mu_{b_m}^2}{\mu_{b_m+1}^2})\|p_{b_m} - v_{b_m}\|^2 + \beta_{b_m}(1-\alpha_{b_m} - \beta_{b_m})\|p_{b_m} - S_{b_m}w_{b_m}\|^2\\
&\quad\le \|p_{b_m} - \bar{p}\|^2 - \|p_{b_m+1} - \bar{p}\|^2 + \alpha_{b_m}\|h(p_{b_m}) - \bar{p}\|^2\\
&\quad\le \alpha_{b_m}\|h(p_{b_m}) - \bar{p}\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, so we get

$$\lim_{m\to\infty}\|p_{b_m} - v_{b_m}\| = \lim_{m\to\infty}\|p_{b_m} - S_{b_m}w_{b_m}\| = 0.$$

Similarly from Case A, we have

$$\limsup_{m\to\infty}\langle h(\bar{p}) - \bar{p}, p_{b_m+1} - \bar{p}\rangle \le 0.$$

Using (3.8) for $n \ge \max\{n^*, n_0\}$, we have the following estimate:

$$\begin{aligned}
&\|p_{b_m+1} - \bar{p}\|^2\\
&\le [1-\alpha_{b_m}(1-\lambda)]\|p_{b_m} - \bar{p}\|^2 + \alpha_{b_m}(1-\lambda)[\frac{2}{1-\lambda}(\beta_{b_m}\|p_{b_m} - S_{b_m}w_{b_m}\|\|p_{b_m+1} - \bar{p}\| + \langle h(\bar{p}) - \bar{p}, p_{b_m+1} - \bar{p}\rangle)]\\
&\le [1-\alpha_{b_m}(1-\lambda)]\|p_{b_m+1} - \bar{p}\|^2 + \alpha_{b_m}(1-\lambda)[\frac{2}{1-\lambda}(\beta_{b_m}\|p_{b_m} - S_{b_m}w_{b_m}\|\|p_{b_m+1} - \bar{p}\| + \langle h(\bar{p}) - \bar{p}, p_{b_m+1} - \bar{p}\rangle)].
\end{aligned}$$

The above estimate yields that

$$\|p_{b_m+1} - \bar{p}\|^2 \le \frac{2}{1-\lambda}(\beta_{b_m}\|p_{b_m} - S_{b_m}w_{b_m}\|\|p_{b_m+1} - \bar{p}\| + \langle h(\bar{p}) - \bar{p}, p_{b_m+1} - \bar{p}\rangle). \tag{3.18}$$

Therefore, $\limsup_{m\to\infty}\|p_{b_m} - \bar{p}\|^2 \le 0$. Therefore, $p_n \to \bar{p} \in \Gamma$ and this completes the proof. □

We now propose a variant of the iterative algorithm (3.1) embedded with the Halpern iterative algorithm [27].

Theorem 3.2. *Let $A \subseteq \mathcal{H}\times\mathcal{H}$ be a maximal monotone operator and let B be a monotone and ρ-Lipschitz operator for some $\rho > 0$ on a real Hilbert space $\mathcal{H}$. Let W_n be the W-operator and let h be a λ-contraction on $\mathcal{H}$ with $\lambda \in [0,1)$. Assume that $\Gamma = (A+B)^{-1}(0) \cap Fix(W) \neq \emptyset$, $(\mu_1) > 0$, $\sigma \in (0,1)$, $\{\bar{\xi}_n\} \subset [0,1)$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$. For given $q, p_0, p_1 \in \mathcal{H}$, let the iterative sequences $\{p_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{p_{n+1}\}$ be generated by*

$$\begin{cases}
u_n = p_n + \bar{\xi}_n(p_n - p_{n-1});\\
v_n = J_{\mu_n}^A(Id - \mu_n B)u_n;\\
s_n = v_n - \mu_n(Bv_n - Bu_n);\\
p_{n+1} = \alpha_n q + (1-\alpha_n - \beta_n)p_n + \beta_n\frac{1}{n}\sum_{i=0}^{n-1} W_i s_n.
\end{cases} \tag{3.19}$$

Assume that the following step size rule

$$\mu_{n+1} = \begin{cases} \min\{\frac{\sigma\|u_n - v_n\|}{\|Bu_n - Bv_n\|}, \mu_n\}, & if \ Bu_n - Bv_n \neq 0, \\ \mu_n, & otherwise, \end{cases}$$

and conditions

(C1) $\sum_{n=1}^{\infty} \bar{\xi}_n \|p_n - p_{n-1}\| < \infty$;

(C2) $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$, $1 - \alpha_n - \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n} = \infty$;

(C3) For each $n \in \mathbb{N}$, $0 < a^* < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < b^* < 1 - \alpha_n$, *where* a^*, b^* *be positive real numbers hold.*

Then the sequence $\{p_n\}$ *generated by (3.19) converges strongly to a point in* Γ.

Remark 3.1. *In order to obtain the desired result, for the iteration (3.19), we have to assume a stopping criteria for (3.19) to be* $n > n_{max}$ *for some sufficiently large number* n_{max}.

Proof. Observe that for each $n \geq 1$, arguing similarly as in the proof of Theorem 3.1 (Steps 1–3), we deduce that Γ is well defined, closed and bounded. Furthermore, the sequence (p_n) is bounded and

$$\lim_{n\to\infty} \|p_{n+1} - p_n\| = 0. \tag{3.20}$$

Let $p_{n+1} = \alpha_n q + (1 - \alpha_n - \beta_n) p_n + \beta_n W_n s_n$. An easy calculation along (3.20), (C2) and (C3) implies that

$$\|W_n s_n - p_n\| \leq \frac{1}{(\beta_n)} \|p_{n+1} - p_n\| + \frac{\alpha_n}{\beta_n} \|q - p_n\|.$$

The above estimate infers that

$$\lim_{n\to\infty} \|W_n s_n - p_n\| = 0.$$

The rest of the proof of Theorem 3.2 is similar to the proof of Theorem 3.1 and is therefore omitted. □

The following remark elaborate how to align condition (C1) in a computer-assisted iterative algorithm.

Remark 3.2. *We remark here that the condition* (C1) *can easily be aligned in a computer-assisted iterative algorithm since the value of* $\|p_n - p_{n-1}\|$ *is quantified before choosing* $\bar{\xi}_n$ *such that* $0 \leq \bar{\xi}_n \leq \widehat{\bar{\xi}_n}$ *with*

$$\widehat{\bar{\xi}_n} = \begin{cases} \min\{\frac{\Theta_n}{\|p_n - p_{n-1}\|}, \bar{\xi}\}, & if \ p_n \neq p_{n-1}; \\ \bar{\xi}, & otherwise. \end{cases}$$

Here $\{\Theta_n\}$ *denotes a sequence of positives* $\sum_{n=1}^{\infty} \Theta_n < \infty$ *and* $\bar{\xi} \in [0, 1)$.

4. Example and numerical results

In this section, we compute a numerical experiment for the viability of the iterative algorithm (3.1).

Example 4.1. Let $\mathcal{H} = \mathbb{R}$. We denote the inner product $\langle s, t\rangle = st$, for all $s, t \in \mathbb{R}$ and induced norm $|s| = \sqrt{\langle s, t\rangle}$. Let the operators $h, A, B : \mathbb{R} \to \mathbb{R}$ be defined as $h(s) = \frac{s}{8}$, $As = 4s$ and $Bs = 3s$ for all $s \in \mathbb{R}$. Observe that, h is a contraction with constant $\lambda \in [0, 1)$, B is a monotone and ρ-Lipschitz operator for some $\rho > 0$ and A is a maximal monotone operator such that $(A + B)^{-1}(0) = \{0\}$. Let the sequence of operators $T_i : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_i(s) = \begin{cases} -\frac{3s}{i}, & s \in (-\infty, 1); \\ s, & s \in (1, \infty). \end{cases}$$

Note that T_i is an infinite family of $\frac{3-i^2}{(3+i)^2}$-demimetric operators with $\bigcap_{i=1}^{\infty} Fix(T_i) = 0 = Fix(W)$. Hence $\Gamma = (A + B)^{-1}(0) \cap Fix(W) = 0$. In order to compute the numerical values of (p_n), we choose $\Theta = 0.5$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2(n+1)}$, $\mu_1 = 7.45$ and $\sigma = 0.785$. Since

$$\begin{cases} \min\{\frac{1}{n^2\|p_n - p_{n-1}\|}, 0.5\}, & if \quad p_n \neq p_{n-1}; \\ 0.5, & \text{otherwise.} \end{cases}$$

We now provide a numerical test for a comparison between accelerated Tseng's type splitting method defined in Theorem 3.1 (i.e., Theorem 3.1, $\bar{\xi}_n \neq 0$), standard Tseng's type splitting method (i.e., Theorem 3.1, $\bar{\xi}_n = 0$), Algorithm 1 [31] and Theorem 2 [23]. The stopping criteria is defined as $E_n = \|v_n - u_n\| < 10^{-5}$. Table 1 summarises the comparison of these algorithm with respect to the following choices of initial inputs:

Choice 1. $p_0 = 4$, $p_1 = 4.5$.

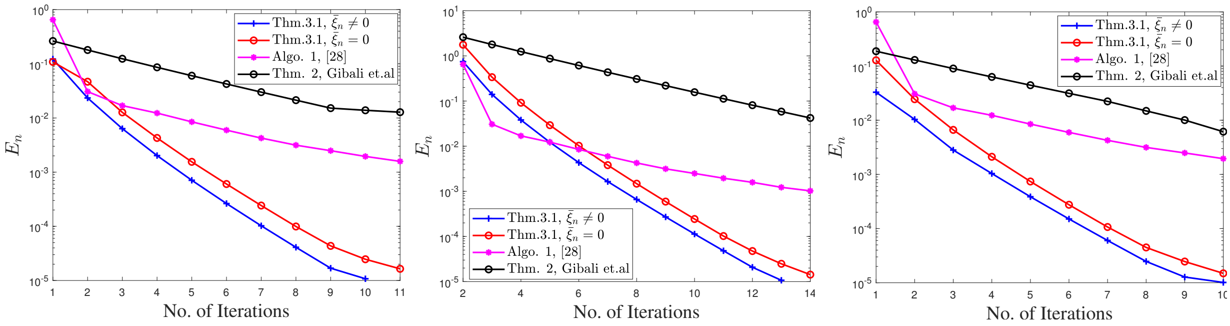
Choice 2. $p_0 = 5$, $p_1 = -3$.

Choice 3. $p_0 = -1.3$, $p_1 = -4.7$.

Table 1. Numerical results for Example 4.1.

	Choice 1		Choice 2		Choice 3	
	Iteration	CPU(s)	Iteration	CPU(s)	Iteration	CPU(s)
(1) Theorem 3.1, $\bar{\xi}_n \neq 0$	11	0.053120	14	0.051362	10	0.048537
(2) Theorem 3.1, $\bar{\xi}_n = 0$	17	0.060018	19	0.058867	16	0.057642
(3) Algorithm 1 [31]	27	0.068117	37	0.069215	33	0.065345
(4) Theorem 2, Gibali et al.	36	0.074537	45	0.077642	38	0.068804

The error plotting E_n of $\bar{\xi}_n \neq 0$ and $\bar{\xi}_n = 0$ for each choice in Table 1 are shown in Figure 1.



(a) Choice 1 (b) Choice 2 (c) Choice 3

Figure 1. Comparison of Theorem 3.1 for $\bar{\xi}_n \neq 0$ and $\bar{\xi}_n = 0$ with Theorem 2 [23].

We can see from Table 1 and Figure 1 that the Theorem 3.1 with $\bar{\xi}_n \neq 0$ performs better as compared to the Theorem 3.1 with $\bar{\xi}_n = 0$, Algorithm 1 [31] and Theorem 2 [23].

5. Applications

In this section, we demonstrate some theoretical as well as applied instances of the main result in Section 3.

5.1. Split feasibility problem

The classical split feasibility problem (SFP), essentially due to Censor and Elfving [16], aims to find $\hat{s} \in \omega := C \cap h^{-1}(Q) = \{\bar{t} \in C : h\bar{t} \in Q\}$, where $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ are nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. In order to derive the result for SFP from Theorem 3.1, we recall the indicator operator of a nonempty, closed and convex subset C of $\mathcal{H}_1$ as

$$\Phi_C(s^*) := \begin{cases} 0, & s^* \in C; \\ \infty, & otherwise. \end{cases}$$

It is well known that the subdifferential $\partial\Phi_C$ associated with Φ_C is a maximal monotone operator. Recall also that $\partial\Phi_C = \mathcal{N}(\mu, C)$, where $\mathcal{N}(\mu, C)$ is the normal cone of C at μ. Utilizing this fact, we conclude that the resolvent operator of $\partial\Phi_C$ is the metric projection operator of $\mathcal{H}_1$ onto C. Setting $B(\bar{x}) = \hbar^*(Id - P_Q)\hbar\bar{x}$, where P_Q is the metric projection onto Q and $A(\bar{x}) = \partial\Phi_C(\bar{x})$ then the SCFP has the inclusion structure as defined in (1.1). Since B is ρ-Lipschitz continuous, where $\rho = \|\hbar\|^2 = 1$ and A is maximal monotone, (see [12]), we, therefore, arrive at the following variant of Theorem 3.1:

Theorem 5.1. *Assume that $\Gamma = \omega \cap Fix(W) \neq \emptyset$. For given $p_0, p_1 \in \mathcal{H}_1$, let the iterative sequence (p_n) be generated by*

$$\begin{cases} u_n = p_n + \bar{\xi}_n(p_n - p_{n-1}); \\ v_n = P_C(Id - \mu_n\hbar^*(Id - P_Q)\hbar)u_n; \\ s_n = v_n - \mu_n((\hbar^*(Id - P_Q)\hbar)v_n - (\hbar^*(Id - P_Q)\hbar)u_n); \\ p_{n+1} = \alpha_n h(p_n) + (1 - \alpha_n - \beta_n)p_n + \beta_n \frac{1}{n}\sum_{i=0}^{n-1} W_i s_n. \end{cases} \tag{5.1}$$

Assume that the following step size rule:

$$\mu_{n+1} = \begin{cases} \min\{\frac{\sigma\|u_n - v_n\|}{\|(\hbar^*(Id-P_Q)\hbar)u_n - (\hbar^*(Id-P_Q)\hbar)v_n\|}, \mu_n\} & if\ (\hbar^*(Id - P_Q)\hbar)u_n - (\hbar^*(Id - P_Q)\hbar)v_n \neq 0; \\ \mu_n, & otherwise, \end{cases}$$

and conditions (C1) and (C2) hold. Then the sequence (p_n) generated by (5.1) converges strongly to an element in Γ.

5.2. Convex minimization problems

Let $\mathfrak{f} : \mathcal{H} \to \mathbb{R} \cup (+\infty)$ and $\mathfrak{g} : \mathcal{H} \to \mathbb{R} \cup (+\infty)$ be two convex, proper and lower semicontinuous functions such that $\mathfrak{f}$ differentiable with ρ-Lipschitz continuous gradient and $\mathfrak{g}$ is such that its proximal map. We consider the following convex minimization problem of finding $\bar{x} \in \mathcal{H}$ such that

$$\mathfrak{f}(\bar{x}) + \mathfrak{g}(\bar{x}) = \min_{x\in\mathcal{H}} \{\mathfrak{f}(x) + \mathfrak{g}(x)\}. \tag{5.2}$$

In view of the Fermat's rule, the problem (5.2) is equivalent to the following problem of finding $\bar{x} \in \mathcal{H}$ such that

$$0 \in \nabla\mathfrak{f}(\bar{x}) + \partial\mathfrak{g}(\bar{x}), \tag{5.3}$$

where the subdifferential $\partial\mathfrak{g}$ is a maximal monotone operator and the gradient $\nabla\mathfrak{f}$ is ρ-Lipschitz continuous [12,37]. Assume that ω is the set of solutions of problem (1.1) and $\omega \neq \emptyset$. In Theorem 3.1, set that $B := \nabla\mathfrak{f}$ and $A := \partial\mathfrak{g}$. Then, we compute the following result.

Theorem 5.2. *Let $\mathfrak{f} : \mathcal{H} \to \mathbb{R} \cup (+\infty)$, $\mathfrak{g} : \mathcal{H} \to \mathbb{R} \cup (+\infty)$ be two proper, convex and lower semicontinuous functions on a real Hilbert space $\mathcal{H}$. Assume that $\Gamma = \omega \cap \bigcap_{i=1}^{\infty} Fix(W_i) \neq \emptyset$ and $\bar{\xi}_n$ is a bounded real sequence. For given $p_0, p_1 \in \mathcal{H}$, let the iterative sequences (p_n) be generated by*

$$\begin{cases} u_n = p_n + \bar{\xi}_n(p_n - p_{n-1}); \\ v_n = J_{\mu_n}^{\partial\mathfrak{g}}(Id - \mu_n\nabla\mathfrak{f})u_n; \\ s_n = v_n - \mu_n(\nabla\mathfrak{f}v_n - \nabla\mathfrak{f}u_n); \\ p_{n+1} = \alpha_n h(p_n) + (1 - \alpha_n - \beta_n)p_n + \beta_n\frac{1}{n}\sum_{i=0}^{n-1} W_i s_n. \end{cases} \tag{5.4}$$

Assume that the following step size rule

$$\mu_{n+1} = \begin{cases} \min\{\frac{\sigma\|u_n-v_n\|}{\|\nabla\mathfrak{f}u_n-\nabla\mathfrak{f}v_n\|}, \mu_n\}, & \text{if } \nabla\mathfrak{f}u_n - \nabla\mathfrak{f}v_n \neq 0; \\ \mu_n, & \text{otherwise}, \end{cases}$$

and the conditions (C1) and (C2) hold. Then the sequence (p_n) generated by (5.4) converges strongly to an element in Γ.

5.3. Application to image processing problems

Let $\mathbf{h} \in \mathbb{R}^{n\times m}$ be a blurring operator, $z \in \mathbb{R}^n$ be the original image and $\mathbf{w} \in \mathbb{R}^m$ be the blurred and noisy image (observed image) with v be the additive noise from $\mathbb{R}^m$. The following structure is known as an image recovery problem:

$$\mathbf{h}z = \mathbf{w} + v.$$

For solving this problem, we make use of the model of Tibshirani [43] which is known as LASSO problem:

$$\min_{z\in\mathbb{R}^n}\Big\{\frac{1}{2}\|\mathbf{h}z - \mathbf{w}\|_2^2 + \Bbbk\|z\|_1\Big\}, \tag{5.5}$$

where $\Bbbk > 0$ is a regularization parameter. Problem (5.5) cannot be used to solve the image deblurring directly, as the image is sparse under some gradient transformation. In order to reconstruct the images from their noisy, blurry and/or incomplete measurements, Guo et al. [26] proposed a novel regularization model for reproducing high-quality images using fewer measurements than the state-of-the-art methods. We therefore use the following model:

$$\min_{z\in\mathbb{R}^n}\left\{\frac{1}{2}\|\mathbf{h}z-\mathbf{w}\|_2^2+\Bbbk\|\nabla z\|_1\right\}. \tag{5.6}$$

The Richardson iteration, which is often called the Landweber method [20, 21, 45], is generally used as an iterative regularization method to solve (5.6). This method is defined as follows:

$$z_{k+1}=z_k+\rho\mathbf{h}^T(\mathbf{w}-\mathbf{h}z_k), \tag{5.7}$$

where ρ step size is constant. To ensure the convergence, the step size satisfy $0<\rho<\frac{2}{\epsilon_{\max}^2}$ and $\epsilon_{\max}$ is the largest singular value of $\mathbf{h}$. We set $\Bbbk = 0.7875$ and $\mu = 0.001$, $\xi_n = \frac{1}{(100*n+1)^2}$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{1}{88n+1}$. The quality of the the restored images are analyzed on the following scale of signal to noise ratio (SNR) defined as $SNR = 20\log_{10}\frac{\|z\|^2}{\|z-z_n\|^2}$, where z and z_n are the original and estimated images at iteration n, respectively. We compare the performance of the algorithms abbreviated as Theorem 5.1, $\bar{\xi}_n \neq 0$, Theorem 5.1, $\bar{\xi}_n = 0$, Algorithm 1 [31] and Theorem 2 of Gibali et al. [23] on the test images (Mona Lisa and Cameraman) via the image restoration experiment for motion operator, respectively.

It can be observed from Figures 3 and 5 that the larger SNR values infer the better restored images. We can see from Table 2, and the corresponding test images in Figures 2 and 4, that the inertial variant of the iterative algorithm in Theorem 5.1 (i.e., $\bar{\xi}_n \neq 0$) performs better as compared to the non-inertial variant (i.e., $\bar{\xi}_n = 0$) Algorithm 1 [31] and Theorem 2 of Gibali and Thong [23].

(a) Original image **(b)** Observed image **(c)** Reconstructed image

Figure 2. (a) Original image (182 × 276) with a motion length 30 and an angle 45; (b) Observed image, degraded by motion; (c) Reconstructed image.

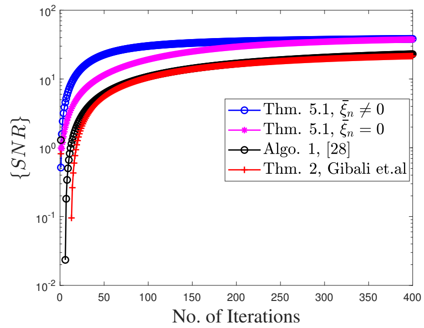


Figure 3. Comparison of (5.1), $\bar{\xi}_n \neq 0$, (5.1), $\bar{\xi}_n = 0$ and Theorem 2 [23].

(a) Original image **(b)** Observed image **(c)** Reconstructed image

Figure 4. (a) Original image (256×256)with Gaussian blur of size 9×9 and standard deviation $\sigma = 6$; (b) Observed image, degraded by Gaussian; (c) Reconstructed image.

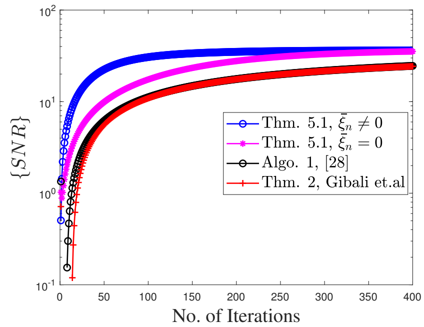


Figure 5. Comparison of (5.1), $\bar{\xi}_n \neq 0$, (5.1), $\bar{\xi}_n = 0$ and Algorithm 2 [23].

Table 2. The SNR in decibel(dB) values and average per iteration computation time of the two optimization algorithms.

	Mona Lisa		Cameraman	
No. of test image	SNR(dB)	CPU(sec)	SNR(dB)	CPU(sec)
(1) Theorem 5.1, $\bar{\xi}_n \neq 0$	38.3032	30.1321	34.9918	20.1326
(2) Theorem 5.1, $\bar{\xi}_n = 0$	37.5156	26.3298	27.7731	17.0077
(3) Algorithm 1 [31]	29.3231	25.4876	21.8794	15.6142
(4) Theorem 2 of Gibali and Thong	22.3231	23.1861	17.6226	13.0051

6. Conclusions

In this paper, we have devised an accelerated Visco-Cesáro means Tseng's type splitting method for computing a common solution of a monotone inclusion problem and the FPP associated with an infinite family of η-demimetric operators in Hilbert spaces. We have incorporated an appropriate numerical example for the viability the iterative algorithm. We have also included some theoretical, as well as applied instances, of the main result in Section 3 that can provide an important future research direction in these theories.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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