A general hybrid relaxed CQ algorithm for solving the fixed-point problem and split-feasibility problem

Yuanheng Wang*, Bin Huang, Bingnan Jiang, Tiantian Xu and Ke Wang

School of Mathematic Science, Zhejiang Normal University, Jinhua 321004, China

* Correspondence: Email: yhwang@zjnu.cn; Tel: +08657982298258.

Abstract: In this paper, we introduce a new hybrid relaxed iterative algorithm with two half-spaces to solve the fixed-point problem and split-feasibility problem involving demicontractive mappings. The strong convergence of the iterative sequence produced by our algorithm is proved under certain weak conditions. We give several numerical experiments to demonstrate the efficiency of the proposed iterative method in comparison with previous algorithms.

Keywords: inertial iterative method; demicontractive mapping; fixed point; hybrid CQ method; split-feasibility problem

Mathematics Subject Classification: 47H04, 47H09, 47H10, 65K10

1. Introduction

The split feasibility problem (SFP) in Euclidean spaces was introduced by Censor and Elfving [1] in 1994. It is depicted as finding a point \( x \) such that

\[
x \in C, \quad A x \in Q,
\]

where \( C \subseteq H_1 \) and \( Q \subseteq H_2 \) are nonempty closed convex sets; \( A \) is a bounded linear operator from a Hilbert space \( H_1 \) onto a Hilbert space \( H_2 \) with \( A \neq 0 \). Denote the set of solutions for (1.1) by \( \Delta \). The SFP plays an important role in the study of medical image reconstruction, signal processing, etc. Many scholars regard the SFP and its generalizations, such as the multiple-set SFP and the split common-fixed-point problem as their research direction (see [2–6]).

It is known that Byrne [7, 8] proposed the famous CQ algorithm (CQA) for solving the problem (1.1). The proposed method is given as follows:

\[
x_{n+1} = P_C(x_n - \tau_n A^*(I - P_Q)Ax_n), \quad \forall n \geq 1,
\]

\[ (1.2) \]
where \( \tau_n \in \left(0, \frac{2}{\|A\|^2}\right) \) is the step size, \( A^* \) is the adjoint operator of \( A \), \( P_C \) is the projection onto \( C \) and \( P_Q \) is the projection onto \( Q \). Regrettably, there are two drawbacks of applying (1.2). On one hand, estimating the operator norm can often be challenging. On the other hand, computing projections onto both \( C \) and \( Q \) can be very difficult.

To overcome the first drawback, Lópe et al. [9] proposed a novel approach for selecting the step size, denoted by \( \tau_n \), which is defined as follows:

\[
\tau_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2},
\]

where \( f(x_n) := \frac{1}{2}\| (I-P_Q)Ax_n \|^2 \), \( \nabla f(x_n) = A^*(I-P_Q)Ax_n \), \( \rho_n \in (0,4) \) and \( \inf_{n \in \mathbb{N}} \rho_n (4 - \rho_n) > 0 \).

To overcome the second drawback, Fukushima [10] introduced the level sets \( C \) and \( Q \):

\[
C = \{ x \in H_1 : \phi(x) \leq 0 \}, \quad Q = \{ y \in H_2 : \varphi(y) \leq 0 \}, \tag{1.3}
\]

where \( \phi : H_1 \to \mathbb{R} \) and \( \varphi : H_2 \to \mathbb{R} \) are convex, subdifferential and weakly lower semi-continuous functions. Motivated by Fukushima’s method, Yang [11] introduced a new relaxed CQA by replacing \( P_C \) with \( P_{C_n} \) and \( P_Q \) with \( P_{Q_n} \). The algorithm has the form

\[
x_{n+1} = P_{C_n}(x_n - \alpha_n A^*(I-P_{Q_n})Ax_n), \quad \forall n \geq 1,
\]

where \( \alpha_n \in (0, \frac{2}{\|A^*\|^2}) \), \( A^* \) is the adjoint operator of \( A \) and \( C_n \) and \( Q_n \) are defined as follows:

\[
C_n = \{ x \in H_1 : \phi(x_n) \leq \langle \vartheta_n, x_n - x \rangle \}, \quad \vartheta_n \in \partial \phi(x_n),
\]

and

\[
Q_n = \{ y \in H_2 : \varphi(Ax_n) \leq \langle \chi_n, Ax_n - y \rangle \}, \quad \chi_n \in \partial \varphi(Ax_n),
\]

where \( \partial \phi \) and \( \partial \varphi \) are bounded operators.

Qin and Wang [12] introduced (1.4) to find a common solution of the SFP (1.1) and fixed-point problem in 2019. Their algorithm has the form

\[
\begin{aligned}
\left\{ \begin{array}{l}
y_n = P_C((1-\delta_n)(x_n - \tau_n A^*(I-P_Q)Ax_n) + \delta_n S x_n), \\
x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \gamma_n y_n, \quad n \geq 1,
\end{array} \right.
\end{aligned}
\tag{1.4}
\]

where \( g \) is a \( k \)-contraction from \( C \) onto \( C \), \( S \) is nonexpansive from \( C \) onto \( C \) and \( \text{Fix}(S) := \{ x \in H : Sx = x \} \). \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \) and \( \{\tau_n\} \) are included in \( (0,1) \); they satisfy the following:

(i) \( \sum_{n=1}^{\infty} \alpha_n = \infty \), \( \lim_{n \to \infty} \alpha_n = 0 \);  
(ii) \( 0 < \lim_{n \to \infty} \inf \beta_n \leq \lim_{n \to \infty} \sup \beta_n < 1 \);  
(iii) \( \alpha_n + \beta_n + \gamma_n = 1 \);  
(iv) \( \lim_{n \to \infty} |\tau_n - \tau_{n+1}| = 0, 0 < \lim_{n \to \infty} \inf \tau_n \leq \lim_{n \to \infty} \sup \tau_n < \frac{2}{\|A\|^2} \);  
(v) \( \lim_{n \to \infty} |\delta_n - \delta_{n+1}| = 0, 0 < \lim_{n \to \infty} \inf \delta_n \leq \lim_{n \to \infty} \sup \delta_n < 1 \).
Then, there is a point \( q \) of \( \Delta \cap \text{Fix}(S) \) with \( x_n \to q \). Furthermore, it satisfies (1.5):

\[
\langle z' - q, g(q) - q \rangle \leq 0, \quad \forall z' \in \text{Fix}(S) \cap \Delta. \tag{1.5}
\]

Motivated by López et al. [9], Wang et al. [13] improved the selection of step size in (1.4). In order to accelerate the convergence rate, they introduced (1.6) to solve the SFP (1.1) and fixed-point problem. Given that \( x_0, x_1 \in H \), their algorithm has the form

\[
\mu_n = \begin{cases} 
\min \left\{ \mu, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|} \right\}, & \text{if } x_{n-1} \neq x_n, \\
\mu, & \text{otherwise,}
\end{cases}
\]

where \( \mu \geq 0 \), \( \tau_n = \frac{\rho_n f(w_n)}{\|\nabla f(w_n)\|^2} \), \( 0 < \rho_n < 4 \), \( f(w_n) := \frac{1}{2}\|I - P_Q\|A w_n \|^2 \) and \( \nabla f(w_n) = A^*(I - P_Q)A x_n \). Assume that \( S : H_1 \to H_1 \) is quasi-nonexpansive, \( I - S \) is demiclosed at zero and \( g : H_1 \to H_1 \) is a \( k \)-contraction. \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\delta_n\} \) are included in \([0, 1]\), which satisfy the following conditions:

(i) \( \sum_{n=1}^{\infty} \alpha_n = \infty \), \( \lim_{n \to \infty} \alpha_n = 0 \);
(ii) \( \limsup_{n \to \infty} \beta_n < 1 \);
(iii) \( \alpha_n + \beta_n + \gamma_n = 1 \);
(iv) \( 0 < \liminf_{n \to \infty} \delta_n \leq \limsup_{n \to \infty} \delta_n < 1 \);
(v) \( \varepsilon_n > 0 \), \( \lim_{n \to \infty} \frac{\varepsilon_n}{\varepsilon_n} = 0 \).

Then, there is a point \( q \) of \( \text{Fix}(S) \cap \Delta \) with \( x_n \to q \). Furthermore, it satisfies (1.7):

\[
\langle z' - q, g(q) - q \rangle \leq 0, \quad \forall z' \in \text{Fix}(S) \cap \Delta. \tag{1.7}
\]

In addition, Tian [14] introduced an iterative form:

\[
x_{n+1} = \alpha_n \lambda g(x_n) + (I - \lambda \alpha_n B) S x_n, \quad \forall n \in \mathbb{N}.
\]

Suppose that \( \text{Fix}(S) \neq \emptyset \), where \( \text{Fix}(S) := \{x \in H : S x = x\} \). Then, there is a point \( q \) in \( \text{Fix}(S) \) with \( x_n \to q \). Furthermore, it satisfies the following:

\[
\langle (\lambda B - \lambda g) q, z' - q \rangle \geq 0, \quad \forall z' \in \text{Fix}(S),
\]

where \( B \) is \( \lambda \)-strongly monotone and \( M \)-Lipschitz continuous from \( H \) onto \( H \) with \( \lambda, M > 0 \), \( S \) is nonexpansive from \( H \) onto \( H \) and \( g \) is a \( k \)-contraction from \( H \) onto \( H \) with \( 0 < k < 1 \). Let \( \lambda, \lambda \in \mathbb{R} \) and \( \{\alpha_n\} \subset (0, 1) \) satisfy the following:

(i) \( 0 < \lambda < \frac{2\lambda}{M^2} \);
(ii) \( 0 < \lambda < \frac{\lambda}{\lambda^2} \frac{M^2}{k} \);
Motivated by the algorithms developed by Qin and Wang [12], Wang et al. [13], Tian [14] and Kwari et al. [15], we propose a new half-space relaxation projection method for solving the SFP and fixed-point problem of demicontractive mappings by expanding the scope of operator $S$, modifying the iteration format and optimizing the selection of step sizes. Our result in this article extends and improves many recent correlative results of other authors. Particularly, our method extends and improves the methods in some papers of other authors in the following ways: (1) We have considered and studied a new modified half-space hybrid method to solve the fixed-point problem and SFP of nonlinear operators at the same time. And, our method can be used more widely; (2) We extend the nonexpansive mapping and the quasi-nonexpansive mapping to the demicontractive mapping, which expands the scope of the study; (3) The inertia is added to accelerate the convergence rate further; (4) The selection of the step size is a multistep and self-adaptive process, so it no longer depends on the operator norm; (5) The projection onto a half-space greatly facilitates the calculation, and under some weaker conditional assumptions, the iterative sequence generated by our new algorithm converges strongly to the common solution of the two problems. At last, we present some numerical experiments to show the effectiveness and feasibility of our new iterative method.

2. Preliminaries

In this paper, let $H$ be a real Hilbert space; the inner product be denoted by $\langle \cdot, \cdot \rangle$ and the norm be denoted by $\| \cdot \|$. We use the symbol $\rightharpoonup$ to indicate weak convergence and $\rightarrow$ to indicate strong convergence.

**Definition 2.1.** A self-mapping $P$ is said to be

(i) nonexpansive if

\[ \| Px - Py \| \leq \| x - y \|, \quad \forall x, y \in H; \]

(ii) quasi-nonexpansive if $\text{Fix}(P) \neq \emptyset$ such that

\[ \| Px - y \| \leq \| x - y \|, \quad \forall x \in H, y \in \text{Fix}(P); \]

(iii) $\alpha$-demicontractive ($0 \leq \alpha < 1$) if $\text{Fix}(P) \neq \emptyset$ such that

\[ \| Px - y \|^2 \leq \alpha \| (I - P)x \|^2 + \| x - y \|^2, \quad \forall x \in H, y \in \text{Fix}(P); \]

(iv) firmly nonexpansive if

\[ \| Px - Py \|^2 \leq \langle Px - Py, x - y \rangle, \quad \forall x, y \in H; \]

(v) $t$-contractive ($0 \leq t < 1$) if

\[ \| Px - Py \| \leq t \| x - y \|, \quad \forall x, y \in H; \]

(vi) $M$-Lipschitz continuous ($M > 0$) if

\[ \| Px - Py \| \leq M \| x - y \|, \quad \forall x, y \in H; \]
(vii) \( \beta \)-strongly monotone (\( \beta \geq 0 \)) if

\[
\langle Px - Py, x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in H.
\]

When \( \text{Fix}(P) \neq \emptyset \), we obtain (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii).

**Definition 2.2.** When the function \( f : H \rightarrow \mathbb{R} \) is convex and subdifferentiable, an element \( d \in H \) is called a subgradient of \( f(x_0) \) if

\[
f(y) \geq f(x_0) + \langle d, y - x_0 \rangle, \quad \forall y \in H.
\]

The set of subgradients of \( f \) at \( x_0 \) is denoted by \( \partial f(x_0) \).

**Definition 2.3.** The function \( f : H \rightarrow \mathbb{R} \) is called weakly lower semi-continuous if \( x_n \rightarrow x_0 \) implies that

\[
f(x_0) \leq \liminf_{n \rightarrow \infty} f(x_n).
\]

**Lemma 2.1** ([16–18]). Let \( D \subset H \) be a nonempty, convex and closed set. For any \( x \in H \), we have

(i) \( \|P_Dx - y\|^2 \leq \|x - y\|^2 - \|x - P_Dx\|^2 \), \( \forall y \in D \);

(ii) \( \langle x - P_Dx, y - P_Dx \rangle \leq 0 \), \( \forall y \in D \);

(iii) \( \|P_Dx - P_Dz\|^2 \leq \langle P_Dx - P_Dz, x - y \rangle \), \( \forall y \in H \);

(iv) \( \|(I - P_D)x - (I - P_D)y\|^2 \leq \langle (I - P_D)x - (I - P_D)y, x - y \rangle \), \( \forall y \in H \).

**Lemma 2.2** ([19, 20]). For any \( x, y \in H \), the following results hold

(i) \( \|x + y\|^2 \leq 2\|x\| + \|y\|^2 \);

(ii) \( \|(1 - m)x + my\|^2 = m\|y\|^2 + (1 - m)\|x\|^2 - (1 - m)m\|x - y\|^2 \), \( m \in \mathbb{R} \).

**Lemma 2.3** ([14]). Assume that \( B \) is an \( M \)-Lipschitz continuous and \( \overline{\lambda} \)-strongly monotone operator with \( \overline{\lambda}, M > 0 \). For \( \mu, \nu > 0 \), they satisfy the following conditions:

\[
0 < \mu < \frac{2\overline{\lambda}}{M^2}, \quad \nu = \overline{\lambda} - \frac{M^2\mu}{2}.
\]

Let \( \{\gamma_n\} \subset (0, 1) \) and \( \lim_{n \rightarrow \infty} \gamma_n = 0 \). If \( n \) is a sufficiently large number, we have

\[
\|(I - \gamma_nB)x - (I - \gamma_nB)y\|^2 \leq (1 - \gamma_n\nu)\|x - y\|^2.
\]

**Proof.**

\[
\|(I - \gamma_nB)x - (I - \gamma_nB)y\|^2 \\
= \|x - y - \gamma_n(Bx - By)\|^2 \\
= \|x - y\|^2 - 2\gamma_n\langle x - y, Bx - By \rangle + \gamma_n^2\|Bx - By\|^2 \\
\leq \|x - y\|^2 - 2\gamma_n\overline{\lambda}\|x - y\|^2 + \gamma_n^2M^2\|x - y\|^2 \\
= (1 - 2\gamma_n\overline{\lambda} + \gamma_n^2M^2)\|x - y\|^2 \\
= (1 - 2\gamma_n\nu - \gamma_nM^2\mu + \gamma_n^2M^2)\|x - y\|^2 \\
\leq (1 - 2\gamma_n\nu - \gamma_nM^2(\mu - \gamma_n) + \gamma_n^2\nu^2)\|x - y\|^2.
\]
Combining \( \lim_{n \to \infty} \gamma_n = 0 \) and \( \mu - \gamma_n > 0 \) \((n \text{ sufficiently large})\), we have

\[
\|(I - \gamma_n B)x - (I - \gamma_n B)y\|^2 \\
\leq (1 - 2\gamma_n \nu + \gamma_n^2 \nu^2)\|x - y\|^2 \\
= (1 - \gamma_n \nu)^2\|x - y\|^2.
\]

Since \( 1 - \gamma_n \nu > 0 \) \((n \text{ sufficiently large})\), we have

\[
\|(I - \gamma_n B)x - (I - \gamma_n B)y\| \leq (1 - \gamma_n \nu)\|x - y\|.
\]

\(\square\)

**Lemma 2.4** ([21]). Let the self-mapping \( S \) be \( \alpha \)-demicontractive in \( H \) and \( \text{Fix}(S) \neq \emptyset \). Define \( S_{\sigma} = (1 - \sigma)I + \sigma S, \sigma \in (0, 1 - \alpha); \) then, we have the following:

(i) \( \text{Fix}(S) = \text{Fix}(S_{\sigma}) \subset H \) is convex and closed;

(ii) \( \|S_{\sigma}x - q\|^2 \leq \|x - q\|^2 - \frac{1}{\sigma}(1 - \alpha - \sigma)\|(I - S_{\sigma})x\|^2 \) \( \forall x \in H, q \in \text{Fix}(S) \).

**Lemma 2.5** ([8]). Let \( f(z) = \frac{1}{2}\|(I - P_Q)Az\|^2 \). We can know that \( \nabla f \) is \( M \)-Lipschitz continuous with \( M = \|A\|^2 \).

**Lemma 2.6** ([22]). Let \( \{a_n\} \) be a sequence of nonnegative numbers such that

\[
a_{n+1} \leq (1 - \Theta_n)a_n + \Theta_n \gamma_n, \quad n \geq n_0,
\]

\[
a_{n+1} \leq a_n - \Pi_n + \Gamma_n, \quad n \geq n_0,
\]

where \( n_0 \) is a sufficiently large integer, the sequence \( \{\Theta_n\} \) is contained in \((0, 1)\), the real sequence \( \{\Pi_n\} \)

is nonnegative \((\gamma_n)\) and \((\Gamma_n)\) are two sequences which are included in \(\mathbb{R} \). If the following conditions hold

(i) \( \sum_{n=0}^{\infty} \Theta_n = \infty \); (ii) \( \lim_{n \to \infty} \Gamma_n = 0 \); (iii) \( \lim_{n \to \infty} \Pi_n = 0 \Rightarrow \lim_{n \to \infty} \gamma_n \leq 0, \forall \{n_i\} \subseteq \{n\} \), then \( \lim_{n \to \infty} a_n = 0 \).

### 3. Results

In this section, \( C \) and \( Q \) are defined by (1.3). We give some conditions for the convergence analysis of our Algorithm 1.

1. \( \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0, 0 < \alpha_n < 1; \)
2. \( \beta_n \in [0, 1], \lim \sup_{n \to \infty} \beta_n < 1; \)
3. \( \epsilon_n > 0, \lim_{n \to \infty} \epsilon_n = 0; \)
4. \( 0 < \lim \inf_{n \to \infty} \delta_n \leq \lim \sup_{n \to \infty} \delta_n < 1; \)
5. \( \alpha_n \in (0, 1 - \alpha], \lim \inf_{n \to \infty} \alpha_n > 0; \)
6. \( \inf_{n \in \mathbb{N}} \rho_n (4 - \rho_n) > 0, \{\rho_n\} \subset (0, 4). \)
Algorithm 1

Initialization: Apply $x_0, x_1 \in H_1, \mu \geq 0, \tau > 0, 0 < \epsilon < \frac{2\lambda}{L^2}, \nu = \frac{\lambda - L^2}{2}, \lambda > 0$ such that $0 \leq k\lambda < \nu$.

Iterative step: Compute $x_{n+1}$ for $n \geq 1$:

Step 1. Compute

$$d_n = x_n + \mu_n(x_n - x_{n-1}),$$

where

$$\mu_n = \begin{cases} \min \left\{ \mu, \frac{\epsilon}{\|x_n - x_{n-1}\|} \right\}, & \text{if } x_{n-1} \neq x_n, \\ \mu, & \text{otherwise}. \end{cases}$$

Step 2. Compute

$$y_n = P_{C_n}(1 - \delta_n)(d_n - \tau_n A^*(I - P_{Q_n})A d_n) + \delta_n((1 - \sigma_n)d_n + \sigma_n S d_n),$$

where

$$\tau_n = \begin{cases} \frac{\rho_n f_n(d_n)}{\|\nabla f_n(d_n)\|^2}, & \text{if } \|\nabla f_n(d_n)\| \neq 0, \\ \rho_n, & \text{if } \|\nabla f_n(d_n)\| = 0, \end{cases}$$

$$f_n(d_n) = \frac{1}{2}\|((I - P_{Q_n})A d_n)^2, \quad \nabla f_n(d_n) = A^*(I - P_{Q_n})A d_n,$$

and

$$C_n = \{x \in H_1 : \phi(x_n) \leq \langle \partial_n, x_n - x \rangle\}, \quad \text{where } \partial_n \in \partial \phi(x_n),$$

$$Q_n = \{y \in H_2 : \varphi(A x_n) \leq \langle \chi_n, A x_n - y \rangle\}, \quad \text{where } \chi_n \in \partial \varphi(A x_n).$$

Step 3. Compute

$$x_{n+1} = \alpha_n \lambda g(x_n) + \beta_n d_n + ((1 - \beta_n)I - \alpha_n B)y_n.$$
Theorem 3.1. Let \( S : H_1 \to H_1 \) be \( \alpha \)-demicontractive, \( I - S \) be demicontracted at zero and \( g : H_1 \to H_1 \) be \( \lambda \)-contractive. Moreover, let operator \( B \) be \( L \)-Lipschitz continuous and \( \bar{\lambda} \)-strongly monotone. Assume that \( \{\alpha_n\}, \{\beta_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\} \) are sequences satisfying \((C_1)-(C_6)\) and \( \text{Fix}(S) \cap \Delta \neq \emptyset \). Then, the \( \{x_n\} \) generated by Algorithm 1 converges strongly to \( x^* \in \text{Fix}(S) \cap \Delta \). Furthermore, it is the unique solution of (3.1):

\[
\langle (B - \lambda g)x^*, z' - x^* \rangle \geq 0, \quad \forall z' \in \text{Fix}(S) \cap \Delta.
\] (3.1)

Proof. Let \( r \in \Delta \cap \text{Fix}(S) \). Since \( C \subset C_n \) and \( Q \subset Q_n \), we have that \( r \in C_n, Ar \in Q_n \) and \( r \in \text{Fix}(S) \). By combining Lemma 2.1 with Lemma 2.2, we get

\[
\begin{align*}
\|y_n - r\|^2 &= \left\|P_{C_n}\left((1 - \delta_n)(d_n - \tau_nA^*(I - P_{Q_n})Ad_n) + \delta_n(1 - \sigma_n)d_n + \sigma_nSd_n\right) - r\right\|^2 \\
&\leq \left\|((1 - \delta_n)(d_n - \tau_nA^*(I - P_{Q_n})Ad_n) + \delta_n(1 - \sigma_n)d_n + \sigma_nSd_n) - r\right\|^2 \\
&\quad - \left\|P_{C_n}\left((1 - \delta_n)(d_n - \tau_nA^*(I - P_{Q_n})Ad_n) + \delta_n(1 - \sigma_n)d_n + \sigma_nSd_n\right)\right\|^2 \\
&\quad + \delta_n(1 - \sigma_n)d_n + \sigma_nSd_n)\right\|^2 \\
&= \left\|\delta_n((1 - \sigma_n)d_n + \sigma_nSd_n - r) + (1 - \delta_n)(d_n - \tau_nA^*(I - P_{Q_n})Ad_n - r)\right\|^2 \\
&\quad - \left\|P_{C_n}\left((1 - \delta_n)(d_n - \tau_nA^*(I - P_{Q_n})Ad_n) + \delta_n(1 - \sigma_n)d_n + \sigma_nSd_n\right)\right\|^2.
\end{align*}
\]

Set

\[ S_{\sigma_n}d_n = (1 - \sigma_n)d_n + \sigma_nSd_n; \]

according to Lemma 2.4, we have

\[
\begin{align*}
\|y_n - r\|^2 &\leq \delta_n\|S_{\sigma_n}d_n - r\|^2 + (1 - \delta_n)\|d_n - \tau_n\nabla f_n(d_n) - r\|^2 \\
&\quad - \delta_n(1 - \delta_n)\|\sigma_n(Sd_n - d_n) + \tau_nA^*(I - P_{Q_n})Ad_n\|^2 \\
&\quad - \left\|P_{C_n}\left((1 - \delta_n)(d_n - \tau_nA^*(I - P_{Q_n})Ad_n) + \delta_n(1 - \sigma_n)d_n + \sigma_nSd_n\right)\right\|^2 \\
&\leq \delta_n\|d_n - r\|^2 + (1 - \delta_n)\|d_n - \tau_n\nabla f_n(d_n) - r\|^2 \\
&\quad - \delta_n(1 - \delta_n)\|\sigma_n(Sd_n - d_n) + \tau_nA^*(I - P_{Q_n})Ad_n\|^2 \\
&\quad - \left\|P_{C_n}\left((1 - \delta_n)(d_n - \tau_nA^*(I - P_{Q_n})Ad_n) + \delta_n(1 - \sigma_n)d_n + \sigma_nSd_n\right)\right\|^2 \\
&\leq \delta_n\|d_n - r\|^2 + (1 - \delta_n)(\|d_n - r\|^2 + \tau_n^2\|\nabla f_n(d_n)\|^2 \\
&\quad - 2\tau_n\langle \nabla f_n(d_n), d_n - r\rangle) \\
&\quad - \delta_n(1 - \delta_n)\|\sigma_n(Sd_n - d_n) + \tau_nA^*(I - P_{Q_n})Ad_n\|^2.
\end{align*}
\]
\[-\left\| (I - P_{C_n}) \left( (1 - \delta_n)(d_n - \tau_n A^*(I - P_{\Omega_n}) A d_n) \right) \right\|^2 - \delta_n((1 - \sigma_n)d_n + \sigma_n S d_n) \right\|^2.

From the definition of \( \nabla f_n(d_n) \), we have
\[
\langle \nabla f_n(d_n), d_n - r \rangle
= \langle A^*(I - P_{\Omega_n}) A d_n - A^*(I - P_{\Omega_n}) A r, d_n - r \rangle
= \langle (I - P_{\Omega_n}) A d_n - (I - P_{\Omega_n}) A r, A d_n - A r \rangle
\geq \| (I - P_{\Omega_n}) A d_n \|^2
= 2 f_n(d_n),
\] (3.3)

which means that when \( \| \nabla f_n(d_n) \| = 0 \), we get that \( f_n(d_n) = 0 \). Substituting (3.3) into (3.2), we have
\[
\| y_n - r \|^2 \leq \| d_n - r \|^2 + \tau_n^2(1 - \delta_n)\| \nabla f_n(d_n) \|^2 - 4(1 - \delta_n)\tau_n f_n(d_n)
- \delta_n((1 - \sigma_n)d_n + \sigma_n S d_n) \left\| (I - P_{C_n}) \left( (1 - \delta_n)(d_n - \tau_n A^*(I - P_{\Omega_n}) A d_n) \right) \right\|^2
\leq \| d_n - r \|^2 - (4 - \rho_n)(1 - \delta_n)\tau_n f_n(d_n)
- \delta_n((1 - \sigma_n)d_n + \sigma_n S d_n) \left\| (I - P_{C_n}) \left( (1 - \delta_n)(d_n - \tau_n A^*(I - P_{\Omega_n}) A d_n) \right) \right\|^2.
\] (3.4)

Noting that \( \{ \delta_n \} \) is a sequence in \((0, 1)\) and \( \rho_n \in (0, 4) \), we derive that
\[
\| y_n - r \| \leq \| d_n - r \|. \tag{3.5}
\]

From the definition of \( d_n \), we have
\[
\| d_n - r \|
= \| x_n + \mu_n(x_n - x_{n-1}) - r \|
\leq \mu_n\| x_n - x_{n-1} \| + \| x_n - r \|
\leq \varepsilon_n + \| x_n - r \|. \tag{3.6}
\]

In the rest of the proof, \( n_0 \) is assumed to be a sufficiently large positive integer and \( n \geq n_0 \).
Now, let \( \gamma_n := \frac{\alpha_n}{1 - \beta_n} \). By Lemma 2.3, (3.5) and (3.6), we estimate \( \|x_{n+1} - r\| \):

\[
\|x_{n+1} - r\| = \|\alpha_n \lambda g(x_n) + \beta_n d_n + ((1 - \beta_n)I - \alpha_n B)y_n - r\| \\
= \|\alpha_n \lambda g(x_n) + \beta_n d_n + ((1 - \beta_n)I - \alpha_n B)y_n - r + \alpha_n Br - \alpha_n Br\| \\
\leq \alpha_n \|\lambda g(x_n) - Br\| + \beta_n \|d_n - r\| + (1 - \beta_n)\|((I - \gamma_n B)y_n - (I - \gamma_n B)r\| \\
\leq \alpha_n \|\lambda g(x_n) - \lambda g(r)\| + \alpha_n \|\lambda g(r) - Br\| + \beta_n \|d_n - r\| \\
+ (1 - \beta_n)(1 - \gamma_n \nu)\|y_n - r\| \\
\leq \alpha_n \lambda k\|x_n - r\| + \alpha_n \|\lambda g(r) - Br\| + \beta_n \|d_n - r\| \\
+ (1 - \beta_n)(1 - \gamma_n \nu)\|y_n - r\| \\
= \alpha_n \lambda k\|x_n - r\| + \alpha_n \|\lambda g(r) - Br\| + \beta_n \|d_n - r\| \\
\leq (1 - \alpha_n(\nu - \lambda k))\|x_n - r\| + \alpha_n \|\lambda g(r) - Br\| + (1 - \alpha_n \nu)\|\nu - r\| \\
\leq (1 - \alpha_n(\nu - \lambda k))\|x_n - r\| + \alpha_n \|\lambda g(r) - Br\| + (1 - \alpha_n \nu)\|\nu - r\| \\
\leq \max \left\{ \|x_n - r\|, \frac{\|\lambda g(r) - Br\| + M}{\nu - \lambda k} \right\} \\
\leq \max \left\{ \|x_n - r\|, \frac{\|\lambda g(r) - Br\| + M}{\nu - \lambda k} \right\}.
\]

We can assume that \( M \) is a suitable positive number such that \( \frac{\alpha_n}{\alpha_n} \leq M \) since \( \lim_{n \to \infty} \frac{\alpha_n}{\alpha_n} = 0 \). Next, we have

\[
\|x_{n+1} - r\| \\
\leq (1 - \alpha_n(\nu - \lambda k))\|x_n - r\| + \alpha_n(\nu - \lambda k)\left( \frac{\|\lambda g(r) - Br\|}{\nu - \lambda k} + \frac{M(1 - \alpha_n \nu)}{\nu - \lambda k} \right) \\
\leq (1 - \alpha_n(\nu - \lambda k))\|x_n - r\| + \alpha_n(\nu - \lambda k)\left( \frac{\|\lambda g(r) - Br\| + M}{\nu - \lambda k} \right) \\
\leq \max \left\{ \|x_n - r\|, \frac{\|\lambda g(r) - Br\| + M}{\nu - \lambda k} \right\} \\
\leq \max \left\{ \|x_n - r\|, \frac{\|\lambda g(r) - Br\| + M}{\nu - \lambda k} \right\}.
\]

Therefore, the sequence \( \{x_n\} \) is bounded.

From the definition of \( x_{n+1} \),

\[
x_{n+1} = \beta_n d_n + (1 - \beta_n)(\gamma_n \lambda g(x_n) + (I - \gamma_n B)y_n);
\]

set

\[
z_n := \gamma_n \lambda g(x_n) + (I - \gamma_n B)y_n; \quad (3.7)
\]

then, we obtain

\[
x_{n+1} = \beta_n d_n + (1 - \beta_n)z_n. \quad (3.8)
\]
In view of the arbitrariness of \( r \), Lemma 2.2, Lemma 2.3, (3.5) and (3.7), we have

\[
\|z_n - x^*\|^2 \\
= \|\gamma_n \lambda g(x_n) + (I - \gamma_n B)y_n - x^*\|^2 \\
= \|\gamma_n \lambda g(x_n) - Bx^*\|^2 + \|(I - \gamma_n B)y_n - (I - \gamma_n B)x^*\|^2 \\
= \gamma_n^2 \|\lambda g(x_n) - Bx^*\|^2 + \|(I - \gamma_n B)y_n - (I - \gamma_n B)x^*\|^2 \\
+ 2\gamma_n \langle \lambda g(x_n) - Bx^*, (I - \gamma_n B)y_n - (I - \gamma_n B)x^* \rangle \\
\leq \gamma_n^2 \|\lambda g(x_n) - Bx^*\|^2 + \|(I - \gamma_n B)y_n - x^*\|^2 \\
+ 2\gamma_n \langle \lambda g(x_n) - Bx^*, y_n - x^* \rangle - 2\gamma_n^2 \|\lambda g(x_n) - Bx^*, B y_n - B x^* \rangle \\
\leq \gamma_n^2 \|\lambda g(x_n) - Bx^*\|^2 + \|(I - \gamma_n B)y_n - x^*\|^2 \\
+ 2\gamma_n \langle \lambda g(x_n) - Bx^*, y_n - x^* \rangle + 2\gamma_n^2 \|\lambda g(x_n) - Bx^*\|\|y_n - Bx^*\| \\
\leq \gamma_n^2 \|\lambda g(x_n) - Bx^*\|^2 + \|(I - \gamma_n B)y_n - x^*\|^2 \\
+ 2\gamma_n \langle \lambda g(x_n) - \lambda g(x^*), y_n - x^* \rangle + 2\gamma_n \langle \lambda g(x^*), y_n - x^* \rangle + 2\gamma_n^2 \|\lambda g(x_n) - Bx^*\|\|y_n - Bx^*\| \\
\leq \gamma_n^2 \|\lambda g(x_n) - Bx^*\|^2 + \|(I - \gamma_n B)y_n - x^*\|^2 \\
+ 2\gamma_n \langle \lambda g(x_n) - Bx^*, y_n - x^* \rangle + 2\gamma_n \lambda k \|x_n - x^*\|\|d_n - x^*\| \\
+ 2\gamma_n \langle \lambda g(x^*), y_n - x^* \rangle + 2\gamma_n^2 \|\lambda g(x_n) - Bx^*\|\|y_n - Bx^*\| \\
= (1 - 2\gamma_n \nu)\|d_n - x^*\|^2 + 2\gamma_n \lambda k \|x_n - x^*\|\|d_n - x^*\| \\
+ 2\gamma_n \langle \lambda g(x^*), y_n - x^* \rangle + 2\gamma_n^2 \|\lambda g(x_n) - Bx^*\|\|y_n - Bx^*\| \\
+ \gamma_n^2 \|\lambda g(x_n) - Bx^*\|^2 + \gamma_n^2 \nu^2 \|d_n - x^*\|^2.
\]

(3.9)

According to the definition of \( \{d_n\} \) and Lemma 2.2, we have

\[
\|d_n - x^*\|^2 \\
= \|x_n + \mu_n(x_n - x_{n-1}) - x^*\|^2 \\
\leq 2\mu_n\|x_n - x_{n-1}, d_n - x^*\| + \|x_n - x^*\|^2 \\
\leq 2\mu_n\|x_n - x_{n-1}\|\|d_n - x^*\| + \|x_n - x^*\|^2 \\
\leq 2\epsilon_n\|d_n - x^*\| + \|x_n - x^*\|^2.
\]

(3.10)

Next, we can estimate \( \|x_{n+1} - x^*\|^2 \). On the one hand, in view of (3.6), (3.8), (3.9), (3.10) and Lemma 2.2, we obtain

\[
\|x_{n+1} - x^*\|^2 \\
= \|\beta_n d_n + (1 - \beta_n)z_n - x^*\|^2 \\
= \|(1 - \beta_n)(z_n - x^*) + \beta_n(d_n - x^*)\|^2 \\
= (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n\|d_n - x^*\|^2 - \beta_n(1 - \beta_n)\|d_n - z_n\|^2 \\
\leq (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n\|d_n - x^*\|^2 \\
\leq \beta_n\|d_n - x^*\|^2 + (1 - \beta_n)(1 - 2\gamma_n \nu)\|d_n - x^*\|^2.
\]
\[ +2(1 - \beta_n)\gamma_n \lambda k \|x_n - x^\star\|d_n - x^\star\| + 2\gamma_n(1 - \beta_n)\langle \lambda g(x^\star) - Bx^\star, y_n - x^\star\rangle \\
+2\gamma_n^2(1 - \beta_n)\|\lambda g(x_n) - Bx^\star\|\|B y_n - Bx^\star\| \\
+\gamma_n^2(1 - \beta_n)\|\lambda g(x_n) - Bx^\star\|^2 + \gamma_n^2(1 - \beta_n)\nu^2\|d_n - x^\star\|^2 \]
\[ = (1 - 2\alpha_n \nu)\|d_n - x^\star\|^2 + 2\alpha_n \nu k \|x_n - x^\star\|d_n - x^\star\| \\
+2\alpha_n \nu \|\lambda g(x^\star) - Bx^\star, y_n - x^\star\| + 2\alpha_n \gamma_n \|\lambda g(x_n) - Bx^\star\|\|B y_n - Bx^\star\| \\
+\alpha_n \gamma_n \|\lambda g(x_n) - Bx^\star\|^2 + \alpha_n \gamma_n \nu^2\|d_n - x^\star\|^2 \leq (1 - 2\alpha_n \nu)\|d_n - x^\star\|^2 + 2\alpha_n \nu \|x_n - x^\star\| + 2\alpha_n \nu k \|x_n - x^\star\|d_n - x^\star\| + 2\epsilon_n \|x_n - x^\star\| \\
+2\alpha_n \nu \|\lambda g(x^\star) - Bx^\star, y_n - x^\star\| + 2\alpha_n \gamma_n \|\lambda g(x_n) - Bx^\star\|\|B y_n - Bx^\star\| \\
+\alpha_n \gamma_n \|\lambda g(x_n) - Bx^\star\|^2 + \alpha_n \gamma_n \nu^2\|d_n - x^\star\|^2 \leq (1 - 2\alpha_n \nu (v - \lambda k))\|x_n - x^\star\|^2 + \alpha_n(2\|\lambda g(x^\star) - Bx^\star, y_n - x^\star\| \\
+2\alpha_n \nu \|\lambda g(x_n) - Bx^\star\|\|B y_n - Bx^\star\| + \gamma_n \|\lambda g(x_n) - Bx^\star\|^2 \\
+\gamma_n \nu^2\|d_n - x^\star\|^2 + \frac{2\epsilon_n}{\alpha_n} \|d_n - x^\star\| + \frac{4\epsilon_n}{\alpha_n} \|x_n - x^\star\| \]
\[ = (1 - 2\alpha_n (v - \lambda k))\|x_n - x^\star\|^2 + 2\alpha_n (v - \lambda k)\left(\frac{\langle \lambda g(x^\star) - Bx^\star, y_n - x^\star\rangle}{v - \lambda k}\right) \\
+\frac{\gamma_n \lambda g(x_n) - Bx^\star\|B y_n - Bx^\star\|}{v - \lambda k} + \frac{\gamma_n \|\lambda g(x_n) - Bx^\star\|^2}{2(v - \lambda k)} \\
+\frac{\gamma_n \nu^2\|d_n - x^\star\|^2}{2(v - \lambda k)} + \frac{\epsilon_n}{\alpha_n(v - \lambda k)} \|d_n - x^\star\| + \frac{\epsilon_n}{\alpha_n(v - \lambda k)} \|x_n - x^\star\| \]. (3.11)

On the other hand, using the definition of \{z_n\}, Lemma 2.2, (3.4) and (3.10), we get
\[ \|x_{n+1} - x^\star\|^2 \]
\[ \leq (1 - \beta_n)\|z_n - x^\star\|^2 + \beta_n \|d_n - x^\star\|^2 \]
\[ = (1 - \beta_n)\|\gamma_n \lambda g(x_n) + (I - \gamma_n B)y_n - x^\star\|^2 + \beta_n \|d_n - x^\star\|^2 \]
\[ = (1 - \beta_n)\|\gamma_n (y_n - x^\star) + \gamma_n(B \lambda g(x_n) - By_n)\|^2 + \beta_n \|d_n - x^\star\|^2 \]
\[ \leq (1 - \beta_n)(\|y_n - x^\star\|^2 + 2\gamma_n \langle \lambda g(x_n) - By_n, z_n - x^\star\rangle) + \beta_n \|d_n - x^\star\|^2 \]
\[ \leq \beta_n \|d_n - x^\star\|^2 + (1 - \beta_n)\left(\|d_n - x^\star\|^2 - (4 - \rho_n)\|\tau_n f_n(d_n)\| \right) \\
-\delta_n(1 - \delta_n)\|\sigma_n(Sd_n - d_n) + \tau_n A^*(I - P_{Q_n})Ad_n\|^2 \\
-\|\|I - P_{C_n}(1 - \delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n \| \right) \\
+\delta_n((1 - \sigma_n)\|d_n + \sigma_n Sd_n\|\|^2) + 2\alpha_n \langle \lambda g(x_n) - By_n, z_n - x^\star\rangle \]
\[ \leq \|x_n - x^\star\|^2 + 2\epsilon_n \|d_n - x^\star\| - (1 - \beta_n)(4 - \rho_n)(1 - \delta_n)\|\tau_n f_n(d_n)\| \]
\[ - (1 - \beta_n)\delta_n (1 - \delta_n)\|\sigma_n(Sd_n - d_n) + \tau_n A^*(I - P_{Q_n})Ad_n\|^2 \\
- (1 - \beta_n)\|\|I - P_{C_n}(1 - \delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n \| \right). \]
\[ + \delta_n (1 - \sigma_n) d_n + \sigma_n S d_n \right\|^2 + 2 \alpha_n (\lambda g(x_n) - B y_n, z_n - x^*) . \] (3.12)

Apply the following:

\[
\begin{align*}
\Theta_n &= 2 \alpha_n (v - \lambda k), \\
\Upsilon_n &= \frac{1}{v - \lambda k} \left( \langle \lambda g(x^*) - B x^*, y_n - x^* \rangle + y_n \| \lambda g(x_n) - B x^* \| \| B y_n - B x^* \| \\
&\quad + \frac{\gamma_n \| \lambda g(x_n) - B x^* \|^2}{2} + \frac{\gamma_n^2 \| d_n - x^* \|^2}{2} + \frac{\varepsilon_n \| d_n - x^* \|}{\alpha_n} \right), \\
\Pi_n &= (1 - \beta_n) (4 - \rho_n) (1 - \delta_n) \tau_n f_n(d_n) \\\n&\quad + (1 - \beta_n) \delta_n (1 - \delta_n) \| \sigma_n (S d_n - d_n) + \tau_n A^* (I - P_{Q_n}) A d_n \|^2 \\
&\quad + (1 - \beta_n) \| (I - P_{C_{Q_n}}) \| (1 - \delta_n) (d_n - \tau_n A^* (I - P_{Q_n}) A d_n) \\\n&\quad + \delta_n (1 - \sigma_n) d_n + \sigma_n S d_n \right\|^2, \\
\Gamma_n &= 2 \alpha_n (\lambda g(x_n) - B y_n, z_n - x^*) + 2 \varepsilon_n \| d_n - x^* \|.
\end{align*}
\]

Thus, we can rewrite (3.11) and (3.12):

\[
\begin{align*}
\| x_{n+1} - x^* \|^2 &\leq (1 - \Theta_n) \| x_n - x^* \|^2 + \Theta_n \Upsilon_n, \\
\| x_{n+1} - x^* \|^2 &\leq \| x_n - x^* \|^2 - \Pi_n + \Gamma_n.
\end{align*}
\]

It is easy for us to know that \( \lim_{n \to \infty} \Theta_n = 0 \), \( \sum_{n=1}^{\infty} \Theta_n = \infty \) and \( \lim \Gamma_n = 0 \). If we prove that \( \lim \sup \nabla \Gamma_{n_i} \leq 0 \) when \( \lim \Pi_{n_i} = 0 \) for any subsequence \( \{n_i\} \subset \{n\} \), we can get that \( \lim_{n \to \infty} \| x_n - x^* \| = 0 \) by Lemma 2.6.

Assume that

\[
\lim_{i \to \infty} \Pi_{n_i} = 0.
\]

Using the conditions of \( \{\beta_n\} \), \( \{\delta_n\} \) and \( \{\sigma_n\} \), we have

\[
\lim_{i \to \infty} (4 - \rho_n) \tau_n f_n(d_{n_i}) = 0, \quad (3.13)
\]

\[
\lim_{i \to \infty} \| \sigma_n (S d_{n_i} - d_{n_i}) + \tau_n A^* (I - P_{Q_{n_i}}) A d_{n_i} \|^2 = 0, \quad (3.14)
\]

\[
\lim_{i \to \infty} \| (I - P_{C_{Q_{n_i}}}) (1 - \delta_{n_i}) (d_{n_i} - \tau_n A^* (I - P_{Q_{n_i}}) A d_{n_i}) \| = 0. \quad (3.15)
\]

From (3.13) and the conditions of \( \{\rho_n\} \), we deduce that

\[
\tau_n f_n(d_{n_i}) \to 0. \quad (3.16)
\]

If \( \| \nabla f_n(d_{n_i}) \| \neq 0 \), we have that \( f_n(d_{n_i}) \neq 0 \) and

\[
\begin{align*}
\tau_n &= \frac{\rho_n f_n(d_{n_i})}{\| \nabla f_n(d_{n_i}) \|^2} \\
&= \frac{\rho_n \| (I - P_{Q_{n_i}}) A d_{n_i} \|^2}{2 \| A^* (I - P_{Q_{n_i}}) A d_{n_i} \|^2}
\end{align*}
\]
\[
\begin{align*}
\frac{\rho_n}{2\|A\|^2} \geq \frac{\rho_n\| (I - P_{Q_n}) Ad_n \|_2^2}{2\|A\|^2\| (I - P_{Q_n}) Ad_n \|_2^2} \\
= \frac{\rho_n}{2\|A\|^2},
\end{align*}
\]
which means that \( \inf_{i \in \mathbb{N}} \tau_n > 0 \). Hence, we obtain \( f_n(d_n) \to 0 \) as \( i \to \infty \). This implies that

\[
\lim_{i \to \infty} \| (I - P_{Q_n}) Ad_n \| = 0.
\]

Using (3.16) and the conditions of \( \{\rho_n\} \), we have

\[
\tau_n\| \nabla f_n(d_n) \| = \sqrt{\rho_n \tau_n f_n(d_n)} \to 0.
\]

Moreover, from (3.14) and the condition of \( \{\sigma_n\} \), we get

\[
\| S d_n - d_n \| \to 0.
\]

Using (3.15) and the definition of \( y_n \), we have

\[
y_n = P_{C_n}(1 - \delta_n)(d_n - \tau_n A^*(I - P_{Q_n}) Ad_n) + \delta_n((1 - \sigma_n)d_n + \sigma_n S d_n);
\]

again using (3.15), we obtain

\[
\| (1 - \delta_n)(d_n - \tau_n \nabla f_n(d_n)) + \delta_n S \sigma_n d_n - y_n \| \to 0,
\]

i.e.,

\[
\| (1 - \delta_n)d_n - (1 - \delta_n)\tau_n \nabla f_n(d_n) + \delta_n S \sigma_n d_n - y_n \| \to 0.
\]

Due to (3.17) and (3.19), we get

\[
\| (1 - \delta_n)d_n + \delta_n((1 - \sigma_n)d_n + \sigma_n S d_n) - y_n \| \to 0,
\]

i.e.,

\[
\| d_n - y_n + \delta_n \sigma_n (S d_n - d_n) \| \to 0.
\]

Then, we have

\[
\| d_n - y_n \|
= \| d_n - y_n + \delta_n \sigma_n (S d_n - d_n) - \delta_n \sigma_n (S d_n - d_n) \|
\leq \| d_n - y_n + \delta_n \sigma_n (S d_n - d_n) \| + \| \delta_n \sigma_n (S d_n - d_n) \|;
\]

combining (3.18) with (3.20), we obtain

\[
\| d_n - y_n \| \to 0.
\]

Moreover, using the definition of \( \{\mu_n\} \) and the boundedness of \( \{x_n\} \), we can have

\[
\| x_n - d_n \|
= \| x_n - x_n - \mu_n(x_n - x_{n-1}) \|.
\]
$$= \mu_n ||x_{n_i} - x_{n_{i-1}}||$$
$$\leq \varepsilon_{n_i} \to 0.$$  (3.22)

Next, combining (3.21) and (3.22), we have
$$||x_n - y_n|| \leq ||x_n - d_n|| + ||d_n - y_n|| \to 0.$$  (3.23)

It is easy for us to know that \(\omega_n(d_n) \subseteq \text{Fix}(S)\) by combining (3.18) and the case that \(I - S\) is demiclosed at zero. Now, we choose a subsequence \(\{d_{n_j}\}\) of \(\{d_n\}\) to satisfy
$$\limsup_{i \to \infty} (\lambda g(x^*) - Bx^*, d_{n_j} - x^*) = \limsup_{j \to \infty} (\lambda g(x^*) - Bx^*, d_{n_j} - x^*) = \liminf_{j \to \infty} (\lambda g(x^*) - Bx^*, d_{n_j} - x^*).$$

Without loss of generality, we suppose that \(d_{n_j} \rightharpoonup \xi^*\). It is easy to get that \(y_{n_j} \rightharpoonup \xi^*\) and \(x_{n_j} \rightharpoonup \xi^*\) by using (3.21) and (3.23). Now, we show that \(\xi^* \in C\). We know that \(y_{n_j} \in C_{n_j}\) by the definition of \(y_{n_j}\); this implies that
$$\phi(x_{n_j}) \leq \langle \theta_{n_j}, x_{n_j} - y_{n_j} \rangle,$$
where \(\theta_{n_j} \in \partial \phi(x_{n_j})\). From (3.23) and the boundedness of \(\partial \phi\), we have
$$\phi(x_{n_j}) \leq ||\theta_{n_j}|| ||x_{n_j} - y_{n_j}|| \to 0$$  (3.25)

as \(n \to \infty\). Because of the weakly lower semi-continuous nature of \(\phi\), \(x_{n_j} \rightharpoonup \xi^*\) and (3.25), the following holds:
$$\phi(\xi^*) \leq \liminf_{j \to \infty} \phi(x_{n_j}) \leq 0.$$

Hence \(\xi^* \in C\).

Next, we show that \(Az^* \in Q\). Since \(P_{Q_{n_j}}(Ax_{n_j}) \in Q_{n_j}\), we have
$$\varphi(Ax_{n_j}) = \langle \chi_{n_j}, Ax_{n_j} - P_{Q_{n_j}}(Ax_{n_j}) \rangle,$$  (3.26)

where \(\chi_{n_j} \in \partial \varphi(Ax_{n_j})\). Given that \(\lim_{j \to \infty} ||(I - P_{Q_{n_j}})Ax_{n_j}|| = 0\) and (3.22), we get that \(\lim_{j \to \infty} ||(I - P_{Q_{n_j}})Ax_{n_j}|| = 0\). By (3.26) and the boundedness of \(\partial \varphi\), we have
$$\varphi(Ax_{n_j}) \leq ||\chi_{n_j}|| ||Ax_{n_j} - P_{Q_{n_j}}(Ax_{n_j})|| \to 0$$  (3.27)

as \(n \to \infty\). In view of \(x_{n_j} \rightharpoonup \xi^*\), \(Ax_{n_j} \rightharpoonup Az^*\), the weakly lower semi-continuous nature of \(\varphi\) and (3.27), we can get
$$\varphi(Az^*) \leq \liminf_{j \to \infty} \varphi(Ax_{n_j}) \leq 0.$$

Hence \(Az^* \in Q\).

From the above proof process, we can derive that \(\xi^* \in \text{Fix}(S) \cap \Delta\). Now, using (3.21), we get
$$\limsup_{i \to \infty} (\lambda g(x^*) - Bx^*, y_n - x^*)$$
$$= \limsup_{i \to \infty} (\lambda g(x^*) - Bx^*, d_n - x^*)$$
$$= \liminf_{j \to \infty} (\lambda g(x^*) - Bx^*, d_{n_j} - x^*)$$
\[= \langle \lambda g(x^*) - Bx^*, z' - x^* \rangle \leq 0.\]

It means that
\[\lim_{i \to \infty} \Upsilon_n \leq 0.\]

\[\square\]

4. Numerical experiments

Now, we present a comparison of Algorithm 1 with other algorithms by describing two numerical experiments. All of the programs were written in Matlab 9.5 and performed on a desktop PC with AMD Ryzen 7 4800U with Radeon Graphics 1.80 GHz, RAM 16.0GB.

Example 4.1. We assume that \(H_1 = H_2 = C = \mathbb{R}^4\) and \(Q = \{b\}\), where \(b \in \mathbb{R}^4\); then, we have that \(\Delta = \{x \in \mathbb{R}^4 : Ax = b\}\). Considering the system of linear equations \(Ax = b\), let

\[
A = \begin{pmatrix}
1 & 1 & 2 & 3 \\
2 & 3 & 5 & 2 \\
3 & -1 & -1 & -2 \\
3 & 5 & 2 & -2
\end{pmatrix}, \quad b = \begin{pmatrix}
5 \\
1 \\
-6 \\
-15
\end{pmatrix}.
\]

By calculation, we can find that \(Ax = b\) has a unique solution \(x^* = (-1, -2, 1, 2)^T\). Let

\[
Sx = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & \frac{1}{4}
\end{pmatrix} x + \begin{pmatrix}
-\frac{1}{4} \\
-\frac{1}{4} \\
-\frac{1}{4} \\
-\frac{1}{2}
\end{pmatrix}, \quad \forall x \in \mathbb{R}^4.
\]

By calculation, we can find that the mapping \(S\) is nonexpansive and \(x^* = (-1, -2, 1, 2)^T \in \text{Fix}(S)\).

In Algorithm 1 and scheme (1.4), we let \(\alpha_n = \frac{1}{100}, \beta_n = 0.2, \delta_n = 0.5, x_1 = (1, 2, 3, 4)^T\) and \(g = 0.2I\); in scheme (1.4), we let \(\tau_n = \frac{7}{40\sqrt{n}}\); in Algorithm 1, we let \(\phi(x) = 0, \forall x \in \mathbb{R}^4, \varphi(y) = \|y - b\|, \forall y \in \mathbb{R}^4, B = I, \lambda = 1, \varepsilon_n = \frac{1}{n^2}, \mu = 1.5, \sigma_n = 1, \rho_n = 3.5, \tau = 1\) and \(x_0 = (1, 2, 3, 4)^T\). The results of numerical experiments are revealed in Table 1 and Figure 1.

From Table 1, we can find that \(x_n\) is closer to the exact solution \(x^*\) with an increase of the number of iterations. We can also see that errors are closer to 0. Therefore, it can be concluded that Algorithm 1 is feasible. From Figure 1, we can find that the number of iterations for Algorithm 1 is less than that for scheme (1.4).
### Table 1. Numerical results of Algorithm 1 for Example 4.1.

<table>
<thead>
<tr>
<th>$n - 1$</th>
<th>$x_n^{(1)}$</th>
<th>$x_n^{(2)}$</th>
<th>$x_n^{(3)}$</th>
<th>$x_n^{(4)}$</th>
<th>$E_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>2.0000</td>
<td>3.0000</td>
<td>4.0000</td>
<td>5.2915E+00</td>
</tr>
<tr>
<td>10</td>
<td>-0.9884</td>
<td>-1.9774</td>
<td>0.9764</td>
<td>1.9905</td>
<td>3.5918E-02</td>
</tr>
<tr>
<td>50</td>
<td>-0.9987</td>
<td>-1.9959</td>
<td>0.9963</td>
<td>1.9969</td>
<td>6.4333E-03</td>
</tr>
<tr>
<td>100</td>
<td>-0.9994</td>
<td>-1.9980</td>
<td>0.9982</td>
<td>1.9985</td>
<td>3.1627E-03</td>
</tr>
<tr>
<td>500</td>
<td>-0.9999</td>
<td>-1.9996</td>
<td>0.9996</td>
<td>1.9997</td>
<td>6.3660E-04</td>
</tr>
<tr>
<td>1000</td>
<td>-0.9999</td>
<td>-1.9998</td>
<td>0.9998</td>
<td>1.9998</td>
<td>3.1853E-04</td>
</tr>
<tr>
<td>5000</td>
<td>-1.0000</td>
<td>-2.0000</td>
<td>1.0000</td>
<td>2.0000</td>
<td>6.3743E-05</td>
</tr>
<tr>
<td>10000</td>
<td>-1.0000</td>
<td>-2.0000</td>
<td>1.0000</td>
<td>2.0000</td>
<td>3.1874E-05</td>
</tr>
</tbody>
</table>

![Figure 1. Comparison of Algorithm 1 and scheme (1.4) for Example 4.1.](image)

#### Example 4.2. Assume that $H_1 = \mathbb{R}$, $H_2 = \mathbb{R}^3$, $C = [-2, 6]$, $Q = \{(y_a, y_b, y_c)^T : |y_a| + |y_b| + |y_c| \leq 3\}$.

\[
A = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix},
\]

\[
S x = \frac{x}{2} \sin x, \quad \forall x \in \mathbb{R}.
\]

We can see that $S$ is not nonexpansive but quasi-nonexpansive [23]. By calculation, we can find that $\text{Fix}(S) = \{0\}$. It is easy to obtain that $0 \in \Delta$. We denote $x^* = 0$. 

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In Algorithm 1 and scheme (1.6), we let $x_0 = x_1 = 1$, $\varepsilon_n = \frac{1}{n^2}$, $\mu = 0.9$, $\beta_n = 0.5$, $\rho_n = 3.5$, $\delta_n = 0.4$, $g(x) = \frac{1}{3} \sin x$ and $\alpha_n = \frac{1}{8n^2}$, where $0 < p \leq 1$; in scheme (1.6), we used the function quadprog to compute the projection over $Q$ by using Matlab 9.5 Optimization Toolbox; In Algorithm 1, we let $\phi(x) = (x - 6)(x + 2)$, $\forall x \in \mathbb{R}$, $\varphi(y) = |y_a| + |y_b| + |y_c| - 3$, $\forall y = (y_a,y_b,y_c)^T \in \mathbb{R}^3$, $B = I$, $\lambda = 1$, $\sigma_n = 1$ and $\tau = 1$. We used $\|x_n - x^*\| \leq 10^{-8}$ as the stopping criterion. The results of numerical experiments for different values of $p$ are shown in Table 2. The convergence behavior for $p = 0.5$ is shown in Figures 2 and 3.

**Table 2.** Numerical results of Algorithm 1 and scheme (1.6) for Example 4.2.

<table>
<thead>
<tr>
<th>$p$</th>
<th>Algorithm 1</th>
<th>Scheme (1.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>86</td>
<td>0.0515</td>
</tr>
<tr>
<td>0.6</td>
<td>86</td>
<td>0.0534</td>
</tr>
<tr>
<td>0.7</td>
<td>73</td>
<td>0.0517</td>
</tr>
<tr>
<td>0.8</td>
<td>80</td>
<td>0.0522</td>
</tr>
<tr>
<td>0.9</td>
<td>80</td>
<td>0.0545</td>
</tr>
<tr>
<td>1</td>
<td>87</td>
<td>0.0522</td>
</tr>
</tbody>
</table>

**Figure 2.** Comparison between the number of iterations of Algorithm 1 and scheme (1.6) for Example 4.2 with $p = 0.5$. 

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5. Conclusions

In this work, we developed a method for solving the SFP and the fixed-point problem involving demicontractive mapping. In Algorithm 1, we have expanded the nonexpansive mapping to the demicontractive mapping, selected a new step size and added the inertial method based on scheme (1.4). Our Algorithm 1 exhibits better convergence behavior than scheme (1.4). Additionally, we have extended the quasi-nonexpansive mapping to the demicontractive mapping, applied projection onto a half-space and selected a new step size in our Algorithm 1 based on scheme (1.6). Our Algorithm 1 also demonstrated superior convergence behavior compared to scheme (1.6). From the above, our algorithm exhibits a superior convergence rate and a wider range for the operator, thus making it applicable in a broader range of scenarios. Furthermore, we have presented two numerical results that demonstrate the feasibility and effectiveness of our method. In the following work, we intend to extend the demicontractive mapping to the quasi-pseudocontractive mapping and extend the SFP to the multiple-set split feasibility problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.
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Conflict of interest

The authors declare that they have no competing interests.

References


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