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Research article

A general hybrid relaxed CQ algorithm for solving the fixed-point problem and split-feasibility problem

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Abstract: In this paper, we introduce a new hybrid relaxed iterative algorithm with two half-spaces to solve the fixed-point problem and split-feasibility problem involving demicontractive mappings. The strong convergence of the iterative sequence produced by our algorithm is proved under certain weak conditions. We give several numerical experiments to demonstrate the efficiency of the proposed iterative method in comparison with previous algorithms.

Keywords: inertial iterative method; demicontractive mapping; fixed point; hybrid CQ method; split-feasibility problem
Mathematics Subject Classification: 47H04, 47H09, 47H10, 65K10

1. Introduction

The split feasibility problem (SFP) in Euclidean spaces was introduced by Censor and Elfving [1] in 1994. It is depicted as finding a point x such that

$$x \in C, \quad Ax \in Q, \tag{1.1}$$

where $C \subset H_1$ and $Q \subset H_2$ are nonempty closed convex sets; A is a bounded linear operator from a Hilbert space H_1 onto a Hilbert space H_2 with $A \neq 0$. Denote the set of solutions for (1.1) by Δ. The SFP plays an important role in the study of medical image reconstruction, signal processing, etc. Many scholars regard the SFP and its generalizations, such as the multiple-set SFP and the split common-fixed-point problem as their research direction (see [2–6]).

It is known that Byrne [7, 8] proposed the famous CQ algorithm (CQA) for solving the problem (1.1). The proposed method is given as follows:

$$x_{n+1} = P_C(x_n - \tau_n A^*(I - P_Q)Ax_n), \quad \forall n \geq 1, \tag{1.2}$$

where $\tau_n \in \left(0, \dfrac{2}{\|A\|^2}\right)$ is the step size, A^* is the adjoint operator of A, P_C is the projection onto C and P_Q is the projection onto Q. Regrettably, there are two drawbacks of applying (1.2). On one hand, estimating the operator norm can often be challenging. On the other hand, computing projections onto both C and Q can be very difficult.

To overcome the first drawback, López et al. [9] proposed a novel approach for selecting the step size, denoted by τ_n, which is defined as follows:

$$\tau_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2},$$

where $f(x_n) := \dfrac{1}{2}\|(I - P_Q)Ax_n\|^2$, $\nabla f(x_n) = A^*(I - P_Q)Ax_n$, $\rho_n \in (0, 4)$ and $\inf_{n\in\mathbb{N}} \rho_n(4 - \rho_n) > 0$.

To overcome the second drawback, Fukushima [10] introduced the level sets C and Q:

$$C = \{x \in H_1 : \phi(x) \le 0\}, \qquad Q = \{y \in H_2 : \varphi(y) \le 0\}, \tag{1.3}$$

where $\phi : H_1 \to \mathbb{R}$ and $\varphi : H_2 \to \mathbb{R}$ are convex, subdifferential and weakly lower semi-continuous functions. Motivated by Fukushima's method, Yang [11] introduced a new relaxed CQA by replacing P_C with P_{C_n} and P_Q with P_{Q_n}. The algorithm has the form

$$x_{n+1} = P_{C_n}(x_n - \alpha_n A^*(I - P_{Q_n})Ax_n), \quad \forall n \ge 1,$$

where $\alpha_n \in (0, \frac{2}{\|AA^*\|})$, A^* is the adjoint operator of A and C_n and Q_n are defined as follows:

$$C_n = \{x \in H_1 : \phi(x_n) \le \langle \vartheta_n, x_n - x\rangle\}, \quad \vartheta_n \in \partial\phi(x_n),$$

and

$$Q_n = \{y \in H_2 : \varphi(Ax_n) \le \langle \chi_n, Ax_n - y\rangle\}, \quad \chi_n \in \partial\varphi(Ax_n),$$

where $\partial\phi$ and $\partial\varphi$ are bounded operators.

Qin and Wang [12] introduced (1.4) to find a common solution of the SFP (1.1) and fixed-point problem in 2019. Their algorithm has the form

$$\begin{cases} y_n = P_C((1 - \delta_n)(x_n - \tau_n A^*(I - P_Q)Ax_n) + \delta_n S x_n), \\ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \gamma_n y_n, \quad n \ge 1, \end{cases} \tag{1.4}$$

where g is a k-contraction from C onto C, S is nonexpansive from C onto C and $\mathrm{Fix}(S) := \{x \in H : Sx = x\}$. $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\tau_n\}$ are included in $(0, 1)$; they satisfy the following:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\alpha_n + \beta_n + \gamma_n = 1$;
(iv) $\lim_{n\to\infty} |\tau_n - \tau_{n+1}| = 0$, $0 < \liminf_{n\to\infty} \tau_n \le \limsup_{n\to\infty} \tau_n < \frac{2}{\|A\|^2}$;
(v) $\lim_{n\to\infty} |\delta_n - \delta_{n+1}| = 0$, $0 < \liminf_{n\to\infty} \delta_n \le \limsup_{n\to\infty} \delta_n < 1$.

Then, there is a point q of $\Delta \cap \mathrm{Fix}(S)$ with $x_n \to q$. Furthermore, it satisfies (1.5):

$$\langle z' - q, g(q) - q\rangle \leq 0, \quad \forall z' \in \mathrm{Fix}(S) \cap \Delta. \tag{1.5}$$

Motivated by López et al. [9], Wang et al. [13] improved the selection of step size in (1.4). In order to accelerate the convergence rate, they introduced (1.6) to solve the SFP (1.1) and fixed-point problem. Given that $x_0, x_1 \in H$, their algorithm has the form

$$\mu_n = \begin{cases} \min\left\{\mu, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_{n-1} \neq x_n, \\ \mu, & \text{otherwise,} \end{cases}$$

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}), \\ y_n = P_C((1-\delta_n)(w_n - \tau_n A^*(I - P_Q)Aw_n) + \delta_n S w_n), \\ x_{n+1} = \alpha_n g(x_n) + \beta_n w_n + \gamma_n y_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\mu \geq 0$, $\tau_n = \dfrac{\rho_n f(w_n)}{\|\nabla f(w_n)\|^2}$, $0 < \rho_n < 4$, $f(w_n) := \dfrac{1}{2}\|(I - P_Q)Aw_n\|^2$ and $\nabla f(w_n) = A^*(I - P_Q)Aw_n$. Assume that $S : H_1 \to H_1$ is quasi-nonexpansive, $I - S$ is demiclosed at zero and $g : H_1 \to H_1$ is a k-contraction. $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are included in $[0, 1]$, which satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\limsup_{n\to\infty} \beta_n < 1$;
(iii) $\alpha_n + \beta_n + \gamma_n = 1$;
(iv) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(v) $\varepsilon_n > 0$, $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

Then, there is a point q of $\mathrm{Fix}(S) \cap \Delta$ with $x_n \to q$. Furthermore, it satisfies (1.7):

$$\langle z' - q, g(q) - q\rangle \leq 0, \quad \forall z' \in \mathrm{Fix}(S) \cap \Delta. \tag{1.7}$$

In addition, Tian [14] introduced an iterative form:

$$x_{n+1} = \alpha_n \lambda g(x_n) + (I - \iota\alpha_n B)S x_n, \quad \forall n \in \mathbb{N}.$$

Suppose that $\mathrm{Fix}(S) \neq \emptyset$, where $\mathrm{Fix}(S) := \{x \in H : Sx = x\}$. Then, there is a point q in $\mathrm{Fix}(S)$ with $x_n \to q$. Furthermore, it satisfies the following:

$$\langle (\iota B - \lambda g)q, z' - q\rangle \geq 0, \quad \forall z' \in \mathrm{Fix}(S),$$

where B is $\overline{\lambda}$-strongly monotone and M-Lipschitz continuous from H onto H with $\overline{\lambda}, M > 0$, S is nonexpansive from H onto H and g is a k-contraction from H onto H with $0 < k < 1$. Let $\iota, \lambda \in \mathbb{R}$ and $\{\alpha_n\} \subset (0, 1)$ satisfy the following:

(i) $0 < \iota < \dfrac{2\overline{\lambda}}{M^2}$;
(ii) $0 < \lambda < \iota\dfrac{\overline{\lambda} - \frac{M^2\iota}{2}}{k}$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

Motivated by the algorithms developed by Qin and Wang [12], Wang et al. [13], Tian [14] and Kwari et al. [15], we propose a new half-space relaxation projection method for solving the SFP and fixed-point problem of demicontractive mappings by expanding the scope of operator S, modifying the iteration format and optimizing the selection of step sizes. Our result in this article extends and improves many recent correlative results of other authors. Particularly, our method extends and improves the methods in some papers of other authors in the following ways: (1) We have considered and studied a new modified half-space hybrid method to solve the fixed-point problem and SFP of nonlinear operators at the same time. And, our method can be used more widely; (2) We extend the nonexpansive mapping and the quasi-nonexpansive mapping to the demicontractive mapping, which expands the scope of the study; (3) The inertia is added to accelerate the convergence rate further; (4) The selection of the step size is a multistep and self-adaptive process, so it no longer depends on the operator norm; (5) The projection onto a half-space greatly facilitates the calculation, and under some weaker conditional assumptions, the iterative sequence generated by our new algorithm converges strongly to the common solution of the two problems. At last, we present some numerical experiments to show the effectiveness and feasibility of our new iterative method.

2. Preliminaries

In this paper, let H be a real Hilbert space; the inner product be denoted by $\langle \cdot, \cdot \rangle$ and the norm be denoted by $\|\cdot\|$. We use the symbol $\rightharpoonup$ to indicate weak convergence and $\to$ to indicate strong convergence.

Definition 2.1. A self-mapping P is said to be

(i) nonexpansive if

$$\|Px - Py\| \leq \|x - y\|, \quad \forall x, y \in H;$$

(ii) quasi-nonexpansive if $\mathrm{Fix}(P) \neq \emptyset$ such that

$$\|Px - y\| \leq \|x - y\|, \quad \forall x \in H, y \in \mathrm{Fix}(P);$$

(iii) α -demicontractive $(0 \leq \alpha < 1)$ if $\mathrm{Fix}(P) \neq \emptyset$ such that

$$\|Px - y\|^2 \leq \alpha\|(I - P)x\|^2 + \|x - y\|^2, \quad \forall x \in H, y \in \mathrm{Fix}(P);$$

(iv) firmly nonexpansive if

$$\|Px - Py\|^2 \leq \langle Px - Py, x - y\rangle, \quad \forall x, y \in H;$$

(v) t-contractive $(0 \leq t < 1)$ if

$$\|Px - Py\| \leq t\|x - y\|, \quad \forall x, y \in H;$$

(vi) M-Lipschitz continuous $(M > 0)$ if

$$\|Px - Py\| \leq M\|x - y\|, \quad \forall x, y \in H;$$

(vii) β-strongly monotone ($\beta \geq 0$) if

$$\langle Px - Py, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in H.$$

When Fix(P) $\neq \emptyset$, we obtain (i)⇒(ii)⇒(iii).

Definition 2.2. When the function $f : H \to \mathbb{R}$ is convex and subdifferentiable, an element $d \in H$ is called a subgradient of $f(x_0)$ if

$$f(y) \geq f(x_0) + \langle d, y - x_0\rangle, \quad \forall y \in H.$$

The set of subgradients of f at x_0 is denoted by $\partial f(x_0)$.

Definition 2.3. The function $f : H \to \mathbb{R}$ is called weakly lower semi-continuous if $x_n \rightharpoonup x_0$ implies that

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

Lemma 2.1 ([16–18])**.** *Let $D \subset H$ be a nonempty, convex and closed set. For any $x \in H$, we have*

(i) $\|P_D x - y\|^2 \leq \|x - y\|^2 - \|x - P_D x\|^2, \quad \forall y \in D;$
(ii) $\langle x - P_D x, y - P_D x\rangle \leq 0, \quad \forall y \in D;$
(iii) $\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y\rangle, \quad \forall y \in H;$
(iv) $\|(I - P_D)x - (I - P_D)y\|^2 \leq \langle (I - P_D)x - (I - P_D)y, x - y\rangle, \quad \forall y \in H.$

Lemma 2.2 ([19, 20])**.** *For any $x, y \in H$, the following results hold*

(i) $\|x + y\|^2 \leq 2\langle y, x + y\rangle + \|x\|^2;$
(ii) $\|(1 - m)x + my\|^2 = m\|y\|^2 + (1 - m)\|x\|^2 - (1 - m)m\|x - y\|^2, \quad m \in \mathbb{R}.$

Lemma 2.3 ([14])**.** *Assume that B is an M-Lipschitz continuous and $\overline{\lambda}$-strongly monotone operator with $\overline{\lambda}, M > 0$. For $\mu, \nu > 0$, they satisfy the following conditions:*

$$0 < \mu < \frac{2\overline{\lambda}}{M^2}, \qquad \nu = \overline{\lambda} - \frac{M^2\mu}{2}.$$

Let $\{\gamma_n\} \subset (0, 1)$ and $\lim_{n\to\infty} \gamma_n = 0$. If n is a sufficiently large number, we have

$$\|(I - \gamma_n B)x - (I - \gamma_n B)y\| \leq (1 - \gamma_n\nu)\|x - y\|.$$

Proof.

$$\begin{aligned}
& \|(I - \gamma_n B)x - (I - \gamma_n B)y\|^2 \\
= \; & \|x - y - \gamma_n(Bx - By)\|^2 \\
= \; & \|x - y\|^2 - 2\gamma_n\langle x - y, Bx - By\rangle + \gamma_n^2\|Bx - By\|^2 \\
\leq \; & \|x - y\|^2 - 2\gamma_n\overline{\lambda}\|x - y\|^2 + \gamma_n^2 M^2\|x - y\|^2 \\
= \; & (1 - 2\gamma_n\overline{\lambda} + \gamma_n^2 M^2)\|x - y\|^2 \\
= \; & (1 - 2\gamma_n\nu - \gamma_n M^2\mu + \gamma_n^2 M^2)\|x - y\|^2 \\
\leq \; & (1 - 2\gamma_n\nu - \gamma_n M^2(\mu - \gamma_n) + \gamma_n^2\nu^2)\|x - y\|^2.
\end{aligned}$$

Combining $\lim_{n\to\infty} \gamma_n = 0$ and $\mu - \gamma_n > 0$ (n sufficiently large), we have

$$\begin{aligned} & \|(I-\gamma_n B)x-(I-\gamma_n B)y\|^2 \\ \leq\ & (1-2\gamma_n\nu+\gamma_n^2\nu^2)\|x-y\|^2 \\ =\ & (1-\gamma_n\nu)^2\|x-y\|^2. \end{aligned}$$

Since $1-\gamma_n\nu > 0$ (n sufficiently large), we have

$$\|(I-\gamma_n B)x-(I-\gamma_n B)y\| \leq (1-\gamma_n\nu)\|x-y\|.$$

□

Lemma 2.4 ([21]). *Let the self-mapping S be α-demicontractive in H and* $\mathrm{Fix}(S) \neq \emptyset$*. Define $S_\sigma = (1-\sigma)I+\sigma S$, $\sigma \in (0, 1-\alpha)$; then, we have the following:*

(i) $\mathrm{Fix}(S) = \mathrm{Fix}(S_\sigma) \subset H$ *is convex and closed;*
(ii) $\|S_\sigma x - q\|^2 \leq \|x-q\|^2 - \frac{1}{\sigma}(1-\alpha-\sigma)\|(I-S_\sigma)x\|^2 \quad \forall x \in H,\ q \in \mathrm{Fix}(S)$.

Lemma 2.5 ([8]). *Let $f(z) = \frac{1}{2}\|(I-P_Q)Az\|^2$. We can know that ∇f is M-Lipschitz continuous with $M = \|A\|^2$.*

Lemma 2.6 ([22]). *Let $\{a_n\}$ be a sequence of nonnegative numbers such that*

$$\begin{aligned} & a_{n+1} \leq (1-\Theta_n)a_n + \Theta_n\Upsilon_n, \quad n \geq n_0, \\ & a_{n+1} \leq a_n - \Pi_n + \Gamma_n, \quad n \geq n_0, \end{aligned}$$

where n_0 is a sufficiently large integer, the sequence $\{\Theta_n\}$ is contained in $(0,1)$*, the real sequence $\{\Pi_n\}$ is nonnegative $\{\Upsilon_n\}$ and $\{\Gamma_n\}$ are two sequences which are included in $\mathbb{R}$. If the following conditions hold*
(i) $\sum_{n=0}^{\infty} \Theta_n = \infty$; (ii) $\lim_{n\to\infty} \Gamma_n = 0$; (iii)$\lim_{i\to\infty} \Pi_{n_i} = 0 \Rightarrow \limsup_{i\to\infty} \Upsilon_{n_i} \leq 0$, $\forall \{n_i\} \subseteq \{n\}$, then $\lim_{n\to\infty} a_n = 0$.

3. Results

In this section, C and Q are defined by (1.3). We give some conditions for the convergence analysis of our Algorithm 1.

(C_1) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $0 < \alpha_n < 1$;
(C_2) $\beta_n \in [0,1]$, $\limsup_{n\to\infty} \beta_n < 1$;
(C_3) $\varepsilon_n > 0$, $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$;
(C_4) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(C_5) $\sigma_n \in (0, 1-\alpha]$, $\liminf_{n\to\infty} \sigma_n > 0$;
(C_6) $\inf_{n\in\mathbb{N}} \rho_n(4-\rho_n) > 0$, $\{\rho_n\} \subset (0,4)$.

Algorithm 1

Initialization: Apply $x_0, x_1 \in H_1, \mu \geq 0, \tau > 0, 0 < \iota < \dfrac{2\bar{\lambda}}{L^2}, \nu = \bar{\lambda} - \dfrac{L^2\iota}{2}, \lambda > 0$ such that $0 \leq k\lambda < \nu$.

Iterative step: Compute x_{n+1} for $n \geq 1$:
Step 1. Compute

$$d_n = x_n + \mu_n(x_n - x_{n-1}),$$

where

$$\mu_n = \begin{cases} \min\left\{\mu, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_{n-1} \neq x_n, \\ \mu, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$y_n = P_{C_n}\Big((1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n) + \delta_n((1-\sigma_n)d_n + \sigma_n S d_n)\Big),$$

where

$$\tau_n = \begin{cases} \dfrac{\rho_n f_n(d_n)}{\|\nabla f_n(d_n)\|^2}, & \text{if } \|\nabla f_n(d_n)\| \neq 0, \\ \tau, & \text{if } \|\nabla f_n(d_n)\| = 0, \end{cases}$$

$$f_n(d_n) = \frac{1}{2}\|(I - P_{Q_n})Ad_n\|^2,$$

$$\nabla f_n(d_n) = A^*(I - P_{Q_n})Ad_n,$$

and

$$C_n = \{x \in H_1 : \phi(x_n) \leq \langle \vartheta_n, x_n - x\rangle\}, \quad \text{where } \vartheta_n \in \partial\phi(x_n),$$

$$Q_n = \{y \in H_2 : \varphi(Ax_n) \leq \langle \chi_n, Ax_n - y\rangle\}, \quad \text{where } \chi_n \in \partial\varphi(Ax_n).$$

Step 3. Compute

$$x_{n+1} = \alpha_n \lambda g(x_n) + \beta_n d_n + ((1-\beta_n)I - \alpha_n B)y_n.$$

Set $n := n + 1$ and go to Step 1.

Theorem 3.1. *Let $S : H_1 \to H_1$ be α-demicontractive, $I - S$ be demiclosed at zero and $g : H_1 \to H_1$ be k-contractive. Moreover, let operator B be L-Lipschitz continuous and $\overline{\lambda}$-strongly monotone. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $\{\rho_n\}$ are sequences satisfying $(C_1) - (C_6)$ and $\mathrm{Fix}(S) \cap \Delta \neq \emptyset$. Then, the $\{x_n\}$ generated by Algorithm 1 converges strongly to $x^* \in \mathrm{Fix}(S) \cap \Delta$. Furthermore, it is the unique solution of (3.1):*

$$\langle (B - \lambda g)x^*, z' - x^* \rangle \geq 0, \quad \forall z' \in \mathrm{Fix}(S) \cap \Delta. \tag{3.1}$$

Proof. Let $r \in \Delta \cap \mathrm{Fix}(S)$. Since $C \subset C_n$ and $Q \subset Q_n$, we have that $r \in C_n$, $Ar \in Q_n$ and $r \in \mathrm{Fix(S)}$. By combining Lemma 2.1 with Lemma 2.2, we get

$$\begin{aligned}
& \|y_n - r\|^2 \\
= \;& \Big\| P_{C_n}\Big((1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n) + \delta_n((1-\sigma_n)d_n + \sigma_n S d_n)\Big) - r\Big\|^2 \\
\leq \;& \Big\|(1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n) + \delta_n((1-\sigma_n)d_n + \sigma_n S d_n) - r\Big\|^2 \\
& -\Big\|(I - P_{C_n})\Big((1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n) \\
& +\delta_n((1-\sigma_n)d_n + \sigma_n S d_n\Big)\Big\|^2 \\
= \;& \Big\|\delta_n((1-\sigma_n)d_n + \sigma_n S d_n - r) + (1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n - r)\Big\|^2 \\
& -\Big\|(I - P_{C_n})\Big((1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n) \\
& +\delta_n((1-\sigma_n)d_n + \sigma_n S d_n)\Big)\Big\|^2.
\end{aligned}$$

Set

$$S_{\sigma_n} d_n = (1-\sigma_n)d_n + \sigma_n S d_n;$$

according to Lemma 2.4, we have

$$\begin{aligned}
& \|y_n - r\|^2 \\
\leq \;& \delta_n \|S_{\sigma_n} d_n - r\|^2 + (1-\delta_n)\|d_n - \tau_n \nabla f_n(d_n) - r\|^2 \\
& -\delta_n(1-\delta_n)\|\sigma_n(S d_n - d_n) + \tau_n A^*(I - P_{Q_n})Ad_n\|^2 \\
& -\Big\|(I - P_{C_n})\Big((1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n) \\
& +\delta_n((1-\sigma_n)d_n + \sigma_n S d_n)\Big)\Big\|^2 \\
\leq \;& \delta_n \|d_n - r\|^2 + (1-\delta_n)\|d_n - \tau_n \nabla f_n(d_n) - r\|^2 \\
& -\delta_n(1-\delta_n)\|\sigma_n(S d_n - d_n) + \tau_n A^*(I - P_{Q_n})Ad_n\|^2 \\
& -\Big\|(I - P_{C_n})\Big((1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n) \\
& +\delta_n((1-\sigma_n)d_n + \sigma_n S d_n)\Big)\Big\|^2 \\
\leq \;& \delta_n \|d_n - r\|^2 + (1-\delta_n)(\|d_n - r\|^2 + \tau_n^2 \|\nabla f_n(d_n)\|^2 \\
& -2\tau_n \langle \nabla f_n(d_n), d_n - r\rangle) \\
& -\delta_n(1-\delta_n)\|\sigma_n(S d_n - d_n) + \tau_n A^*(I - P_{Q_n})Ad_n\|^2
\end{aligned}$$

$$-\Big\|(I-P_{C_n})\Big((1-\delta_n)(d_n-\tau_n A^*(I-P_{Q_n})Ad_n)$$
$$+\delta_n((1-\sigma_n)d_n+\sigma_n Sd_n)\Big)\Big\|^2. \qquad (3.2)$$

From the definition of $\nabla f_n(d_n)$, we have

$$\begin{aligned}
&\langle \nabla f_n(d_n), d_n-r\rangle\\
=\;& \langle A^*(I-P_{Q_n})Ad_n - A^*(I-P_{Q_n})Ar, d_n-r\rangle\\
=\;& \langle (I-P_{Q_n})Ad_n-(I-P_{Q_n})Ar, Ad_n-Ar\rangle\\
\geq\;& \|(I-P_{Q_n})Ad_n\|^2\\
=\;& 2f_n(d_n), \qquad (3.3)
\end{aligned}$$

which means that when $\|\nabla f_n(d_n)\|=0$, we get that $f_n(d_n)=0$. Substituting (3.3) into (3.2), we have

$$\begin{aligned}
&\|y_n-r\|^2\\
\leq\;& \|d_n-r\|^2+\tau_n^2(1-\delta_n)\|\nabla f_n(d_n)\|^2-4(1-\delta_n)\tau_n f_n(d_n)\\
&-\delta_n(1-\delta_n)\|\sigma_n(Sd_n-d_n)+\tau_n A^*(I-P_{Q_n})Ad_n\|^2\\
&-\Big\|(I-P_{C_n})\Big((1-\delta_n)(d_n-\tau_n A^*(I-P_{Q_n})Ad_n)\\
&+\delta_n((1-\sigma_n)d_n+\sigma_n Sd_n)\Big)\Big\|^2\\
=\;& \|d_n-r\|^2+(1-\delta_n)\tau_n\rho_n f_n(d_n)-4(1-\delta_n)\tau_n f_n(d_n)\\
&-\delta_n(1-\delta_n)\|\sigma_n(Sd_n-d_n)+\tau_n A^*(I-P_{Q_n})Ad_n\|^2\\
&-\Big\|(I-P_{C_n})\Big((1-\delta_n)(d_n-\tau_n A^*(I-P_{Q_n})Ad_n)\\
&+\delta_n((1-\sigma_n)d_n+\sigma_n Sd_n)\Big)\Big\|^2\\
\leq\;& \|d_n-r\|^2-(4-\rho_n)(1-\delta_n)\tau_n f_n(d_n)\\
&-\delta_n(1-\delta_n)\|\sigma_n(Sd_n-d_n)+\tau_n A^*(I-P_{Q_n})Ad_n\|^2\\
&-\Big\|(I-P_{C_n})\Big((1-\delta_n)(d_n-\tau_n A^*(I-P_{Q_n})Ad_n)\\
&+\delta_n((1-\sigma_n)d_n+\sigma_n Sd_n)\Big)\Big\|^2. \qquad (3.4)
\end{aligned}$$

Noting that $\{\delta_n\}$ is a sequence in $(0,1)$ and $\rho_n\in(0,4)$, we derive that

$$\|y_n-r\|\leq\|d_n-r\|. \qquad (3.5)$$

From the definition of d_n, we have

$$\begin{aligned}
&\|d_n-r\|\\
=\;& \|x_n+\mu_n(x_n-x_{n-1})-r\|\\
\leq\;& \mu_n\|x_n-x_{n-1}\|+\|x_n-r\|\\
\leq\;& \varepsilon_n+\|x_n-r\|. \qquad (3.6)
\end{aligned}$$

In the rest of the proof, n_0 is assumed to be a sufficiently large positive integer and $n\geq n_0$.

Now, let $\gamma_n := \frac{\alpha_n}{1-\beta_n}$. By Lemma 2.3, (3.5) and (3.6), we estimate $\|x_{n+1} - r\|$:

$$\begin{aligned}
& \|x_{n+1} - r\| \\
= & \left\|\alpha_n \lambda g(x_n) + \beta_n d_n + ((1-\beta_n)I - \alpha_n B)y_n - r\right\| \\
= & \left\|\alpha_n \lambda g(x_n) + \beta_n d_n + ((1-\beta_n)I - \alpha_n B)y_n - r + \alpha_n Br - \alpha_n Br\right\| \\
= & \Big\|\alpha_n \lambda g(x_n) - \alpha_n Br + \beta_n d_n - \beta_n r + ((1-\beta_n)I - \alpha_n B)y_n \\
& -((1-\beta_n)I - \alpha_n B)r\Big\| \\
\leq & \alpha_n\|\lambda g(x_n) - Br\| + \beta_n\|d_n - r\| + (1-\beta_n)\|(I - \gamma_n B)y_n - (I - \gamma_n B)r\| \\
\leq & \alpha_n\|\lambda g(x_n) - \lambda g(r)\| + \alpha_n\|\lambda g(r) - Br\| + \beta_n\|d_n - r\| \\
& +(1-\beta_n)(1-\gamma_n \nu)\|y_n - r\| \\
\leq & \alpha_n \lambda k\|x_n - r\| + \alpha_n\|\lambda g(r) - Br\| + \beta_n\|d_n - r\| \\
& +(1-\beta_n)(1-\gamma_n \nu)\|y_n - r\| \\
= & \alpha_n \lambda k\|x_n - r\| + \alpha_n\|\lambda g(r) - Br\| + \beta_n\|d_n - r\| + (1-\beta_n - \alpha_n \nu)\|y_n - r\| \\
\leq & \alpha_n \lambda k\|x_n - r\| + \alpha_n\|\lambda g(r) - Br\| + (1-\alpha_n \nu)\|d_n - r\| \\
\leq & (1-\alpha_n(\nu - \lambda k))\|x_n - r\| + \alpha_n\|\lambda g(r) - Br\| + (1-\alpha_n \nu)\varepsilon_n \\
= & (1-\alpha_n(\nu - \lambda k))\|x_n - r\| + \alpha_n(\nu - \lambda k)\Big(\frac{\|\lambda g(r) - Br\|}{\nu - \lambda k} + \frac{(1-\alpha_n \nu)\varepsilon_n}{\alpha_n(\nu - \lambda k)}\Big).
\end{aligned}$$

We can assume that M is a suitable positive number such that $\frac{\varepsilon_n}{\alpha_n} \leq M$ since $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$. Next, we have

$$\begin{aligned}
& \|x_{n+1} - r\| \\
\leq & (1-\alpha_n(\nu - \lambda k))\|x_n - r\| + \alpha_n(\nu - \lambda k)\Big(\frac{\|\lambda g(r) - Br\|}{\nu - \lambda k} + \frac{M(1-\alpha_n \nu)}{\nu - \lambda k}\Big) \\
\leq & (1-\alpha_n(\nu - \lambda k))\|x_n - r\| + \alpha_n(\nu - \lambda k)\Big(\frac{\|\lambda g(r) - Br\| + M}{\nu - \lambda k}\Big) \\
\leq & \max\left\{\|x_n - r\|, \frac{\|\lambda g(r) - Br\| + M}{\nu - \lambda k}\right\} \\
& \vdots \\
\leq & \max\left\{\|x_{n_0} - r\|, \frac{\|\lambda g(r) - Br\| + M}{\nu - \lambda k}\right\}.
\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded.

From the definition of x_{n+1},

$$x_{n+1} = \beta_n d_n + (1-\beta_n)(\gamma_n \lambda g(x_n) + (I - \gamma_n B)y_n);$$

set

$$z_n := \gamma_n \lambda g(x_n) + (I - \gamma_n B)y_n; \tag{3.7}$$

then, we obtain

$$x_{n+1} = \beta_n d_n + (1-\beta_n)z_n. \tag{3.8}$$

In view of the arbitrariness of r, Lemma 2.2, Lemma 2.3, (3.5) and (3.7), we have

$$\begin{aligned}
&\|z_n - x^*\|^2\\
=\;& \|\gamma_n\lambda g(x_n) + (I-\gamma_n B)y_n - x^*\|^2\\
=\;& \Big\|\gamma_n(\lambda g(x_n) - Bx^*) + \big((I-\gamma_n B)y_n - (I-\gamma_n B)x^*\big)\Big\|^2\\
=\;& \gamma_n^2\|\lambda g(x_n) - Bx^*\|^2 + \|(I-\gamma_n B)y_n - (I-\gamma_n B)x^*\|^2\\
& +2\gamma_n\langle \lambda g(x_n) - Bx^*, (I-\gamma_n B)y_n - (I-\gamma_n B)x^*\rangle\\
\le\;& \gamma_n^2\|\lambda g(x_n) - Bx^*\|^2 + (1-\gamma_n\nu)^2\|y_n - x^*\|^2\\
& +2\gamma_n\langle \lambda g(x_n) - Bx^*, y_n - x^*\rangle - 2\gamma_n^2\langle \lambda g(x_n) - Bx^*, By_n - Bx^*\rangle\\
\le\;& \gamma_n^2\|\lambda g(x_n) - Bx^*\|^2 + (1-\gamma_n\nu)^2\|y_n - x^*\|^2\\
& +2\gamma_n\langle \lambda g(x_n) - Bx^*, y_n - x^*\rangle + 2\gamma_n^2\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\|\\
\le\;& \gamma_n^2\|\lambda g(x_n) - Bx^*\|^2 + (1-\gamma_n\nu)^2\|y_n - x^*\|^2\\
& +2\gamma_n\langle \lambda g(x_n) - \lambda g(x^*), y_n - x^*\rangle + 2\gamma_n\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle\\
& +2\gamma_n^2\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\|\\
\le\;& \gamma_n^2\|\lambda g(x_n) - Bx^*\|^2 + (1-\gamma_n\nu)^2\|y_n - x^*\|^2 + 2\gamma_n\lambda k\|x_n - x^*\|\|y_n - x^*\|\\
& +2\gamma_n\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle + 2\gamma_n^2\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\|\\
\le\;& \gamma_n^2\|\lambda g(x_n) - Bx^*\|^2 + (1-\gamma_n\nu)^2\|d_n - x^*\|^2 + 2\gamma_n\lambda k\|x_n - x^*\|\|d_n - x^*\|\\
& +2\gamma_n\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle + 2\gamma_n^2\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\|\\
=\;& (1-2\gamma_n\nu)\|d_n - x^*\|^2 + 2\gamma_n\lambda k\|x_n - x^*\|\|d_n - x^*\|\\
& +2\gamma_n\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle + 2\gamma_n^2\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\|\\
& +\gamma_n^2\|\lambda g(x_n) - Bx^*\|^2 + \gamma_n^2\nu^2\|d_n - x^*\|^2.
\end{aligned} \tag{3.9}$$

According to the definition of $\{d_n\}$ and Lemma 2.2, we have

$$\begin{aligned}
&\|d_n - x^*\|^2\\
=\;& \|x_n + \mu_n(x_n - x_{n-1}) - x^*\|^2\\
\le\;& 2\mu_n\langle x_n - x_{n-1}, d_n - x^*\rangle + \|x_n - x^*\|^2\\
\le\;& 2\mu_n\|x_n - x_{n-1}\|\|d_n - x^*\| + \|x_n - x^*\|^2\\
\le\;& 2\varepsilon_n\|d_n - x^*\| + \|x_n - x^*\|^2.
\end{aligned} \tag{3.10}$$

Next, we can estimate $\|x_{n+1} - x^*\|^2$. On the one hand, in view of (3.6), (3.8), (3.9), (3.10) and Lemma 2.2, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2\\
=\;& \|\beta_n d_n + (1-\beta_n)z_n - x^*\|^2\\
=\;& \|(1-\beta_n)(z_n - x^*) + \beta_n(d_n - x^*)\|^2\\
=\;& (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|d_n - x^*\|^2 - \beta_n(1-\beta_n)\|d_n - z_n\|^2\\
\le\;& (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|d_n - x^*\|^2\\
\le\;& \beta_n\|d_n - x^*\|^2 + (1-\beta_n)(1-2\gamma_n\nu)\|d_n - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&+2(1-\beta_n)\gamma_n\lambda k\|x_n - x^*\|\|d_n - x^*\| + 2\gamma_n(1-\beta_n)\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle \\
&+2\gamma_n^2(1-\beta_n)\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\| \\
&+\gamma_n^2(1-\beta_n)\|\lambda g(x_n) - Bx^*\|^2 + \gamma_n^2(1-\beta_n)\nu^2\|d_n - x^*\|^2 \\
= \;& (1-2\alpha_n\nu)\|d_n - x^*\|^2 + 2\alpha_n\lambda k\|x_n - x^*\|\|d_n - x^*\| \\
&+2\alpha_n\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle + 2\alpha_n\gamma_n\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\| \\
&+\alpha_n\gamma_n\|\lambda g(x_n) - Bx^*\|^2 + \alpha_n\gamma_n\nu^2\|d_n - x^*\|^2 \\
\leq \;& (1-2\alpha_n\nu)(\|x_n - x^*\|^2 + 2\varepsilon_n\|d_n - x^*\|) + 2\alpha_n\lambda k\|x_n - x^*\|(\|x_n - x^*\| + \varepsilon_n) \\
&+2\alpha_n\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle + 2\alpha_n\gamma_n\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\| \\
&+\alpha_n\gamma_n\|\lambda g(x_n) - Bx^*\|^2 + \alpha_n\gamma_n\nu^2\|d_n - x^*\|^2 \\
\leq \;& (1-2\alpha_n\nu)\|x_n - x^*\|^2 + 2\varepsilon_n\|d_n - x^*\| + 2\alpha_n\lambda k\|x_n - x^*\|^2 + 2\varepsilon_n\|x_n - x^*\| \\
&+2\alpha_n\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle + 2\alpha_n\gamma_n\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\| \\
&+\alpha_n\gamma_n\|\lambda g(x_n) - Bx^*\|^2 + \alpha_n\gamma_n\nu^2\|d_n - x^*\|^2 \\
= \;& (1-2\alpha_n(\nu - \lambda k))\|x_n - x^*\|^2 + \alpha_n(2\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle \\
&+2\gamma_n\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\| + \gamma_n\|\lambda g(x_n) - Bx^*\|^2 \\
&+\gamma_n\nu^2\|d_n - x^*\|^2 + \frac{2\varepsilon_n}{\alpha_n}\|d_n - x^*\| + \frac{2\varepsilon_n}{\alpha_n}\|x_n - x^*\|) \\
= \;& (1-2\alpha_n(\nu - \lambda k))\|x_n - x^*\|^2 + 2\alpha_n(\nu - \lambda k)\Big(\frac{\langle \lambda g(x^*) - Bx^*, y_n - x^*\rangle}{\nu - \lambda k} \\
&+\frac{\gamma_n\|\lambda g(x_n) - Bx^*\|\|By_n - Bx^*\|}{\nu - \lambda k} + \frac{\gamma_n\|\lambda g(x_n) - Bx^*\|^2}{2(\nu - \lambda k)} \\
&+\frac{\gamma_n\nu^2\|d_n - x^*\|^2}{2(\nu - \lambda k)} + \frac{\varepsilon_n}{\alpha_n(\nu - \lambda k)}\|d_n - x^*\| + \frac{\varepsilon_n}{\alpha_n(\nu - \lambda k)}\|x_n - x^*\|\Big).
\end{aligned} \tag{3.11}
$$

On the other hand, using the definition of $\{z_n\}$, Lemma 2.2, (3.4) and (3.10), we get

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\leq \;& (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|d_n - x^*\|^2 \\
= \;& (1-\beta_n)\|\gamma_n\lambda g(x_n) + (I - \gamma_n B)y_n - x^*\|^2 + \beta_n\|d_n - x^*\|^2 \\
= \;& (1-\beta_n)\|(y_n - x^*) + \gamma_n(\lambda g(x_n) - By_n)\|^2 + \beta_n\|d_n - x^*\|^2 \\
\leq \;& (1-\beta_n)(\|y_n - x^*\|^2 + 2\gamma_n\langle \lambda g(x_n) - By_n, z_n - x^*\rangle) + \beta_n\|d_n - x^*\|^2 \\
\leq \;& \beta_n\|d_n - x^*\|^2 + (1-\beta_n)\Big(\|d_n - x^*\|^2 - (4-\rho_n)(1-\delta_n)\tau_n f_n(d_n) \\
&-\delta_n(1-\delta_n)\|\sigma_n(Sd_n - d_n) + \tau_n A^*(I - P_{Q_n})Ad_n\|^2 \\
&-\Big\|(I - P_{C_n})\Big((1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n) \\
&+\delta_n((1-\sigma_n)d_n + \sigma_n Sd_n)\Big)\Big\|^2\Big) + 2\alpha_n\langle \lambda g(x_n) - By_n, z_n - x^*\rangle \\
\leq \;& \|x_n - x^*\|^2 + 2\varepsilon_n\|d_n - x^*\| - (1-\beta_n)(4-\rho_n)(1-\delta_n)\tau_n f_n(d_n) \\
&-(1-\beta_n)\delta_n(1-\delta_n)\|\sigma_n(Sd_n - d_n) + \tau_n A^*(I - P_{Q_n})Ad_n\|^2 \\
&-(1-\beta_n)\Big\|(I - P_{C_n})\Big((1-\delta_n)(d_n - \tau_n A^*(I - P_{Q_n})Ad_n)
\end{aligned}
$$

$$+\delta_n((1-\sigma_n)d_n+\sigma_n Sd_n)\Big)\Big\|^2+2\alpha_n\langle\lambda g(x_n)-By_n,z_n-x^*\rangle. \tag{3.12}$$

Apply the following:

$$\begin{aligned}
\Theta_n &= 2\alpha_n(\nu-\lambda k),\\
\Upsilon_n &= \frac{1}{\nu-\lambda k}\Big(\langle\lambda g(x^*)-Bx^*,y_n-x^*\rangle+\gamma_n\|\lambda g(x_n)-Bx^*\|\|By_n-Bx^*\|\\
&\quad+\frac{\gamma_n\|\lambda g(x_n)-Bx^*\|^2}{2}+\frac{\gamma_n\nu^2\|d_n-x^*\|^2}{2}+\frac{\varepsilon_n\|d_n-x^*\|}{\alpha_n}+\frac{\varepsilon_n\|x_n-x^*\|}{\alpha_n}\Big),\\
\Pi_n &= (1-\beta_n)(4-\rho_n)(1-\delta_n)\tau_n f_n(d_n)\\
&\quad+(1-\beta_n)\delta_n(1-\delta_n)\|\sigma_n(Sd_n-d_n)+\tau_n A^*(I-P_{Q_n})Ad_n\|^2\\
&\quad+(1-\beta_n)\Big\|(I-P_{C_n})\Big((1-\delta_n)(d_n-\tau_n A^*(I-P_{Q_n})Ad_n)\\
&\quad+\delta_n((1-\sigma_n)d_n+\sigma_n Sd_n)\Big)\Big\|^2,\\
\Gamma_n &= 2\alpha_n\langle\lambda g(x_n)-By_n,z_n-x^*\rangle+2\varepsilon_n\|d_n-x^*\|.
\end{aligned}$$

Thus, we can rewrite (3.11) and (3.12):

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\Theta_n)\|x_n-x^*\|^2+\Theta_n\Upsilon_n,\\
\|x_{n+1}-x^*\|^2 &\le \|x_n-x^*\|^2-\Pi_n+\Gamma_n.
\end{aligned}$$

It is easy for us to know that $\lim_{n\to\infty}\Theta_n=0$, $\sum_{n=1}^{\infty}\Theta_n=\infty$ and $\lim_{n\to\infty}\Gamma_n=0$. If we prove that $\limsup_{i\to\infty}\Upsilon_{n_i}\le 0$ when $\lim_{i\to\infty}\Pi_{n_i}=0$ for any subsequence $\{n_i\}\subset\{n\}$, we can get that $\lim_{n\to\infty}\|x_n-x^*\|=0$ by Lemma 2.6.

Assume that

$$\lim_{i\to\infty}\Pi_{n_i}=0.$$

Using the conditions of $\{\beta_n\}$, $\{\delta_n\}$ and $\{\sigma_n\}$, we have

$$\lim_{i\to\infty}(4-\rho_{n_i})\tau_{n_i}f_{n_i}(d_{n_i})=0, \tag{3.13}$$

$$\lim_{i\to\infty}\|\sigma_{n_i}(Sd_{n_i}-d_{n_i})+\tau_{n_i}A^*(I-P_{Q_{n_i}})Ad_{n_i}\|^2=0, \tag{3.14}$$

$$\lim_{i\to\infty}\Big\|(I-P_{C_{n_i}})\Big((1-\delta_{n_i})(d_{n_i}-\tau_{n_i}\nabla f_{n_i}(d_{n_i}))+\delta_{n_i}S_{\sigma_{n_i}}d_{n_i}\Big)\Big\|^2=0. \tag{3.15}$$

From (3.13) and the conditions of $\{\rho_n\}$, we deduce that

$$\tau_{n_i}f_{n_i}(d_{n_i})\to 0. \tag{3.16}$$

If $\|\nabla f_{n_i}(d_{n_i})\|\ne 0$, we have that $f_{n_i}(d_{n_i})\ne 0$ and

$$\begin{aligned}
\tau_{n_i} &= \frac{\rho_{n_i}f_{n_i}(d_{n_i})}{\|\nabla f_{n_i}(d_{n_i})\|^2}\\
&= \frac{\rho_{n_i}\|(I-P_{Q_{n_i}})Ad_{n_i}\|^2}{2\|A^*(I-P_{Q_{n_i}})Ad_{n_i}\|^2}
\end{aligned}$$

$$\begin{aligned} &\geq \frac{\rho_{n_i}\|(I-P_{Q_{n_i}})Ad_{n_i}\|^2}{2\|A\|^2\|(I-P_{Q_{n_i}})Ad_{n_i}\|^2} \\ &= \frac{\rho_{n_i}}{2\|A\|^2}, \end{aligned}$$

which means that $\inf_{i\in\mathbb{N}} \tau_{n_i} > 0$. Hence, we obtain $f_{n_i}(d_{n_i}) \to 0$ as $i \to \infty$. This implies that

$$\lim_{i\to\infty} \|(I-P_{Q_{n_i}})Ad_{n_i}\| = 0.$$

Using (3.16) and the conditions of $\{\rho_n\}$, we have

$$\tau_{n_i}\|\nabla f_{n_i}(d_{n_i})\| = \sqrt{\rho_{n_i}\tau_{n_i}f_{n_i}(d_{n_i})} \to 0. \tag{3.17}$$

Moreover, from (3.14) and the condition of $\{\sigma_n\}$, we get

$$\|Sd_{n_i} - d_{n_i}\| \to 0. \tag{3.18}$$

Using (3.15) and the definition of y_n, we have

$$y_{n_i} = P_{C_{n_i}}\Big((1-\delta_{n_i})(d_{n_i} - \tau_{n_i}A^*(I-P_{Q_{n_i}})Ad_{n_i}) + \delta_{n_i}((1-\sigma_{n_i})d_{n_i} + \sigma_{n_i}Sd_{n_i})\Big);$$

again using (3.15), we obtain

$$\Big\|(1-\delta_{n_i})(d_{n_i} - \tau_{n_i}\nabla f_{n_i}(d_{n_i})) + \delta_{n_i}S_{\sigma_{n_i}}d_{n_i} - y_{n_i}\Big\| \to 0,$$

i.e.,

$$\|(1-\delta_{n_i})d_{n_i} - (1-\delta_{n_i})\tau_{n_i}\nabla f_{n_i}(d_{n_i}) + \delta_{n_i}S_{\sigma_{n_i}}d_{n_i} - y_{n_i}\| \to 0. \tag{3.19}$$

Due to (3.17) and (3.19), we get

$$\Big\|(1-\delta_{n_i})d_{n_i} + \delta_{n_i}((1-\sigma_{n_i})d_{n_i} + \sigma_{n_i}Sd_{n_i}) - y_{n_i}\Big\| \to 0,$$

i.e.,

$$\|d_{n_i} - y_{n_i} + \delta_{n_i}\sigma_{n_i}(Sd_{n_i} - d_{n_i})\| \to 0. \tag{3.20}$$

Then, we have

$$\begin{aligned} &\|d_{n_i} - y_{n_i}\| \\ = \;& \|d_{n_i} - y_{n_i} + \delta_{n_i}\sigma_{n_i}(Sd_{n_i} - d_{n_i}) - \delta_{n_i}\sigma_{n_i}(Sd_{n_i} - d_{n_i})\| \\ \leq \;& \|d_{n_i} - y_{n_i} + \delta_{n_i}\sigma_{n_i}(Sd_{n_i} - d_{n_i})\| + \|\delta_{n_i}\sigma_{n_i}(Sd_{n_i} - d_{n_i})\|; \end{aligned}$$

combining (3.18) with (3.20), we obtain

$$\|d_{n_i} - y_{n_i}\| \to 0. \tag{3.21}$$

Moreover, using the definition of $\{\mu_n\}$ and the boundedness of $\{x_n\}$, we can have

$$\begin{aligned} &\|x_{n_i} - d_{n_i}\| \\ = \;& \|x_{n_i} - x_{n_i} - \mu_{n_i}(x_{n_i} - x_{n_i-1})\| \end{aligned}$$

$$
\begin{aligned}
&= \mu_{n_i}\|x_{n_i} - x_{n_i-1}\| \\
&\leq \varepsilon_{n_i} \to 0. \qquad (3.22)
\end{aligned}
$$

Next, combining (3.21) and (3.22), we have

$$\|x_{n_i} - y_{n_i}\| \leq \|x_{n_i} - d_{n_i}\| + \|d_{n_i} - y_{n_i}\| \to 0. \qquad (3.23)$$

It is easy for us to know that $\omega_w(d_{n_i}) \subset \mathrm{Fix}(S)$ by combining (3.18) and the case that $I - S$ is demiclosed at zero. Now, we choose a subsequence $\{d_{n_{i_j}}\}$ of $\{d_{n_i}\}$ to satisfy

$$\limsup_{i\to\infty}\langle \lambda g(x^*) - Bx^*, d_{n_i} - x^*\rangle = \lim_{j\to\infty}\langle \lambda g(x^*) - Bx^*, d_{n_{i_j}} - x^*\rangle.$$

Without loss of generality, we suppose that $d_{n_{i_j}} \rightharpoonup z'$. It is easy to get that $y_{n_{i_j}} \rightharpoonup z'$ and $x_{n_{i_j}} \rightharpoonup z'$ by using (3.21) and (3.23). Now, we show that $z' \in C$. We know that $y_{n_{i_j}} \in C_{n_{i_j}}$ by the definition of $y_{n_{i_j}}$; this implies that

$$\phi(x_{n_{i_j}}) \leq \langle \vartheta_{n_{i_j}}, x_{n_{i_j}} - y_{n_{i_j}}\rangle, \qquad (3.24)$$

where $\vartheta_{n_{i_j}} \in \partial\phi(x_{n_{i_j}})$. From (3.23) and the boundedness of $\partial\phi$, we have

$$\phi(x_{n_{i_j}}) \leq \|\vartheta_{n_{i_j}}\|\|x_{n_{i_j}} - y_{n_{i_j}}\| \to 0 \qquad (3.25)$$

as $n \to \infty$. Because of the weakly lower semi-continuous nature of ϕ, $x_{n_{i_j}} \rightharpoonup z'$ and (3.25), the following holds:

$$\phi(z') \leq \liminf_{j\to\infty} \phi(x_{n_{i_j}}) \leq 0.$$

Hence $z' \in C$.

Next, we show that $Az' \in Q$. Since $P_{Q_{n_{i_j}}}(Ax_{n_{i_j}}) \in Q_{n_{i_j}}$, we have

$$\varphi(Ax_{n_{i_j}}) \leq \langle \chi_{n_{i_j}}, Ax_{n_{i_j}} - P_{Q_{n_{i_j}}}(Ax_{n_{i_j}})\rangle, \qquad (3.26)$$

where $\chi_{n_{i_j}} \in \partial\varphi(Ax_{n_{i_j}})$. Given that $\lim_{i\to\infty}\|(I - P_{Q_{n_i}})Ad_{n_i}\| = 0$ and (3.22), we get that $\lim_{i\to\infty}\|(I - P_{Q_{n_i}})Ax_{n_i}\| = 0$. By (3.26) and the boundedness of $\partial\varphi$, we have

$$\varphi(Ax_{n_{i_j}}) \leq \|\chi_{n_{i_j}}\|\|Ax_{n_{i_j}} - P_{Q_{n_{i_j}}}(Ax_{n_{i_j}})\| \to 0 \qquad (3.27)$$

as $n \to \infty$. In view of $x_{n_{i_j}} \rightharpoonup z'$, $Ax_{n_{i_j}} \rightharpoonup Az'$, the weakly lower semi-continuous nature of φ and (3.27), we can get

$$\varphi(Az') \leq \liminf_{j\to\infty} \varphi(Ax_{n_{i_j}}) \leq 0.$$

Hence $Az' \in Q$.

From the above proof process, we can derive that $z' \in \mathrm{Fix}(S) \cap \Delta$. Now, using (3.21), we get

$$
\begin{aligned}
&\limsup_{i\to\infty}\langle \lambda g(x^*) - Bx^*, y_{n_i} - x^*\rangle \\
&= \limsup_{i\to\infty}\langle \lambda g(x^*) - Bx^*, d_{n_i} - x^*\rangle \\
&= \lim_{j\to\infty}\langle \lambda g(x^*) - Bx^*, d_{n_{i_j}} - x^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle \lambda g(x^*) - Bx^*, z' - x^* \rangle \\
&\leq 0.
\end{aligned}
$$

It means that

$$\lim_{i\to\infty} \Upsilon_{n_i} \leq 0.$$

□

4. Numerical experiments

Now, we present a comparison of Algorithm 1 with other algorithms by describing two numerical experiments. All of the programs were written in Matlab 9.5 and performed on a desktop PC with AMD Ryzen 7 4800U with Radeon Graphics 1.80 GHz, RAM 16.0GB.

Example 4.1. We assume that $H_1 = H_2 = C = \mathbb{R}^4$ and $Q = \{b\}$, where $b \in \mathbb{R}^4$; then, we have that $\Delta = \{x \in \mathbb{R}^4 : Ax = b\}$. Considering the system of linear equations $Ax = b$, let

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 2 \\ 3 & -1 & -1 & -2 \\ 3 & 5 & 2 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 1 \\ -6 \\ -15 \end{pmatrix}.$$

By calculation, we can find that $Ax = b$ has a unique solution $x^* = (-1, -2, 1, 2)^{\mathrm{T}}$. Let

$$Sx = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{3}{4} \end{pmatrix} x + \begin{pmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}, \quad \forall x \in \mathbb{R}^4.$$

By calculation, we can find that the mapping S is nonexpansive and $x^* = (-1, -2, 1, 2)^{\mathrm{T}} \in \mathrm{Fix}(S)$.

In Algorithm 1 and scheme (1.4), we let $\alpha_n = \frac{1}{10n}, \beta_n = 0.2$, $\delta_n = 0.5$, $x_1 = (1, 2, 3, 4)^{\mathrm{T}}$ and $g = 0.2I$; in scheme (1.4), we let $\tau_n = \frac{7}{4\|A\|^2}$; in Algorithm 1, we let $\phi(x) = 0$, $\forall x \in \mathbb{R}^4$, $\varphi(y) = \|y - b\|$, $\forall y \in \mathbb{R}^4$, $B = I$, $\lambda = 1$, $\varepsilon_n = \frac{1}{n^2}$, $\mu = 1.5$, $\sigma_n = 1$, $\rho_n = 3.5$, $\tau = 1$ and $x_0 = (1, 2, 3, 4)^{\mathrm{T}}$. The results of numercial experiments are revealed in Table 1 and Figure 1.

From Table 1, we can find that x_n is closer to the exact solution x^* with an increase of the number of iterations. We can also see that errors are closer to 0. Therefore, it can be concluded that Algorithm 1 is feasible. From Figure 1, we can find that the number of iterations for Algorithm 1 is less than that for scheme (1.4).

Table 1. Numerical results of Algorithm 1 for Example 4.1.

$n-1$	$x_n^{(1)}$	$x_n^{(2)}$	$x_n^{(3)}$	$x_n^{(4)}$	E_n
0	1.0000	2.0000	3.0000	4.0000	5.2915E+00
10	-0.9884	-1.9774	0.9764	1.9905	3.5918E-02
50	-0.9987	-1.9959	0.9963	1.9969	6.4333E-03
100	-0.9994	-1.9980	0.9982	1.9985	3.1627E-03
500	-0.9999	-1.9996	0.9996	1.9997	6.3660E-04
1000	-0.9999	-1.9998	0.9998	1.9998	3.1853E-04
5000	-1.0000	-2.0000	1.0000	2.0000	6.3743E-05
10000	-1.0000	-2.0000	1.0000	2.0000	3.1874E-05

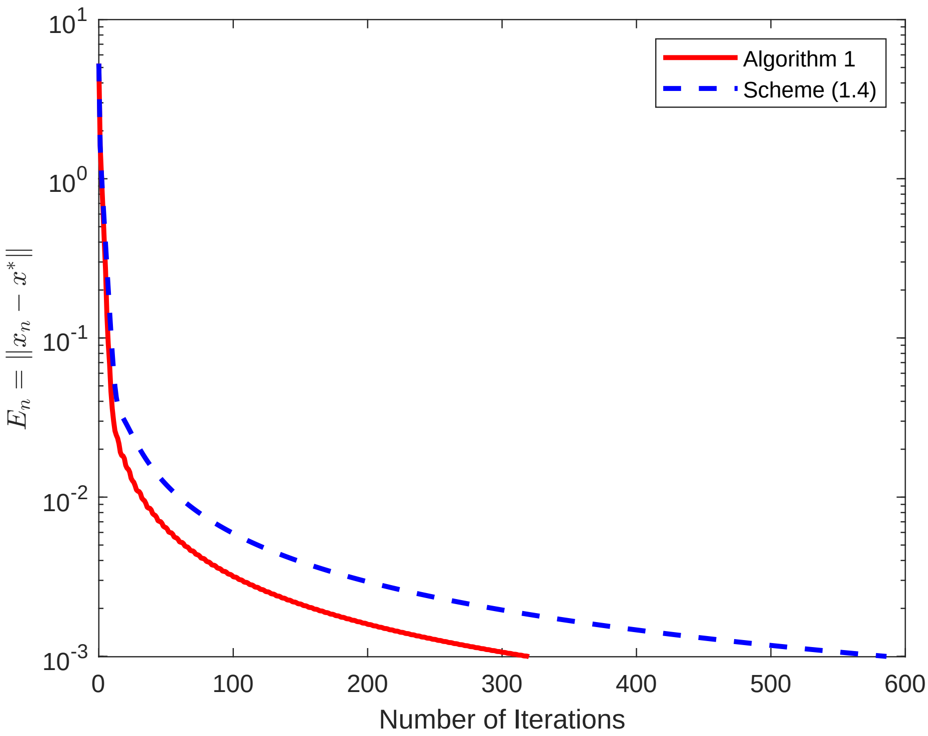

Figure 1. Comparison of Algorithm 1 and scheme (1.4) for Example 4.1.

Example 4.2. Assume that $H_1 = \mathbb{R}$, $H_2 = \mathbb{R}^3$, $C = [-2, 6]$, $Q = \{(y_a, y_b, y_c)^{\mathrm{T}} : |y_a| + |y_b| + |y_c| \leq 3\}$,

$$A = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix},$$

$$Sx = \frac{x}{2} \sin x, \quad \forall x \in \mathbb{R}.$$

We can see that S is not nonexpansive but quasi-nonexpansive [23]. By calculation, we can find that $\mathrm{Fix}(S) = \{0\}$. It is easy to obtain that $0 \in \Delta$. We denote $x^* = 0$.

In Algorithm 1 and scheme (1.6), we let $x_0 = x_1 = 1$, $\varepsilon_n = \frac{1}{n^2}$, $\mu = 0.9$, $\beta_n = 0.5$, $\rho_n = 3.5$, $\delta_n = 0.4$, $g(x) = \frac{1}{3}\sin x$ and $\alpha_n = \frac{1}{8n^p}$, where $0 < p \leq 1$; in scheme (1.6), we used the function *quadprog* to compute the projection over Q by using Matlab 9.5 Optimization Toolbox; In Algorithm 1, we let $\phi(x) = (x-6)(x+2)$, $\forall x \in \mathbb{R}$, $\varphi(y) = |y_a| + |y_b| + |y_c| - 3$, $\forall y = (y_a, y_b, y_c)^{\mathrm{T}} \in \mathbb{R}^3$, $B = I$, $\lambda = 1$, $\sigma_n = 1$ and $\tau = 1$. We used $\|x_n - x^*\| \leq 10^{-8}$ as the stopping criterion. The results of numerical experiments for different values of p are shown in Table 2. The convergence behavior for $p = 0.5$ is shown in Figures 2 and 3.

Table 2. Numerical results of Algorithm 1 and scheme (1.6) for Example 4.2.

p	Algorithm 1		Scheme (1.6)	
	Iter.	Time [sec]	Iter.	Time [sec]
0.5	86	0.0515	86	0.7586
0.6	86	0.0534	86	0.7643
0.7	73	0.0517	73	0.7125
0.8	80	0.0522	80	0.7473
0.9	80	0.0545	80	0.7242
1	87	0.0522	87	0.8208

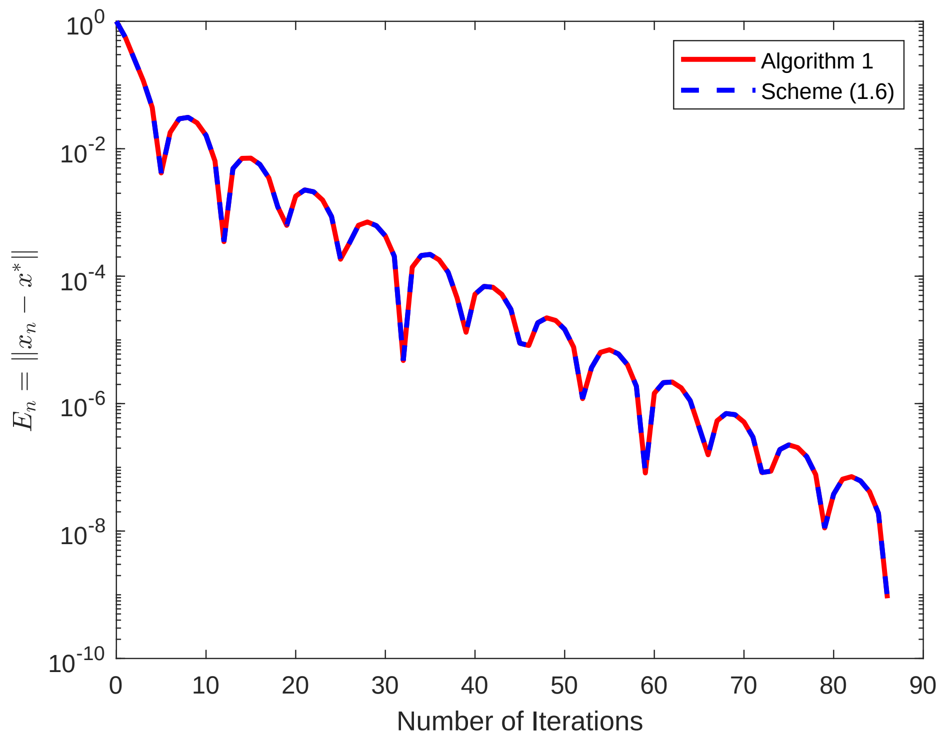

Figure 2. Comparison between the number of iterations of Algorithm 1 and scheme (1.6) for Example 4.2 with $p = 0.5$.

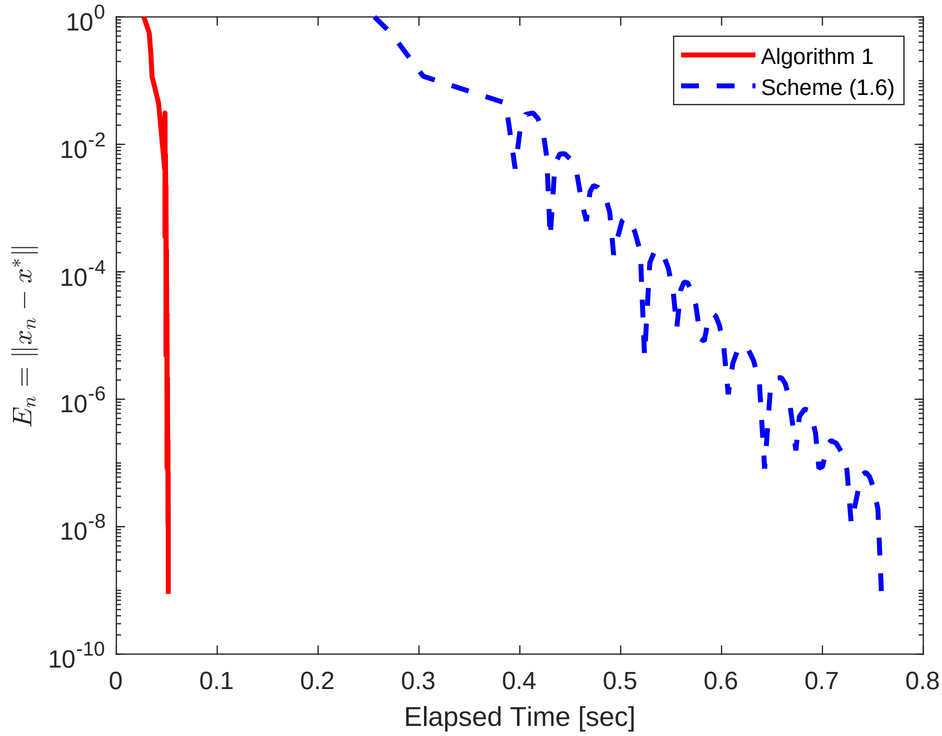

Figure 3. Comparison between elapsed time of Algorithm 1 and scheme (1.6) for Example 4.2 with $p = 0.5$.

5. Conclusions

In this work, we developed a method for solving the SFP and the fixed-point problem involving demicontractive mapping. In Algorithm 1, we have expanded the nonexpansive mapping to the demicontractive mapping, selected a new step size and added the inertial method based on scheme (1.4). Our Algorithm 1 exhibits better convergence behavior than scheme (1.4). Additionally, we have extended the quasi-nonexpansive mapping to the demicontractive mapping, applied projection onto a half-space and selected a new step size in our Algorithm 1 based on scheme (1.6). Our Algorithm 1 also demonstrated superior convergence behavior compared to scheme (1.6). From the above, our algorithm exhibits a superior convergence rate and a wider range for the operator, thus making it applicable in a broader range of scenarios. Furthermore, we have presented two numerical results that demonstrate the feasibility and effectiveness of our method. In the following work, we intend to extend the demicontractive mapping to the quasi-pseudocontractive mapping and extend the SFP to the multiple-set split feasibility problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algor.*, **8** (1994), 221–239. https://doi.org/10.1007/BF02142692
2. Y. Yao, M. Postolache, X. Qin, J. Yao, Iterative algorithms for the proximal split feasibility problem, *U. P. B. Sci. Bull. A*, **80** (2018), 37–44.
3. A. E. Al-Mazrooei, A. Latif, X. Qin, J. C. Yao, Fixed point algorithms for split feasibility problems, *Fixed Point Theor.*, **20** (2019), 245–254. https://doi.org/10.24193/fpt-ro.2019.1.16
4. Q. Dong, S. He, Y. Zhao, On global convergence rate of two acceleration projection algorithms for solving the multiple-sets split feasibility problem, *Filomat*, **30** (2016), 3243–3252. https://doi.org/10.2298/FIL1612243D
5. J. Zhao, D. Hou, A self-adaptive iterative algorithm for the split common fixed problems, *Numer Algor.*, **82** (2019), 1047–1063. https://doi.org/10.1007/s11075-018-0640-x
6. D. Hou, J. Zhao, X. Wang, Weak convergence of a primal-dual algorithm for split common fixed-point problems in Hilbert spaces, *J. Appl. Numer. Optim.*, **2** (2020), 187–197.
7. C. Byrne, Iterative oblique projection onto convex set and the split feasibility problem, *Inverse Probl.*, **18** (2002), 441–453. https://doi.org/10.1088/0266-5611/18/2/310
8. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Probl.*, **20** (2004), 103–120. https://doi.org/10.1088/0266-5611/20/1/006
9. G. López, V. Martín-Márquez, F. Wang, H. K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, *Inverse Probl.*, **28** (2012), 085004. https://doi.org/10.1088/0266-5611/28/8/085004
10. M. Fukushima, A relaxed projection method for variational inequalities, *Math. Program.*, **35** (1986), 58–70. https://doi.org/10.1007/BF01589441
11. Q. Yang, The relaxed CQ algorithm for solving the split feasibility problem, *Inverse Probl.*, **20** (2004), 1261–1266. https://doi.org/10.1088/0266-5611/20/4/014
12. X. Qin, L. Wang, A fixed point method for solving a split feasibility problem in Hilbert spaces, *RACSAM*, **13** (2019), 215–325. https://doi.org/10.1007/s13398-017-0476-6
13. Y. H. Wang, T. T. Xu, J. C. Yao, B. N. Jiang, Self-adaptive method and inertial modification for solving the split feasibility problem and fixed-point problem of quasi-nonexpansive mapping, *Mathematics*, **10** (2022), 1612. https://doi.org/10.3390/math10091612

14. M. Tian, A general iterative algorithm for nonexpansive mapping in Hilbert spaces, *Nonlinear Anal.*, **73** (2010), 689–694. https://doi.org/10.1016/j.na.2010.03.058
15. L. J. Kwari, J. Sunday, J. N. Ndam, A. Shokri, Y. Wang, On the Simulations of Second-Order Oscillatory Problems with Applications to Physical Systems, *Axioms*, **12** (2013), 282. https://doi.org/10.2139/ssrn.4183287 https://doi.org/10.3390/axioms12030282
16. H. H. Bauschke, P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert space*, New York: Springer, 2011.
17. S. Kesornprom, N. Pholasa, P. Cholamjiak, On the convergence analysis of the gradient-CQ algorithms for the split feasibility problem, *Numer. Alogr.*, **84** (2020), 997–1017. https://doi.org/10.1007/s11075-019-00790-y
18. S. Suantai , S. Kesornprom , N. Pholasa , Y. J. Cho, P. Cholamjiak, A relaxed projections method using a new linesearch for the split feasibility problem, *AIMS Mathematics*, **6** (2021), 2690–2703.
19. Y. H. Wang, M. Y. Yuan, B. N. Jiang, Multi-step inertial hybrid and shrinking Tseng's algorithm with Meir-Keeler contractions for variational inclusion problems, *Mathematics*, **9** (2021), 1548. https://doi.org/10.3390/math9131548
20. W. L. Sun, G. Lu, Y. F. Jin, C. Park, Self-adaptive algorithms for the split problem of the quasi-pseudocontractive operators in Hilbert spaces, *AIMS Mathematics*, **7** (2022), 8715–8732. https://doi.org/10.3934/math.2022487
21. D. V. Thong, Viscosity approximation methods for solving fixed point problems and split common fixed point problems, *J. Fixed Point Theory Appl.*, **19** (2017), 1481–1499. https://doi.org/10.1007/s11784-016-0323-y
22. S. He, C. Yang, Solving the variational inequality problem defined on intersection of finite level sets, *Abstr. Appl. Anal.*, **2013** (2013), 942315. https://doi.org/10.1155/2013/942315
23. M. Tian, M. Tong, Self-adaptive subgradient extragradient method with inertial modification for solving monotone variational inequality problems and quasi-nonexpansive fixed point problems, *J. Inequal. Appl.*, **2019** (2019), 7. https://doi.org/10.1186/s13660-019-1958-1