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Research article

On the asymptotic stability of linear difference equations with time-varying coefficients

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Abstract: Issues of the asymptotic stability for linear difference equations with time-varying coefficients are discussed. It is shown that, in contrast to equations with constant coefficients, the condition of Schur stability of the characteristic polynomial for a linear difference equation with time-varying coefficients is neither necessary nor sufficient for the asymptotic stability of the difference equation. It is proved that the analog of Kharitonov's theorem on robust stability and the edge theorem do not hold for a difference equation if the coefficients of the equation are not constant.

Keywords: linear difference equations; asymptotic stability; time-varying system; Schur stability; robust stability
Mathematics Subject Classification: 39A06, 39A30

1. Introduction

A big amount of works are devoted to the study of the Schur stability of a family of interval polynomials and interval matrices. Reviews of well-known results on the Schur stability of uncertain systems with constant coefficients can be found in articles [1–6] and monographs [7–9]. Necessary conditions and sufficient conditions for the robust stability of linear discrete-time equations and systems of equations with discrete time are presented in the papers [10–14].

Consider a linear homogeneous difference equation of the nth order with constant coefficients

$$x(t+n) + p_1 x(t+n-1) + \ldots + p_n x(t) = 0, \qquad (1.1)$$
$$t \in \mathbb{Z}, \quad x \in \mathbb{R}, \quad p_j \in \mathbb{R}, \quad j = \overline{1,n}.$$

Using the Eq (1.1), we construct the characteristic polynomial

$$\chi(\lambda) = \lambda^n + p_1\lambda^{n-1} + \ldots + p_n. \qquad (1.2)$$

For a linear system of difference equations with constant coefficients

$$z(t+1) = Az(t), \quad t \in \mathbb{Z}, \quad z \in \mathbb{R}^n, \tag{1.3}$$

the characteristic polynomial has the form $\chi(\lambda) = \det(\lambda I - A)$.

Recall that a polynomial

$$\varphi(\lambda) = s_0\lambda^n + s_1\lambda^{n-1} + \ldots + s_{n-1}\lambda + s_n \tag{1.4}$$

is said to be *Schur stable* if all its roots lie in the open unit disc of the complex plane.

The well-known necessary and sufficient conditions for the Schur stability of the polynomial $\varphi(\lambda)$ provide, for example, the Schur-Kohn criterion [15, Ch. 6, p. 498], the Jury stability test [16, Sect. 4.3, p. 185] or the method based on transforming the unit disc into the left half-plane [16, Sect. 4.3, p. 191].

The condition of the Schur stability of the characteristic polynomial $\chi(\lambda)$ is a necessary and sufficient condition for the asymptotic stability of the difference equation (1.1) or the system (1.3), respectively.

Similarly, consider a linear differential equation of the nth order with constant coefficients

$$x^{(n)} + q_1x^{(n-1)} + q_2x^{(n-2)} + \ldots + q_nx = 0, \tag{1.5}$$
$$t \in \mathbb{R}, \quad x \in \mathbb{R}, \quad q_j \in \mathbb{R}, \quad j = \overline{1, n}.$$

Some polynomial (1.4) is said to be *Hurwitz stable* if all its roots lie in the open left half-plane of the complex plane. The condition of the Hurwitz stability of the characteristic polynomial $\psi(\lambda) = \lambda^n + q_1\lambda^{n-1} + \ldots + q_n$ of Eq (1.5) is a necessary and sufficient condition for the asymptotic stability of the differential equation (1.5).

If the coefficients q_j of Eq (1.5) are not constant, then there is no such connection between the roots of the characteristic polynomial and the asymptotic stability of (systems of) differential equations (see, for example, [17, Ch. IV, § 9], [18, p. 3], [19]). These issues were discussed in a recent work [20]. In addition, in [20] it was proved that the analog of Kharitonov's theorem on robust stability does not hold if the coefficients of the differential equation (1.5) are not constant.

In the present work, we show that there is no such connection for (systems of) difference equations either. In addition, it is known that some analogs of Kharitonov's theorems on robust stability take place for systems with discrete time. We will prove that these assertions do not hold if the coefficients of the equation are not constant, i.e., the condition for the coefficients of the equation (system) to be constant is essential in these robust stability theorems.

The paper is organized as follows. In Section 2, we prove that, for a linear time-varying system with discrete time, the condition of the Schur stability of the characteristic polynomial of the matrix of the system is neither sufficient nor necessary for the asymptotic stability of the system. In Section 3, we formulate similar Theorems 1 and 2 for a linear difference equation with time-varying coefficients saying that the condition of the Schur stability of the characteristic polynomial of the equation is neither sufficient nor necessary for the asymptotic stability of the equation. In Section 4, we formulate Theorem 5 saying that, for a linear difference equation with time-varying coefficients, the condition of the Schur stability of every characteristic polynomial in the family of the edge polynomials is not sufficient for the asymptotic stability of the family of interval polynomials. In Section 5, we give the proof of Theorems 5 and 1. In Section 6, we give the proof of Theorem 2. In Section 7, we show that the (counter)examples given in the previous sections are not valid when the dimension of the equation is equal to 1. In Section 8, conclusions are given.

2. Asymptotic stability of a linear time-varying system with discrete time

Consider a linear discrete-time system of equations with time-varying coefficients

$$z(t+1) = A(t)z(t), \quad t \in \mathbb{Z}, \quad z \in \mathbb{R}^n. \tag{2.1}$$

Proposition 1. *There exists a system* (2.1) *such that the following properties hold:*

(1) System (2.1) *is unstable.*

(2) The characteristic polynomial of the matrix of the system (2.1) *has constant coefficients and is Schur stable.*

Proof. Take a system

$$\xi(t+1) = B\xi(t), \quad t \in \mathbb{Z}, \quad \xi \in \mathbb{R}^n, \tag{2.2}$$

where $n = 2$, with the constant matrix

$$B = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}. \tag{2.3}$$

The characteristic polynomial of the matrix (2.3) is $\lambda^2 - 2\lambda + 3/4$. Its roots are $\lambda_1 = 1/2$, $\lambda_2 = 3/2$. We have $\lambda_2 > 1$. Hence, the system (2.2), (2.3) is unstable.

Construct the matrix

$$L(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ t = 2k; \quad L(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ t = 2k+1; \quad k \in \mathbb{Z}. \tag{2.4}$$

Then, the matrix $L(t)$ is periodic and $\sup_{t\in\mathbb{Z}} \left(\|L(t)\| + \|L^{-1}(t)\| \right) < +\infty$. Let us make a transformation

$$z(t) = L(t)\xi(t). \tag{2.5}$$

Then, this transformation is a Lyapunov transformation (see [21, p. 15]). Therefore, it preserves the properties of (asymptotic) stability and instability. The transformation (2.5) reduces system (2.2) to system (2.1) where

$$A(t) = L(t+1)BL^{-1}(t). \tag{2.6}$$

Calculating the matrix $A(t)$ by the formula (2.6), from (2.3) and (2.4) we get

$$A(t) = \begin{bmatrix} -1/2 & -1 \\ 1 & 1/2 \end{bmatrix}, \ t = 2k; \quad A(t) = \begin{bmatrix} -1/2 & 1 \\ -1 & 1/2 \end{bmatrix}, \ t = 2k+1; \quad k \in \mathbb{Z}. \tag{2.7}$$

The system (2.1) with the matrix (2.7) is unstable because the system (2.2), (2.3) is unstable. For any $t \in \mathbb{Z}$, the characteristic polynomial of the matrix (2.7) is equal to $\lambda^2 + 3/4$. Its roots are $\lambda_{1,2} = \pm i\sqrt{3}/2$. We have $|\lambda_1| = |\lambda_2| = \sqrt{3}/2 < 1$, i.e., the characteristic polynomial is Schur stable. □

Proposition 2. *There exists a system* (2.1) *such that the following properties hold:*

(1) System (2.1) *is asymptotically stable.*

(2) The characteristic polynomial of the matrix of the system (2.1) *has constant coefficients and is not Schur stable.*

Proof. Take a system (2.2), where $n = 2$ with the constant matrix

$$B = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}. \tag{2.8}$$

The characteristic polynomial of the matrix (2.8) is $\lambda^2 - \lambda + 1/2$. Its roots are $\lambda_1 = (1+i)/2$, $\lambda_2 = (1-i)/2$. We have $|\lambda_1| = |\lambda_2| = 1/\sqrt{2} < 1$. Hence, the system (2.2), (2.8) is asymptotically stable.

Construct the matrix

$$L(t) = \begin{bmatrix} 0 & 1/2 \\ 2 & 0 \end{bmatrix}, \ t = 2k; \quad L(t) = \begin{bmatrix} 0 & 2 \\ 1/2 & 0 \end{bmatrix}, \ t = 2k+1; \quad k \in \mathbb{Z}. \tag{2.9}$$

Then, the matrix $L(t)$ is periodic and $\sup_{t\in\mathbb{Z}} \left(\|L(t)\| + \|L^{-1}(t)\| \right) < +\infty$. Let us make a transformation (2.5). Then, this transformation is a Lyapunov transformation. The transformation (2.5) reduces system (2.2) to system (2.1) where $A(t)$ is defined by the equality (2.6). Calculating the matrix $A(t)$ by the formula (2.6), from (2.8) and (2.9) we get

$$A(t) = \begin{bmatrix} 2 & 1/2 \\ -1/2 & 1/8 \end{bmatrix}, \ t = 2k; \quad A(t) = \begin{bmatrix} 1/8 & 1/2 \\ -1/2 & 2 \end{bmatrix}, \ t = 2k+1; \quad k \in \mathbb{Z}. \tag{2.10}$$

The system (2.1) with the matrix (2.10) is asymptotically stable because the system (2.2), (2.8) is asymptotically stable. For any $t \in \mathbb{Z}$, the characteristic polynomial of the matrix (2.10) is equal to $\lambda^2 - (17/8)\lambda + (1/2)$. Its roots are $\lambda_1 = (17 + \sqrt{161})/16$, $\lambda_2 = (17 - \sqrt{161})/16$. We have $\lambda_1 > 1$, i.e., the characteristic polynomial is not Schur stable. □

Remark 1. *Similar examples for systems of differential equations are given in [20, Sect. 1].*

Remark 2. *Examples in Propositions 1 and 2 can be constructed for any $n \geq 2$. To do this, it suffices to take as the matrix $A(t)$ the block $n \times n$-matrix $A_n(t) = \begin{bmatrix} 0 & 0 \\ 0 & A_2(t) \end{bmatrix}$, where $A_2(t)$ is 2×2-matrix constructed in the corresponding Proposition 1 or 2. For $n = 1$ such examples cannot be constructed (see Sect. 7).*

3. Asymptotic stability of a linear difference equation with time-varying coefficients

Now, consider the linear difference equation of order n with time-varying coefficients

$$x(t+n) + p_1(t)x(t+n-1) + \ldots + p_n(t)x(t) = 0, \tag{3.1}$$
$$t \in \mathbb{Z}, \quad x \in \mathbb{R}, \quad p_j(t) \in \mathbb{R}, \quad j = \overline{1,n}.$$

We suppose that the functions $p_j(\cdot)$, $j = \overline{1,n}$, are bounded: $a_j \leq p_j(t) \leq b_j$, $j = \overline{1,n}$, $t \in \mathbb{Z}$.

Using the Eq (3.1), we construct the characteristic polynomial

$$\chi(t,\lambda) = \lambda^n + p_1(t)\lambda^{n-1} + \ldots + p_n(t). \tag{3.2}$$

Obviously, it also has time-varying coefficients. Suppose that for every $t \in \mathbb{Z}$ the polynomial (3.2) is Schur stable. Will this condition be necessary and/or sufficient for the asymptotic stability of the Eq (3.1)? Note that the above examples do not answer this question for the Eq (3.1). We will prove that the answer to this question is negative. The following theorems hold.

Theorem 1. *Let $n \geq 2$. There exist bounded functions $p_j : \mathbb{Z} \to \mathbb{R}$, $j = \overline{1,n}$, such that for every $t \in \mathbb{Z}$, the polynomial* (3.2) *is Schur stable and Eq* (3.1) *is unstable.*

Theorem 2. *Let $n \geq 2$. There exist bounded functions $p_j : \mathbb{Z} \to \mathbb{R}$, $j = \overline{1,n}$, such that for every $t \in \mathbb{Z}$, the polynomial* (3.2) *is not Schur stable and Eq* (3.1) *is asymptotically stable.*

The proofs of Theorems 1, 2 will be given below.

Remark 3. *Similar theorems for linear differential equations are established in [20, Theorems 1, 2].*

4. Robust stability

Consider a linear differential equation (1.5) with constant coefficients. Suppose that the coefficients of the equation (1.5) are unknown and satisfy the constraints $a_j \leq q_j \leq b_j$, $j = \overline{1,n}$. The famous Kharitonov's theorem [22] on robust stability (the stability of a family of interval polynomials) says that every polynomial in the family of interval polynomials is Hurwitz stable if and only if the four (Kharitonov's) polynomials located at the vertices of the family of interval polynomials are Hurwitz stable (see the formulation in [8, Ch. 5, Sect. 5.2, Theorem 5.1] as well). It was shown in [20] that if the coefficients of the differential equation (1.5) are not constant, then Kharitonov's theorem on robust stability does not hold.

Let us consider similar issues for difference equations. For equations with discrete time, the analog of Kharitonov's theorem on robust stability does not hold in the general case. The Schur stability of four vertex polynomials and even of all vertex polynomials of an interval family of polynomials does not imply the Schur stability of all polynomials of the interval family [23], see also [8, Ch. 5, Sect. 5.8, Example 5.8] and [8, Ch. 5, Sect. 5.8, Example 5.9]. Nevertheless, certain analogs of the stability theorem for an interval family of polynomials hold for systems with discrete time. Let us recall the formulations of these theorems.

Consider a family $\mathcal{I}(\lambda)$ of real polynomials of degree n of the form

$$\varphi(\lambda) = s_0\lambda^n + s_1\lambda^{n-1} + \ldots + s_{n-1}\lambda + s_n,$$

whose coefficients belong to the box

$$\Pi := \{s := (s_0, \ldots, s_n) \in \mathbb{R}^{n+1} : a_j \leq s_j \leq b_j,\ j = \overline{0,n}\}.$$

Let us introduce vertices V and edges E of the box Π:

$$V := \{s := (s_0, \ldots, s_n) \in \mathbb{R}^{n+1} : s_j = a_j \text{ or } s_j = b_j,\ j = \overline{0,n}\},$$

$$E_k := \{s := (s_0, \ldots, s_n) \in \mathbb{R}^{n+1} : s_j = a_j \text{ or } s_j = b_j,\ j = \overline{0,n},\ j \neq k,\ s_k \in [a_k, b_k]\},$$

and $E = \bigcup_{k=0}^{n} E_k$. The corresponding family of vertex and edge polynomials will be denoted by

$$\mathcal{I}_V(\lambda) := \{\varphi(\lambda) = s_0\lambda^n + s_1\lambda^{n-1} + \ldots + s_{n-1}\lambda + s_n,\ (s_0, \ldots, s_n) \in V\},$$
$$\mathcal{I}_E(\lambda) := \{\varphi(\lambda) = s_0\lambda^n + s_1\lambda^{n-1} + \ldots + s_{n-1}\lambda + s_n,\ (s_0, \ldots, s_n) \in E\}.$$

The following theorem is true (see [24, 25], [7, Sect. 13.3]).

Theorem 3. *Let $n \leq 3$ and $a_0 = b_0 = 1$. Every polynomial in the family $\mathcal{I}(\lambda)$ is Schur stable if and only if every polynomial in the family $\mathcal{I}_V(\lambda)$ is Schur stable.*

Thus, the analog of Kharitonov's theorem for difference equations expressed in terms of vertex polynomials holds only for $n \leq 3$. For $n \geq 4$ there are corresponding counterexamples [8, 23].

For an arbitrary n, the Schur stability is characterized not by vertex polynomials but by edge polynomials [26], [8, Theorem 5.13].

Theorem 4. *Let $0 \notin [a_0, b_0]$. Every polynomial in the family $\mathcal{I}(\lambda)$ is Schur stable if and only if every polynomial in the family $\mathcal{I}_E(\lambda)$ is Schur stable.*

Note that Theorem 4 can be strengthened (see [8, Theorem 5.14]), and the number of edge polynomials that need to be examined for Schur stability in Theorem 4 can be reduced.

Consider a linear difference equation of the nth order (1.1) with constant coefficients. Suppose that the exact values of the coefficients p_j, $j = \overline{1,n}$, are unknown, but their upper and lower bounds $a_j \leq p_j \leq b_j$, $j = \overline{1,n}$, are known. The question of the asymptotic stability of Eq (1.1) (or a family of Eq (1.1)) belongs to the problem of robust stability. Since for the equation with constant coefficients (1.1) its asymptotic stability reduces to the Schur stability of its characteristic polynomial (1.2), Theorems 3 and 4 give a criterion for the asymptotic stability of the family of Eq (1.1).

Consider now the linear difference equation of the nth order with unknown time-varying coefficients

$$x(t+n) + p_1(t)x(t+n-1) + \ldots + p_n(t)x(t) = 0, \qquad (4.1)$$
$$t \in \mathbb{Z}, \quad x \in \mathbb{R}, \quad p_j(t) \in \mathbb{R}, \quad j = \overline{1,n}.$$

Suppose that the functions $p_j(\cdot)$, $j = \overline{1,n}$, are bounded:

$$a_j \leq p_j(t) \leq b_j, \quad j = \overline{1,n}, \quad t \in \mathbb{R}. \qquad (4.2)$$

Set $a_0 := b_0 := 1$. Consider the set of vertices $\mathcal{I}_V(\lambda)$ and edges $\mathcal{I}_E(\lambda)$ of the family $\mathcal{I}(\lambda)$. Let us pose the following questions. Is it a sufficient condition for the asymptotic stability of the Eq (4.1):

(a) the condition of the Schur stability for any polynomial in the family $\mathcal{I}_V(\lambda)$ (for $n \leq 3$);

(b) the condition of the Schur stability for any polynomial in the family $\mathcal{I}_E(\lambda)$?

We will prove that the answers to these questions are negative. This is proved by the following theorem.

Theorem 5. *For any $n \geq 2$ there exist numbers a_j, b_j and functions $p_j : \mathbb{Z} \to \mathbb{R}$, $j = \overline{1,n}$, such that the following properties hold:*

(1) Inequalities (4.2) are satisfied.

(2) Every polynomial in the family $\mathcal{I}_E(\lambda)$ is Schur stable.

(3) The Eq (4.1) is unstable.

Remark 4. *A similar theorem for a linear differential equation was established in [20, Theorem 5].*

5. Proof of Theorem 5

(**n = 2**). Set $a_1 := -\sqrt{2}$, $b_1 := \sqrt{2}$, $a_2 := b_2 := 1/2$. Then

$$\mathcal{I}(\lambda) = \mathcal{I}_E(\lambda) = \{\varphi(\lambda) = \lambda^2 + s_1\lambda + s_2 : \ s_1 \in [-\sqrt{2}, \sqrt{2}], \ s_2 = 1/2\}.$$

Let λ_1, λ_2 be the roots of the quadratic polynomial $\lambda^2 + s_1\lambda + 1/2$. Its discriminant is $D = s_1^2 - 2$. If $s_1 \in [-\sqrt{2}, \sqrt{2}]$, then $D \in [-2, 0]$. Therefore, $\lambda_{1,2} = (-s_1 \pm i\sqrt{-D})/2$. Hence, $|\lambda_1| = |\lambda_2| = 1/\sqrt{2}$. Therefore, every polynomial from $\mathcal{I}_E(\lambda)$ is Schur stable.

Define the functions

$$p_1(t) := \begin{cases} -\sqrt{2}, & t = 2k, \\ \sqrt{2}, & t = 2k+1, \end{cases} \quad k \in \mathbb{Z}, \qquad p_2(t) := 1/2, \quad t \in \mathbb{Z}. \tag{5.1}$$

Then inequalities (4.2) are satisfied.

Let us prove that the equation

$$x(t+2) + p_1(t)x(t+1) + p_2(t)x(t) = 0 \tag{5.2}$$

with coefficients (5.1) is unstable. We make the change of variables $z_1(t) = x(t)$, $z_2(t) = x(t+1)$, $t \in \mathbb{Z}$, in the Eq (5.2); denote $z := [z_1, z_2]^T \in \mathbb{R}^2$. The Eq (5.2) is transformed into a system of difference equations

$$z(t+1) = A(t)z(t), \quad t \in \mathbb{Z}, \quad z \in \mathbb{R}^2, \tag{5.3}$$

with the matrix

$$A(t) = \begin{cases} A_0, & t = 2k, \\ A_1, & t = 2k+1, \end{cases} \quad k \in \mathbb{Z}, \qquad A_0 = \begin{bmatrix} 0 & 1 \\ -1/2 & \sqrt{2} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1/2 & -\sqrt{2} \end{bmatrix}.$$

The system (5.3) is periodic with the period $\omega = 2$. Denote by $Z(t, s)$ the Cauchy matrix of system (5.3), i.e., a solution of the matrix initial value problem $Z(t+1) = A(t)Z(t)$, $Z(s) = I$ ($I \in M_{2,2}$ is the identity matrix; here and below we denote by $M_{p,q}$ the space of real $p \times q$-matrices). Calculate the monodromy matrix $\Psi := Z(2, 0)$. We get

$$\Psi = A_1 A_0 = \begin{bmatrix} 0 & 1 \\ -1/2 & -\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1/2 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{2} \\ \sqrt{2}/2 & -5/2 \end{bmatrix}.$$

The characteristic polynomial of the matrix Ψ is $\lambda^2 + 3\lambda + 1/4$. Its roots are $\lambda_1 = -(3 + 2\sqrt{2})/2$, $\lambda_2 = -(3 - 2\sqrt{2})/2$. We have $|\lambda_1| > 1$. Hence, the system (5.3) is unstable. Therefore, the Eq (5.2) is unstable.

(**n > 2**). Set $a_1 := -\sqrt{2}$, $b_1 := \sqrt{2}$, $a_2 := b_2 := 1/2$, $a_j := b_j := 0$, $j = \overline{3, n}$. Then

$$\mathcal{I}(\lambda) = \mathcal{I}_E(\lambda) = \{\varphi(\lambda) = \lambda^n + s_1\lambda^{n-1} + s_2\lambda^{n-2} : \ s_1 \in [-\sqrt{2}, \sqrt{2}], \ s_2 = 1/2\}.$$

We have $\lambda^n + s_1\lambda^{n-1} + \lambda^{n-2}/2 = \lambda^{n-2}(\lambda^2 + s_1\lambda + 1/2)$. Then, for roots λ_j, $j = \overline{1, n}$, of this polynomial, we have: $|\lambda_1| = |\lambda_2| = 1/\sqrt{2}$, $\lambda_3 = \ldots = \lambda_n = 0$. Therefore, every polynomial from $\mathcal{I}_E(\lambda)$ is Schur stable.

Define the functions

$$p_1(t) := \begin{cases} -\sqrt{2}, & t = 2k, \\ \sqrt{2}, & t = 2k+1, \end{cases} \quad k \in \mathbb{Z}, \qquad p_2(t) := 1/2, \quad p_j(t) := 0, \quad j = \overline{3,n}, \quad t \in \mathbb{Z}. \tag{5.4}$$

Then, inequalities (4.2) are satisfied.

Let us prove that the Eq (4.1) with coefficients (5.4) is unstable. We make the change of variables $y_1(t) = x(t), y_2(t) = x(t+1), \dots, y_n(t) = x(t+n-1), t \in \mathbb{Z}$, in the Eq (4.1); denote $y := [y_1, \dots, y_n]^T \in \mathbb{R}^n$. The Eq (4.1) is transformed into a system of difference equations

$$y(t+1) = B(t)y(t), \quad t \in \mathbb{Z}, \quad y \in \mathbb{R}^n, \tag{5.5}$$

with the matrix

$$B(t) = \begin{cases} B_0, & t = 2k, \\ B_1, & t = 2k+1, \end{cases} \quad k \in \mathbb{Z}, \quad \text{where} \quad B_0 = \begin{bmatrix} F_1 & F_2 \\ F_3 & P \end{bmatrix}, \quad B_1 = \begin{bmatrix} F_1 & F_2 \\ F_3 & Q \end{bmatrix},$$

$$F_1 = 0 \in M_{n-1,1}, \quad F_2 = I \in M_{n-1,n-1}, \quad F_3 = 0 \in M_{1,n-2},$$

$$P = \begin{bmatrix} -1/2 & \sqrt{2} \end{bmatrix} \in M_{1,2}, \quad Q = \begin{bmatrix} -1/2 & -\sqrt{2} \end{bmatrix} \in M_{1,2}.$$

The system (5.5) is periodic with the period $\omega = 2$. Denote by $Y(t, s)$ the Cauchy matrix of system (5.5). Calculate the monodromy matrix $\Phi := Y(2, 0)$. We get

$$\Phi = B_1 B_0 = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix},$$

$$G_1 = 0 \in M_{n-2,2}, \quad G_2 = I \in M_{n-2,n-2}, \quad G_3 = 0 \in M_{2,n-2}, \quad G_4 = \Psi \in M_{2,2}.$$

The characteristic polynomial of the matrix Φ is $(\lambda^2 + 3\lambda + 1/4)\lambda^{n-2}$. Its roots are $\lambda_1 = -(3 + 2\sqrt{2})/2$, $\lambda_2 = -(3 - 2\sqrt{2})/2$, $\lambda_j = 0$, $j = \overline{3,n}$. We have $|\lambda_1| > 1$. Hence, the system (5.5) is unstable. Therefore, the Eq (4.1) is unstable.

Remark 5. *Theorem 5 proves that the answer to the question* (*b*) *is negative for every* $n \geq 2$. *Furthermore, since* $\mathcal{I}_V(\lambda) \subset \mathcal{I}_E(\lambda)$, *it follows from Theorem 5 that the answer to the question* (*a*) *is also negative for* $n = 2$ *and* $n = 3$.

Remark 6. *The example of a system given in the proof of Theorem 5 obviously provides an example of a system for Theorem 1. Thus, together with the proof of Theorem 5, Theorem 1 is also proved.*

6. Proof of Theorem 2

(**n** = **2**). Define the functions

$$p_1(t) := \begin{cases} -1, & t = 2k, \\ 1, & t = 2k+1, \end{cases} \quad k \in \mathbb{Z}, \qquad p_2(t) := -1/2, \quad t \in \mathbb{Z}. \tag{6.1}$$

If $t = 2k$, then the polynomial (3.2) is $\lambda^2 - \lambda - 1/2$. Its roots are $\lambda_1 = (1 + \sqrt{3})/2$, $\lambda_2 = (1 - \sqrt{3})/2$. We have $|\lambda_1| > 1$. Hence, the polynomial (3.2) is not Schur stable.

If $t = 2k + 1$, then the polynomial (3.2) is $\lambda^2 + \lambda - 1/2$. Its roots are $\lambda_1 = (-1 + \sqrt{3})/2$, $\lambda_2 = (-1 - \sqrt{3})/2$. We have $|\lambda_2| > 1$. Hence, the polynomial (3.2) is not Schur stable.

Let us prove that the Eq (3.1) with coefficients (6.1) is asymptotically stable. We make the change of variables $z_1(t) = x(t)$, $z_2(t) = x(t + 1)$, $t \in \mathbb{Z}$, in the Eq (3.1); denote $z := [z_1, z_2]^T \in \mathbb{R}^2$. The Eq (3.1) is transformed into a system of difference equations

$$z(t + 1) = A(t)z(t), \quad t \in \mathbb{Z}, \quad z \in \mathbb{R}^2, \tag{6.2}$$

with the matrix

$$A(t) = \begin{cases} A_0, & t = 2k, \\ A_1, & t = 2k + 1, \end{cases} \quad k \in \mathbb{Z}, \qquad A_0 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1/2 & -1 \end{bmatrix}.$$

The system (6.2) is periodic with the period $\omega = 2$. Denote by $Z(t, s)$ the Cauchy matrix of system (6.2). Calculate the monodromy matrix $\Psi := Z(2, 0)$. We get

$$\Psi = A_1A_0 = \begin{bmatrix} 0 & 1 \\ 1/2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ -1/2 & -1/2 \end{bmatrix}.$$

The characteristic polynomial of the matrix Ψ is $\lambda^2 + 1/4$. Its roots are $\lambda_{1,2} = \pm i/2$. We have $|\lambda_1| = |\lambda_2| < 1$. Hence, the system (6.2) is asymptotically stable. Therefore, the Eq (3.1) is asymptotically stable.

(**n** > **2**). Define the functions

$$p_1(t) := \begin{cases} -1, & t = 2k, \\ 1, & t = 2k + 1, \end{cases} \quad k \in \mathbb{Z}, \qquad p_2(t) := -1/2, \quad p_j(t) := 0, \quad j = \overline{3, n}, \quad t \in \mathbb{Z}. \tag{6.3}$$

If $t = 2k$, then the polynomial (3.2) is $(\lambda^2 - \lambda - 1/2)\lambda^{n-2}$. Its roots are $\lambda_1 = (1 + \sqrt{3})/2$, $\lambda_2 = (1 - \sqrt{3})/2$, $\lambda_j = 0$, $j = \overline{3, n}$. We have $|\lambda_1| > 1$. Hence, the polynomial (3.2) is not Schur stable.

If $t = 2k + 1$, then the polynomial (3.2) is $(\lambda^2 + \lambda - 1/2)\lambda^{n-2}$. Its roots are $\lambda_1 = (-1 + \sqrt{3})/2$, $\lambda_2 = (-1 - \sqrt{3})/2$, $\lambda_j = 0$, $j = \overline{3, n}$. We have $|\lambda_2| > 1$. Hence, the polynomial (3.2) is not Schur stable.

Let us prove that the Eq (3.1) with coefficients (6.3) is asymptotically stable. We make the change of variables $y_1(t) = x(t)$, $y_2(t) = x(t + 1)$, $\ldots$, $y_n(t) = x(t + n - 1)$, $t \in \mathbb{Z}$, in the Eq (3.1) and denote $y := [y_1, \ldots, y_n]^T \in \mathbb{R}^n$. The Eq (3.1) is transformed into a system of difference equations

$$y(t + 1) = B(t)y(t), \quad t \in \mathbb{Z}, \quad y \in \mathbb{R}^n, \tag{6.4}$$

with the matrix

$$B(t) = \begin{cases} B_0, & t = 2k, \\ B_1, & t = 2k + 1, \end{cases} \quad k \in \mathbb{Z}, \quad \text{where} \quad B_0 = \begin{bmatrix} F_1 & F_2 \\ F_3 & P \end{bmatrix}, \quad B_1 = \begin{bmatrix} F_1 & F_2 \\ F_3 & Q \end{bmatrix},$$

$$F_1 = 0 \in M_{n-1,1}, \quad F_2 = I \in M_{n-1,n-1}, \quad F_3 = 0 \in M_{1,n-2},$$

$$P = \begin{bmatrix} 1/2 & 1 \end{bmatrix} \in M_{1,2}, \quad Q = \begin{bmatrix} 1/2 & -1 \end{bmatrix} \in M_{1,2}.$$

The system (6.4) is periodic with the period $\omega = 2$. Denote by $Y(t, s)$ the Cauchy matrix of system (6.4). Calculate the monodromy matrix $\Phi := Y(2, 0)$. We get

$$\Phi = B_1 B_0 = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix},$$

$$G_1 = 0 \in M_{n-2,2}, \quad G_2 = I \in M_{n-2,n-2}, \quad G_3 = 0 \in M_{2,n-2}, \quad G_4 = \Psi \in M_{2,2}.$$

The characteristic polynomial of the matrix Φ is $(\lambda^2 + 1/4)\lambda^{n-2}$. Its roots are $\lambda_{1,2} = \pm i/2$, $\lambda_j = 0$, $j = \overline{3, n}$. We have $|\lambda_j| < 1$, $j = \overline{1, n}$. Hence, the system (6.4) is asymptotically stable. Therefore, the Eq (3.1) is asymptotically stable.

7. The case $n = 1$

Let us discuss whether it is possible to construct the corresponding examples in Theorems 1, 2, 5 for $n = 1$. Consider an equation

$$x(t + 1) + p_1(t)x(t) = 0, \quad t \in \mathbb{Z}, \quad x \in \mathbb{R}, \quad p_1(t) \in \mathbb{R}. \tag{7.1}$$

A general solution of Eq (7.1) has the form

$$x(t) = C \prod_{\tau=t_0}^{t-1} p_1(\tau). \tag{7.2}$$

Assume that the function $p_1(t)$ satisfies the condition

$$a_1 \le p_1(t) \le b_1, \quad t \in \mathbb{Z}. \tag{7.3}$$

Let every polynomial from $\mathcal{I}_E(\lambda)$ (and hence, from $\mathcal{I}(\lambda)$) be Schur stable. Then,

$$-1 < a_1 \le b_1 < 1. \tag{7.4}$$

Then, it follows from (7.2)–(7.4) that $|x(t)| \to 0$ as $t \to +\infty$. Hence, the Eq (7.1) is necessarily asymptotically stable. Thus, it is impossible to construct the corresponding example in Theorem 5 for $n = 1$.

If for every $t \in \mathbb{Z}$ the polynomial

$$\lambda + p_1(t) \tag{7.5}$$

is Schur stable, then $-1 < p_1(t) < 1$ for all $t \in \mathbb{Z}$. Then, $\left|\prod_{\tau=t_0}^{\infty} p_1(\tau)\right| \le 1$, i.e., the Eq (7.1) is Lyapunov stable. Thus, it is impossible to construct the corresponding example in Theorem 1 for $n = 1$.

Further, if, for every $t \in \mathbb{Z}$, the polynomial (7.5) is not Schur stable, then $|p_1(t)| \ge 1$. Then, the general solution (7.2) of Eq (7.1) satisfies the inequality

$$|x(t)| = |C| \left|\prod_{\tau=t_0}^{t-1} p_1(\tau)\right| = |C| \prod_{\tau=t_0}^{t-1} |p_1(\tau)| \ge |C|, \quad t \ge t_0.$$

Consequently, Eq (7.1) is not asymptotically stable. Hence, it is also impossible to construct the corresponding example in Theorem 2 for $n = 1$.

8. Conclusions

For linear difference equations and systems, a number of (counter) examples are given showing that the condition for the coefficients of the equation (system) to be constant is an essential condition in theorems on the relationship between the property of asymptotic stability for the difference equation (system) and the property of the Schur stability for the characteristic polynomial of the equation (system). Also, we have proved that the analog of Kharitonov's theorem on robust stability and the edge theorem do not hold for a difference equation if the coefficients of the equation are time-varying. The results of the paper are new. In particular, they answer questions (a) and (b) before Theorem 5. In earlier works, there were no answers to these questions. We believe that the results obtained can be useful in the problems of robust stabilization of discrete-time control systems with time-varying coefficients.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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