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Research article

Fixed-point results via α_i^j-$\left(\mathbf{D}_{\mathscr{C}}\left(\mathfrak{P}_{\hat{E}}\right)\right)$-contractions in partial $\flat$-metric spaces

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Abstract: In this study, we characterize a novel contraction mapping referred to as α_i^j-$(\mathbf{D}_{\mathscr{C}}(\mathfrak{P}_{\hat{E}}))$-contraction in light of $\mathbf{D}_{\mathscr{C}}$-contraction mappings associated with the Geraghty-type contraction and E-type contraction. Besides, a novel common fixed-point theorem providing such mappings is demonstrated in the context of partial $\flat$-metric spaces. It is stated that the main theorem is a generalization of the existing literature, and its comparisons with the results are expressed. Additionally, the efficiency of the result of this study is demonstrated through some examples and an application to homotopy theory.

Keywords: common fixed-point; partial $\flat$-metric space; α-admissibility; Geraghty contraction; E-contraction; comparison function; $\mathcal{F}$-contraction
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1. Introduction

Metric fixed-point theory is a prominent and essential topic of functional analysis. As expected, this field has had a flood of scientific activity. Because of its elegance, simplicity, and ease of application in various mathematics disciplines, the Banach fixed-point theorem [1], also known as the Banach contraction principle, is unquestionably crucial for the metric fixed-point theory. The existence and uniqueness of a fixed-point of contraction mappings on a complete metric space are guaranteed by this theorem, which also gives a constructive approach to finding the fixed-point.

One beneficial aspect of studying the metric fixed-point theory is generalizing the metric space structure under consideration. Furthermore, generalizing the metric function has emerged as one of the most intriguing and profitable research topics. New structures have arisen through the modification of certain aspects of the distance function or the addition of some new features to this function, and many new topological structures have been added to the literature. Additionally, researchers successfully apply these novel metric functions by studying summability theory, sequence spaces, Banach space geometry, fuzzy theory, and so on. The $\flat$-metric function, which has a constant in the triangle inequality, has been the most visible extension of the metric function in the previous 40 years. Bakhtin [2] and Czerwik [3] expressed the $\flat$-metric function as an expanded form of metric functions. The triangle inequality of the known metric function has been replaced by a more general inequality by using a constant $\varsigma \geq 1$. It corresponds to the standard metric function with $\varsigma = 1$. Some researchers have studied fixed-point theorems in $\flat$-metric spaces, as detailed in [4–13].

In 1992, Matthews [14] proposed a partial metric function, which is an intriguing expansion of the metric function. Partial metric spaces are metric space extensions in which any of the points has a non-zero self-distance. This topic has a wide range of applications in various branches of mathematics, computer science and semantics and has also become a thriving area in metric fixed-point theory. Currently, a number of researchers depend on the partial metric as a crucial idea for investigating the presence and uniqueness of a fixed-point for mappings that satisfy various contractive conditions; for details, please see [15–23].

Many new metric functions have appeared in the literature as a result of combining various distance functions. These structures have significance in the study of metric fixed-point theory, summability theory, fuzzy theory and other related topics. The study conducted by Brzdek et al. [24] in 2018 is recognized as an example of one of the most significant studies in which metric frameworks are employed simultaneously.

The partial $\flat$-metric, which is a new concept combining the above-mentioned partial metric and $\flat$-metric structures, was developed in 2013 by Mustafa et al. [25], and its modification was introduced by Shukla [26] in 2014. The properties of this newly defined space were examined, and generalizations of the metric fixed-point theory were obtained, see [27, 28].

On the other hand, many authors have attempted to generalize the Banach contraction principle by applying auxiliary functions in various abstract spaces to gain more constructive results in fixed-point theory. This research is still garnering attention today.

The Geraghty-type contraction, one of the most significant variations of the Banach contraction, was highlighted in 1973 by Geraghty [29]. Like the Banach contraction principle, the Geraghty contraction principle appealed to the researchers. It has found a place in the literature in several investigations, notably, the ones mentioned in [30–32].

Samet et al. [33] proposed the ideas of α-admissibility and α-ψ-contraction mappings and some fixed-point insights for these contractions were put forward. Soon after, various researchers' evaluations appeared in the literature [34–36]. Cho et al. founded diverse fixed-point theorems by combining α-admissibility and Geraghty contractions in [37]. Afshari et al. [38] focused on these results by implementing the idea of generalized α-ψ-Geraghty contraction mappings and investigating the existence and uniqueness of a fixed-point for such mappings in the context of $\flat$-metric spaces.

Meanwhile, Fulga and Proca [39] proposed the premise of an E-contraction in 2017. In the following year, Fulga and Proca [40] setup a fixed-point theorem for the $\mathfrak{P}_E$-Geraghty contraction, and some findings were brought about by implementing this concept; see [41–43]. In 2018, Alqahtani et al. [44] confirmed a common fixed-point theorem on complete metric spaces by applying the Geraghty contraction of type $E_{\varsigma,O}$. The following year, Aydi et al. [45] presented the α-$\mathfrak{P}_E$-Geraghty contraction on a $\flat$-metric space, and some fixed-point findings were achieved. Lang and

Guan [46] recently introduced α_i^j-$\mathfrak{P}_{E_{\Xi,O}}$-Geraghty contraction mappings and established common fixed-point findings for generalized α_i^j-$\mathfrak{P}_{E_{\Xi,O}}$-Geraghty contraction mappings in $\flat$-metric spaces.

Liu et al. [47] identified the $\mathbf{D}_{\mathscr{C}}$-class, which encompasses the F-contractions given by Wardowski [48]. See [49, 50] for additional knowledge on $\mathbf{D}_{\mathscr{C}}$-contractions.

This study aims to expound upon the generalized α_i^j-($\mathbf{D}_{\mathscr{C}}(\mathfrak{P}_{\hat{E}})$)-contraction by using the aforementioned concepts and to prove some fixed-point and common fixed-point theorems for such a contraction in partial $\flat$-metric spaces. It is seen that the obtained results generalize and improve many of the already existing results in the literature. At the same time, the presented examples support the accuracy of the results obtained. Finally, the application of the homotopy theory also illustrates the conclusion that the proposed study is multifaceted.

2. Preliminaries

The fundamental principles that are essential to this study are set forth below.

Definition 2.0.1. *([14]) A function $\wp : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$, where $\mathscr{U}$ is a nonempty set, is called a partial metric if the following assertions are provided:*

$\wp_1$. $\wp(\jmath, \jmath) = \wp(\jmath, \rho) = \wp(\rho, \rho) \Leftrightarrow \jmath = \rho$;
$\wp_2$. $\wp(\jmath, \jmath) \leq \wp(\jmath, \rho)$;
$\wp_3$. $\wp(\jmath, \rho) = \wp(\rho, \jmath)$;
$\wp_4$. $\wp(\jmath, \rho) \leq \wp(\jmath, r) + \wp(r, \rho) - \wp(r, r)$

for all $\jmath, \rho, r \in \mathscr{U}$. The pair $(\mathscr{U}, \wp)$ denotes a partial metric space.
It is evident from ($\wp_1$) and ($\wp_2$) that $\jmath = \rho$ provided that $\wp(\jmath, \rho) = 0$. However, $\wp(\jmath, \rho) = 0$ might not hold for each $\jmath \in \mathscr{U}$. This means that every metric space is a partial metric space, but the converse is not necessarily accurate.

Remark 2.0.2. *([17]) In $(\mathscr{U}, \wp)$, for all $\jmath \in \mathscr{U}$ and $\varepsilon > 0$, the ensuing sets*

$$B_{\wp}(\jmath, \varepsilon) = \{\rho \in \mathscr{U} : \wp(\jmath, \rho) < \wp(\jmath, \jmath) + \varepsilon\}$$

and

$$B_{\wp}[\jmath, \varepsilon] = \{\rho \in \mathscr{U} : \wp(\jmath, \rho) \leq \wp(\jmath, \jmath) + \varepsilon\}$$

denote $\wp$-open balls and $\wp$-closed balls, respectively. A T_0 topology arises over $(\mathscr{U}, \wp)$ via a base family of $\wp$-open balls

$$\{B_{\wp}(\jmath, \varepsilon) : \jmath \in \mathscr{U},\ \varepsilon > 0\}.$$

Example 2.0.3. *([20]) Let $\mathscr{U} = [0, 1]$ and $\wp(\jmath, \rho) = \max\{\jmath, \rho\}$ for all $\jmath, \rho \in \mathscr{U}$. Then, $(\mathscr{U}, \wp)$ denotes a partial metric space. However, this is not a metric space.*

Example 2.0.4. *([14]) Let $\mathscr{U} = [0, 1] \cup [2, 3]$ and define $\wp : \mathscr{U} \times \mathscr{U} \to [0, \infty)$ by*

$$\wp(\jmath, \rho) = \begin{cases} \max\{\jmath, \rho\}, & \{\jmath, \rho\} \cap [2, 3], \\ |\jmath - \rho|, & \{\jmath, \rho\} \subset [0, 1]. \end{cases}$$

Then, $\wp$ fulfills all partial metric conditions. As a result, $(\mathscr{U}, \wp)$ is a partial metric space.

Definition 2.0.5. *([3]) Assume that $\mathscr{U}$ is a nonempty set. A $\flat$-metric has been identified as the function $\flat : \mathscr{U} \times \mathscr{U} \to [0, \infty)$, which possesses the subsequent attributes for any $\jmath, \rho, r \in \mathscr{U}$:*

$\flat_1$. $\flat(\jmath, \rho) = 0 \Leftrightarrow \jmath = \rho$;
$\flat_2$. $\flat(\jmath, \rho) = \flat(\rho, \jmath)$;
$\flat_3$. *there is a real constant* $\varsigma \geq 1$ *that implements* $\flat(\jmath, \rho) \leq \varsigma [\flat(\jmath, r) + \flat(r, \rho)]$.

$(\mathscr{U}, \flat)$ *stands for a* $\flat$*-metric space with the coefficient* ς.

If $\varsigma = 1$, the function $\flat$ is an ordinary metric. In this circumstance, each metric is a $\flat$-metric. Nevertheless, the reverse is not generally accurate.

Example 2.0.6. *([10]) Let $(\mathscr{U}, d)$ be a metric space. The function $\flat : \mathscr{U} \times \mathscr{U} \to [0, \infty)$, which is defined as $\flat(\jmath, \rho) = [d(\jmath, \rho)]^{\mu}$, is a $\flat$-metric for all $\jmath, \rho, r \in \mathscr{U}$ and $\mu > 1$. So, $(\mathscr{U}, \flat)$ is a $\flat$-metric space with $\varsigma = 2^{\mu-1}$.*

In 2013, Mustafa et al. [25] introduced the idea of a partial $\flat$-metric, which is considered an improvement of the partial metric and $\flat$-metric, which was subsequently refined by Shukla [26] in 2014.

Definition 2.0.7. *([25]) Assume that $\mathscr{U}$ is a nonempty set. If the succeeding characteristics are met for all $\jmath, \rho, r \in \mathscr{U}$, $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ is referred to as a partial $\flat$-metric:*

$(\wp_\flat 1)$ $\wp_\flat(\jmath, \rho) = \wp_\flat(\jmath, \jmath) = \wp_\flat(\rho, \rho) \Leftrightarrow \jmath = \rho$;
$(\wp_\flat 2)$ $\wp_\flat(\jmath, \jmath) \leq \wp_\flat(\jmath, \rho)$;
$(\wp_\flat 3)$ $\wp_\flat(\jmath, \rho) = \wp_\flat(\rho, \jmath)$;
$(\wp_\flat 4)$ a real number $\varsigma \geq 1$ exists such that

$$\wp_\flat(\jmath, \rho) \leq \varsigma [\wp_\flat(\jmath, r) + \wp_\flat(r, \rho) - \wp_\flat(r, r)] + \frac{1-\varsigma}{2} (\wp_\flat(\jmath, \jmath) + \wp_\flat(\rho, \rho)). \tag{2.1}$$

Thus, the pair $(\mathscr{U}, \wp_\flat)$ specifies a partial $\flat$-metric space via the coefficient ς.

Shukla [26] revised the partial $\flat$-metric approach by using the property $\left(\wp_\flat 4'\right)$ instead of $(\wp_\flat 4)$, which is given below:

$\left(\wp_\flat 4'\right)$ for all $\jmath, \rho, r \in \mathscr{U}$

$$\wp_\flat(\jmath, \rho) \leq \varsigma [\wp_\flat(\jmath, r) + \wp_\flat(r, \rho)] - \wp_\flat(r, r). \tag{2.2}$$

Throughout the paper, we use the definition of a partial $\flat$-metric in the sense of Shukla [26]. The notations $\mathcal{P}_\flat$M and $\mathcal{P}_\flat$M$\mathcal{S}$, will be used throughout this work to designate the ideas of the partial $\flat$-metric and partial $\flat$-metric space, respectively.

Remark 2.0.8. *([25–28]) With the same coefficient $\varsigma \geq 1$, every $\flat$-metric space is a $\mathcal{P}_\flat$M$\mathcal{S}$; taking the coefficient $\varsigma = 1$, every partial metric space is a $\mathcal{P}_\flat$M$\mathcal{S}$. Moreover, a $\mathcal{P}_\flat$M on $\mathscr{U}$ is not a partial metric, nor a $\flat$-metric, in general. As far as we comprehend, a $\mathcal{P}_\flat$M$\mathcal{S}$ includes the set of a $\flat$-metric space and partial metric space.*

Example 2.0.9. *([26])*

i. Let $\mathscr{U} = [0, \infty)$ and $\omega > 1$ be a fixed element and define $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ by $\wp_\flat(\jmath, \rho) = \{\max\{\jmath, \rho\}\}^{\omega} + |\jmath - \rho|^{\omega}$ for all $\jmath, \rho \in \mathscr{U}$. Then, $(\mathscr{U}, \wp_\flat)$ is a $\mathcal{P}_\flat$M$\mathcal{S}$ with the coefficient $\varsigma = 2^{\omega} > 1$. It is not a $\flat$-metric or a partial metric space.

ii. *Assume that* $a > 0$ *is a fixed element and* $\mathscr{U} = [0, \infty)$. *Define* $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ *in order to become* $\wp_\flat(\jmath, \rho) = \max\{\jmath, \rho\} + a$ *for all* $\jmath, \rho \in \mathscr{U}$. *Then,* $(\mathscr{U}, \wp_\flat)$ *is a* $\mathcal{P}_\flat \mathrm{M}\mathcal{S}$ *with the coefficient* $\varsigma \geq 1$.

iii. *Let* $q \geq 1$ *and* $(\mathscr{U}, \wp)$ *be a partial metric space.* $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ *is a* $\mathcal{P}_\flat\mathrm{M}$ *with the coefficient* $\varsigma = 2^{q-1}$ *if this mapping is defined by* $\wp_\flat(\jmath, \rho) = [\wp(\jmath, \rho)]^q$.

iv. *Let* $\mathscr{U} = \{1, 2, 3, 4\}$ *and* $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ *be a* $\mathcal{P}_\flat\mathrm{M}$ *with the coefficient* $\varsigma = 4$*, where* $\mathcal{P}_\flat\mathrm{M}$ *is given by*

$$\wp_\flat(\jmath, \rho) = \begin{cases} |\jmath - \rho|^2 + \max\{\jmath, \rho\}, & \jmath \neq \rho; \\ \jmath, & \jmath = \rho \neq 1; \\ 0, & \jmath = \rho = 1. \end{cases}$$

It is clear that $\wp_\flat(2, 2) = 2 \neq 0$*, so it is not a* $\flat$*-metric. Also,*

$$\wp_\flat(3, 1) = 7 > 5 = \wp_\flat(3, 2) + \wp_\flat(2, 1) - \wp_\flat(2, 2)$$

is obtained. As seen here, $\wp_\flat$ *is not a partial metric.*

v. *With* $\varsigma \geq 1$*, assume that* $\wp$ *is a partial metric and* $\flat$ *is a* $\flat$*-metric on a nonempty set* $\mathscr{U}$. $\wp_\flat$ *is a* $\mathcal{P}_\flat\mathrm{M}$ *with the coefficient* $\varsigma \geq 1$ *on* $\mathscr{U}$*, where* $\wp_\flat$ *is characterized with* $\wp_\flat(\jmath, \rho) = \wp(\jmath, \rho) + \flat(\jmath, \rho)$.

We confer two novel instances to diversify the partial $\flat$-metric idea.

Example 2.0.10. i. *Let* $\mathscr{U} = (0, 1] \bigcup \{2, 3, 4, \ldots\}$*, and for* $q \geq 1$*,* $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$*is a* $\mathcal{P}_\flat\mathrm{M}$ *with the coefficient* $\varsigma = 2^{q-1}$*, where* $\wp_\flat$ *is specified by the subsequent expression:*

$$\wp_\flat(\jmath, \rho) = \begin{cases} \left|\frac{1}{\jmath} - \frac{1}{\rho}\right|^q + 1, & \jmath, \rho \in (0, 1], \quad \jmath \neq \rho; \\ e^{\max\{\jmath, \rho\} - \min\{\jmath, \rho\}}, & \jmath, \rho \in \{2, 3, 4, \ldots\}, \quad \jmath \neq \rho; \\ 1, & \jmath = \rho. \end{cases} \tag{2.3}$$

ii. *Let* $\mathscr{U} = \mathbb{N}$ *and* $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ *be a* $\mathcal{P}_\flat\mathrm{M}$ *with the coefficient* $\varsigma = 4$*, where* $\wp_\flat$ *is defined as follows:*

$$\wp_\flat(\jmath, \rho) = \begin{cases} e^{\max\left\{|\jmath - \rho|, \frac{\jmath + \rho}{2}\right\}}, & \jmath \leq \rho; \\ e^{\frac{|\jmath - \rho|^2 + \jmath + \rho}{2}}, & \jmath > \rho. \end{cases} \tag{2.4}$$

Proof. i. The axioms $(\wp_\flat 1)$–$(\wp_\flat 3)$ are apparent. In order to show the validity of $(\wp_\flat 4)$, the following six cases will be examined.

Case 1: For $\jmath, \rho, r \in (0,1]$ and $\jmath \neq \rho \neq r$, since $q \geq 1$, $2^q - 1 \geq 1$; we have

$$\begin{aligned}
\wp_\flat(\jmath,\rho) &= 1 + \left|\tfrac{1}{\jmath} - \tfrac{1}{\rho}\right|^q = 1 + \left|\tfrac{1}{\jmath} - \tfrac{1}{r} + \tfrac{1}{r} - \tfrac{1}{\rho}\right|^q \\
&\leq (2^q - 1) + 2^{q-1}\left[\left|\tfrac{1}{\jmath} - \tfrac{1}{r}\right|^q + \left|\tfrac{1}{r} - \tfrac{1}{\rho}\right|^q\right] \\
&\leq 2^{q-1}\left[\left|\tfrac{1}{\jmath} - \tfrac{1}{r}\right|^q + \left|\tfrac{1}{r} - \tfrac{1}{\rho}\right|^q + 2\right] - 1 \\
&\leq 2^{q-1}\left[\left|\tfrac{1}{\jmath} - \tfrac{1}{r}\right|^q + 1 + \left|\tfrac{1}{r} - \tfrac{1}{\rho}\right|^q + 1\right] - 1 - \left|\tfrac{1}{r} - \tfrac{1}{r}\right|^q \\
&= 2^{q-1}\left[\wp_\flat(\jmath, r) + \wp_\flat(r,\rho)\right] - \wp_\flat(r,r) \\
&= \varsigma\left[\wp_\flat(\jmath, r) + \wp_\flat(r,\rho)\right] - \wp_\flat(r,r).
\end{aligned}$$

Case 2: For $\jmath, \rho \in (0,1]$, $\jmath \neq \rho$ and $\jmath = r$, we conclude that

$$\begin{aligned}
\wp_\flat(\jmath,\rho) &= 1 + \left|\tfrac{1}{\jmath} - \tfrac{1}{\rho}\right|^q = 1 + \left|\tfrac{1}{r} - \tfrac{1}{\rho}\right|^q + 1 - 1 - \left|\tfrac{1}{r} - \tfrac{1}{r}\right|^q \\
&\leq 2^{q-1}\left[1 + \left|\tfrac{1}{r} - \tfrac{1}{\rho}\right|^q + 1\right] - \left[1 + \left|\tfrac{1}{r} - \tfrac{1}{r}\right|^q\right] \\
&= 2^{q-1}\left[\wp_\flat(\jmath, r) + \wp_\flat(r,\rho)\right] - \wp_\flat(r,r) \\
&= \varsigma\left[\wp_\flat(\jmath, r) + \wp_\flat(r,\rho)\right] - \wp_\flat(r,r).
\end{aligned}$$

Case 3: A similar consequence is true for $\jmath, r \in (0,1]$, $\jmath \neq r$ and $\jmath = \rho$.
Case 4: For $\jmath, \rho, r \in \{2,3,4,...\}$ and $\jmath \neq \rho \neq r$, owing to fact that

$$\begin{aligned}
\max\{\jmath,\rho\} - \min\{\jmath,\rho\} &= \max\{\jmath + r - r, \rho + r - r\} - \min\{\jmath + r - r, \rho + r - r\} \\
&\leq \max\{\jmath, r\} + \max\{r,\rho\} - \max\{r,r\} - \min\{\jmath, r\} - \min\{r,\rho\} + \min\{r,r\} \\
&= \max\{\jmath, r\} - \min\{\jmath, r\} + \max\{r,\rho\} - \min\{r,\rho\} - [\max\{r,r\} - \min\{r,r\}],
\end{aligned}$$

we derive

$$e^{\max\{\jmath,\rho\}-\min\{\jmath,\rho\}} \leq e^{\max\{\jmath,r\}-\min\{r,\rho\}} + e^{\max\{r,\rho\}-\min\{r,\rho\}} - e^{\max\{r,r\}-\min\{r,r\}},$$

which is the desired inequality.
Case 5: For $\jmath, \rho \in \{2,3,4,...\}$, $\jmath \neq \rho$ and $\jmath = r$, since

$$\begin{aligned}
\max\{\jmath,\rho\} - \min\{\jmath,\rho\} &= \max\{r,\rho\} - \min\{r,\rho\} \\
&= 0 + \max\{r,\rho\} - \min\{r,\rho\} - 0,
\end{aligned}$$

we have

$$e^{\max\{\jmath,\rho\}-\min\{\jmath,\rho\}} = e^{0} + e^{\max\{r,\rho\}-\min\{r,\rho\}} - e^{0}$$

$$\begin{aligned}\wp_\flat(\jmath,\rho) &= 1 + \wp_\flat(r,\rho) - 1\\ &\leq \varsigma\left[\wp_\flat(\jmath,r) + \wp_\flat(r,\rho)\right] - \wp_\flat(r,r).\end{aligned}$$

Case 6: This case is provided for $\jmath, r \in \{2,3,4,...\}$, $\jmath \neq r$ and $\jmath = \rho$, as in Case 5.

Consequently, since $(\wp_\flat 1)$–$(\wp_\flat 4)$ are provided, the mapping indicated in (2.3) is a $\mathcal{P}_\flat$M with $\varsigma = 2^{q-1}$.

However, for $\jmath = \rho$, by definition, since $\wp_\flat(\jmath,\rho) = 1$, $\wp_\flat(\jmath,\rho) = 0$ does not occur. The first axiom of the $\flat$-metric does not hold. So, a partial $\flat$-metric need not be a $\flat$-metric.

ii. The axioms of $(\wp_\flat 1)$–$(\wp_\flat 3)$ are evident. To demonstrate $(\wp_\flat 4)$, we need to examine the six cases mentioned below.

Case 1: Let $\jmath \leq r \leq \rho$. Then,

$$\begin{aligned}\max\left\{|\jmath-\rho|, \tfrac{\jmath+\rho}{2}\right\} &= \max\left\{|\jmath-r+r-\rho|, \tfrac{\jmath+r+r+\rho}{2} - r\right\}\\ &\leq \max\left\{|\jmath-r| + |r-\rho|, \tfrac{\jmath+r}{2} + \tfrac{r+\rho}{2} - r\right\}\\ &= \max\left\{|\jmath-r|, \tfrac{\jmath+r}{2}\right\} + \max\left\{|r-\rho|, \tfrac{r+\rho}{2}\right\} - \max\left\{|r-r|, \tfrac{r+r}{2}\right\};\end{aligned}$$

we have

$$e^{\max\left\{|\jmath-\rho|, \frac{\jmath+\rho}{2}\right\}} \leq e^{\max\left\{|\jmath-r|, \frac{\jmath+r}{2}\right\}} + e^{\max\left\{|r-\rho|, \frac{r+\rho}{2}\right\}} - e^{\max\left\{|r-r|, \frac{r+r}{2}\right\}}.$$

Thereby, the following expression is ensured for all $\varsigma \geq 1$:

$$\begin{aligned}\wp_\flat(\jmath,\rho) &\leq \wp_\flat(\jmath,r) + \wp_\flat(r,\rho) - \wp_\flat(r,r)\\ &\leq \varsigma\left[\wp_\flat(\jmath,r) + \wp_\flat(r,\rho)\right] - \wp_\flat(r,r).\end{aligned}$$

Case 2: Let $\jmath \leq \rho \leq r$. Because

$$\begin{aligned}\max\left\{|\jmath-\rho|, \tfrac{\jmath+\rho}{2}\right\} &= \max\left\{|\jmath-r+r-\rho|, \tfrac{\jmath+r+r+\rho}{2} - r\right\}\\ &\leq \max\left\{|\jmath-r| + |r-\rho|, \tfrac{\jmath+r}{2} + \tfrac{r+\rho}{2} - r\right\}\\ &= \max\left\{|\jmath-r|, \tfrac{\jmath+r}{2}\right\} + \max\left\{|r-\rho|, \tfrac{r+\rho}{2}\right\} - \max\left\{|r-r|, \tfrac{r+r}{2}\right\}\\ &\leq \max\left\{|\jmath-r|, \tfrac{\jmath+r}{2}\right\} + \tfrac{|r-\rho|^2+r+\rho}{2} - \max\left\{|r-r|, \tfrac{r+r}{2}\right\},\end{aligned}$$

we have

$$e^{\max\left\{|\jmath-\rho|, \frac{\jmath+\rho}{2}\right\}} \leq e^{\max\left\{|\jmath-r|, \frac{\jmath+r}{2}\right\}} + e^{\frac{|r-\rho|^2+r+\rho}{2}} - e^{\max\left\{|r-r|, \frac{r+r}{2}\right\}}.$$

Then, for all $\varsigma \geq 1$, we procure

$$\begin{aligned}\wp_b(\jmath,\rho) &\leq \wp_b(\jmath,r) + \wp_b(r,\rho) - \wp_b(r,r)\\ &\leq \varsigma\left[\wp_b(\jmath,r) + \wp_b(r,\rho)\right] - \wp_b(r,r).\end{aligned}$$

Case 3: Let $r \leq \jmath \leq \rho$. Then, for all $\varsigma \geq 1$, we achieve

$$\begin{aligned}\max\left\{|\jmath-\rho|, \tfrac{\jmath+\rho}{2}\right\} &= \max\left\{|\jmath-r+r-\rho|, \tfrac{\jmath+r+r+\rho}{2} - r\right\}\\ &\leq \max\left\{|\jmath-r| + |r-\rho|, \tfrac{\jmath+r}{2} + \tfrac{r+\rho}{2} - r\right\}\\ &= \max\left\{|\jmath-r|, \tfrac{\jmath+r}{2}\right\} + \max\left\{|r-\rho|, \tfrac{r+\rho}{2}\right\} - \max\left\{|r-r|, \tfrac{r+r}{2}\right\}\\ &\leq \tfrac{|\jmath-r|^2+\jmath+r}{2} + \max\left\{|r-\rho|, \tfrac{r+\rho}{2}\right\} - \max\left\{|r-r|, \tfrac{r+r}{2}\right\},\end{aligned}$$

and

$$e^{\max\left\{|\jmath-\rho|, \frac{\jmath+\rho}{2}\right\}} \leq e^{\frac{|\jmath-r|^2+\jmath+r}{2}} + e^{\max\left\{|r-\rho|, \frac{r+\rho}{2}\right\}} - e^{\max\left\{|r-r|, \frac{r+r}{2}\right\}}.$$

Case 4: Let $\jmath > r > \rho$. Because

$$\begin{aligned}\tfrac{|\jmath-\rho|^2+\jmath+\rho}{2} &= \tfrac{|\jmath-r+r-\rho|^2+\jmath+r-r+\rho}{2} - r\\ &\leq 2^2\left[\tfrac{|\jmath-r|^2}{2} + \tfrac{|r-\rho|^2}{2}\right] + \tfrac{\jmath+r}{2} + \tfrac{r+\rho}{2} - r\\ &\leq 2^2\left[\tfrac{|\jmath-r|^2+\jmath+r}{2} + \tfrac{|r-\rho|^2+r+\rho}{2}\right] - \max\left\{|r-r|, \tfrac{r+r}{2}\right\},\end{aligned}$$

for $\varsigma = 4$, we gain

$$\begin{aligned}e^{\frac{|\jmath-\rho|^2+\jmath+\rho}{2}} &\leq 2^2\left[e^{\frac{|\jmath-r|^2+\jmath+r}{2}} + e^{\frac{|r-\rho|^2+r+\rho}{2}}\right] - e^{\max\left\{|r-r|, \frac{r+r}{2}\right\}},\\ \wp_b(\jmath,\rho) &\leq 4\left[\wp_b(\jmath,r) + \wp_b(r,\rho)\right] - \wp_b(r,r).\end{aligned}$$

Case 5: Let $\jmath > \rho > r$. Then, for all $\varsigma \geq 1$, we attain

$$\begin{aligned}\tfrac{|\jmath-\rho|^2+\jmath+\jmath}{2} &= \tfrac{|\jmath-r|^2+\jmath+r-r+\jmath}{2} - r\\ &= \tfrac{|\jmath-r|^2+\jmath+r}{2} + \tfrac{r+\jmath}{2} - r\\ &\leq \tfrac{|\jmath-r|^2+\jmath+r}{2} + \max\left\{|r-\jmath|, \tfrac{r+\jmath}{2}\right\} - \max\left\{|r-r|, \tfrac{r+r}{2}\right\},\end{aligned}$$

and

$$\begin{aligned}e^{\frac{|\jmath-\rho|^2+\jmath+\rho}{2}} &\leq e^{\frac{|\jmath-r|^2+\jmath+r}{2}} + e^{\max\left\{|r-\rho|, \frac{r+\rho}{2}\right\}} - e^{\max\left\{|r-r|, \frac{r+r}{2}\right\}}\\ \wp_b(\jmath,\rho) &\leq \wp_b(\jmath,r) + \wp_b(r,\rho) - \wp_b(r,r)\\ &\leq \varsigma\left[\wp_b(\jmath,r) + \wp_b(r,\rho)\right] - \wp_b(r,r).\end{aligned}$$

Case 6: Let $r > \jmath > \rho$. Since

$$\frac{|\jmath-\rho|^2+\jmath+\rho}{2} \leq \frac{|r-\rho|^2+r+\rho}{2} + 2\max\left\{|\jmath - r|, \frac{\jmath+r}{2}\right\} - \max\left\{|r - r|, \frac{r+r}{2}\right\}$$

$$\leq 2\left[\max\left\{|\jmath - r|, \frac{\jmath+r}{2}\right\} + \frac{|r-\rho|^2+r+\rho}{2}\right] - \max\left\{|r - r|, \frac{r+r}{2}\right\},$$

and

$$e^{\frac{|\jmath-\rho|^2+\jmath+\rho}{2}} \leq 2\left[e^{\max\left\{|\jmath-r|, \frac{\jmath+r}{2}\right\}} + e^{\frac{|r-\rho|^2+r+\rho}{2}}\right] - e^{\max\left\{|r-r|, \frac{r+r}{2}\right\}},$$

for $\varsigma = 2$, we derive

$$\begin{aligned}\wp_\flat(\jmath,\rho) &\leq 2\left[\wp_\flat(\jmath, r) + \wp_\flat(r,\rho)\right] - \wp_\flat(r, r)\\ &\leq \varsigma\left[\wp_\flat(\jmath, r) + \wp_\flat(r,\rho)\right] - \wp_\flat(r, r).\end{aligned}$$

Contemplating the above six cases, we conclude that the mapping given in (2.4) is a $\mathcal{P}_\flat$M with $\varsigma = 4$. □

Proposition 2.0.11. *([25]) Every partial $\flat$-metric $\wp_\flat$ defines a $\flat$-metric $d_{\wp_\flat}$, where*

$$d_{\wp_\flat}(\jmath,\rho) = 2\wp_\flat(\jmath,\rho) - \wp_\flat(\jmath, \jmath) - \wp_\flat(\rho,\rho)$$

for all $\jmath, \rho \in \mathscr{U}$.

Remark 2.0.12. *Let $\varepsilon > 0$ $\left\{B_{\wp_\flat}(\jmath,\varepsilon) : \jmath \in \mathscr{U}, \varepsilon > 0\right\}$ be the family of $\wp_\flat$-open balls, where $B_{\wp_\flat}(\jmath,\varepsilon) = \{\rho \in \mathscr{U} : \wp_\flat(\jmath,\rho) < \wp_\flat(\jmath, \jmath) + \varepsilon\}$ for all $\jmath \in \mathscr{U}$. Each $\mathcal{P}_\flat$M brings on a T_0 topology $T_{\wp_\flat}$ on $\mathscr{U}$; however, it need not be T_1. To explain this, in Example 2.0.10 (i), for $1 = \jmath \neq \rho = 3$ and $q = 2$, we currently possess*

$$\begin{aligned}B_{\wp_\flat}\left(\tfrac{1}{2}, 1\right) &= \left\{\rho \in \mathscr{U} : \wp_\flat\left(\tfrac{1}{2},\rho\right) < \wp_\flat\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + 1\right\}\\ &= \left\{\rho \in \mathscr{U} : \left|2 - \tfrac{1}{\rho}\right|^2 + 1 < 1 + 1\right\}\\ &= \left\{\rho \in \mathscr{U} : \left|2 - \tfrac{1}{\rho}\right|^2 < 1\right\}\\ &= \left\{\rho \in \mathscr{U} : \tfrac{1}{2} \leq \rho < 1\right\} = \left[\tfrac{1}{2}, 1\right)\end{aligned}$$

and

$$\begin{aligned}B_{\wp_\flat}\left(\tfrac{1}{4}, 4\right) &= \left\{\rho \in \mathscr{U} : \wp_\flat\left(\tfrac{1}{4},\rho\right) < \wp_\flat\left(\tfrac{1}{4}, \tfrac{1}{4}\right) + 4\right\}\\ &= \left\{\rho \in \mathscr{U} : \left|4 - \tfrac{1}{\rho}\right|^2 + 1 < 1 + 4\right\}\\ &= \left\{\rho \in \mathscr{U} : \left|4 - \tfrac{1}{\rho}\right|^2 < 4\right\}\\ &= \left\{\rho \in \mathscr{U} : 0 \leq 4 - \tfrac{1}{\rho} < 2\right\} = \left[\tfrac{1}{4}, \tfrac{1}{2}\right).\end{aligned}$$

$\frac{1}{2} \in B_{\wp_\flat}\left(\frac{1}{2}, 1\right)$, but $\frac{1}{4} \notin B_{\wp_\flat}\left(\frac{1}{2}, 1\right)$, and $\frac{1}{4} \in B_{\wp_\flat}\left(\frac{1}{4}, 4\right)$, but $\frac{1}{2} \notin B_{\wp_\flat}\left(\frac{1}{4}, 4\right)$. Thus, it is deduced that a $\mathcal{P}_\flat$M on a set $\mathscr{U}$ need not to be T_1.

Definition 2.0.13. *([25]) Let $(\mathscr{U}, \wp_\flat)$ be a $\mathcal{P}_\flat$M$\mathcal{S}$ with a coefficient ς and $\{\jmath_n\}_{n\in\mathbb{N}}$ be a sequence in $(\mathscr{U}, \wp_\flat)$.*

i. Provided that the equality $\lim_{n\to\infty} \wp_\flat(\jmath_n, \jmath) = \wp_\flat(\jmath, \jmath)$ holds, $\{\jmath_n\}_{n\in\mathbb{N}}$ is called a $\wp_\flat$-convergent sequence in $\mathscr{U}$ and $\jmath$ is termed the $\wp_\flat$-limit of $\{\jmath_n\}_{n\in\mathbb{N}}$.

ii. $\{\jmath_n\}_{n\in\mathbb{N}}$ is a $\wp_\flat$-Cauchy sequence if $\lim_{n,m\to\infty} \wp_\flat(\jmath_n, \jmath_m)$ exists and is finite.

iii. If, for each $\wp_\flat$-Cauchy sequence $\{\jmath_n\}_{n\in\mathbb{N}}$, a point $\jmath$ exists in $\in \mathscr{U}$ such that

$$\lim_{n,m\to\infty} \wp_\flat(\jmath_n, \jmath_m) = \lim_{n\to\infty} \wp_\flat(\jmath_n, \jmath) = \wp_\flat(\jmath, \jmath),$$

then $(\mathscr{U}, \wp_\flat)$ is called a $\wp_\flat$-complete space.

Remark 2.0.14. *In a* $\mathcal{P}_\flat$M$\mathcal{S}$*, a convergent sequence does not require a unique limit. In Example 2.0.10 (ii), consider*

$$(\jmath_n)_{n\in\mathbb{N}} = \left(\frac{1}{n^2+1}\right)_{n\in\mathbb{N}} \in \mathscr{U}.$$

Then,

$$\lim_{n\to\infty} \wp_\flat\left(\tfrac{1}{n^2+1}, 0\right) = \lim_{n\to\infty} e^{\frac{\left|\frac{1}{n^2+1}-0\right|^2+\frac{1}{n^2+1}+0}{2}} = 1 = \wp_\flat(0,0) = e^{\max\left\{|0-0|, \frac{0+0}{2}\right\}},$$
$$\lim_{n\to\infty} \wp_\flat\left(\tfrac{1}{n^2+1}, 1\right) = \lim_{n\to\infty} e^{\max\left\{\left|\frac{1}{n^2+1}-1\right|, \frac{\frac{1}{n^2+1}+1}{2}\right\}} = e^1 = \wp_\flat(1,1) = e^{\max\left\{|1-1|, \frac{1+1}{2}\right\}}.$$

As can be seen, the sequence $\frac{1}{n^2+1}$ has more than one limit.

Example 2.0.15. *([20]) Let* $\mathscr{U} = \mathbb{R}^+$ *and* $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ *denote a mapping determined by* $\wp_\flat(\jmath, \rho) = \max\{\jmath, \rho\}$ *for all* $\jmath, \rho \in \mathscr{U}$*. In such a case,* $(\mathscr{U}, \wp_\flat)$ *has become a* $\mathcal{P}_\flat$M$\mathcal{S}$ *with the coefficient* $\varsigma \geq 1$*. We obtain the following for the sequence* $\left\{\frac{1}{n+1}\right\}_{n\in\mathbb{N}}$ *in* $(\mathscr{U}, \wp_\flat)$ *:*

$$\lim_{n\to\infty} \wp_\flat\left(1, \tfrac{1}{1+n}\right) = 1 = \wp_\flat(1,1),$$
$$\lim_{n\to\infty} \wp_\flat\left(2, \tfrac{1}{1+n}\right) = 2 = \wp_\flat(2,2).$$

The sequence $\left\{\frac{1}{n+1}\right\}_{n\in\mathbb{N}}$ owns two limits in $\mathscr{U}$.

Lemma 2.0.16. *([25]) Presume that* $(\mathscr{U}, \wp_\flat)$ *and* $\left(\mathscr{U}, d_{\wp_\flat}\right)$ *are a* $\mathcal{P}_\flat$M$\mathcal{S}$ *and* $\flat$*-metric space, respectively.*

1) $\{\jmath_n\}_{n\in\mathbb{N}}$ is a $\wp_\flat$-Cauchy sequence in $(\mathscr{U}, \wp_\flat)$ if and only if it is also a $\flat$-Cauchy sequence in $\left(\mathscr{U}, d_{\wp_\flat}\right)$.
2) $(\mathscr{U}, \wp_\flat)$ is $\wp_\flat$-complete if and only if $\left(\mathscr{U}, d_{\wp_\flat}\right)$ is $\flat$-complete. As well, $\lim_{n\to\infty} d_{\wp_\flat}(\jmath, \jmath_n) = 0$ if and only if

$$\lim_{m\to\infty} \wp_\flat(\jmath, \jmath_m) = \lim_{n\to\infty} \wp_\flat(\jmath, \jmath_n) = \wp_\flat(\jmath, \jmath).$$

Remark 2.0.17. *([27]) The mapping* $\wp_\flat$ *is not continuous in general.*

Presume that $\mathscr{U} = \mathbb{N} \cup \{\infty\}$ and $\wp_\flat : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ is a mapping defined as follows:

$$\wp_\flat(\jmath,\rho) = \begin{cases} 0, & \jmath = \rho; \\ \left|\frac{1}{\jmath} - \frac{1}{\rho}\right|, & \text{if one of } \jmath \text{ and } \rho \text{ odd and the other is odd or } \jmath\rho = \infty; \\ 5, & \text{if one of } \jmath \text{ and } \rho \text{ even and the other is even } \jmath \neq \rho \text{ or } \infty; \\ 2, & \text{otherwise.} \end{cases}$$

$(\mathscr{U}, \wp_\flat)$ is a $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$ with $\varsigma = 3$. For each $n \in \mathbb{N}$, consider that $\jmath_n = 2n + 3$. Then,

$$\wp_\flat(2n+3, \infty) = \frac{1}{2n+3} \to 0$$

as $n \to \infty$; so, $\jmath_n \to \infty$; however, $\lim_{n\to\infty} \wp_\flat(\jmath_n, 4) = 2 \neq 5 = \wp_\flat(\infty, 4)$, which means that $\wp_\flat$ does not have the continuity property.

The subsequent lemma is essential in regarding the $\wp_\flat$-convergent sequences as the confirmation of our findings because a $\mathcal{P}_\flat\mathrm{M}$ is not continuous in general.

Lemma 2.0.18. *([27]) Ensure that $(\mathscr{U}, \wp_\flat)$ is a $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$ with the coefficient $\varsigma > 1$ and the sequences $\{\jmath_n\}_{n\in\mathbb{N}}$ and $\{\rho_n\}_{n\in\mathbb{N}}$ are $\wp_\flat$-convergent to $\jmath$ and ρ, respectively. Afterward, we obtain*

$$\begin{aligned} \frac{1}{\varsigma^2}\wp_\flat(\jmath,\rho) - \frac{1}{\varsigma}\wp_\flat(\jmath,\jmath) - \wp_\flat(\rho,\rho) &\leq \lim_{n\to\infty} \inf \wp_\flat(\jmath_n,\rho_n) \\ &\leq \lim_{n\to\infty} \sup \wp_\flat(\jmath_n,\rho_n) \\ &\leq \varsigma\wp_\flat(\jmath,\jmath) + \varsigma^2\wp_\flat(\rho,\rho) + \varsigma^2\wp_\flat(\jmath,\rho). \end{aligned}$$

Wardowski [48] proposed a novel notion associated with the $\mathcal{F}$-contraction in 2012. As a result, several investigations have been conducted to obtain more extended contractive mappings on metric spaces and other generalized metric spaces.

Definition 2.0.19. *([48]) Let $(\mathscr{U}, d)$ be a metric space. The mapping $O : \mathscr{U} \to \mathscr{U}$ is an $\mathcal{F}$-contraction provided that $\mathcal{F} \in \mathfrak{F}$ and $\kappa > 0$ exist such that, for all $\jmath, \rho \in \mathscr{U}$,*

$$d(O\jmath, O\rho) > 0 \Rightarrow \kappa + \mathcal{F}(d(O\jmath, O\rho)) \leq \mathcal{F}(d(\jmath,\rho)),$$

where $\mathfrak{F}$ is the set of functions $\mathcal{F} : (0,\infty) \to \mathbb{R}$ fulfilling the subsequent statements:

$(\mathcal{F}_1)$ $\mathcal{F}$ is strictly increasing, i.e., for all $\mathfrak{a}, \mathfrak{b} \in (0,\infty)$, such that $\mathfrak{a} < \mathfrak{b}$, $\mathcal{F}(\mathfrak{a}) < \mathcal{F}(\mathfrak{b})$;
$(\mathcal{F}_2)$ for each $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \mathfrak{a}_n = 0 \Leftrightarrow \lim_{n\to\infty} \mathcal{F}(\mathfrak{a}_n) = -\infty$;
$(\mathcal{F}_3)$ a constant $\mathfrak{c} \in (0,1)$ exists such that $\lim_{\mathfrak{a}\to 0^+} \mathfrak{a}^{\mathfrak{c}}\mathcal{F}(\mathfrak{a}) = 0$.

Subsequently, Wardowski attested that in [48], any $\mathcal{F}$-contraction enjoys a unique fixed-point in a complete metric space $(\mathscr{U}, d)$.

As an extension of the family $\mathfrak{F}$, Piri and Kumam [51] put forth a new set of functions $\mathcal{F} \in \mathfrak{F}^*$ by substituting the term $(\mathcal{F}_3)$ with $(\mathcal{F}_3')$ in Definition 2.0.19, as noted below:
$(\mathcal{F}_3')$ $\mathcal{F}$ is continuous.

Briefly, set $\mathfrak{F}^* = \{\mathcal{F} : (0,\infty) \to (-\infty, +\infty) : \mathcal{F} \text{ holds } (\mathcal{F}_1), (\mathcal{F}_2) \text{ and } (\mathcal{F}_3')\}$.

In 2014, Jleli and Samet [52] introduced the concept of a θ-contraction, and the class $\Theta = \{\theta : (0,\infty) \to (1,\infty)\}$ represents all such functions provided to fulfill the below circumstances:

(θ_1) θ is non-decreasing;
(θ_2) for each $\{\mathfrak{a}_{\mathfrak{z}}\} \subset (0,\infty)$, $\lim_{\mathfrak{z}\to\infty} \theta(\mathfrak{a}_{\mathfrak{z}}) = 1 \Leftrightarrow \lim_{\mathfrak{z}\to\infty} \mathfrak{a}_{\mathfrak{z}} = 0^+$;
(θ_3) a constant $r \in (0,1)$ and $\mathfrak{s} \in (0,\infty]$ exist such that $\lim_{\mathfrak{a}\to 0^+} \frac{\theta(\mathfrak{a})}{\mathfrak{a}^r} = \mathfrak{s}$.

Theorem 2.0.20. *Let $O : \mathscr{U} \to \mathscr{U}$ be a mapping on a complete metric space $(\mathscr{U}, d)$. Provided that a function $\theta \in \Theta$ and a constant $\tau \in (0,1)$ exist such that*

$$d(O\jmath, O\rho) \neq 0 \quad \Rightarrow \quad \theta(d(O\jmath, O\rho)) \leq [\theta(d(\jmath,\rho))]^{\tau}$$

for all $\jmath, \rho \in \mathscr{U}$, then O owns a unique fixed-point.

Furthermore, Liu et al. [47] identified the set $\tilde{\Theta} = \{\theta : (0,\infty) \to (1,\infty) : \ \theta \text{ holds } (\theta_1{}') \text{ and } (\theta_2{}')\}$, where

($\theta_1{}'$) θ is non-decreasing and continuous;
($\theta_2{}'$) $\inf_{\mathfrak{a}\in(0,\infty)} \theta(\mathfrak{a}) = 1$.

Theorem 2.0.21. *[47] Let $O : \mathscr{U} \to \mathscr{U}$ be a self-mapping on a complete metric space $(\mathscr{U}, d)$. Thereby, the following statements are equivalent:*

i. the mapping O is a θ−contraction with $\theta \in \tilde{\Theta}$;
ii. the mapping O is an $\mathcal{F}$−contraction with $\mathcal{F} \in \mathfrak{F}^*$.

The concept of a $\mathbf{D}_{\mathscr{C}}$-contraction was proposed by Liu et al. [47] as follows.

Presume that $\mathbf{D} : (0,\infty) \to (0,\infty)$ is a function and fulfills the terms $(\mathbf{D}1) - (\mathbf{D}3)$.

($\mathbf{D}1$) $\mathbf{D}$ is non-decreasing;
($\mathbf{D}2$) $\lim_{n\to\infty} \mathbf{D}(\mathfrak{a}_n) = 0 \Leftrightarrow \lim_{n\to\infty} \mathfrak{a}_n = 0$;
($\mathbf{D}3$) $\mathbf{D}$ is continuous.

Set $\Delta = \{\mathbf{D} : (0,\infty) \to (0,\infty) : \mathbf{D} \ satisfies \ (\mathbf{D}1) - (\mathbf{D}3)\}$.
$\mathscr{C} : (0,\infty) \to (0,\infty)$ is a comparison function that has the features $(\mathscr{C}_1)$ and $(\mathscr{C}_2)$.

($\mathscr{C}_1$) $\mathscr{C}$ is monotonically increasing, that is, $\mathfrak{a} < \mathfrak{b} \Rightarrow \mathscr{C}(\mathfrak{a}) < \mathscr{C}(\mathfrak{b})$.
($\mathscr{C}_2$) $\lim_{n\to\infty} \mathscr{C}^n(\mathfrak{a}) = 0$ for all $\mathfrak{a} > 0$, where $\mathscr{C}^n$ denotes the n^{th}-iteration of $\mathscr{C}$.

If $\mathscr{C}$ is a comparison function, then $\mathscr{C}(\mathfrak{a}) < \mathfrak{a}$ for all $\mathfrak{a} > 0$. The mappings

- $\mathscr{C}_x(\mathfrak{a}) = \chi\mathfrak{a}, \ 0 < \chi < 1, \ \mathfrak{a} > 0,$
- $\mathscr{C}_y(\mathfrak{a}) = \frac{\mathfrak{a}}{1+\mathfrak{a}}$

can be given as examples of comparison functions.

Definition 2.0.22. *([47]) Let $(\mathscr{U}, d)$ be a metric space and O be a self-mapping on this space. Let $\mathfrak{I} = \left\{(\jmath,\rho) \in \mathscr{U}^2 : d(O\jmath, O\rho) > 0\right\}$. O is named a $\mathbf{D}_{\mathscr{C}}$-contraction if it ensures the following expression:*

$$\mathbf{D}(d(O\jmath, O\rho)) \leq \mathscr{C}(\mathbf{D}(d(\jmath,\rho))) \tag{2.5}$$

for all $\jmath, \rho \in \mathfrak{I}$.

In 2021, Nazam et al. [50] presented a new definition of the $\mathbf{D}_{\mathscr{C}}$-contraction, including two self-mappings with a binary relation in a $\mathcal{P}_{\mathfrak{b}}M\mathcal{S}$, referring to Liu's definition of a $\mathbf{D}_{\mathscr{C}}$-contraction.

Definition 2.0.23. *([50]) Let O and $\mathfrak{S}$ be two self-mappings on a $\mathcal{P}_\flat$MS and $\mathfrak{R}$ be a binary relation on $\mathscr{U}$. Define the set $\mathfrak{I} = \{(\jmath,\rho) \in \mathfrak{R} : \wp_\flat (O\jmath, \mathfrak{S}\rho) > 0\}$. The mappings O and $\mathfrak{S}$ form a $\mathbf{D}_{\mathscr{C}}$-contraction if there exist $\mathbf{D} \in \Delta$ and a continuous comparison function $\mathscr{C}$ such that*

$$\mathbf{D}\left(\varsigma^2 \wp_\flat (O(\jmath), \mathfrak{S}(\rho))\right) \leq \mathscr{C}(\mathbf{D}(\wp_\flat(\jmath,\rho))) \tag{2.6}$$

for all $\jmath, \rho \in \mathfrak{I}$.

Definition 2.0.24. *([46]) Let $\mathscr{U}$ be a nonempty set and $O : \mathscr{U} \to \mathscr{U}$ and $\alpha : \mathscr{U}^2 \to \mathbb{R}$ be given mappings. O is α-orbital-admissible provided that the expression given below is true:*

$$\alpha(\jmath, O\jmath) \geq 1 \Rightarrow \alpha\left(O\jmath, O^2\jmath\right) \geq 1$$

for all $\jmath \in \mathscr{U}$.

Definition 2.0.25. *([35]) Let $\mathscr{U}$ be a nonempty set and $\alpha : \mathscr{U}^2 \to \mathbb{R}$ be a given function.*

i. O is an α-orbital-admissible mapping;
ii. $\alpha(\jmath,\rho) \geq 1$ and $\alpha(\rho, O\rho) \geq 1 \Rightarrow \alpha(\jmath, O\rho) \geq 1,\ \jmath,\rho \in \mathscr{U}$.

A mapping $O : \mathscr{U} \to \mathscr{U}$ that satisfies the above features is named a triangular α-orbital-admissible mapping.

Definition 2.0.26. *([46]) Let $(\mathscr{U}, d)$ be a complete metric space. The mapping $O : \mathscr{U} \to \mathscr{U}$ is a Geraghty-type contraction if a function $\mathfrak{P} : [0,\infty) \to [0,1)$ exists, which ensures the following term:*

$$\lim_{n\to\infty} \mathfrak{P}(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0 \tag{2.7}$$

such that

$$d(O\jmath, O\rho) \leq \mathfrak{P}(d(\jmath,\rho))\, d(\jmath,\rho)$$

for all $\jmath,\rho \in \mathscr{U}$. The family of $\mathfrak{P} : [0,\infty) \to [0,1)$ satisfying (2.7) is represented as $\mathbf{B}$.

Definition 2.0.27. *([46]) In a metric space $(\mathscr{U}, d)$, if the mappings $O, \mathfrak{S} : \mathscr{U} \to \mathscr{U}$ satisfy the below statements for all $\jmath,\rho \in \mathscr{U}$:*

$$d(O\jmath, \mathfrak{S}\rho) \leq \mathfrak{P}(E_{\mathfrak{S},O}(\jmath,\rho))\, E_{\mathfrak{S},O}(\jmath,\rho),$$

where

$$E_{\mathfrak{S},O}(\jmath,\rho) = d(\jmath,\rho) + |d(\jmath, O\jmath) - d(\rho, \mathfrak{S}\rho)|$$

whenever a function $\mathfrak{P} \in \mathbf{B}$ exists, then O, $\mathfrak{S}$ are called Geraghty contractions of type $E_{\mathfrak{S},O}$.

Definition 2.0.28. *([46]) The set of $\mathfrak{P} : [0,\infty) \to \left[0, \frac{1}{\varsigma}\right)$ functions fulfilling the constraint $\lim_{n\to\infty} \mathfrak{P}(t_n) = \frac{1}{\varsigma}$ for $\varsigma \geq 1$ implies that $\lim_{n\to\infty} t_n = 0$ and it is stated as $\mathbf{B}_\varsigma$.*

Definition 2.0.29. *([46]) Let $\alpha : \mathscr{U}^2 \to \mathbb{R}$ be a function in $(\mathscr{U}, d)$. Provided that the subsequent statement*

$$\alpha(\jmath,\rho) \geq 1 \Rightarrow d(\mathcal{O}\jmath,\mathcal{O}\rho) \leq \mathfrak{P}(E_{\mathcal{O}}(\jmath,\rho))\,E_{\mathcal{O}}(\jmath,\rho),$$

where

$$E_{\mathcal{O}}(\jmath,\rho) = d(\jmath,\rho) + |d(\jmath,\mathcal{O}\jmath) - d(\rho,\mathcal{O}\rho)|$$

is satisfied for all $\jmath,\rho \in \mathscr{U}$ whenever $\mathfrak{P} \in \mathbf{B}_{\varsigma}$, then $\mathcal{O} : \mathscr{U} \to \mathscr{U}$ is called an α-$\mathfrak{P}_E$-Geraghty contraction.

Unless otherwise stated, i and j will be treated as arbitrary positive integers throughout this study.

Definition 2.0.30. *([46]) Let $\alpha_i^j : \mathscr{U}^2 \to \mathbb{R}$ be a function in a $\flat$-metric space $(\mathscr{U},\flat)$ with the coefficient $\varsigma \geq 1$. $\mathcal{O},\mathfrak{S} : \mathscr{U} \to \mathscr{U}$ are two self-mappings which are called α_i^j-$\mathfrak{P}_{\hat{E}_{\mathfrak{S},\mathcal{O}}}$-Geraghty contractions if a function $\mathfrak{P} \in \mathbf{B}_{\varsigma}$ exists, which ensures the following expressions:*

$$\alpha_i^j(\jmath,\rho) \geq \varsigma^f \Rightarrow \alpha_i^j(\jmath,\rho)\,\flat\left(\mathcal{O}^i\jmath,\mathfrak{S}^j\rho\right) \leq \mathfrak{P}(E_{\mathfrak{S},\mathcal{O}}(\jmath,\rho))\,E_{\mathfrak{S},\mathcal{O}}(\jmath,\rho),$$

where

$$\hat{E}_{\mathfrak{S},\mathcal{O}}(\jmath,\rho) = \flat(\jmath,\rho) + \left|\flat\left(\jmath,\mathcal{O}^i\jmath\right) - \flat\left(\rho,\mathfrak{S}^j\rho\right)\right|$$

for all $\jmath,\rho \in \mathscr{U}$, and $f \geq 2$ is a constant.

Remark 2.0.31. *([46])*

i. If $\varsigma = 1$, $\alpha_i^j(\jmath,\rho) = 1$ and $i = j = 1$, we obtain the Geraghty contraction of type $E_{\mathfrak{S},\mathcal{O}}$.
ii. If $\mathfrak{S} = \mathcal{O}$ and $i = j = 1$, we obtain the α-$\mathfrak{P}_E$-Geraghty contraction.

Definition 2.0.32. *([46]) Let $\alpha_i^j : \mathscr{U}^2 \to [0,\infty)$ be a function in a $\flat$-metric space $(\mathscr{U},\flat)$ with the coefficient $\varsigma \geq 1$. The mappings $\mathcal{O},\mathfrak{S} : \mathscr{U} \to \mathscr{U}$ are called α_i^j-orbital-admissible if the following circumstances hold:*

$$\alpha_i^j\left(\jmath,\mathcal{O}^i\jmath\right) \geq \varsigma^f \Rightarrow \alpha_i^j\left(\mathcal{O}^i\jmath,\mathfrak{S}^j\mathcal{O}^i\jmath\right) \geq \varsigma^f,$$
$$\alpha_i^j\left(\jmath,\mathfrak{S}^j\jmath\right) \geq \varsigma^f \Rightarrow \alpha_i^j\left(\mathfrak{S}^j\jmath,\mathcal{O}^i\mathfrak{S}^j\jmath\right) \geq \varsigma^f$$

for all $\jmath \in \mathscr{U}$, where $f \geq 2$ is a constant.

Definition 2.0.33. *([46]) Let $\alpha_i^j : \mathscr{U}^2 \to [0,\infty)$ be a function in a complete $\flat$-metric space $(\mathscr{U},\flat)$ with the coefficient $\varsigma \geq 1$. When $\mathcal{O},\mathfrak{S} : \mathscr{U} \to \mathscr{U}$ are two self-mappings, the pair $(\mathcal{O},\mathfrak{S})$ is a triangular α_i^j-orbital-admissible pair if*

i. $\mathcal{O},\mathfrak{S}$ are α_i^j-orbital-admissible,
ii. $\alpha_i^j(\jmath,\rho) \geq \varsigma^f$, $\alpha_i^j\left(\rho,\mathcal{O}^i\rho\right) \geq \varsigma^f$ and $\alpha_i^j\left(\rho,\mathfrak{S}^j\rho\right) \geq \varsigma^f$ imply that $\alpha_i^j\left(\jmath,\mathcal{O}^i\rho\right) \geq \varsigma^f$ and $\alpha_i^j\left(\jmath,\mathfrak{S}^j\rho\right) \geq \varsigma^f$, where $f \geq 2$ is a constant.

Lemma 2.0.34. *([46]) Consider that $\mathcal{O},\mathfrak{S} : \mathscr{U} \to \mathscr{U}$ are two mappings in a complete $\flat$-metric space $(\mathscr{U},\flat)$ with the coefficient $\varsigma \geq 1$. Assume that $(\mathcal{O},\mathfrak{S})$ is a triangular α_i^j-orbital-admissible pair and an element $\jmath_0$ in $\mathscr{U}$ exists with the property $\alpha_i^j\left(\jmath_0,\mathcal{O}^i\jmath_0\right) \geq \varsigma^f$. Setup a sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ in $(\mathscr{U},b)$ as follows:*

$$\jmath_{2n} = \mathfrak{S}^j \jmath_{2n-1},$$
$$\jmath_{2n+1} = O^i \jmath_{2n},$$

where $n = 0, 1, 2, ...$; afterward, for $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$, $\alpha_i^j(\jmath_n, \jmath_m) \geq \varsigma^f$ exists.

3. Main results

This is the leading part of the study, and it contains a novel definition and a common fixed-point theorem. Furthermore, some relevant results will be presented. Illustrative examples are provided to indicate the accuracy and validity of the findings.

In the course of the study, the set of fixed-points of O and the set of common fixed-points of O and $\mathfrak{S}$ will be denoted with the notations $Fix(O)$ and $C_{Fix}(O, \mathfrak{S})$, respectively.

Definition 3.0.1. *Let* $(\mathscr{U}, \wp_\flat)$ *be a* $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$ *with the coefficient* $\varsigma \geq 1$ *and* $\alpha_i^j : \mathscr{U}^2 \to [0, \infty)$ *be a function. The mappings* $O, \mathfrak{S} : \mathscr{U} \to \mathscr{U}$ *are called generalized* α_i^j*-*$(\mathbf{D}_{\mathscr{C}}(\mathfrak{P}_{\hat{E}}))$*-contractions if* $\mathbf{D} \in \Delta$ *and a continuous comparison function* $\mathscr{C}$ *and* $\mathfrak{P} \in \mathbf{B}_\varsigma$ *exist such that, for all* $\jmath, \rho \in \mathscr{U}$, $\alpha_i^j(\jmath, \rho) \geq \varsigma^f$ *and* $\wp_\flat\left(O^i \jmath, \mathfrak{S}^j \rho\right) > 0$*,*

$$\mathbf{D}\left(\alpha_i^j(\jmath,\rho)\,\wp_\flat\left(O^i \jmath, \mathfrak{S}^j \rho\right)\right) \leq \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{O,\mathfrak{S}}(\jmath,\rho)\right)\hat{E}_{O,\mathfrak{S}}(\jmath,\rho)\right)\right), \tag{3.1}$$

where
$$\hat{E}_{\mathfrak{S},O}(\jmath,\rho) = \wp_\flat(\jmath,\rho) + \left|\wp_\flat\left(\jmath, O^i \jmath\right) - \wp_\flat\left(\rho, \mathfrak{S}^j \rho\right)\right|$$

and $f \geq 2$ is a constant.

Theorem 3.0.2. *Ensure that* $(\mathscr{U}, \wp_\flat)$ *is a* $\wp_\flat$*-complete* $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$*,* $O, \mathfrak{S} : \mathscr{U} \to \mathscr{U}$ *are two mappings and* $\alpha_i^j : \mathscr{U}^2 \to [0, \infty)$ *is a function. Assume that the subsequent statements are satisfied:*

i. $O, \mathfrak{S}$ are generalized α_i^j-$(\mathbf{D}_{\mathscr{C}}(\mathfrak{P}_{\hat{E}}))$-contractions mappings;
ii. the pair $(O, \mathfrak{S})$ is triangular α_i^j-orbital-admissible;
iii. $\jmath_0 \in \mathscr{U}$ exists, satisfying $\alpha_i^j\left(\jmath_0, O^i \jmath_0\right) \geq \varsigma^f$;
iv. one of the below terms is provided:
  iv$_a$. O^i and $\mathfrak{S}^j$ are $\wp_\flat$-continuous,
  or
  iv$_b$. if $\{\jmath_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{U}$ such that $\alpha_i^j(\jmath_n, \jmath_{n+1}) \geq \varsigma^f$ for each $n \in \mathbb{N}$ and $\jmath_n \to r \in \mathscr{U}$ as $n \to \infty$, then a subsequence $\{\jmath_{n_k}\}$ of $\{\jmath_n\}$ exists such that $\alpha_i^j(\jmath_{n_k}, r) \geq \varsigma^f$ for each $k \in \mathbb{N}$;
v. for all $r \in Fix\left(O^i\right)$ or $w \in Fix\left(\mathfrak{S}^j\right)$ we have $\alpha_i^j(r, w) \geq \varsigma^f$.

Then, the set of $C_{Fix}(O, \mathfrak{S})$ consists of a unique element belonging to $(\mathscr{U}, \wp_\flat)$.

Proof. An initial point $\jmath_0 \in \mathscr{U}$ with the property $\alpha_i^j\left(\jmath_0, O^i \jmath_0\right) \geq \varsigma^f$ exists from (iii). Consider $\{\jmath_n\}_{n\in\mathbb{N}}$ in $\mathscr{U}$, which is constructed as

$$\jmath_{2n+2} = O^i \jmath_{2n+1} \text{ and } \jmath_{2n+1} = \mathfrak{S}^j \jmath_{2n}$$

for each $n \in \mathbb{N}$. Hypothetically, a natural number n_0 exists such that $\jmath_{n_0} = \jmath_{n_0+1}$. Let $\jmath_{2n_0} = \jmath_{2n_0+1}$; then, $\jmath_{2n_0} = \jmath_{2n_0+1} = \Im^j \jmath_{2n_0}$. Hence, $Fix\left(\Im^j\right) = \{\jmath_{2n_0}\}$. Suppose that $\wp_\flat\left(\jmath_{2n_0+2}, \jmath_{2n_0+1}\right) > 0$ and $\mathcal{O}^i \jmath_{2n_0+1} \neq \Im^j \jmath_{2n_0}$. According to Lemma 2.0.34, we have

$$\alpha_i^j\left(\jmath_{2n_0}, \jmath_{2n_0+1}\right) = \alpha_i^j\left(\jmath_{2n_0+1}, \jmath_{2n_0}\right) \geq \varsigma^f.$$

In (3.1), writing $\jmath = o_{2n_0+1}$ and $\rho = o_{2n_0}$, we attain

$$\begin{aligned}\mathbf{D}\left(\wp_\flat\left(\mathcal{O}^i \jmath_{2n_0+1}, \Im^j \jmath_{2n_0}\right)\right) &\leq \mathbf{D}\left(\alpha_{i,j}\left(\jmath_{2n_0+1}, \jmath_{2n_0}\right) \wp_\flat\left(\mathcal{O}^i \jmath_{2n_0+1}, \Im^j \jmath_{2n_0}\right)\right)\\ &\leq \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\Im,\mathcal{O}}\left(\jmath_{2n_0+1}, \jmath_{2n_0}\right)\right)\hat{E}_{\Im,\mathcal{O}}\left(\jmath_{2n_0+1}, \jmath_{2n_0}\right)\right)\right)\\ &< \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}\hat{E}_{\Im,\mathcal{O}}\left(\jmath_{2n_0+1}, \jmath_{2n_0}\right)\right)\right),\end{aligned}$$

where

$$\begin{aligned}\hat{E}_{\Im,\mathcal{O}}\left(\jmath_{2n_0+1}, \jmath_{2n_0}\right) &= \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0}\right) + \left|\wp_\flat\left(\jmath_{2n_0+1}, \mathcal{O}^i \jmath_{2n_0+1}\right) - \wp_\flat\left(\jmath_{2n_0}, \Im^j \jmath_{2n_0}\right)\right|\\ &= \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+1}\right) + \left|\wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+2}\right) - \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+1}\right)\right|\\ &= \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+1}\right) + \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+2}\right) - \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+1}\right)\\ &= \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+2}\right).\end{aligned}$$

Hence, employing the property of $\mathscr{C}(\mathfrak{a}) < \mathfrak{a}$, we obtain the following contradictory expression:

$$\begin{aligned}\mathbf{D}\left(\wp_\flat\left(\jmath_{2n_0+2}, \jmath_{2n_0+1}\right)\right) &= \mathbf{D}\left(\wp_\flat\left(\mathcal{O}^i \jmath_{2n_0+1}, \Im^j \jmath_{2n_0}\right)\right)\\ &< \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}\wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+2}\right)\right)\right) < \mathbf{D}\left(\tfrac{1}{\varsigma}\wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+2}\right)\right).\end{aligned}$$

In this case, we obtain $\wp_\flat\left(\jmath_{2n_0+2}, \jmath_{2n_0+1}\right) = 0$, with the property (**D**1) of the function **D**. By ($\wp_\flat 2$), we have that $\wp_\flat\left(\jmath_{2n_0+2}, \jmath_{2n_0+2}\right) = \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+1}\right) = 0$, and it yields that

$$\wp_\flat\left(\jmath_{2n_0+2}, \jmath_{2n_0+2}\right) = \wp_\flat\left(\jmath_{2n_0+1}, \jmath_{2n_0+1}\right) = \wp_\flat\left(\jmath_{2n_0+2}, \jmath_{2n_0+1}\right).$$

By ($\wp_\flat 1$), $\jmath_{2n_0+2} = \jmath_{2n_0+1}$ and $\mathcal{O}^i \jmath_{2n_0+1} = \jmath_{2n_0+1}$. Therefore, $\jmath_{2n_0+1}$ is a fixed-point of $\mathcal{O}^i$, and then $\jmath_{2n_0} = \jmath_{2n_0+1}$ belongs to $C_{Fix}\left(\mathcal{O}^i, \Im^j\right)$. For some natural numbers n_0, we achieve a similar result for $\jmath_{2n_0} = \jmath_{2n_0-1}$. Hereafter, we presume that $\jmath_n \neq \jmath_{n+1}$ for all $n \in \mathbb{N}$. It is needed to investigate the subsequent two cases:

Case 1: Presume that $\jmath_{2n} \neq \jmath_{2n-1}$ for all $n \in \mathbb{N}$. $\wp_\flat\left(\mathcal{O}^i \jmath_{2n-1}, \Im^j \jmath_{2n}\right) > 0$ and, moreover, $\alpha_i^j\left(\jmath_{2n-1}, \jmath_{2n}\right) \geq \varsigma^f$. Therefore, employing (3.1), we gain

$$\begin{aligned}\mathbf{D}\left(\wp_\flat\left(\jmath_{2n}, \jmath_{2n+1}\right)\right) &= \mathbf{D}\left(\wp_\flat\left(\mathcal{O}^i \jmath_{2n-1}, \Im^j \jmath_{2n}\right)\right)\\ &\leq \mathbf{D}\left(\alpha_i^j\left(\jmath_{2n-1}, \jmath_{2n}\right) \wp_\flat\left(\mathcal{O}^i \jmath_{2n-1}, \Im^j \jmath_{2n}\right)\right)\\ &\leq \mathscr{C}\left(\mathbf{D}\left(\left(\hat{E}_{\Im,\mathcal{O}}\left(\jmath_{2n-1}, \jmath_{2n}\right)\right)\hat{E}_{\Im,\mathcal{O}}\left(\jmath_{2n-1}, \jmath_{2n}\right)\right)\right)\\ &< \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}\hat{E}_{\Im,\mathcal{O}}\left(\jmath_{2n-1}, \jmath_{2n}\right)\right)\right),\end{aligned}$$

where

$$\begin{aligned}\hat{E}_{\Im,\mathcal{O}}\left(\jmath_{2n-1}, \jmath_{2n}\right) &= \wp_\flat\left(\jmath_{2n-1}, \jmath_{2n}\right) + \left|\wp_\flat\left(\jmath_{2n-1}, \mathcal{O}^i \jmath_{2n-1}\right) - \wp_\flat\left(\jmath_{2n}, \Im^j \jmath_{2n}\right)\right|\\ &= \wp_\flat\left(\jmath_{2n-1}, \jmath_{2n}\right) + \left|\wp_\flat\left(\jmath_{2n-1}, \jmath_{2n}\right) - \wp_\flat\left(\jmath_{2n}, \jmath_{2n+1}\right)\right|.\end{aligned}$$

If $\wp_\flat(\jmath_{2n}, \jmath_{2n+1}) \geq \wp_\flat(\jmath_{2n-1}, \jmath_{2n})$, the following expression is obtained:

$$\hat{E}_{\Im,O}(\jmath_{2n-1}, \jmath_{2n}) = \wp_\flat(\jmath_{2n-1}, \jmath_{2n}) - \wp_\flat(\jmath_{2n-1}, \jmath_{2n}) + \wp_\flat(\jmath_{2n}, \jmath_{2n+1}).$$

Thereby, the inequality given below is obtained:

$$\begin{aligned}\mathbf{D}(\wp_\flat(\jmath_{2n}, \jmath_{2n+1})) &< \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}\wp_\flat(\jmath_{2n}, \jmath_{2n+1})\right)\right)\\ &< \mathbf{D}\left(\tfrac{1}{\varsigma}\wp_\flat(\jmath_{2n}, \jmath_{2n+1})\right).\end{aligned}$$

However, this produces a discrepancy. Accordingly, the following statement must be true:

$$\wp_\flat(\jmath_{2n}, \jmath_{2n+1}) < \wp_\flat(\jmath_{2n-1}, \jmath_{2n}).$$

Case 2: Presume that $\jmath_{2n} \neq \jmath_{2n+1}$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, $\wp_\flat\left(O^i \jmath_{2n}, \Im^j \jmath_{2n+1}\right) > 0$ and $\alpha_i^j(\jmath_{2n}, \jmath_{2n+1}) \geq \varsigma^f$, we procure

$$\begin{aligned}\mathbf{D}(\wp_\flat(\jmath_{2n+1}, \jmath_{2n+2})) &= \mathbf{D}\left(\wp_\flat\left(O^i \jmath_{2n}, \Im^j \jmath_{2n+1}\right)\right)\\ &\leq \mathbf{D}\left(\alpha_i^j(\jmath_{2n}, \jmath_{2n+1})\wp_\flat\left(O^i \jmath_{2n}, \Im^j \jmath_{2n+1}\right)\right)\\ &\leq \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\Im,O}(\jmath_{2n}, \jmath_{2n+1})\right)\hat{E}_{\Im,O}(\jmath_{2n}, \jmath_{2n+1})\right)\right)\\ &< \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}\hat{E}_{\Im,O}(\jmath_{2n}, \jmath_{2n+1})\right)\right),\end{aligned}$$

where

$$\begin{aligned}\hat{E}_{\Im,O}(\jmath_{2n}, \jmath_{2n+1}) &= \wp_\flat(\jmath_{2n}, \jmath_{2n+1}) + \left|\wp_\flat\left(\jmath_{2n}, O^i \jmath_{2n}\right) - \wp_\flat\left(\jmath_{2n+1}, \Im^j \jmath_{2n+1}\right)\right|\\ &= \wp_\flat(\jmath_{2n}, \jmath_{2n}) + \left|\wp_\flat(\jmath_{2n}, \jmath_{2n+1}) - \wp_\flat(\jmath_{2n+1}, \jmath_{2n+2})\right|.\end{aligned}$$

If $\wp_\flat(\jmath_{2n+1}, \jmath_{2n+2}) \geq \wp_\flat(\jmath_{2n}, \jmath_{2n+1})$, the following equation is obtained:

$$\hat{E}_{\Im,O}(\jmath_{2n}, \jmath_{2n+1}) = \wp_\flat(\jmath_{2n}, \jmath_{2n+1}) - \wp_\flat(\jmath_{2n}, \jmath_{2n+1}) + \wp_\flat(\jmath_{2n+1}, \jmath_{2n+2}).$$

Thus, we conclude that

$$\begin{aligned}\mathbf{D}(\wp_\flat(\jmath_{2n+1}, \jmath_{2n+2})) &< \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}\wp_\flat(\jmath_{2n+1}, \jmath_{2n+2})\right)\right)\\ &< \mathbf{D}\left(\tfrac{1}{\varsigma}\wp_\flat(\jmath_{2n+1}, \jmath_{2n+2})\right).\end{aligned}$$

This indicates a contradiction. Thus, the subsequent situation is provided:

$$\wp_\flat(\jmath_{2n+1}, \jmath_{2n+2}) < \wp_\flat(\jmath_{2n}, \jmath_{2n+1}).$$

As can be seen from the above two cases, $\{\wp_\flat(\jmath_n, \jmath_{n+1})\}_{n\in\mathbb{N}}$ is a non-increasing sequence. Therefore, $L \geq 0$ exists such that $\lim_{n\to\infty} \wp_\flat(\jmath_n, \jmath_{n+1}) = L$. Suppose that $L > 0$. If the limit is taken on both sides of the subsequent inequality owing to the fact that $\mathbf{D}$ and $\mathscr{C}$ are continuous functions, we gain

$$\mathbf{D}\left(\lim_{n\to\infty} \wp_\flat(\jmath_{2n}, \jmath_{2n+1})\right) < \mathscr{C}\left(\mathbf{D}\left(\frac{1}{\varsigma}\lim_{n\to\infty} \hat{E}_{\Im,O}(\jmath_{2n-1}, \jmath_{2n})\right)\right),$$

where

$$\begin{aligned}\hat{E}_{\Im,O}(\jmath_{2n-1},\jmath_{2n}) &= \wp_\flat(\jmath_{2n-1},\jmath_{2n}) + \left|\wp_\flat\left(\jmath_{2n-1},O^i\jmath_{2n-1}\right) - \wp_\flat\left(\jmath_{2n},\Im^j\jmath_{2n}\right)\right| \\ &= \wp_\flat(\jmath_{2n-1},\jmath_{2n}) + |\wp_\flat(\jmath_{2n-1},\jmath_{2n}) - \wp_\flat(\jmath_{2n},\jmath_{2n+1})| \\ &= 2\wp_\flat(\jmath_{2n-1},\jmath_{2n}) - \wp_\flat(\jmath_{2n},\jmath_{2n+1})\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\inf \hat{E}_{\Im,O}(\jmath_{2n-1},\jmath_{2n}) &= \liminf_{n\to\infty}\ [2\wp_\flat(\jmath_{2n-1},\jmath_{2n}) - \wp_\flat(\jmath_{2n},\jmath_{2n+1})] \\ &\le \limsup_{n\to\infty}\ [2\wp_\flat(\jmath_{2n-1},\jmath_{2n}) - \wp_\flat(\jmath_{2n},\jmath_{2n+1})] \\ &= L.\end{aligned}$$

Hence, we attain

$$\begin{aligned}\mathbf{D}(L) = \mathbf{D}\left(\lim_{n\to\infty}\wp_\flat(\jmath_{2n},\jmath_{2n+1})\right) &< \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}\lim_{n\to\infty}\ \sup \hat{E}_{\Im,O}(\jmath_{2n-1},\jmath_{2n})\right)\right) \\ &< \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}L\right)\right) < \mathbf{D}\left(\tfrac{L}{\varsigma}\right),\end{aligned}$$

which is a contradiction. The same argument is true for $\wp_\flat(\jmath_{2n},\jmath_{2n-1})$. Therefore,

$$\lim_{n\to\infty}\wp_\flat(\jmath_n,\jmath_{n+1}) = 0.$$

Because of ($\wp_\flat$2), we have that $\wp_\flat(\jmath_n,\jmath_n) \le \wp_\flat(\jmath_n,\jmath_{n+1})$ and $\wp_\flat(\jmath_{n+1},\jmath_{n+1}) \le \wp_\flat(\jmath_n,\jmath_{n+1})$; then,

$$\lim_{n\to\infty}\wp_\flat(\jmath_n,\jmath_n) = \lim_{n\to\infty}\wp_\flat(\jmath_{n+1},\jmath_{n+1}) = 0.$$

Using Proposition 2.0.11, we write

$$d_{\wp_\flat}(\jmath_n,\jmath_{n+1}) = 2.\wp_\flat(\jmath_n,\jmath_{n+1}) - \wp_\flat(\jmath_n,\jmath_n) - \wp_\flat(\jmath_{n+1},\jmath_{n+1}).$$

If we take the limit in the above equation as n tends to infinity, we get

$$\begin{aligned}0 \le \lim_{n\to\infty} d_{\wp_\flat}(\jmath_n,\jmath_{n+1}) &= \lim_{n\to\infty} 2[\wp_\flat(\jmath_n,\jmath_{n+1}) - \wp_\flat(\jmath_n,\jmath_n) - \wp_\flat(\jmath_{n+1},\jmath_{n+1})] \\ &\le \lim_{n\to\infty}\ \sup 2[\wp_\flat(\jmath_n,\jmath_{n+1}) - \wp_\flat(\jmath_n,\jmath_n) - \wp_\flat(\jmath_{n+1},\jmath_{n+1})] \\ &= 0.\end{aligned}$$

Thus, $\lim_{n\to\infty} d_{\wp_\flat}(\jmath_{2n},\jmath_{2n+1}) = 0$ for all $n \in \mathbb{N}$. Furthermore, we currently possess the subsequent expression for all $n, m \ge 1$:

$$\lim_{n,m\to\infty} d_{\wp_\flat}(\jmath_{2m},\jmath_{2n}) = 2\lim_{n,m\to\infty}\ \sup \wp_\flat(\jmath_{2m},\jmath_{2n}).$$

It is necessary to indicate that the sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ is a $\wp_\flat$-Cauchy sequence in $(\mathscr{U},\wp_\flat)$. Instead, it is needed to verify that $\{\jmath_{2n}\}_{n\in\mathbb{N}}$ is a $\wp_\flat$-Cauchy sequence. According to Lemma 2.0.16 (1), it is required to explicitly state that $\{\jmath_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\left(\mathscr{U},d_{\wp_\flat}\right)$. Assume that $\{\jmath_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence in $\left(\mathscr{U},d_{\wp_\flat}\right)$. In this instance, two subsequences of positive numbers $\{\jmath_{2m_k}\}$ and $\{\jmath_{2n_k}\}$ exist such that $n(k) > m(k) > k$ and the number $\varepsilon > 0$, which yield that

$$d_{\wp_\flat}(\jmath_{2m_k},\jmath_{2n_k}) \ge \varepsilon, \tag{3.2}$$

and, for $k \in \mathbb{N}$,

$$d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k-2}\right) < \varepsilon. \tag{3.3}$$

Applying $\flat_3$, we deduce that

$$\varepsilon \leq d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k}\right) \leq \varsigma d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2m_k+1}\right) + \varsigma d_{\wp_\flat}\left(\jmath_{2m_k+1}, \jmath_{2n_k}\right).$$

By taking the limit in the above as $k \to \infty$, we write

$$\frac{\varepsilon}{\varsigma} \leq \lim_{k\to\infty} \inf d_{\wp_\flat}\left(\jmath_{2m_k+1}, \jmath_{2n_k}\right) \leq \lim_{k\to\infty} \sup d_{\wp_\flat}\left(\jmath_{2m_k+1}, \jmath_{2n_k}\right). \tag{3.4}$$

Also, employing $\flat_3$, we gain

$$d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right) \leq \varsigma d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k-2}\right) + \varsigma d_{\wp_\flat}\left(\jmath_{2n_k-2}, \jmath_{2n_k-1}\right);$$

again, taking the limit in the above as $k \to \infty$, we get

$$\lim_{k\to\infty} \sup d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right) \leq \varsigma\varepsilon. \tag{3.5}$$

Moreover, the following statement is derived:

$$\begin{aligned} d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k}\right) &\leq \varsigma d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k-2}\right) + \varsigma d_{\wp_\flat}\left(\jmath_{2n_k-2}, \jmath_{2n_k}\right) \\ &\leq \varsigma d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k-2}\right) + \varsigma^2 d_{\wp_\flat}\left(\jmath_{2n_k-2}, \jmath_{2n_k-1}\right) + \varsigma^2 d_{\wp_\flat}\left(\jmath_{2n_k-1}, \jmath_{2n_k}\right). \end{aligned}$$

Similarly, if the limit is taken for $k \to \infty$, we obtain

$$\lim_{k\to\infty} \sup d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k}\right) \leq \varsigma\varepsilon. \tag{3.6}$$

Again, from $\flat_3$, it is achievable to produce

$$d_{\wp_\flat}\left(\jmath_{2m_k+1}, \jmath_{2n_k-1}\right) \leq \varsigma d_{\wp_\flat}\left(\jmath_{2m_k+1}, \jmath_{2m_k}\right) + \varsigma d_{\wp_\flat}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right).$$

Likewise, taking the limit as $k \to \infty$, it is concluded that

$$\lim_{k\to\infty} \sup d_{\wp_\flat}\left(\jmath_{2m_k+1}, \jmath_{2n_k-1}\right) \leq \varsigma^2\varepsilon. \tag{3.7}$$

From the inequalities (3.4)–(3.7), the subsequent expressions are attained:

$$\frac{\varepsilon}{2\varsigma} \leq \lim_{k\to\infty} \inf \wp_\flat\left(\jmath_{2m_k+1}, \jmath_{2n_k}\right) \leq \lim_{k\to\infty} \sup \wp_\flat\left(\jmath_{2m_k+1}, \jmath_{2n_k}\right), \tag{3.8}$$

$$\lim_{k\to\infty} \sup \wp_\flat\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right) \leq \frac{\varsigma\varepsilon}{2}, \tag{3.9}$$

$$\lim_{k\to\infty} \sup \wp_\flat\left(\jmath_{2m_k}, \jmath_{2n_k}\right) \leq \frac{\varsigma\varepsilon}{2}, \tag{3.10}$$

$$\lim_{k\to\infty} \sup \wp_\flat\left(\jmath_{2m_k+1}, \jmath_{2n_k-1}\right) \leq \frac{\varsigma^2\varepsilon}{2}. \tag{3.11}$$

Due to Lemma 2.0.18, it is known that $\alpha_i^j\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\geq\varsigma^f$ and

$$\begin{aligned}\mathbf{D}\left(\wp_\flat\left(\jmath_{2m_k+1},\jmath_{2n_k}\right)\right)&\leq\mathbf{D}\left(\alpha_i^j\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\wp_\flat\left(\mathcal{O}^i\jmath_{2m_k},\mathfrak{S}^j\jmath_{2n_k-1}\right)\right)\\&\leq\mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\right),\end{aligned}$$

where

$$\begin{aligned}\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)&=\wp_\flat\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)+\left|\wp_\flat\left(\jmath_{2m_k},\mathcal{O}^i\jmath_{2m_k}\right)-\wp_\flat\left(\jmath_{2n_k-1},\mathfrak{S}^j\jmath_{2n_k-1}\right)\right|\\&=\wp_\flat\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)+\left|\wp_\flat\left(\jmath_{2m_k},\jmath_{2m_k+1}\right)-\wp_\flat\left(\jmath_{2n_k-1},\jmath_{2n_k}\right)\right|,\end{aligned}$$

and, taking the limit in the above as $k\to\infty$, we procure

$$\begin{aligned}\lim_{k\to\infty}\ \inf\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)&=\lim_{k\to\infty}\ \inf\wp_\flat\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\\&\leq\lim_{k\to\infty}\ \sup\wp_\flat\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\\&=\lim_{k\to\infty}\ \sup\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\\&\leq\tfrac{\varsigma\varepsilon}{2}.\end{aligned}$$

In light of the results obtained above, using (3.1), we write

$$\begin{aligned}\mathbf{D}\left(\tfrac{\varepsilon}{2}\right)&=\mathbf{D}\left(\varsigma\tfrac{\varepsilon}{2\varsigma}\right)\\&\leq\mathbf{D}\left(\varsigma\lim_{k\to\infty}\ \inf\wp_\flat\left(\jmath_{2m_k+1},\jmath_{2n_k}\right)\right)\\&\leq\mathbf{D}\left(\varsigma^p\lim_{k\to\infty}\ \inf\wp_\flat\left(\jmath_{2m_k+1},\jmath_{2n_k}\right)\right)\\&\leq\mathbf{D}\left(\lim_{k\to\infty}\ \inf\alpha_i^j\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\wp_\flat\left(\mathcal{O}^i\jmath_{2m_k},\mathfrak{S}^j\jmath_{2n_k-1}\right)\right)\\&\leq\mathscr{C}\left(\mathbf{D}\left(\lim_{k\to\infty}\ \inf\mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\right)\\&\leq\mathbf{D}\left(\lim_{k\to\infty}\ \inf\tfrac{1}{\varsigma}\tfrac{\varsigma\varepsilon}{2}\right)\\&=\mathbf{D}\left(\tfrac{\varepsilon}{2}\right).\end{aligned}\tag{3.12}$$

Then,

$$\begin{aligned}\mathbf{D}\left(\tfrac{\varepsilon}{2}\right)&\leq\mathbf{D}\left(\alpha_{i,j}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\lim_{k\to\infty}\ \sup\wp_\flat\left(\mathcal{O}^i\jmath_{2m_k},\mathfrak{S}^j\jmath_{2n_k-1}\right)\right)\\&\leq\lim_{k\to\infty}\ \sup\mathbf{D}\left(\alpha_i^j\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\wp_\flat\left(\mathcal{O}^i\jmath_{2m_k},\mathfrak{S}^j\jmath_{2n_k-1}\right)\right)\\&\leq\lim_{k\to\infty}\ \sup\mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\right)\\&=\mathscr{C}\left(\mathbf{D}\left(\lim_{k\to\infty}\ \sup\mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\right)\\&<\mathbf{D}\left(\lim_{k\to\infty}\ \sup\tfrac{1}{\varsigma}\tfrac{\varsigma\varepsilon}{2}\right)\\&=\mathbf{D}\left(\tfrac{\varepsilon}{2}\right).\end{aligned}\tag{3.13}$$

Therefore, from (3.12) and (3.13), we obtain

$$\lim_{k\to\infty}\mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)\right)\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k},\jmath_{2n_k-1}\right)=\frac{\varepsilon}{2}.\tag{3.14}$$

Similarly,

$$\begin{aligned}\mathbf{D}\left(\tfrac{\varsigma\varepsilon}{2}\right) &= \mathbf{D}\left(\varsigma^2 \tfrac{\varepsilon}{2\varsigma}\right)\\ &\leq \mathbf{D}\left(\varsigma^2 \lim_{k\to\infty} \inf \wp_\flat \left(\jmath_{2m_k+1}, \jmath_{2n_k}\right)\right)\\ &\leq \mathbf{D}\left(\varsigma^p \lim_{k\to\infty} \inf \wp_\flat \left(\jmath_{2m_k+1}, \jmath_{2n_k}\right)\right)\\ &\leq \mathbf{D}\left(\lim_{k\to\infty} \inf \alpha_i^j \left(\jmath_{2m_k}, \jmath_{2n_k-1}\right) \wp_\flat \left(\mathcal{O}^i \jmath_{2m_k}, \mathfrak{S}^j \jmath_{2n_k-1}\right)\right)\\ &\leq \mathscr{C}\left(\mathbf{D}\left(\lim_{k\to\infty} \inf \mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right) \hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right)\right)\\ &< \mathbf{D}\left(\lim_{k\to\infty} \inf \hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right)\\ &\leq \mathbf{D}\left(\tfrac{\varsigma\varepsilon}{2}\right),\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\mathbf{D}\left(\tfrac{\varsigma\varepsilon}{2}\right) &\leq \mathbf{D}\left(\alpha_i^j \left(\jmath_{2m_k}, \jmath_{2n_k-1}\right) \lim_{k\to\infty} \sup \wp_\flat \left(\mathcal{O}^i \jmath_{2m_k}, \mathfrak{S}^j \jmath_{2n_k-1}\right)\right)\\ &\leq \lim_{k\to\infty} \sup \mathbf{D}\left(\alpha_i^j \left(\jmath_{2m_k}, \jmath_{2n_k-1}\right) \wp_\flat \left(\mathcal{O}^i \jmath_{2m_k}, \mathfrak{S}^j \jmath_{2n_k-1}\right)\right)\\ &\leq \lim_{k\to\infty} \sup \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right) \hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right)\right)\\ &= \mathscr{C}\left(\mathbf{D}\left(\lim_{k\to\infty} \sup \mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right) \hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right)\right)\\ &< \mathbf{D}\left(\lim_{k\to\infty} \sup \hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right)\\ &\leq \mathbf{D}\left(\tfrac{\varsigma\varepsilon}{2}\right).\end{aligned} \tag{3.16}$$

From (3.15) and (3.16), it is obtained that

$$\lim_{k\to\infty} \hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right) = \frac{\varsigma\varepsilon}{2}. \tag{3.17}$$

Because of (3.14) and (3.17), we conclude that

$$\lim_{k\to\infty} \mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right)\right) = \frac{1}{\varsigma}, \tag{3.18}$$

and, in (3.18), it is deduced by using the property of $\mathfrak{P}$ that

$$\lim_{k\to\infty} \hat{E}_{\mathfrak{S},\mathcal{O}}\left(\jmath_{2m_k}, \jmath_{2n_k-1}\right) = 0.$$

This contradiction brings about the sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ as a Cauchy sequence in $\left(\mathscr{U}, d_{\wp_\flat}\right)$. The completeness of $\left(\mathscr{U}, d_{\wp_\flat}\right)$ provides that the sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ converges to $r \in \mathscr{U}$. Thus,

$$\lim_{n\to\infty} d_{\wp_\flat}\left(\jmath_n, r\right) = 0.$$

By Lemma 2.0.16, we derive

$$\wp_\flat (r, r) = \lim_{n\to\infty} \wp_\flat \left(\jmath_n, r\right) = \lim_{n,m\to\infty} \wp_\flat \left(\jmath_n, \jmath_m\right).$$

Then, $\{\jmath_n\}_{n\in\mathbb{N}}$ converges to $r \in (\mathscr{U}, \wp_\flat)$. Additionally, by $\flat_3$, the following expression is evident:

$$d_{\wp_\flat}(\jmath_n, \jmath_m) \leq \varsigma\left[d_{\wp_\flat}(\jmath_n, r) + d_{\wp_\flat}(r, \jmath_m)\right].$$

If the limit for $n, m \to \infty$ is taken in the above equation, we have

$$0 \leq \lim_{n,m\to\infty} \sup d_{\wp_\flat}(\jmath_n, \jmath_m) \leq \lim_{n,m\to\infty} \sup \varsigma\left[d_{\wp_\flat}(\jmath_n, r) + d_{\wp_\flat}(r, \jmath_m)\right]$$

and $\lim_{n,m\to\infty} d_{\wp_\flat}(\jmath_n, \jmath_m) = 0$. If we consider the fact that $\lim_{n\to\infty} \wp_\flat(\jmath_n, \jmath_n) = 0$, and, taking the limit as $n, m \to \infty$ in the following equations, we procure

$$d_{\wp_\flat}(\jmath_n, \jmath_m) = 2\wp_\flat(\jmath_n, \jmath_m) - \wp_\flat(\jmath_n, \jmath_n) - \wp_\flat(\jmath_m, \jmath_m) \tag{3.19}$$

and

$$\lim_{n,m\to\infty} \wp_\flat(\jmath_n, \jmath_m) = 0.$$

So,

$$\wp_\flat(r, r) = \lim_{n\to\infty} \wp_\flat(\jmath_n, r) = \lim_{n,m\to\infty} \wp_\flat(\jmath_n, \jmath_m) = 0. \tag{3.20}$$

Because of the hypothesis (iv), a subsequence $\{\jmath_{2n_k}\}$ of $\{\jmath_n\}_{n\in\mathbb{N}}$ exists such that $\alpha_i^j(\jmath_{2n_k}, r) \geq \varsigma^f$ and

$$\begin{aligned}\mathbf{D}\left(\wp_\flat\left(\jmath_{2n_k+1}, \mathfrak{S}^j r\right)\right) &\leq \mathbf{D}\left(\alpha_i^j(\jmath_{2n_k}, r)\,\wp_\flat\left(O^i \jmath_{2n_k}, \mathfrak{S}^j r\right)\right) \\ &\leq \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\mathfrak{S},O}(\jmath_{2n_k}, r)\right)\hat{E}_{\mathfrak{S},O}(\jmath_{2n_k}, r)\right)\right) \\ &< \mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\mathfrak{S},O}(\jmath_{2n_k}, r)\right)\hat{E}_{\mathfrak{S},O}(\jmath_{2n_k}, r)\right) \\ &< \mathbf{D}\left(\tfrac{1}{\varsigma}\hat{E}_{\mathfrak{S},O}(\jmath_{2n_k}, r)\right),\end{aligned}$$

where

$$\begin{aligned}\hat{E}_{\mathfrak{S},O}(\jmath_{2n_k}, r) &= \wp_\flat(\jmath_{2n_k}, r) + \left|\wp_\flat\left(\jmath_{2n_k}, O^i \jmath_{2n_k}\right) - \wp_\flat\left(r, \mathfrak{S}^j r\right)\right| \\ &= \wp_\flat(\jmath_{2n_k}, r) + \left|\wp_\flat(\jmath_{2n_k}, \jmath_{2n_k+1}) - \wp_\flat\left(r, \mathfrak{S}^j r\right)\right|.\end{aligned}$$

In the last equality, by taking the limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \sup \hat{E}_{\mathfrak{S},O}(\jmath_{2n_k}, r) = \wp_\flat\left(r, \mathfrak{S}^j r\right), \tag{3.21}$$

and, using this, we get the following expression:

$$\mathbf{D}\left(\lim_{k\to\infty} \sup \wp_\flat\left(\jmath_{2n_k+1}, \mathfrak{S}^j r\right)\right) < \mathbf{D}\left(\frac{1}{\varsigma}\wp_\flat\left(r, \mathfrak{S}^j r\right)\right). \tag{3.22}$$

Further, since $\frac{1}{\varsigma}\wp_\flat\left(r, \mathfrak{S}^j r\right) \leq \lim_{k\to\infty} \sup \wp_\flat\left(\jmath_{2n_k+1}, \mathfrak{S}^j r\right)$, we attain

$$\mathbf{D}\left(\frac{1}{\varsigma}\wp_\flat\left(r, \mathfrak{S}^j r\right)\right) \leq \mathbf{D}\left(\lim_{k\to\infty} \sup \wp_\flat\left(\jmath_{2n_k+1}, \mathfrak{S}^j r\right)\right). \tag{3.23}$$

Therefore, using the expressions (3.22) and (3.23), we obtain

$$\begin{aligned}\mathbf{D}\left(\tfrac{1}{\varsigma}\wp_{\flat}\left(r,\mathfrak{S}^{j}r\right)\right) &\leq \mathbf{D}\left(\lim_{k\to\infty}\ \sup\wp_{\flat}\left(\jmath_{2n_k+1},\mathfrak{S}^{j}r\right)\right)\\ &\leq \mathbf{D}\left(\lim_{k\to\infty}\ \mathfrak{P}\left(\hat{E}_{\mathfrak{S},O}\left(\jmath_{2n_k},r\right)\right)\hat{E}_{\mathfrak{S},O}\left(\jmath_{2n_k},r\right)\right)\\ &\leq \mathbf{D}\left(\tfrac{1}{\varsigma}\wp_{\flat}\left(r,\mathfrak{S}^{j}r\right)\right).\end{aligned}$$

From here, one can deduce that

$$\lim_{k\to\infty}\mathfrak{P}\left(\hat{E}_{\mathfrak{S},O}\left(\jmath_{2n_k},r\right)\right)\hat{E}_{\mathfrak{S},O}\left(\jmath_{2n_k},r\right)=\frac{1}{\varsigma}\wp_{\flat}\left(r,\mathfrak{S}^{j}r\right). \tag{3.24}$$

Because of (3.21) and (3.24), it is concluded that

$$\lim_{k\to\infty}\mathfrak{P}\left(\hat{E}_{\mathfrak{S},O}\left(\jmath_{2n_k},r\right)\right)=\frac{1}{\varsigma}. \tag{3.25}$$

In (3.25), due to the property of $\mathfrak{P}$, we obtain that $\lim_{k\to\infty}\hat{E}_{\mathfrak{S},O}\left(\jmath_{2n_k},r\right)=0$. Thus, $\wp_{\flat}\left(r,\mathfrak{S}^{j}r\right)=0$. From the axiom ($\wp_{\flat}2$), $\wp_{\flat}(r,r)\leq\wp_{\flat}\left(r,\mathfrak{S}^{j}r\right)$ and $\wp_{\flat}\left(\mathfrak{S}^{j}r,\mathfrak{S}^{j}r\right)\leq\wp_{\flat}\left(r,\mathfrak{S}^{j}r\right)$. Obviously,

$$\wp_{\flat}(r,r)=\wp_{\flat}\left(r,\mathfrak{S}^{j}r\right)=0.$$

Therefore, we get that $\wp_{\flat}(r,r)=\wp_{\flat}\left(r,\mathfrak{S}^{j}r\right)=\wp_{\flat}\left(\mathfrak{S}^{j}r,S^{j}r\right)$. Because of ($\wp_{\flat}1$), $r=\mathfrak{S}^{j}r$. This also yields that r is the fixed-point of $\mathfrak{S}^{j}$. In the same way, one can see that $\wp_{\flat}\left(r,O^{i}r\right)=0$. In this instance, r is the fixed-point of O^{i}. So, r is the common fixed-point of $\mathfrak{S}^{j}$ and O^{i}. Now, to demonstrate the uniqueness of the fixed-point, let $\mathfrak{S}^{j}$ have another fixed-point, provided that $w\in\mathscr{U}$ satisfies $\mathfrak{S}^{j}w=w\neq r$. Because of the fifth hypothesis of the theorem, we have that $\alpha_i^j(r,w)\geq\varsigma^{f}$. By (3.1), we obtain

$$\begin{aligned}\mathbf{D}\left(\wp_{\flat}(r,w)\right) &\leq \mathbf{D}\left(\alpha_i^j(r,w)\wp_{\flat}(r,w)\right)\\ &= \mathbf{D}\left(\alpha_{i,j}(r,w)\wp_{\flat}\left(O^{i}r,\mathfrak{S}^{j}w\right)\right)\\ &\leq \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\mathfrak{S},O}(r,w)\right)\hat{E}_{\mathfrak{S},O}(r,w)\right)\right)\\ &\leq \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{\varsigma}\hat{E}_{\mathfrak{S},O}(r,w)\right)\right)\\ &< \mathbf{D}\left(\tfrac{1}{\varsigma}\hat{E}_{\mathfrak{S},O}(r,w)\right),\end{aligned}$$

where

$$\begin{aligned}\hat{E}_{\mathfrak{S},O}(r,w) &= \wp_{\flat}(r,w)+\left|\wp_{\flat}\left(r,O^{i}r\right)-\wp_{\flat}\left(w,\mathfrak{S}^{j}w\right)\right|\\ &= \wp_{\flat}(r,w)+\left|\wp_{\flat}(r,r)-\wp_{\flat}(w,w)\right|=\wp_{\flat}(r,w).\end{aligned}$$

Hence, this indicates that $\mathbf{D}\left(\wp_{\flat}(r,w)\right)<\mathbf{D}\left(\tfrac{1}{\varsigma}\wp_{\flat}(r,w)\right)$. However, this causes a contradiction. So, assuming that $\mathfrak{S}^{j}$ has a different fixed-point is inaccurate. Thus, $Fix\left(\mathfrak{S}^{j}\right)=\{r\}$. With the same method, it turns out that $Fix\left(O^{i}\right)=\{r\}$. Accordingly, $C_{Fix}\left(\mathfrak{S}^{j},O^{i}\right)=\{r\}$. Due to the fact that

$$\mathfrak{S}r=\mathfrak{S}\mathfrak{S}^{j}r=\mathfrak{S}^{j}\mathfrak{S}r$$

and

$$\mathcal{O}r = \mathcal{O}\mathcal{O}^j r = \mathcal{O}^j\mathcal{O}r,$$

it is obvious to verify that $\mathfrak{S}$ and $\mathcal{O}$ own a unique common fixed-point owing to the uniqueness of the common fixed-point of $\mathfrak{S}^j$ and $\mathcal{O}^i$. □

Example 3.0.3. *Assume that* $\mathscr{U} = [0,1]$ *and* $\wp_\flat : \mathscr{U}\times\mathscr{U} \to \mathbb{R}^+$ *is determined as* $\wp_\flat(\jmath,\rho) = (\jmath-\rho)^2$. *Then,* $(\mathscr{U},\wp_\flat)$ *is a complete* $\mathcal{P}_\flat$M$\mathcal{S}$ *with the coefficient* $\varsigma = 2$. $\alpha_i^j : \mathscr{U}^2 \to [0,\infty)$ *is defined as follows:*

$$\alpha_i^j(\jmath,\rho) = \begin{cases} \varsigma^f, & \jmath,\rho \in [0,1], \\ 0, & \text{otherwise,} \end{cases}$$

with $f \geq 2$. Let $\mathcal{O}, \mathfrak{S} : \mathscr{U} \to \mathscr{U}$ be defined by $\mathcal{O}(\jmath) = \frac{\jmath}{2}$, $\mathfrak{S}(\jmath) = \frac{\jmath}{4}$ and take $\mathfrak{P}(t) = \frac{1}{32}, t > 0, \mathfrak{P} \in \mathbf{B}_\varsigma$. The pair $(\mathcal{O},\mathfrak{S})$ is triangular α_i^j-orbital-admissible. Define $\mathbf{D} : (0,\infty) \to (0,\infty)$ by $\mathbf{D}(\theta) = \theta e^\theta$ for each $\theta > 0$; then, $\mathbf{D} \in \Delta$. $\mathscr{C} : (0,\infty) \to (0,\infty)$, which is defined by $\mathscr{C}(r) = \frac{r}{2}$ for all $r \in (0,\infty)$, is a continuous comparison function. It shall be determined that $\mathcal{O}$ and $\mathfrak{S}$ constitute α_i^j-$(\mathbf{D}_{\mathscr{C}}(\mathfrak{P}_{\hat{E}}))$-contraction mappings. If $f = 2$, $i = 4$ and $j = 2$, then we have that $\alpha_i^j(\jmath,\rho) = 4$. The above choices ensure that

$$\begin{aligned}\wp_\flat\left(\mathcal{O}^i\jmath, \mathfrak{S}^j\rho\right) &= \wp_\flat\left(\frac{\jmath}{16}, \frac{\rho}{16}\right) = \frac{1}{16^2}(\jmath-\rho)^2, \\ \hat{E}_{\mathfrak{S},\mathcal{O}}(\jmath,\rho) &= \wp_\flat(\jmath,\rho) + \left|\wp_\flat\left(\jmath,\mathcal{O}^i\jmath\right) - \wp_\flat\left(\rho,\mathfrak{S}^j\rho\right)\right| \\ &= (\jmath-\rho)^2 + \left|\left(\tfrac{15\jmath}{16}\right)^2 - \left(\tfrac{15\rho}{16}\right)^2\right|,\end{aligned}$$

and

$$\begin{aligned}\mathbf{D}\left(\alpha_{i,j}(\jmath,\rho)\,\wp_\flat\left(\mathcal{O}^i\jmath, \mathfrak{S}^j\rho\right)\right) &= \mathbf{D}\left(4\tfrac{1}{16^2}(\jmath-\rho)^2\right) = \mathbf{D}\left(\tfrac{1}{64}(\jmath-\rho)^2\right) \\ &= \tfrac{1}{64}(\jmath-\rho)^2 e^{\frac{1}{64}(\jmath-\rho)^2}.\end{aligned}$$

Then,

$$\begin{aligned}\mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\mathfrak{S},\mathcal{O}}(\jmath,\rho)\right)\hat{E}_{\mathfrak{S},\mathcal{O}}(\jmath,\rho)\right)\right) &= \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{32}\left[(\jmath-\rho)^2 + \left|\left(\tfrac{15\jmath}{16}\right)^2 - \left(\tfrac{15\rho}{16}\right)^2\right|\right]\right)\right) \\ &= \tfrac{1}{2}\tfrac{1}{32}\left[(\jmath-\rho)^2 + \left|\left(\tfrac{15\jmath}{16}\right)^2 - \left(\tfrac{15\rho}{16}\right)^2\right|\right] e^{\frac{1}{32}\left[(\jmath-\rho)^2+\left|\left(\frac{15\jmath}{16}\right)^2-\left(\frac{15\rho}{16}\right)^2\right|\right]}.\end{aligned}$$

Therefore, (3.1) is achieved; so, $C_{Fix}(\mathcal{O},\mathfrak{S}) = \{0\}$, where 0 is the unique common fixed-point of $\mathcal{O}$ and $\mathfrak{S}$.

Example 3.0.4. *Assume that* $\mathscr{U} = [0,1]$ *and* $\wp_\flat : \mathscr{U}\times\mathscr{U} \to \mathbb{R}^+$ *is determined as* $\wp_\flat(\jmath,\rho) = (\max\{\jmath,\rho\})^2$. *Then,* $(\mathscr{U},\wp_\flat)$ *is a complete* $\mathcal{P}_\flat$M$\mathcal{S}$ *with the coefficient* $\varsigma = 2$. $\alpha_i^j : \mathscr{U}^2 \to [0,\infty)$ *is defined as follows:*

$$\alpha_i^j(\jmath,\rho) = \begin{cases} \varsigma^f, & \jmath,\rho \in [0,1], \\ 0, & \text{otherwise,} \end{cases}$$

with $f \geq 2$. Let $\mathcal{O}, \mathfrak{S} : \mathscr{U} \to \mathscr{U}$ be respectively defined by $\mathcal{O}(\jmath) = \frac{\jmath}{16}$, $\mathfrak{S}(\jmath) = \frac{\jmath}{4}$, and take $\mathfrak{P}(t) = \frac{1}{256}$, $t > 0, \mathfrak{P} \in \mathbf{B}_\varsigma$. The pair $(\mathcal{O},\mathfrak{S})$ is triangular α_i^j-orbital-admissible. Define $\mathbf{D} : (0,\infty) \to (0,\infty)$

by $\mathbf{D}(\vartheta) = \vartheta$ for each $\vartheta > 0$; then, $\mathbf{D} \in \Delta$. $\mathscr{C} : (0, \infty) \to (0, \infty)$, which is defined by $\mathscr{C}(\jmath) = \frac{\jmath}{16}$ for all $\jmath \in (0, \infty)$, is a continuous comparison function. Now, it is shown that $O, \mathfrak{S}$ are α_i^j-$(\mathbf{D}_{\mathscr{C}}(\mathfrak{P}_{\hat{E}}))$-contraction mappings. If $f = 4, i = 2$ and $j = 4$, then we currently have that $\alpha_i^j(\jmath, \rho) = 16$. Based on these, we acquire

$$\wp_\flat\left(O^i\jmath, \mathfrak{S}^j\rho\right) = \wp_\flat\left(\frac{\jmath}{16^2}, \frac{\rho}{16^2}\right) = \frac{1}{16^4}(\max\{\jmath, \rho\})^2,$$

$$\begin{aligned}\hat{E}_{\mathfrak{S},O}(\jmath, \rho) &= \wp_\flat(\jmath, \rho) + \left|\wp_\flat\left(\jmath, O^i\jmath\right) - \wp_\flat\left(\rho, \mathfrak{S}^j\rho\right)\right| \\ &= (\max\{\jmath, \rho\})^2 + |\jmath - \rho|,\end{aligned}$$

and

$$\begin{aligned}\mathbf{D}\left(\alpha_{i,j}(\jmath, \rho)\wp_\flat\left(O^i\jmath, \mathfrak{S}^j\rho\right)\right) &= \mathbf{D}\left(16\tfrac{1}{16^4}(\max\{\jmath, \rho\})^2\right) = \mathbf{D}\left(\tfrac{1}{16^3}(\max\{\jmath, \rho\})^2\right) \\ &= \tfrac{1}{16^3}(\max\{\jmath, \rho\})^2.\end{aligned}$$

Thus, we derive that

$$\begin{aligned}\mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(\hat{E}_{\mathfrak{S},O}(\jmath, \rho)\right)\hat{E}_{\mathfrak{S},O}(\jmath, \rho)\right)\right) &= \mathscr{C}\left(\mathbf{D}\left(\tfrac{1}{256}\left[(\max\{\jmath, \rho\})^2 + |\jmath - \rho|\right]\right)\right) \\ &= \tfrac{1}{16}\tfrac{1}{256}\left[(\max\{\jmath, \rho\})^2 + |\jmath - \rho|\right].\end{aligned}$$

As a result, (3.1) becomes apparent; then, $C_{Fix}(O, \mathfrak{S}) = \{0\}$.

4. Consequences

In this section, contemplating our main theorem, we list some conclusions involving an E-contraction endowed with various auxiliary functions.

Initially, the subsequent corollary generalizes Theorem 5 in [44].

Corollary 4.0.1. *Consider that* $(\mathscr{U}, \wp_\flat)$ *indicates a* $\wp_\flat$*-complete* $\mathcal{P}_\flat$M$\mathcal{S}$ *and* $O, \mathfrak{S}$ *represent two self-mappings on this space. Presume that* $\mathscr{C} : (0, \infty) \to (0, \infty)$ *is a continuous comparison function,* $\mathbf{D} : (0, \infty) \to (0, \infty)$*,* $\mathbf{D} \in \Delta$*, and* $\mathfrak{P} : [0, \infty) \to \left[0, \frac{1}{\varsigma}\right)$*,* $\mathfrak{P} \in \mathbf{B}_\varsigma$*. If the pair* $(O, \mathfrak{S})$ *provides the following statement:*

$$\wp_\flat(O\jmath, \mathfrak{S}\rho) > 0, \mathbf{D}(\wp_\flat(O\jmath, \mathfrak{S}\rho)) \leq \mathscr{C}(\mathbf{D}(\mathfrak{P}(E_{\mathfrak{S},O}(\jmath, \rho))E_{\mathfrak{S},O}(\jmath, \rho)))$$

for all $\jmath, \rho \in \mathscr{U}$, then the set of $C_{Fix}(O, \mathfrak{S})$ has a unique element.

Proof. Consider that $\alpha_i^j(\jmath, \rho) = \varsigma^f = 1$ and $i = j = 1$ in Theorem 3.0.2; then, the result is obvious. □

Moreover, by taking $\mathfrak{S} = O$ in Corollary 4.0.1, we gain the below consequence.

Corollary 4.0.2. *Consider that* $(\mathscr{U}, \wp_\flat)$ *indicates a* $\wp_\flat$*-complete* $\mathcal{P}_\flat$M$\mathcal{S}$ *and* O *represents a self-mapping on this space. Suppose that* $\mathscr{C} : (0, \infty) \to (0, \infty)$ *is a continuous comparison function,* $\mathbf{D} : (0, \infty) \to (0, \infty)$*,* $\mathbf{D} \in \Delta$*, and* $\mathfrak{P} : [0, \infty) \to \left[0, \frac{1}{\varsigma}\right)$*,* $\mathfrak{P} \in \mathbf{B}_\varsigma$*. If* O *provides the following statement:*

$$\wp_\flat(O\jmath, O\rho) > 0, \mathbf{D}(\wp_\flat(O\jmath, O\rho)) \leq \mathscr{C}(\mathbf{D}(\mathfrak{P}(E_O(\jmath, \rho))E_O(\jmath, \rho)))$$

for all $\jmath, \rho \in \mathscr{U}$, then $Fix(O)$ consists of a unique element which belongs to $(\mathscr{U}, d)$.

The following consequence is an analysis of Theorem 2.1 presented by Aydi et al. in [45].

Corollary 4.0.3. *Consider that* $(\mathscr{U}, \wp_\flat)$ *indicates a* $\wp_\flat$*-complete* $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$*,* $O : \mathscr{U} \to \mathscr{U}$ *represents a mapping and* $\alpha : \mathscr{U} \times \mathscr{U} \to [0, \infty)$ *is a function. Let* $\mathscr{C} : (0, \infty) \to (0, \infty)$ *be a continuous comparison function,* $\mathbf{D} : (0, \infty) \to (0, \infty)$*,* $\mathbf{D} \in \Delta$*, and* $\mathfrak{P} : [0, \infty) \to \left[0, \frac{1}{\varsigma}\right)$*,* $\mathfrak{P} \in \mathbf{B}_\varsigma$*. Presume that* O *fulfills the following conditions for* $\wp_\flat (O\jmath, O\rho) > 0$ *:*

i. $\alpha (\jmath, \rho) \geq 1 \Rightarrow \mathbf{D} (\wp_\flat (O\jmath, O\rho)) \leq \mathscr{C} (\mathbf{D} (\mathfrak{P} (E_O (\jmath, \rho)) E_O (\jmath, \rho)))$ for all $\jmath, \rho \in \mathscr{U}$.
ii. O is a triangular α-orbital-admissible mapping.
iii. $\jmath_0$ exists in $\mathscr{U}$ satisfying $\alpha (\jmath_0, O\jmath_0) \geq 1$.
iv. O is $\wp_\flat$-continuous, or, if $\{\jmath_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{U}$ such that $\alpha (\jmath_n, \jmath_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\jmath_n \to \jmath$ as $n \to \infty$, then a subsequence $\{\jmath_{n_k}\}$ of $\{\jmath_n\}$ exists such that $\alpha (\jmath_{n_k}, \jmath) \geq 1$ for all $k \in \mathbb{N}$.

Then, $Fix (O)$ consists of a unique element which belongs to $(\mathscr{U}, d)$.

Proof. By selecting $\alpha_i^j (\jmath, \rho) \geq \varsigma^f \geq 1$ with $i = j = 1$ and $\mathfrak{S} = O$ in Theorem 3.0.2, the proof is completed. □

The subsequent conclusion is an enhancement of the Banach fixed-point theorem [1] by taking the E-contraction into account.

Corollary 4.0.4. *Consider that* $(\mathscr{U}, \wp_\flat)$ *indicates a* $\wp_\flat$*-complete* $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$ *and* O *represents a self-mapping. Suppose that, for all* $\jmath, \rho \in \mathscr{U}$ *and* $\hbar \in [0, 1)$*, the following statement holds:*

$$\wp_\flat (O\jmath, O\rho) \leq \hbar (E_O (\jmath, \rho)). \tag{4.1}$$

Then, $Fix (O)$ consists of a unique element which belongs to $(\mathscr{U}, d)$.

Proof. By taking $\mathscr{C} (t) = \hbar t$ and $\mathbf{D} \in \Delta$ with $\mathbf{D} (t) = t$, as well as $\alpha (\jmath, \rho) = 1$, and keeping $\mathfrak{P} : [0, \infty) \to \left[0, \frac{1}{\varsigma}\right)$ in mind, in Corollary 4.0.3, we achieve the desired conclusion. □

Now, we state a new concept.

Definition 4.0.5. *Consider that* $(\mathscr{U}, \wp_\flat)$ *indicates a* $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$ *and* $O : \mathscr{U} \to \mathscr{U}$ *represents a mapping. Presume that* $\mathfrak{P} \in \mathbf{B}_\varsigma$*,* $\theta \in \tilde{\Theta}$ *and* $\tau \in (0, 1)$ *exist such that*

$$\wp_\flat (O\jmath, O\rho) \neq 0 \quad \Rightarrow \quad \theta (\wp_\flat (O\jmath, O\rho)) \leq [\theta (\mathfrak{P} (E_O (\jmath, \rho)) E_O (\jmath, \rho))]^\tau \tag{4.2}$$

for all $\jmath, \rho \in \mathscr{U}$. Thus, $O : \mathscr{U} \to \mathscr{U}$ is termed as a Geraghty θ_E-contraction.

Theorem 4.0.6. *Consider that* $(\mathscr{U}, \wp_\flat)$ *indicates a* $\wp_\flat$*-complete* $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$ *and* $O : \mathscr{U} \to \mathscr{U}$ *represents a Geraghty* θ_E*-contraction mapping. Thus,* $Fix (O)$ *consists of a unique element which belongs to* $(\mathscr{U}, d)$*.*

Proof. It is enough to take in Corollary 4.0.2 $\mathscr{C} (t) = (\ln k) t$ and $\mathbf{D} \in \Delta$ with $\mathbf{D} (t) = \ln \theta : (0, \infty) \to (0, \infty)$; then, the proof is evident. □

On the other hand, we characterize a new notation, which is an extension of [39], introduced by Fulga and Proca in 2017.

Definition 4.0.7. *Consider that* $(\mathscr{U}, \wp_\flat)$ *indicates a* $\mathcal{P}_\flat\mathrm{M}\mathcal{S}$ *and* $O : \mathscr{U} \to \mathscr{U}$ *represents a mapping. If* $\mathfrak{P} \in \mathbf{B}_\varsigma$*,* $\mathcal{F} \in \mathfrak{F}$ *and* $\kappa > 0$ *exist satisfying the inequality*

$$\kappa + \mathcal{F}\left(\wp_\flat\left(O\jmath, O\rho\right)\right) \leq \mathcal{F}\left(\mathfrak{P}\left(E_O\left(\jmath,\rho\right)\right) E_O\left(\jmath,\rho\right)\right) \tag{4.3}$$

for all $\jmath, \rho \in \mathcal{U}$, then O is a Geraghty $\mathcal{F}_E$-contraction.

Theorem 4.0.8. *Let $(\mathcal{U}, \wp_\flat)$ be a $\wp_\flat$-complete $\mathcal{P}_\flat$MS and $O : \mathcal{U} \to \mathcal{U}$ be a Geraghty $\mathcal{F}_E$-contraction. Thus, O has a unique fixed point.*

Proof. By selecting $\mathscr{C}(t) = e^{-\kappa}t$ and $\mathbf{D} \in \Delta$ with $\mathbf{D}(t) = e^{\mathcal{F}} : (0, \infty) \to (0, \infty)$ in Corollary 4.0.2, we attain the claim. □

5. An application to homotopy theory

We verify the following theorem, which offers an application of Corollary 4.0.2 to homotopy theory.

Theorem 5.0.1. *Let $(\mathcal{U}, \wp_\flat)$ be a $\wp_\flat$-complete $\mathcal{P}_\flat$MS and Υ, Λ be open and closed subsets of $\mathcal{U}$, respectively. Presume that $\mathcal{R} : \Lambda \times [0,1] \to \mathcal{U}$ is an operator ensuring the subsequent statements.*

i. $\jmath \neq \mathcal{R}(\jmath, \iota)$ for every $\jmath \in \Lambda \backslash \Upsilon$ and $\iota \in [0,1)$.

ii. For all $\jmath, \rho \in \Lambda$ and $\iota, \hbar \in [0,1)$, we have

$$\mathbf{D}\left(\wp_\flat\left(\mathcal{R}(\jmath,\iota), \mathcal{R}(\rho,\iota)\right)\right) \leq \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(E_O(\jmath,\rho)\right) E_O(\jmath,\rho)\right)\right),$$

where

$$E_O(\jmath,\rho) = \wp_\flat(\jmath,\rho) + |\wp_\flat(\jmath, \mathcal{R}(\jmath,\iota)) - \wp_\flat(\rho, \mathcal{R}(\rho,\iota))|.$$

iii. A function $\psi : [0,1] \to \mathbb{R}$, which has the continuity property, exists such that

$$\varsigma \wp_\flat\left(\mathcal{R}(\jmath,\iota), \mathcal{R}(\jmath,\iota^*)\right) \leq |\psi(\iota) - \psi(\iota^*)|$$

for all $\iota, \iota^* \in [0,1)$ and $\forall o \in \Lambda$.

Then, $\mathcal{R}(.,0)$ holds a fixed-point $\Leftrightarrow$ $\mathcal{R}(.,1)$ holds a fixed-point.

Proof. Define the subsequent set

$$\mathfrak{X} = \{\iota \in [0,1] : \jmath = \mathcal{R}(\jmath,\iota) \text{ for some } o \in \Upsilon\}.$$

($\Rightarrow$:) Presume that $\mathcal{R}(.,0)$ has a fixed-point. Then, $\mathfrak{X}$ is nonempty, which signifies that $0 \in \mathfrak{X}$. Our claim is that $\mathfrak{X}$ is both open and closed in $[0,1]$. As a result of utilizing the connectedness aspect, we derive $\mathfrak{X} = [0,1]$. In this case, $\mathcal{R}(.,1)$ allows a fixed-point in Υ.

Initially, the closedness of $\mathfrak{X}$ in $[0,1]$ is verified. Assume that $\{\iota_n\}_{n=1}^{\infty} \subseteq \mathfrak{X}$ with $\iota_n \to \iota \in [0,1]$ as $n \to \infty$. It is essential to point out that $\iota \in \mathfrak{X}$. For this reason, $\iota_n \in \mathfrak{X}$ for $n = 1,2,3,\ldots$; $\jmath_n \in \Upsilon$ exists with $\jmath_n = \mathcal{R}(\jmath_n, \iota_n)$. Also, for $n, m \in \{1,2,3,\ldots\}$, we have

$$\begin{aligned} \wp_\flat(\jmath_n, \jmath_m) &= \wp_\flat\left(\mathcal{R}(\jmath_n,\iota_n), \mathcal{R}(\jmath_m,\iota_m)\right) \\ &\leq \varsigma \wp_\flat\left(\mathcal{R}(\jmath_n,\iota_n), \mathcal{R}(\jmath_n,\iota_m)\right) + \varsigma \wp_\flat\left(\mathcal{R}(\jmath_n,\iota_m), \mathcal{R}(\jmath_m,\iota_m)\right). \end{aligned} \tag{5.1}$$

Also, by considering the function $\mathscr{C}_x$ with $\hbar \in (0,1)$, from (b), we get

$$\begin{aligned}\mathbf{D}\left(\wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota_m\right),\mathcal{R}\left(\jmath_m,\iota_m\right)\right)\right) &\leq \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(E_O\left(\jmath_n,\jmath_m\right)\right)E_O\left(\jmath_n,\jmath_m\right)\right)\right)\\ &= \hbar\mathbf{D}\left(\mathfrak{P}\left(E_O\left(\jmath_n,\jmath_m\right)\right)E_O\left(\jmath_n,\jmath_m\right)\right)\\ &\leq \hbar\mathbf{D}\left(\tfrac{1}{\varsigma}\left[\wp_\flat\left(\jmath_n,\jmath_m\right)+\left|\wp_\flat\left(\jmath_n,\mathcal{R}\left(\jmath_n,\iota_m\right)\right)-\wp_\flat\left(\jmath_m,\mathcal{R}\left(\jmath_m,\iota_m\right)\right)\right|\right]\right)\\ &= \hbar\mathbf{D}\left(\tfrac{1}{\varsigma}\left[\wp_\flat\left(\jmath_n,\jmath_m\right)+\wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota_n\right),\mathcal{R}\left(\jmath_n,\iota_m\right)\right)\right]\right),\end{aligned}$$

which, by ($\mathbf{D}$1), implies that

$$\varsigma\wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota_m\right),\mathcal{R}\left(\jmath_m,\iota_m\right)\right) < \hbar\left[\wp_\flat\left(\jmath_n,\jmath_m\right)+\wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota_n\right),\mathcal{R}\left(\jmath_n,\iota_m\right)\right)\right].$$

So, by using the above inequality and (*iii*), (5.1) becomes

$$\wp_\flat\left(\jmath_n,\jmath_m\right) \leq \left|\psi\left(\iota_n\right)-\psi\left(\iota_m\right)\right| + \hbar\left[\wp_\flat\left(\jmath_n,\jmath_m\right)+\frac{1}{\varsigma}\left|\psi\left(\iota_n\right)-\psi\left(\iota_m\right)\right|\right]$$

such that

$$\wp_\flat\left(\jmath_n,\jmath_m\right) \leq \left(\frac{1+\varsigma}{\varsigma\left(1-\hbar\right)}\right)\left|\psi\left(\iota_n\right)-\psi\left(\iota_m\right)\right|.$$

Then, employing the convergence of $\{\iota_n\}_{n\in\mathbb{N}}$ with $n,m\to\infty$, we procure

$$\lim_{n,m\to\infty}\wp_\flat\left(\jmath_n,\jmath_m\right) = 0.$$

This confirms that $\{\jmath_n\}$ is a $\wp_\flat$-Cauchy sequence in $\mathscr{U}$. $\wp_\flat$-completeness of $(\mathscr{U},\wp_\flat)$ ensures that $\jmath^*\in\Lambda$ exists such that

$$\wp_\flat\left(\jmath^*,\jmath^*\right) = \lim_{n\to\infty}\wp_\flat\left(\jmath^*,\jmath_n\right) = \lim_{n,m\to\infty}\wp_\flat\left(\jmath_n,\jmath_m\right) = 0.$$

Moreover,

$$\begin{aligned}\wp_\flat\left(\jmath_n,\mathcal{R}\left(\jmath^*,\iota\right)\right) &= \wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota_n\right),\mathcal{R}\left(\jmath^*,\iota\right)\right)\\ &\leq \varsigma\wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota_n\right),\mathcal{R}\left(\jmath_n,\iota\right)\right)+\varsigma\wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota\right),\mathcal{R}\left(\jmath^*,\iota\right)\right).\end{aligned} \tag{5.2}$$

Likewise, we have

$$\begin{aligned}\mathbf{D}\left(\wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota\right),\mathcal{R}\left(\jmath^*,\iota\right)\right)\right) &\leq \mathscr{C}\left(\mathbf{D}\left(\mathfrak{P}\left(E_O\left(\jmath_n,\jmath^*\right)\right)E_O\left(\jmath_n,\jmath^*\right)\right)\right)\\ &\leq \hbar\mathbf{D}\left(\tfrac{1}{\varsigma}E_O\left(\jmath_n,\jmath^*\right)\right)\\ &= \hbar\mathbf{D}\left(\tfrac{1}{\varsigma}\left[\wp_\flat\left(\jmath_n,\jmath^*\right)+\left|\wp_\flat\left(\jmath_n,\mathcal{R}\left(\jmath_n,\iota\right)\right)-\wp_\flat\left(\jmath^*,\mathcal{R}\left(\jmath^*,\iota\right)\right)\right|\right]\right)\end{aligned}$$

such that

$$\varsigma\wp_\flat\left(\mathcal{R}\left(\jmath_n,\iota\right),\mathcal{R}\left(\jmath^*,\iota\right)\right) < \hbar\left[\wp_\flat\left(\jmath_n,\jmath^*\right)+\left|\wp_\flat\left(\jmath_n,\mathcal{R}\left(\jmath_n,\iota\right)\right)-\wp_\flat\left(\jmath^*,\mathcal{R}\left(\jmath^*,\iota\right)\right)\right|\right].$$

Then, by (5.2), we obtain

$$\wp_\flat\left(\jmath_n,\mathcal{R}\left(\jmath^*,\iota\right)\right) \leq \left|\psi\left(\iota_n\right)-\psi\left(\iota\right)\right| + \hbar\wp_\flat\left(\jmath_n,\jmath^*\right).$$

By taking the limit as $n\to\infty$ in the above equation, we have that $\lim_{n\to\infty}\wp_\flat\left(\jmath_n,\mathcal{R}\left(\jmath^*,\iota\right)\right)=0$; hence,

$$\wp_\flat\left(\jmath^*,\mathcal{R}\left(\jmath^*,\iota\right)\right) = \lim_{n\to\infty}\wp_\flat\left(\jmath_n,\mathcal{R}\left(\jmath_n,\iota\right)\right) = 0.$$

This means that $\jmath^* = \mathcal{R}(\jmath^*, \iota)$. As (i) is provided, we obtain $\jmath^* \in \Upsilon$. Thus, $\iota \in \mathfrak{X}$ and $\mathfrak{X}$ is closed in $[0, 1]$.

Second, the openness of $\mathfrak{X}$ in $[0, 1]$ will be verified. Let $\iota_0 \in \mathfrak{X}$. Then, $\jmath_0 \in \Upsilon$ exists with $\jmath_0 = \mathcal{R}(\jmath_0, \iota_0)$. Because Υ is open, a non-negative δ exists such that $B_{\wp_b}(\jmath_0, \delta) \subseteq \Upsilon$ in $\mathscr{U}$. Considering that $\varepsilon = \frac{\varsigma(1-\hbar)}{\varsigma+\hbar}(\wp_b(\jmath_0, \jmath_0) + \delta) > 0$ with $\hbar \in [0, 1)$ and $\varsigma \geq 1$, there exists $\vartheta(\varepsilon) > 0$ such that $|\psi(\iota) - \psi(\iota_0)| < \varepsilon$ for all $\iota \in (\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon))$ owing to fact of the continuity of ψ on ι_0.

Let $\iota \in (\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon))$; for

$$p \in \overline{B_{\wp_b}(\jmath_0, \delta)} = \{\jmath \in \mathscr{U} : \wp_b(\jmath, \jmath_0) \leq \wp_b(\jmath_0, \jmath_0) + \delta\},$$

we obtain

$$\begin{aligned} \wp_b(\mathcal{R}(\jmath, \iota), \jmath_0) &= \wp_b(\mathcal{R}(\jmath, \iota), \mathcal{R}(\jmath_0, \iota_0)) \\ &\leq \varsigma\wp_b(\mathcal{R}(\jmath, \iota), \mathcal{R}(\jmath, \iota_0)) + \varsigma\wp_b(\mathcal{R}(\jmath, \iota_0), \mathcal{R}(\jmath_0, \iota_0)). \end{aligned} \tag{5.3}$$

Furthermore,

$$\begin{aligned} \mathbf{D}(\wp_b(\mathcal{R}(\jmath, \iota_0), \mathcal{R}(\jmath_0, \iota_0))) &\leq \mathscr{C}(\mathbf{D}(\mathfrak{P}(E_O(\jmath, \jmath_0))E_O(\jmath, \jmath_0))) \\ &\leq \hbar\mathbf{D}\left(\frac{1}{\varsigma}E_O(\jmath, \jmath_0)\right) \\ &= \hbar\mathbf{D}\left(\frac{1}{\varsigma}\left[\wp_b(\jmath, \jmath_0) + |\wp_b(\jmath, \mathcal{R}(\jmath, \iota_0)) - \wp_b(\jmath_0, \mathcal{R}(\jmath_0, \iota_0))|\right]\right), \end{aligned}$$

which implies that

$$\varsigma\wp_b(\mathcal{R}(\jmath, \iota_0), \mathcal{R}(\jmath_0, \iota_0)) < \hbar\left[\wp_b(\jmath, \jmath_0) + \wp_b(\mathcal{R}(\jmath, \iota), \mathcal{R}(\jmath, \iota_0))\right].$$

Finally, taking the above inequalities into account, from (5.3), we gain

$$\begin{aligned} \wp_b(\mathcal{R}(\jmath, \iota), \jmath_0) &\leq |\psi(\iota) - \psi(\iota_0)| + \hbar\left[\wp_b(\jmath, \jmath_0) + \frac{1}{\varsigma}|\psi(\iota) - \psi(\iota_0)|\right] \\ &\leq \left(1 + \frac{\hbar}{\varsigma}\right)|\psi(\iota) - \psi(\iota_0)| + \hbar(\wp_b(\jmath_0, \jmath_0) + \delta) \\ &\leq \left(1 + \frac{\hbar}{\varsigma}\right)\varepsilon + \hbar(\wp_b(\jmath_0, \jmath_0) + \delta) \\ &\leq \wp_b(\jmath_0, \jmath_0) + \delta, \end{aligned}$$

and $\mathcal{R}(\jmath, \iota) \in \overline{B_{\wp_b}(\jmath_0, \delta)}$. Therefore,

$$\mathcal{R}(., \iota) : \overline{B_{\wp_b}(\jmath_0, \delta)} \to \overline{B_{\wp_b}(\jmath_0, \delta)}$$

holds for every fixed $\iota \in (\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon))$. We can now apply Corollary 4.0.2, contemplating the function $\mathscr{C}$ as $\mathscr{C}_x$; then, $\mathcal{R}(., \iota)$ has a fixed-point in Λ. However, this point belongs to Υ, as (i) is true. Therefore, $(\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon)) \subseteq \mathfrak{X}$, and we induce that $\mathfrak{X}$ is open in $[0, 1]$. □

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

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